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On Value Efficiency

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ABSTRACT

Data Envelopment Analysis (DEA) is a linear programming method which evaluates the efficiency of Decision Making Units (DMU)s with multiple inputs and outputs. DEA is non-parametric method which estimates production frontier in economics and Operational Research (OR). This study is based on searching the Most Preferred Solution (MPS) that is the combination of inputs and outputs of DMUs which introduces by Decision Maker (DM) and is DEA efficient DMU.

After pointing out MPS, it is assumed that this MPS optimizes value function which is unknown. So, the countor of value function at MPS should be approximated. This approximation will be done by its tangent cone at MPS. Then, it is possible to evaluate Value Efficiency (VE) scores for each DMU.

In last chapter of this study, the value efficiency is developed by introducing two refinement. First, The upper and lower bound for VE is introduced. Second, a more precise way of evaluating VE is proposed.

Contents

1. DATA ENVELOPMENT ANALYSIS

1.1 Introduction

Data Envelopment Analysis (DEA) is an approach for comparing the efficiency of organization units. Variety of applications of DEA used in evaluating the performance of many kinds of entities engaged in many activities, many different contexts and in many different countries.
DEA has also been used to supply new insights into activities that have previously been evaluated by other methods. In DEA, the units under study are called Decision Making Units (DMU).Generically, a DMU is regarded as entity responsible for converting inputs to outputs and whose performance is to be evaluated. In marginal applications, DMUs may include banks, department stores and extended to car-makers, hospitals, etc. The conventional DEA model measures the performance of a DMU in terms of efficiency. Data Envelopment Analysis (DEA) is used for evaluation of relative efficiency for Decision Making Units with various inputs and outputs. Relative phrase is due to the comparing of units with each other; therefore, the obtained efficiency is relative not absolute. DEA considers a set of decision making units in which any of DMUs may consumes different special inputs for producing a collection of outputs.
With regard to production flow, a DMU is considered as black box. This black box consumes input to produce output without considering the role of inner part.
DEA models may be input-oriented or output-oriented or input/output-oriented which is according to the analyser idea. Those models with input-oriented may specify the decreasing amount of inputs which make a situation for inefficient DMU become efficient one. Similarly, the models with output-oriented may specify the increase amount of outputs in order to have an efficient DMU. An input/output-oriented model may specify at most reduction amount in inputs while maximize the amount of outputs.
DEA builds an efficient frontier in accordance with the best observed function then may evaluate the efficiency of DMUs in compliance with this frontier. Those DMUs which are located on efficient frontier are called relative efficient.
The efficiency of a DMU which is not on the the frontier would be evaluated

against a positive linear composition of efficient DMUs. This DMU is not efficient and all efficient DMUs with positive weights in linear composition may create an efficient reference set for a non- efficient DMU.

DEA model is formulated on a linear problem basis for a special DMU. Such a problem will be solved for each DMU.

Definition 1.1.1: **Production:**

Production means any direct changes for increasing the suitability of goods. Product (Output) is the result of production activity resulted from any changes. Production resources (Input)are the materials and items used for obtaining a product.

Definition 1.1.2: **Production function:**

Production function is a relation between the used production resources (Inputs) and goods (Outputs)of producing items, in a period of time, without any consideration of prices. Following states production function relation:

$$Y = f(U, V)$$

where:

V stands for unknown factors. The vector U may include controllable and uncontrollable elements. $Y = (y_1, ..., y_s)$ is called output vector which is provided by input vectors $U$and V.

Definition 1.1.3: **Production Possibility Frontier-PPF:**

Production Possibility Frontier (PPF) is a curve which depicting all maximum output possibilities for two or more inputs. The PPF assumes that all inputs are used efficiently.

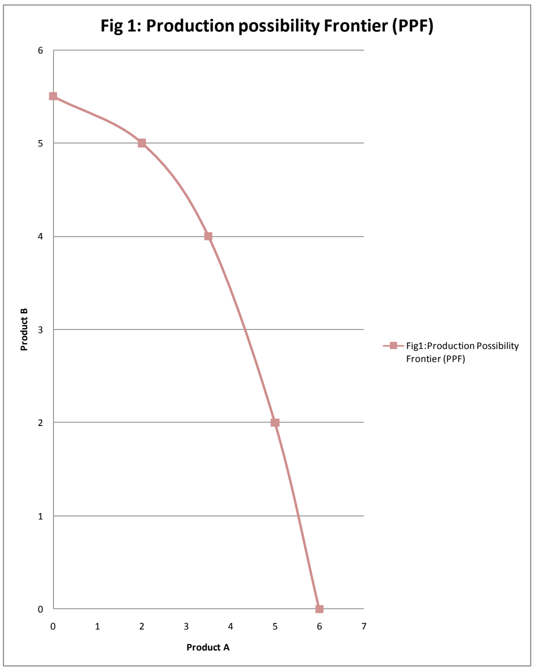

Fig. 1.1: Production possibility frontier

As indicated on the chart above, points which are located in the production possibility frontier represent most efficient points.

1.2 Different types of production function

A production function may have different mathematical forms. For example, it would be explained as a linear and/or as function of production factors. Two basic forms of these functions with more usage are functions with variable coefficients and functions with constant coefficients.
A production function with variable coefficient is a function for obtaining a specified amount of product through different production factors. In this method we can change the production coefficient in a specific period of time. In mentioned method, it is possible to replace widely any production factors. Following is one of the flexible production function:

$$Y = A\prod_{i=1}^{m} x_i^{A_i}, A_i > 0, i = 1, ..., m \tag{1.2.1}$$

A production function with constant coefficients may not accept any substitution of factors.

1.2.1 Production curves

Production curves include three types such as: "Total production curves", "Average production curves" and "Marginal production curves". Production curves can be gained from drawing the production functions.
Total production function: This is the total product obtained through applying of production. In other word $y = f(x)$ is a sign of total production function.
Average production function: Average production function means a radial gradient that may connect all coordinates of origin to different point of production curve that means $\frac{y}{x}$ is the sign of average production function.
Marginal production function: Marginal production function means the additional amount of obtained output resulted from one unit increase of inputs provided that other resources are fixed. In other words, the marginal production function is the same $f(x)$ in x point That means:

$$\frac{dx}{dy}$$

For more information see ([1])

1.2.2 Production Possibility set

Production Possibility Set is a set of all inputs outputs that may show all production amounts (outputs) with respect of different resources (inputs)

and/or all possible compositions of inputs and outputs.
The production possibility set of n derision making unit ($DMU_j, j = 1, ..., n$) with m different inputs $x_1, ...x_m$ and s outputs $y_1, ..., y_s$ that may be shown with T as follows:

T={(X,Y): vector Y would be produced by the vector X}

We may accept different principles for providing a production possibility set. These include a structural base for explaining different DEA models. Following are these specified principles:

1. ***Non-empty axiom T:***

   This is also known as "including the observation". This may explain that all observation belong to T. In other words:

$$\forall j | j = 1, ..., n, (x_j, y_j) \in T$$

2. ***Possibility axiom:***

   A) If we have $(x, y) \in T$ and $x^- > x$, then we have $(x^-, y) \in T$. It means that if we can produce y with x amount of inputs, then with an amount more than x for example x^-, it is possible to produce y.

   B)If we have $(x, y) \in T$ and $y^- < y$,then we have $(x, y^-) \in T$. It means that if x amount of input is use to produce y amount of output, then the same amount of input can produce y^- amount of output.

   This property is named as "possibility" or "Monotonicity" as well. In other words, it is possible to write it as follows:

$$\forall (x, y) \forall x^- \forall y^- [(x, y) \in T, x^- \geq x, y^- \leq y \Rightarrow (x^-, y^-) \in T]$$

3. ***Unbounded ray axiom (constant return to scale):***

   If we have $(x, y) \in T$, then for each $\lambda \geq 0$, we will have:

   $$(\lambda x, \lambda y) \in T$$

   The property unbounded ray is also known as "Constant Returns-to-Scale". In other word, if X could produce Y, then any multiple of X could produce the same multiple of Y. It means any increase (decrease) in input may be resulted in equal increase (decrease)of output.

4. ***convexity axiom:***

   This principle explains that if we have $(r = 1, ..., s)$, $(x_r, y_r) \in T$, then any convex combination of (x_r, y_r) is also belongs to T. In other words, if

   $$(x_r, y_r) \in T, \lambda_r \geq 0, (r = 1, ..., s), \sum_{r=1}^{s} \lambda_r = 1$$

   then

   $$(\sum_{r=1}^{s} \lambda_r x_r, \sum_{r=1}^{s} \lambda_r y_r) \in T$$

   It means that T is convex set. Convexity principle explains that if x_r could produce y_r where $r = 1, ..., s$, then the input of $\sum_{r=1}^{s} \lambda_r x_r$ could produce the output of $\sum_{r=1}^{s} \lambda_r y_r$ in which we have $\sum_{r=1}^{s} \lambda_r = 1$ and $\lambda_r \geq 0$ for $r = 1, ..., s$.

5. ***Minimality axiom:***
   T is the smallest set which satisfies first, second, third and fourth principles.

The production possibility set which is satisfying in principle 1-5 is shown by T_B. Set T_B is defined as follow:

$$T_B = \left\{ (x, y)|x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, (j = 1, ..., n) \right\} \quad (1.2.2)$$

The production possibility set (1.2.2) is named as production possibility set of BCC (Banker, Charnes and Cooper,1984) model.

Theorem 1.2.1
T_B is the the minimal set satisfying axiom 1-4.
Proof: Let, T_B be a set which satisfying in axiom 1-4. now we are going to prove $T_B \subseteq T$. Let $(x, y) \in T_B$, Then there is $\bar{\lambda} = (\bar{\lambda_1}, \ldots, \bar{\lambda_n}) \geq 0$ such that:

$$x \geq \sum_{j=1}^{n} \bar{\lambda}_j x_j$$

$$y \leq \sum_{j=1}^{n} \bar{\lambda}_j y_j$$

Then we put: $\lambda_j^* = \frac{\bar{\lambda}}{\sum_{j=1}^{n} \lambda_j} = \frac{\bar{\lambda}_j}{d}, j = 1, \ldots, n$. As $\lambda_j \geq 0$ then $\sum_{j=1}^{n} \bar{\lambda}_j \geq 0$ and also $d \geq 0$. It can be say that $\lambda_j^* \geq 0$ and $\sum_{j=1}^{n} \lambda_j^* = 1, (j = 1, \ldots, n)$. As T is satisfying in axiom 1-4 :

$$(x_j, y_j) \in T, j = 1, \ldots, n$$

and as T is satisfying in convexity axiom, then we have:

$$(\sum_{j=1}^{n} \lambda_j^* x_j, \sum_{j=1}^{n} \lambda_j^* y_j) \in T$$

Considering that T is satisfying in ray unbound axiom:

$$(d \sum_{j=1}^{n} \lambda_j^* x_j, d \sum_{j=1}^{n} \lambda_j^* y_j) \in T$$

i.e:

$$\sum_{j=1}^{n} \bar{\lambda}_j x_j, \sum_{j=1}^{n} \bar{\lambda}_j y_j \in T$$

As $x \geq \sum_{j=1}^{n} \bar{\lambda}_j x_j$ and $y \leq \sum_{j=1}^{n} \bar{\lambda}_j y_j$ and satisfying in possibility axiom then $(x, y) \in T$ and that is what we want.

We may have the following production possibility set which is known as production possibility set of CCR (Charnes,Cooper and Rhodes,1978) model.

$$T_C = \left\{ (x,y)|x \geq \sum_{j=1}^{n} \lambda_j x_j, y \leq \sum_{j=1}^{n} \lambda_j y_j, \lambda_j \geq 0, (j = 1, ..., n) \right\} \quad (1.2.3)$$

Note! CCR and BCC models are discussed later in this chapter.

1.3 Efficiency

Following we shortly review two kinds of efficiencies.

1.3.1 Economic Efficiency

The expression economic efficiency in economic field is used, when one is supposed to maximize the production of goods and services. In economic when comparing two units, we say that one economic system is more efficient than the others when it can provides more goods and services without using so much resources.
A unit is called economically efficient, if:
-We can make a better efficiency score for one unit only by making the efficiency of another unit worse. (Pareto efficiency)
-The additional output can be achieve only by increasing the amount of inputs.
Not these two definitions are exactly equivalent, but they mean that we can evaluate the efficiency of producing method according to the obtained value of products.

1.3.2 Technical Efficiency

The effectiveness while the set of inputs consume to produce a set of outputs is called Technical Efficiency. One unit under evaluation is called technically efficient if it can produce the maximum output with using the minimum amount of inputs.
there are many difference in Technical efficiency and economic efficiency.
Economic efficiency is mostly involves with the prices related to the factors of production. Technical efficiency is said that there maybe some units which are technically efficient but not economically efficient. Technical efficiency is defined when we do not have any possibility to increase the output without increasing the input. In fact the Economic efficiency have been defined when the production cost of an output is as low as possible.
The prerequisite for allocative or economic efficiency is Technical efficiency.

1.3.3 Measuring methods of Technical efficiency

There are different methods for measuring of technical efficiency of units. These methods may generally divided into two groups as: "Parametric Methods" and "Non-parametric Methods".
A. Parametric methods: Production function is estimated by the use of different statistical and economic methods in this item. Then it is necessary to determine efficiency by applying this function. One of the most well known production functions in micro economic is Cobb-Dauglas with a general form as follow:

$$y = A_0 \prod_{i=1}^{m} x_i^{A_i}, \sum_{i=1}^{m} A_i = 1, i = 1, ..., m \tag{1.3.1}$$

$A_0, A_1, ..., A_m$ are different parameters which should be determined where $x_1, ..., x_m$ are inputs and y is output. One of the greatest defects of parameter methods is their applicable situation only for single-output case. It is impossible to apply them for multi-output case while in real world we are involved with multi amounts functions.
B. Non parametric methods: In these methods there is no need to do any estimation of production function. Data Envelopment Analysis is a non-parametric method that may evaluate relative efficiency of units when comparing with each other. There is no need to recognition of production function from with any further limitation for the number of inputs and outputs.

1.4 Data Envelopment Analysis: preliminaries

Considering all definitions which was defined in previous sections, here the pre-requisite concepts of Data Envelopment Analysis (DEA) are defined:

Definition 1.4.1: **DMU**

The under-evaluation unit in DEA is named as Decision Making Unit (DMU). DMUs may have different forms in different branches. For example, a DMU in marginal application is bank, hospital, library, and/or school while in field it can be an air plane or even its parts (like motor) under the title of DMU.

Data Envelopment Analysis is a method that may calculate the efficiency score of different considered units. In other words, this method will specify which unit has a better function in comparison with other units. As a result it is possible to specify estimated (not absolute) efficiency.

Definition 1.4.2: **Input**

Input is a factor that in case of its increase and maintenance of all other factors we will have a reduction in efficiency and by its reduction and keeping all other factors fix, we will have an increase in efficiency.

Definition 1.4.3: **Output**

Output is a factor that in a case of its increase and maintenance of all other factors, we will have a increasing in efficiency and by its increase and keeping all other factors fix, we will have an increase in efficiency.

Definition 1.4.4: **Dominate vector**

For two vectors V and V^-, V dominates V^-, if:

$$V \geq V^-, V \neq V^-$$

Definition 1.4.5: **Pure efficiency**

Assume that a DMU has m inputs and s outputs and $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_s)$ show the input and output vectors, respectively. Then pure efficiency of this unit is as follow:

$$\frac{u_1 y_1 + u_2 y_2, ..., u_s y_s}{v_1 x_1 + v_2 x_2 + ... + v_m x_m} \tag{1.4.1}$$

where u_r, $r = (1, 2, ..., s)$ and v_i, $i = (1, 2, ..., m)$ include weights related to outputs and inputs, respectively.

Assume that n decision making units $(DMU_1, DMU_2, ..., DMU_n)$ are under evaluation. The inputs and outputs of these DMUs would be selected in a way that:

1. Inputs and outputs are semi-positive data.

2. Smaller amounts of inputs and larger amounts of outputs are preferred.

3. Data and DMUs should be selected through the idea of manager and/or evaluator.

4. It is possible to have non-equal measuring units for different inputs and outputs.

Assume that we have selected m inputs and s outputs according to the above-mentioned rules. In addition, assume that for $DMU_j, (j = 1, \dots, n)$ under evaluation,$(x_{1j}, x_{2j}, ..., x_{mj})$ is vector of inputs and $(y_{1j}, y_{2j}, ..., y_{sj})$ the vector of outputs.

Following is the matrix of input data X and the matrix of output data Y:

$$X = \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{x_{mn}} \end{bmatrix}, Y = \begin{bmatrix} y_{11} & \cdots & y_{1n} \\ \vdots & \ddots & \vdots \\ y_{s1} & \cdots & x_{y_{sn}} \end{bmatrix} \tag{1.4.2}$$

where X is a $(m \times n)$ matrix and y is a $(s \times n)$ matrix.

Definition 1.4.6: **Dominance**

DMU_k dominates DMU_h if:

$$-x_k \geq -x_h, y_k \geq y_h \tag{1.4.3}$$

and inequalities (1.4.3) holds for at least one element.

Definition 1.4.7: **Virtual Inputs and Outputs**

Assume we have n decision making units with $(x_{1j}, x_{2j}, ..., x_{mj}), (j = 1, 2, ..., n)$ as input vector and $(y_{1j}, y_{2j}, ..., y_{sj}), (j = 1, 2, ..., n)$ as output vector and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is a non-negative vector. Following describes virtual inputs and output:

Virtual input:$\sum_{j=1}^{n} x_{ij}\lambda_j$, $(i = 1, 2, ..., n)$

Virtual output:$\sum_{j=1}^{n} y_{rj}\lambda_j$, $(r = 1, 2, ..., s)$

A virtual DMU is a unit with virtual input and output.

Definition 1.4.8: **Relatively efficient**

Assume that we have n number of Decision Making Units (DMU) and DMU_O,$O \in (1, 2, ..., n)$ is called relatively efficient, if and only if there is no more virtual and real DMU dominating DMU_O.

1.5 Basic DEA models

Measuring the efficiency was always important for researchers due to its importance in evaluation of operation in a company and/or organization. In

1957, Farrell started to measure the efficiency for a production unit utilizing of traditional method like measuring of efficiency in engineering discussions. Farrell considered the input and output while measuring the efficiency.
Farrell used his method for estimation of efficiency in agricultural section of USA in comparison with other countries. By the way, he was not successful in presenting a method including various inputs and outputs. Charnes, Cooper and Rhodes developed the opinion of Farrell and presented a model for measuring the efficiency of different inputs and outputs of a unit with making comparison with other units (CCR model). It was named "Data Envelopment Analysis" and was applied for the first time in doctrine of "Edward Rhodes" and by opinion of "Cooper" under the title of "Evaluation of academic progress of students at national school of U.S.A" in 1976 at Carnegie university and presented in another essay under the title of "measuring the efficiency of decision making units" in 1978.
The philosophy of data envelopment analysis means creation of virtual unit for comparing of considered unit and measuring its efficiency. A virtual unit is a combination DMU. Those $DMUs$ on so called frontier would be named as an efficient DMU. If there is a DMU which is not on the frontier, it is possible to lead them towards the frontier with different methods such as:

-Decreasing in inputs
-Increasing in outputs

Those methods for evaluation of efficiency of DMU by reducing the inputs are named as Input Oriented models and those by increasing into outputs are named as Output Oriented models. Also, there are different models which may evaluate $DMUs$ by combining the above-mentioned models. Additive model is an instant of these models.
BCC and CCR models (will defined later in this chapter) include in basic DEA models for evaluating of data either through the input or output oriented. The considered DMU in such models would be drawn on efficiency frontier by decreasing inputs to θx_o or increasing outputs to ηy_o.
Sometimes it is possible to decrease the input and/or increase the output of a DMU even after drawing a DMU on efficient frontier. Then it is possible to say that DMU is on weak frontier and could obtain the efficiency of considered unit by introducing the slack variables.In the next parts we will explained it more detailed.

1.5.1 Input-oriented CCR model

Assume that we have we have n DMUs that $DMU_j, (j = 1, \ldots, n)$ will uses $x_{1j}, ..., x_{mj}$,$(j = 1, ..., n)$ as inputs to produce $y_{1j}, ..., y_{sj}$, $(j = 1, ..., n)$ outputs. The DEA optimization model can be solved n times, one time for each DMU. The allocated weight to i^{th} input is shown by v_i,$(i = 1, 2, ..., m)$ and for r^th outputs is shown by u_r,$(r = 1, 2, ..., s)$, then the fractional form of CCR model would be as follow:

$$\begin{aligned} \text{Maximize} \quad & \theta = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_i o} \leq 1 \\ \text{subject to} \quad & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_i j} \leq 1, j = 1, \ldots, n \\ & u_r \geq 0, r = 1, \ldots, s \\ & v_i \geq 0, i = 1, \ldots, m. \end{aligned} \tag{1.5.1}$$

The above-mentioned limitations show that the virtual input rate to virtual output for DMU should not be more than 1.
It is possible to have very large amount of u_r and/or very small amount of v_i.
For prevention of this problem, we should consider the above mentioned limitations somehow smaller or equal to 1. This is necessary to mention that it is possible to put any other digits such as k instead of 1 in the mentioned model.
By manipulation of above mentioned model, we have linear form of CCR model as follows. This model is named as CCR model :

$$\begin{aligned} & \text{Maximize} \quad z = \sum_{r=1}^{s} u_r y_{ro} \\ & \text{subject to} \\ & \sum_{i=1}^{m} v_i x_{io} = 1 \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1, \ldots, n \\ & v_i \geq 0, i = 1, \ldots, m \\ & u_r \geq 0, r = 1, \ldots, s. \end{aligned} \tag{1.5.2}$$

The above model is called Linear programming (LP) version of previous problem. If we consider θ and λ respectively as dual proportional variables corresponding to the first and second constraints, by writing the dual form of previous model, the envelopment form of CCR model would be introduced as follow:

$$\begin{aligned} \text{Minimize} \quad & \theta \\ \text{subject to} \quad & \theta x_O - X\lambda \geq 0 \\ & Y\lambda \geq y_O \\ & \lambda \geq 0. \end{aligned} \tag{1.5.3}$$

Needless to say that, The production possibility set for CCR model (which is defined later in this chapter), is as follow:

$$T_C = \{(x, y)|x \geq X\lambda, y \leq Y\lambda, \lambda \geq 0\}$$

Model (1.5.3) is called dual for the of LP form and will be shown by (DLP).

Definition 1.5.1: **CCR efficient DMU**

DMU_O is called as a CCR efficient unit if $z* = 1$ and there is at least one optimal solution $(u*, v*)$ while $u* > 0$ and $v* > 0$. Otherwise, DMU_O is CCR non-efficient.

We are looking for a DMU in PPS in (DLP) model which may produce the maximum output of y_O and simultaneously reduce the x_O of input radially. Therefore when $\theta^* < 1$ then $(X\lambda, Y\lambda)$ would act better than (x_O, y_O). It is possible to describe slack variable of input and output be $s^- \in \mathbb{R}^m$ and $s^+ \in \mathbb{R}^s$ respectively which is as follow in (DLP).

$$s^- = \theta x_O - X\lambda \geq 0 \tag{1.5.4}$$

$$s^+ = Y\lambda - y_O \geq 0 \tag{1.5.5}$$

The multiple form of CCR model can be solve in two phases:
Phase I: The DLP problem will solve and the solution will be found.
Phase II: The following LP problem with variables λ, s^-, s^+ will be solved:

$$\begin{aligned}
&\text{Maximize} \quad w = es^- + es^+ \\
&\text{subject to} \\
&s^- = \theta x_O - X\lambda, \\
&s^+ = Y\lambda - y_O, \\
&s^- \geq 0, \\
&s^+ \geq 0, \\
&\lambda \geq 0.
\end{aligned} \tag{1.5.6}$$

While:

$$es^- = \sum_{i=1}^{m} s_i^-$$

$$es^{+} = \sum_{r=1}^{m} s_r^{-}$$

.

Definition 1.5.2: **Pareto efficiency**

DMU_O is called Pareto efficient, if in all optimal solution of envelopment model, $\theta^* = 1$ and all slack variable in solution are Zero.

1.5.2 Farrell efficiency

DMU_O is Farrell efficient if and only if in input-oriented envelopment of CCR model $\theta^* = 1$.

Definition 1.5.3: **Weak efficiency**

If $\theta^* = 1$ and some of the slack variables are not zero while solving the Multiple of CCR model, Then we say that DMU_O is weak efficient.

Definition 1.5.4: **Reference Set**

All sets of $DMUs$, in evaluating DMU_O by envelopment form of CCR model, which in one of their optimal solutions λ is not zero called Reference set. In the other word, the reference set which is shown by E_O is defined as follow:

$$E_O = \{j|\lambda_j > 0\}, j \in 1, \ldots, n$$

Definition 1.5.5: **Extreme Efficient**

A DMU_O is an extreme efficient if and only if $E_O = \{DMU_O\}$. it means that a DMU is an extreme efficient if it is its own reference point, and it is not extreme efficient if it has a meaning of Pareto efficient and its reference set has at least two members.

1.5.3 Output oriented CCR model

The output oriented CCR model will be introduced as:

$$\begin{aligned}
&\text{Minimize} \quad px_0 \\
&\text{subject to} \\
&qy_0 = 1, \\
&-pX + qY \leq 0, \\
&q \geq 0, \\
&p \geq 0.
\end{aligned} \tag{1.5.7}$$

Similar to the previous part, here the envelopment form of output-oriented CCR model is also introduced, the model is as follow:

$$\begin{aligned}
&\text{Maximize} \quad \phi \\
&\text{subject to} \\
&x_0 - X\mu \geq 0, \\
&\phi y_0 - Y\mu \leq 0, \\
&\mu \geq 0.
\end{aligned} \tag{1.5.8}$$

1.5.4 BCC model (Input and output oriented)

Banker, Charnes and Cooper published a paper in 1984 [2] in which the production possibility set of BCC model has been described as follow:

$$T_B = \{(x, y) | x \geq X\lambda, y \leq Y\lambda, \mathbf{e}\lambda = 1, \lambda \geq 0\}$$

Where $x \in \mathbb{R}^{m \times n}$ and $y \in \mathbb{R}^{s \times n}$ are the data sets and $\lambda \in \mathbb{R}^n$. In addition $\mathbf{e}$ is a lineal vector in which all components are equal with all parameters are equal to 1. It is clear that the difference between PPS in CCR model and BCC model is in the condition $\mathbf{e}\lambda = \sum_{j=1}^{n} \lambda_j = 1$. To get more in detail, consider figure (1.2).

Figure (1.2) shows 4 decision making units ($DMUs$) A,B,C and D with one input and one output. straight line stands for efficiency frontier of CCR model and dark broken line is that of BCC efficiency model. It is obvious that DMU_B is the only DMU in evaluating the unit with CCR model which is efficient, while in evaluating with BCC model DMU_A, DMU_B and

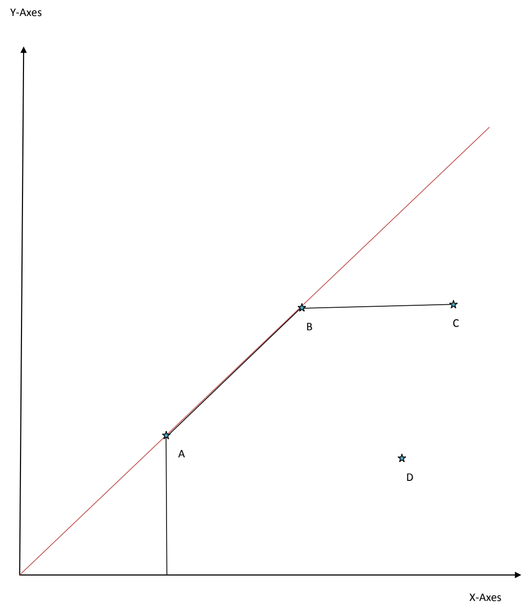

Fig. 1.2: CCR and BCC frontier

DMU_C are efficient $DMUs$. In general condition, the efficiency of CCR is not greater than that of BCC. The envelopment form of BCC model in input orientation when evaluating DMU_O is as follow:

$$\begin{aligned} &\text{Minimize} \quad \theta \\ &\text{subject to} \\ &\theta x_O \geq X\lambda, \\ &Y\lambda \geq y_O, \\ &\mathbf{e}\lambda = 1, \\ &\lambda \geq 0. \end{aligned} \tag{1.5.9}$$

where θ is a scaler.
The dual form of the above-mentioned model is named as multiplier form

of BCC model and formulate as follow:

$$
\begin{aligned}
&\text{Maximize} \quad z = uy_O - u_0 \\
&\text{subject to} \\
&vx_O = 1, \\
&-vX + uY - u_0\mathbf{e} \leq 0, \\
&v \geq 0, \\
&u \geq 0, \\
&u_0 free,
\end{aligned}
\tag{1.5.10}
$$

where u_0 and z are scaler and the relevant fractional form of BCC model is formulate as follow:

$$
\begin{aligned}
&\text{Maximize} \quad \frac{uy_O - u_0}{vx_O} \\
&\text{subject to} \\
&\frac{uy_j - u_0}{vx_j} \leq 1, \\
&v \geq 0, \\
&u \geq 0, \\
&u_0 free,
\end{aligned}
\tag{1.5.11}
$$

The BCC model like CCR model is solve in two phases.

Definition 1.5.6: **BCC efficiency**

A DMU is called BCC efficient, if in optimal solution of BCC model, the following results are achieved:

$$\theta^* = 1, s^{+*} = s^{-*} = 0$$

. Otherwise it is called as BCC non-efficient.

As mentioned before, the reference set according to solution λ^* is defined as follow:

$$E_O = \{j|\lambda^* > 0, j \in 1, 2, \ldots, n\}$$

After evaluating DMU_O by envelopment BCC model, The projection of DMU_O on efficiency frontier will be defined as follow:

$$X = \theta^* x_O - s^{-*} = \textstyle\sum_{j \in E_O} \lambda_j^* x_j \tag{1.5.12}$$

$$Y = y_O + s^{+*} = \textstyle\sum_{j \in E_O} \lambda_j^* y_j \tag{1.5.13}$$

The envelopment output-oriented BCC model is as follow:

$$\begin{aligned}
&\text{Maximize} \quad \phi \\
&\text{subject to} \\
&X\lambda - x_O \leq 0, \\
&\phi y_O - Y\lambda \leq 0, \\
&\mathbf{e}\lambda = 1, \\
&\lambda \geq 0.
\end{aligned} \tag{1.5.14}$$

The dual of above-mentioned model as follow:

$$\begin{aligned}
&\text{Minimize} \quad u_0 + vx_0 \\
&\text{subject to} \\
&uy_0 = 1, \\
&vX - uY + u_0 \geq 0, \\
&\mathbf{e}\lambda = 1, \\
&u \geq 0, \\
&v \geq 0.
\end{aligned} \tag{1.5.15}$$

It is very important to keep in mind that the solution of input/output oriented CCR model and solution of input/output oriented BCC model may not be equivalent.

1.6 Modification of DEA models

The v_i and u_r are non negative variables, therefore it is possible to be zero. For instant if the solution of a CCR model with two inputs and one output is as follows:

$$u_1^* = 2, v_1^* = 0, v_2^* = \frac{3}{2}$$

Then, the presence of $v_1^* = 0$ may cause any lack of attention to first input for determining of efficiency and to be omitted in calculations. Therefore one year after publishing Chernes, Cooper and Rhoods essay (1978) that means 1979 [3], they proposed to consider decision model (u_r, v_i) greater than very small positive amount ε. $\varepsilon > 0$ is a non-Archimedes number. $\varepsilon > 0$ is a real and very small positive number. As a result, by applying the above mentioned modification in envelopment form and multiple form of CCR and BCC models in input oriented we have the new models. For instant, The modified form fo CCR model is as follow:

$$
\begin{aligned}
&\text{Maximize} \quad \theta = \sum_{r=1}^{s} u_r y_{ro} \\
&\text{subject to} \\
&\sum_{i=1}^{m} v_i xio = 1, \\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{r=1}^{m} v_i x_{ij} \geq 0, j = 1, \ldots, n, \\
&u_r \geq \varepsilon, \\
&v_i \geq \varepsilon.
\end{aligned}
\tag{1.6.1}
$$

According to this idea, all of the DEA models can be updated using this idea. (For more information about DEA, please see [4])

2. VALUE EFFICIENCY ANALYSIS

2.1 ***Introduction***

Before getting through Value Efficiency concept, we need to introduce some definitions first. These notions are classified in following order.

Definition 2.1.1: **PseudoConcave function**

Consider differentiable real value function f and also is defined on X which is a convex open set in finite-dimensional Euclidean-space $\mathbb{R}^n$. This function is said to be pseudoconcave if:$\forall x, y$,

$$f(y) > f(x) \Rightarrow \nabla f(x)(y - x) > 0$$

where:

$$\nabla f(x) = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n})$$

(see [5])

2.1.1 **Multi Objective Linear Programming (MOLP)**

As mentioned before, from all available alternatives the process of selecting the best course of action is called decision making and it can normally done by Decision Maker(DM).
In real world problems, the intensity of standards to discuss and make the decision about an another case is so wide. This normally happens when it is more desirable for the DM to achieve more than one objective while she/he is

trying to satisfy the constraints. The Multi Objective Linear Programming (MOLP) model can be formulated as:

$$\begin{aligned}
&\text{Maximize}(or Minimize) \quad \{f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_k(\mathbf{x})\}\\
&\text{subject to}\\
&\mathbf{x} \in X.
\end{aligned} \tag{2.1.1}$$

where, $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ stands for n-dimensional vector of decision variable.

A MOLP problem may present as follow:

$$\begin{aligned}
&\text{Maximize} \quad \begin{array}{r} C_{11}x_1 + C_{12}x_2 + \ldots + C_1nx_n \\ C_{21}x_1 + C_{22}x_2 + \ldots + C_{2n}x_n \\ \vdots \\ C_{k1}x_1 + C_{k2}x_2 + \ldots + C_{kn}x_n \end{array}\\
&\text{subject to}\\
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1\\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2\\
&\vdots\\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m\\
&x_j \geq 0, \forall j = 1, 2, \ldots, n
\end{aligned} \tag{2.1.2}$$

The point $x^* \in X$ can be defined as an efficient solution for problem (2.1.2), if it does not exist an alternative feasible solution $x \in X$ for which $i = 1, \ldots, k, f_i^*(x^*) \leq f_i(x)$. In this case, point $x^* \in X$ can be introduce as an efficient solution and we can write $f_i(x) < f_i^*(x^*)$. for more information you can look at [6] and [7].

2.1.2 The similarity in structure between MOLP and General DEA models

Let, there exists n number of DMUs, where each DMU consumes m inputs to produce p outputs. Also, consider $X \in \mathbb{R}_+^{m\times n}$ and $Y \in \mathbb{R}_+^{p\times n}$ be an input output matrix, respectively. An input/output vector is denoted by:

$u = \left[\begin{smallmatrix} y \\ -x \end{smallmatrix}\right]$ and $U = \left[\begin{smallmatrix} Y \\ -X \end{smallmatrix}\right]$, when it is not necessary to emphasize different role of inputs or outputs then vector U can be used as an input/output vector.

Definition 2.1.2:

Consider $T = \{u|u = U\lambda, \lambda \in \Lambda\}$ and $\Lambda = \left\{\lambda|\lambda \in \mathbb{R}^n_+ and A\lambda \leq b\right\}$ where, $A \in \mathbb{R}^{k\times n}$ is a full rank matrix and $b \in \mathbb{R}^K$. T is called feasible region which is a set of value vector $u \in \mathbb{R}^{m+p}$. [11]

Needless to say that, all efficient DMUs lie on the efficient frontier which is defined as a subset of the points of set T, which satisfy in the efficiency conditions:
cond1:$u^* \in T$ is an **Efficient** point iff $\nexists u \in T$ such that $u \geq u^*$.
cond2:$u^* \in T$ is a **Weakly efficient** point iff $\nexists u \in T$ such that $u > u^*$.
Here we remind that traditionally in DEA, the efficiency of DMU was calculated by:

$$\begin{aligned} &\text{Maximize} \quad \text{(Output)} \\ &\text{subject to} \\ &\text{(Given input level).} \end{aligned} \tag{2.1.3}$$

Or

$$\begin{aligned} &\text{Minimize} \quad \text{(Input)} \\ &\text{subject to} \\ &\text{(Given output level).} \end{aligned} \tag{2.1.4}$$

Then in 1985 a model considering both input minimization and output maximization was firstly introduced by Charnes et.al [[8]]:

$$\begin{aligned} &\text{Maximize} \quad u = U\lambda \\ &\text{subject to} \\ &\lambda \in \Lambda. \end{aligned} \tag{2.1.5}$$

It is clear that this MOLP model has no unique solution then it is desirable to find a linear combination of input/output vectors of existing DMUs which are feasible and simultaneously maximizes all outputs and minimizes all inputs. The goal of this DEA model is to determine which of the existing units $u_j = Ue_j$,($u_j \in T, j = 1, 2, \ldots, n$) are efficient and how efficient the

other DMUs are.
The original CCR DEA model which was discussed in chapter I, is Constant Return to Scale (CRS), i.e set Λ is substitute by $\Lambda = \{\lambda | \lambda \in \mathbb{R}^n_+\}$, and in the original BCC DEA model which is work under Variable Return to Scale (VRS) assumption, the set Λ is replaced by $\Lambda = \{\lambda | \lambda \in \mathbb{R}^n_+, 1^T\lambda = 1\}$.
The most common models in DEA are CCR and BCC models. To combine the expression, we formulate a **General** model, which includes both CCR-BCC model in input-output orientation.
General DEA-model (primal)(see [11]):

$$\begin{aligned}
&\text{Maximize} \quad z = \delta + \epsilon(1^T s^+ + 1^T s^-)\\
&\text{subject to}\\
&Y\lambda - \delta w^y - s^+ = g^y,\\
&X\lambda + \delta w^x + s^- = g^x,\\
&A\lambda + \mu = b,\\
&s^- \geq 0,\\
&s^+ \geq 0,\\
&\epsilon \geq 0,\\
&\lambda \geq 0.
\end{aligned} \tag{2.1.6}$$

General DEA model (Dual):

$$\begin{aligned}
&\text{Minimize} \quad w = \nu^T g^x - \mu^T g^y + \eta^T b\\
&\text{subject to}\\
&-\mu^T Y + \nu^T X + \eta^T A - \gamma^T = 0^T,\\
&\mu^T w^y + \nu^T w^x = 1,\\
&\gamma, \eta \geq 0,\\
&\mu, \nu \geq \epsilon,\\
&\epsilon > 0.
\end{aligned} \tag{2.1.7}$$

where, DMU_O is DMU under evaluation with $DMU_O = (g^x, g^y) \in \mathbb{R}^{m+p}$, (which is aspiration level of input and output), and $w = (w^x, w^y) \geq 0$, and weighted vector for input is w^x and weighted vector for output is w^y.

*2.1.3 **Value Efficiency approach to unify information in DEA***

The procedure brings forward in this section, begins by introducing most preferred combination of inputs and outputs of DMUs, shortly Most Preferred Solution (MPS) given by Decision Maker (DM), which is efficient in

DEA. In this process the resulting value efficiency scores are optimistic ones of the true scores. So, in following definition, MPS is defined.

Definition 2.1.3: **MPS**: In the first sight, it may come to everybody's mind that " the greater the importance- the larger the weights", but not always this idea can be true. The idea which is suggested in this research area is DM's priorities which are mixed in efficiency analysis by locating his/her most preferred input/output vector on the efficient frontier. This vector is named as Most Preferred Solution (MPS).In other explanation we can say that, MPS is a vector on efficiency frontier that the DM prefers to all other vectors on efficient frontier or the ones which are near to efficient frontier.(see [11])

In practice, it is considered MPS is a point that DM's value function $v(u) : \mathbb{R}^{m+p} \to \mathbb{R}$ obtains its maximum over T. Then it is clear that, how DM chooses the MPS which is based on the DM's value function $v(u)$,$u = (y, -x)$ which is strictly increasing and has local maximim value $v(u^*)$ over T where $u^* = (y^*, -x^*) \in \mathbb{R}^{m+p}$. Value function is pseudoconcave so local maximum is global. (see Bazaraa and Sherali 1993 [[9]]).

Definition 2.1.4: **Value function**: The value function which is defined as $v(u) = v(y, -x)$ is assumed to be as a function of situation that means the function of input/output vector u.

The VF, that is proposed for evaluating value efficiency problems, is considered to be pseudoconcave.(see [5])

Consider the problem :

$$\begin{aligned} &\text{Minimize} \quad f(x) \\ &\text{subject to} \\ &x \in X. \end{aligned} \tag{2.1.8}$$

If X is an open convex set and f is differentiable function on X and also f is pseudoconvex function then every local optimum is global. Need less to say that if f is pseudoconvex then $-f$ is pseudocancave.(see [12])

2.2 ***What is Value Efficiency?***

As mentioned before, the purpose of DEA is approximation the efficient frontier with DMUs which are given in problem. DEA also, evaluates efficient and inefficient units and their score. Traditional DEA studies consider that there is no input or no output more important than the others. In real word cases this claim can not be true. To clarify this inscription assume following example:

Example 2.2.1: Consider diagram (2.1). The diagram (2.1) consists of five DMUs, each of these DMUs producing two outputs and using one input. Consider that DM would rather output 1 is more important than output 2. In our example problem, as it is clear in diagram as well, DMU_1 is the DMU which is more preferred than DMU_3. In this case DMU_5 even considering that it is inefficient is preferred to DMU_3. Needless to say that, DMU_1, DMU_2, DMU_3 are efficient and DMU_4 and DMU_5 are inefficient.(see [11])

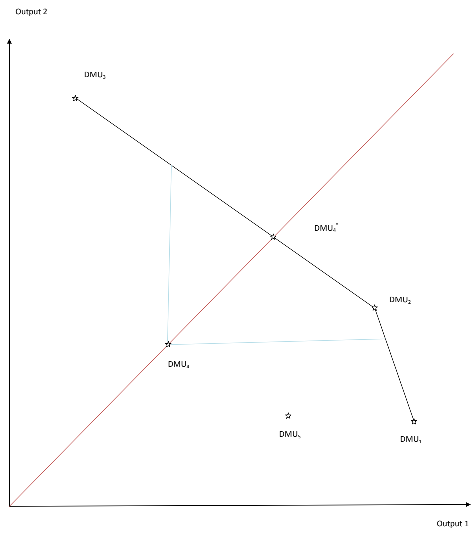

Fig. 2.1: value efficiency example

2.2.1 *Value Efficiency Analysis (VEA)*

After selecting MPS which is an input/output vector by DM, VEA can be defined as an approach to combine the value judgement in DEA. Then we need to insert these information into efficiency analysis and modify the original model. This modification will change the efficient frontier. It is important to know that in VEA, the DM does not exactly assume the weights but chooses the MPS among all efficient units. (for more information see [10]).

The idea of VEA was suggested in order to help the DM to evaluate the value of each vector $u = (y, -x) \in T$. This evaluation could be considered so easy if we could explicitly guess DM's value function. In practice, it is not possible and also realistic if we want to assume that value function is known or even can be precisely estimated. That is the reason that we use all possible approaches to incorporating a DM's priority in efficiency analysis. As mentioned above, we start the approach by the idea of substituting DM's MPS. The only assumption that we are allow to consider is that the value function is pseudoconcave .

The approach is that, first one specify all of tangent hyperplanes of the value function at MPS. This specification should be done for all possible pseudo-concave functions. Then we use this information to evaluate the value of each DMU for DM in the body of DEA.

Definition 2.2.1: **Weighed true value efficiency**

Weighted true value efficiency can be define as follow:

$$E_t^w(u^0) = \delta^t$$

where δ^t is the optimal value of:

$$\begin{aligned}
&\text{Sup} \quad \delta \\
&\text{subject to} \\
&u - \delta w \geq u^0, \\
&u \in V = \{u | v(u) \leq v(u^*)\}, \\
&w > 0.
\end{aligned} \tag{2.2.1}$$

(see [11])

Note: As we do not know if V is closed and we did not assume the continuity of function v, we used "Sup" instead of "Max".
Theorem 2.2.1
Consider value function v be strictly increasing function, following conditions satisfy.
(1) $v(u^0) = v(u^*) \Rightarrow \delta^t = 0$
(2) $\delta^t > 0 \Rightarrow v(u^*) > v(u^0)$
(3) $\delta^t < 0 \Rightarrow v(u^*) < v(u^0)$
Proof:
(1) Consider problem (2.2.1) and let $S = \{(u, \delta) | u - \delta w \geq u^0, u \in V\}$. As $v(u^0) = v(u^*) \Rightarrow (u, \delta) = (u^0, 0) \in S$, then it can be conclude that $\delta^t \geq 0$. Now by contradiction suppose that: $\delta^t = sup\{\delta | (u, \delta) \in S\} > 0$. In this condition there is $(\bar{u}, \bar{\delta}) \in S$ such that $\bar{\delta} > 0$.
According to constraint $u - \delta w \geq u^0 \Rightarrow u^0 < u^0 + \bar{\delta} w \leq \bar{u}$. On the other hand v is strictly increasing which can be conclude that:
$v(u^*) = v(u^0) < v(\bar{u}) \Rightarrow v(u^*) < v(\bar{u})$ and this is in contradict with second constraint of problem (2.2.1). Then the contradiction assumption can not be hold and we can conclude that $\delta^t = 0$.
(2) if $\delta^t > 0 \Rightarrow sup\{\delta | (u, \delta) \in S\} > 0 \Rightarrow \exists (\bar{u}, \bar{\delta}) \in S$ such that $\bar{\delta} > 0$. On the other hand according to $u - \delta w \geq u^0$, we have $u^0 < u^0 + \bar{\delta} w \leq \bar{u}$ and as v is strictly increasing, it can be concluded that $v(u^0) < v(\bar{u})$ and according to $u \in V = \{u | v(u) \leq v(u^*)\}$ we have $v(u^0) < v(u^*)$.
(3) (By contradiction)
Let $v(u^0) \leq v(u^*)$ then we can write $(u, \delta) \leq (u^0, 0) \in S$ which concludes that $\delta^t \geq 0$ and it is in contradiction with the assumption. so, $\delta^t < 0$. For more information please see ([15])

Model (2.2.1) has finite solution. Following we prove this claim.

Lemma 2.2.2:

Let $v(u)$ be strictly increasing. Then for any finite $u^*, u^0, w > 0$ problem Supδ has a finite solution δ^t corresponding to finite input/output point: $u^s = u^0 + \delta^t w$.

Proof:

let $v(u^0) < v(u^*)$ (it includes that $u^0 \in T, v(u^0) \neq v(u^*)$).

As $w > 0 \Rightarrow \exists \delta^1$ such that $u^0 + w\delta^1 > u^*$. Therefore, as v is strictly increasing, then $v(u^0 + w\delta^1) > v(u^*)$.

v is strictly increasing and, $u^*, u^0, w > 0$, then it is evident that: $v(u^0 + w\delta^1)$ is strictly increasing in δ. So, we can say that: $\exists \delta^t < \delta$ where $\delta^t = \text{Sup}\{\delta | v(u^0 + w\delta^t) \leq v(u^*)\}$. The proof for the case $v(u^0) > v(u^*)$ is the same and for the case $v(u^0) = v(u^*)$ the amount $\delta = 0$ can be achieved.(see [11])

2.2.2 *The comparison among value efficiency, technical and overall efficiency*

Interesting result can be achieved by comparing value efficiency with overall and technical efficiency. (Farrel 1957 [14](Norman and Stoker 1991 [16]) Consider graph (2.2), classical efficiency is illustrated be figure (a), the downward sloping line through DMU_0^O stands for profit equation. As can be seen in figure (a), only efficient DMU is DMU_1. Technical efficiency for DMU_O is the ratio $\frac{O-DMU_O}{O-DMU_O^T}$ and ratio $\frac{O-DMU_O}{O-DMU_0^O}$ stands for overall efficiency. Classical overall efficiency is based on the idea of max(min) profit(cost) function. More general unknown pseudoconcave value function is substituted for profit function in VEA. More over, it is assumed that the maximum of this function is known while its prices is unknown. The "overall efficiency" is estimated based on this information. The contour of pseudoconcave value function lies above their tangent hyperplane. As a linear approximation of $v(u)$ the tangent hyperplane at the MPS is used.

In figure (b), the ratio $\frac{O-DMU_O}{O-DMU_O^T}$ shows the "technical efficiency". For true

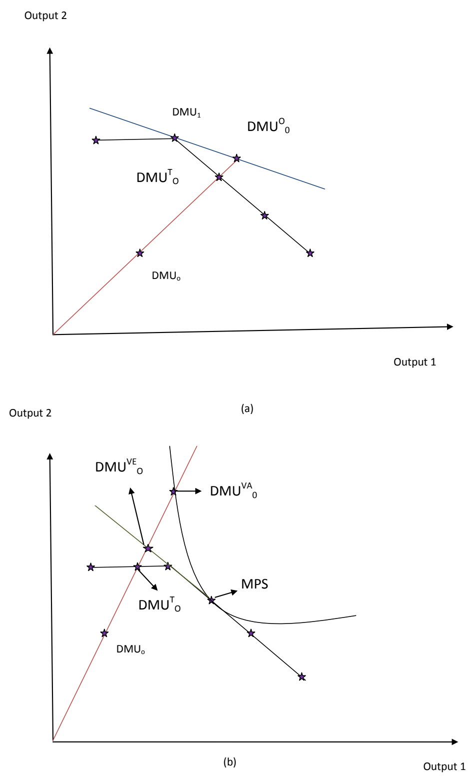

Fig. 2.2: Efficiency examples

value efficiency The ratio $\frac{O-DMU_O}{O-DMU_O^{VE}}$ can be an approximation.
It is desirable to evaluate $\frac{O-DMU_O}{O-DMU_O^{VA}}$ in value efficiency, but as we do not know value function that is impossible. So, we try to know the tangent of value function at MPS. On the other hand, we consider all possible tangent of value function as we can not consider that all tangents are known. It is important to keep in mind that this approximation of value efficiency score is optimistic and it provides the lower bound for real value efficiency scores. To know value efficiency model formulation precisely, we need to know some mathematical points. In next section we try to introduce what we need to know about the body of mathematical modelling of VEA.

2.3 Some mathematical considerations

As mentioned above, in this section the prerequisite of mathematical theory is introduced which helps to formulate a model for computing value efficiency scores.

Definition 2.3.1: **Cone**

A set G is called a cone, if for every $x \in G$ and $\lambda \geq 0$ we have $\lambda x \in G$. Figure (2.3) illustrates cone.

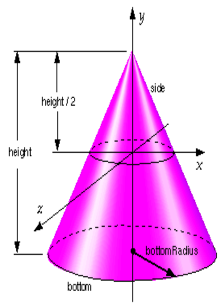

Fig. 2.3: Cone

Definition 2.3.2: **The cone of feasible direction**

Consider the problem:

$$\begin{aligned} &\text{Minimum} \quad f(x) \\ &\text{subject to} \\ &x \in X. \end{aligned} \tag{2.3.1}$$

where $X = \{x : g_i(x) \leq 0, i = 1, \ldots, m\} \subset \mathbb{R}^n$ is a non-empty set. The cone of feasible direction of X at x is defined as:

$D = \{d : d \neq 0, x + \lambda d \in X, \forall \lambda \in (0, \delta)$,for some,$\delta > 0\}$ In figure (2.4), the cone of feasible directions is depicted if the space between d_1 and d_2 stands for the space X.

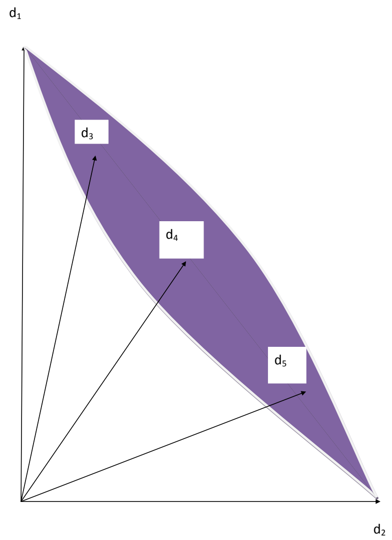

Fig. 2.4: The cone of feasible direction

Definition 2.3.3: **The Tangent cone and Augmented tangent cone**

The cone $G_x = \{y | y = x + d, d \in D(x)\}$ for $x \in X$ is called the tangent cone of X at x and $d \in D(x), d \neq 0$, is called feasible direction.

And:

$W_x = \left\{ s | s = y + z, y \in G_x, z \in \mathbb{R}^n_- \right\}$ for $x \in X$ is called augmented tangent cone of X at x. Both G_x and W_x are closed and convex. For any $s \in w_x$

there is an $y \in G_x$ such that $s \leq y$ and all points $z \leq s$ are in W_x. The tangent cone G_x is illustrated by vectors a and b and augmented cone W_x by vectors a and c in figure (2.5).

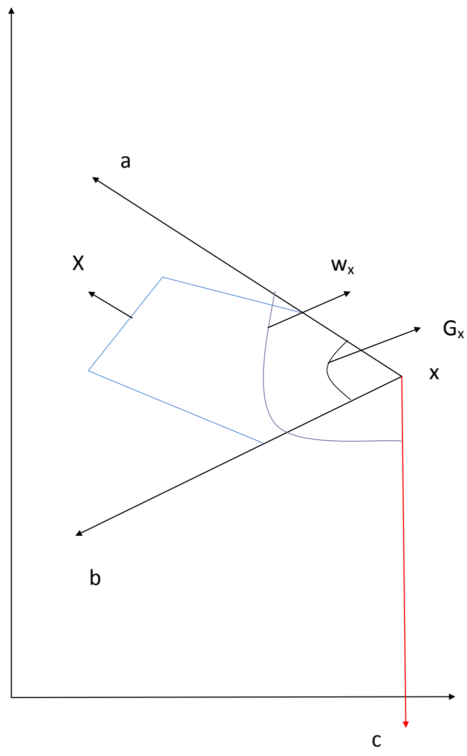

Fig. 2.5: Tangent and augmented cone

To be more precise in cones see [9].

Lemma 2.3.1: let $X = \{x|Ax = b, x \geq 0\}$ is a non empty polytope. Where $A \in \mathbb{R}^{k\times n}, b \in \mathbb{R}^k$ and $x^0 \in X$ is an arbitrary point, then $G_{x^0} = X^0 = \big\{x|Ax = b, x_j \geq 0 \ if \ , x^0_J = 0, \ otherwise \ x_j = \text{free}, j = 1, \ldots, n\big\}$.

Proof:

Clearly the tangent cone of an affine set $X_a = \{x|Ax = b\}$ at x_0 is X_a it self $(G_{X_a^0} = X_a)$.

In addition, tangent cone of close half space $H_j = \{x|x_j \geq 0\}$ at x^0 is $\mathbb{R}^n$ if $x_j^0 > 0$, and is H_j if $x_j^0, j = 1, \ldots, n$. As X is the intersection of X_a and the half space $H_j, j = 1, \ldots, n$ then tangent cone of X at x^0 is the intersection of their tangent cones, ,i.e, set X^0. [11]

Lemma 2.3.2: Let:

$$U = \left\{u \in \mathbb{R}^m | u = Bx, x \in X, B \in \mathbb{R}^{m \times n}\right\}$$

$$X = \{x|Ax = b, x \geq 0\}$$

And also consider: $u_0 \in U, x^0 \in X$ such that $u_0 = Bx^0$. Then:

$$G_{u_0} = BG_{x^0} = \{u|u = Bx, x \in G_{x^0}\}$$

Proof:

By definition of tangent cone:

$$G_{u_0} = \{y|y = u_0 + d, \text{ such that }, d \in D(x)\}$$

Which is called tangent cone of U at u_0. On the other hand:

$\forall u \in G_{u_0}$; it can be defined a feasible direction $u - u_0, (u \neq u_0)$ for U at u_0.

By considering the definition of U, it is obvious that this feasible direction should be generated by $x - x_0$ for X at x_0.

$\Rightarrow G_{u_0} \subseteq BG_{x_0}$ (1)

The same as above:

$\forall x^0 \in G_{x^0}$; it can be defined a feasible direction $x - x_0, (x \neq x^0)$ for X at x^0.

This feasible direction $x - x^0$ define a feasible direction $u - u_0$ for U at u_0.

$\Rightarrow BG_{x_0} \subseteq G_{u_0}$ (2)

From: $G_{u_0} \subseteq BG_{x_0}$ (1) and $BG_{x_0} \subseteq G_{u_0}$ (2)

we can conclude that: $G_{u_0} = BG_{x_0}$. [11]

Remind: A differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is pseudoconcave on convex set S if and only if:

$$\forall x_1, x_2 \in S, \text{ such that } \nabla^T f(x_1)(x_2 - x_1) \leq 0 \Rightarrow f(x_2) \leq f(x_1)$$

Note that by definition pseudo concave functions are by definition differentiable and therefore continuous.

Definition 2.3.4: Let $X \subseteq \mathbb{R}^n$ be a non-empty polytope and $x^* \in X$. Define $\mathbf{E}(x^*)$ as a set of increasing pseudoconcave functions $\xi : \mathbb{R}^n \to \mathbb{R}$, which obtains their max in X at $x^* \in X$. [11]

Definition 2.3.5: Let S_1 and S_2 are non-empty sets in $\mathbb{R}^n$. A hyperplane $H = \{x : p^t x = \alpha\}$ is said can separate s_1 from s_2, if $p^t x \geq \alpha$ for each $x \in s_1$ and $p^t x < \alpha$ for each $x \in s_2$.

Also, let s is a non-empty close convex set in $\mathbb{R}^n$ and $y \notin s$, then there exists $p \neq 0$ and scaler α such that $p^t y > \alpha$ and $p^t x \leq \alpha$ for each $x \in s$.(see [9])

Lemma 2.3.3: Let set $s \neq \emptyset$ and $y \notin$ cl conv (s)(cl conv s=closure of convex set s), then we can say y is strongly separated from set s.

Proof:

Let s is non-empty close convex set and $y \notin$ cl conv (s) then there exits unique minimality point $\bar{x} \in s$ such that for each $x \in s$,

$(x - \bar{x})(y - \bar{x}) \leq 0$.

Now let, $p = y - \bar{x} \neq \emptyset$ and $\alpha = \bar{x}^T(y - \bar{x}) = p^T\bar{x}$. We get $p^T x \leq \alpha$ for each $x \in s$.

Now we want to compute $p^T y - \alpha$:

$p^T y - \alpha = (y - x)^T y - p^T\bar{x} = (y - \bar{x})^T y - (y - \bar{x})^T\bar{x} = (y - \bar{x})^T(y - \bar{x}) = ||y - \bar{x}||^2 > 0 \Rightarrow p^T y - \alpha > 0 \Rightarrow p^t y > \alpha$.

What we get is:

$p^T y - \alpha$ and $p^t y > \alpha$. and this is what we wanted.

Lemma 2.3.4: Let $x^* \in X$ and $\mathbf{E}(x^*) \neq \emptyset$ and W_{x^*} is augmented tangent cone of X at x. Then $x \in W_{x^*}$ if and only if $\xi(x) \leq \xi(x^*)$ for all $\xi \in \mathbb{E}(x^*)$.

Proof:

$(\Rightarrow)$Let: $x \in Wx^* \Rightarrow \exists y \in G_{x^*}$ such that $x \leq y$.

As ξ is increasing $\Rightarrow \xi(x) \leq \xi(y), (1)$

Following I prove why $\nabla^T\xi(x^*)(y - x^*) \leq 0$:

$\mathbf{E(x)}$ is defined as a set of increasing pseudoconcave function $\xi : \mathbb{R}^n \to \mathbb{R}$ which obtain its maximum in X at $x^* \in X$.

In the assumption of lemma(2.3.3) we have $\xi \in \mathbf{E(x)}$, then ξ is increasing pseudoconcave function that obtains its maximum in X at $x \in X$.

As ξ obtains its maximum in $x^* \Rightarrow \nabla^T \xi(x^*) \leq 0$ (*)

On the other hand, $y - x^* \in D(x^*)$ and $(y - x^*) \geq 0(**)$

(*),(**)$\Rightarrow \nabla^T \xi(x^*)(y - x^*) \leq 0$.

As mentioned above: $y - x^* \in D(x^*)$ and ξ is pseudoconcave function, as obtains its maximum in X at x^*. Then:

$$\nabla^T \xi(x^*)(y - x^*) \leq 0 \Rightarrow \xi(y) \leq \xi(x^*), (2)$$

By (1),(2) we get:

$$\xi(x) \leq \xi(x^*), \forall \xi \in \mathbf{E}(x^*)$$

$\Leftarrow$)To prove this part we use one axiom in Logic:

$$(\sim A \Rightarrow \sim B) \Rightarrow (B \Rightarrow A)$$

(Where $\sim A$ means negative A)

Let $x \notin W_{x^*}$ then according to lemma (2.3.3), point x can be strongly separated from W_{x^*}.

i.e. $\exists p \in \mathbb{R}^n$ such that $p^T x > p^T y \forall y \in W_x^*$ i.e. for any $y = x^* + d + z$ such that $d \in D(x)$ and $z \leq 0 \Rightarrow p^T d \leq 0$, $p^T z \leq 0$ then $p \geq 0$ because otherwise $p^T d$ and $p^T z$ could be positive and arbitrary large.

As $p > 0$, we can write:

$$p^T y = p^T x^* + p^T d + p^T z$$

Also we had $p^T y < p^T x$ and we define: $\xi(x) = p^T x$. Then we can write:

$$\xi(y) = \xi(x^*) + \xi(d) + \xi(z) < \xi(x)$$

Which gives us:

$$\xi(x^*) < \xi(x)$$

we proved:

$$\sim (x \in W_{x^*}) \Rightarrow \sim (\xi(x) \leq \xi(x^*)$$

for all $\xi \in \mathbb{E}(x^*)$.

Which means that we proved the second part of lemma by the mentioned axiom in algebra.(see [11])

The mentioned lemma will be used to formulate VE. Before going through theorems, the following substitution in lemmas (2.3.1), (2.3.2) and (2.3.4) is done:

$$\mathbb{R}^n \to \mathbb{R}^{m+p}$$

$$X \to \Lambda$$

$$U \to T$$

$$\mathbf{E}(x^*) \to \mathbf{E}(u^*)$$

In the last substitution, $\mathbf{E}(u^*)$ is a set of pseudoconcave increasing function $v(u)$, which obtains its maximum in T at x^*. Lemma (2.3.4) is used when approximating the set,

$$\{u = (y, -x) | v(u) \leq v(u^*)\}$$

Where,$v(u)$ can be any function in $\mathbf{E}(u^*)$.

Lemma 2.3.5: $W_{u^*} \subset V = \{u | v(u) \leq v(u^*), \forall v \in \mathbf{E}(u^*)\}$.

Proof:

By using previous lemma it is completely evident. To prove this lemma it is enough to prove that $\forall u, u \in W_{u^*}$ Then $u \in V$. We consider for all u which $u \in W_{u^*}$, then by last lemma, $v(u) \leq v(u^*)$ which this means that $u \in V$ and that is what we wanted to prove.(see [11])

Theorem 2.3.1:
Let,$u^* = (y^*, -x^*) \in T$ be MPS. Then,$u = (y, -x) \in \mathbb{R}^{m+p}$ (an arbitrary point in input/output space), is value inefficient with respect to any strictly increasing pseudoconcave function $v(u)$, with max at u^*, if the optimum value (z^*) of the following problem is positive.

$$\begin{aligned}
&\text{Maximum} \quad z = \delta + \epsilon(1^T s^+ + 1^T s^-) \\
&\text{subject to} \\
&Y\lambda - \delta w^y - s^+ = g^y, \\
&X\lambda + \delta w^x + s^- = g^x, \\
&A\lambda + \mu = b, \\
&s^-, s^+ \geq 0, \\
&\epsilon > 0, \\
&\lambda_j \geq 0, \qquad \text{if} \quad \lambda_j^* = 0, j = 1, \ldots, n, \\
&\mu_j \geq 0, \qquad \text{if} \quad \mu_j^* = 0, j = 1, \ldots, k,
\end{aligned} \tag{2.3.2}$$

Where $\lambda^* \in \Lambda$ and μ^* corresponds to Most Preferred Solution (MPS),

$$y^* = Y\lambda^*$$
$$x^* = X\lambda^*$$

Proof: According to lemmas which were stated before, it can be written:
-The tangent cone of T at u^*:

$$G_{u^*} = \{(v, -z) | v = y\lambda, z = x\lambda . \lambda \in G_{\lambda^*}\}$$

(By $\lambda \in G_{\lambda^*}$, I mean that each λ can be written as $(\lambda + d)$).
-The tangent cone of Λ at λ^*:

$$G_{\lambda^*} = \{\lambda | A\lambda + \mu = b, \lambda_j \geq 0, \text{if}, \lambda_j{}^* = 0, (j = 1, \ldots, n)\text{and}, \mu_j \geq 0, \text{if}, \mu_j{}^* = 0, (j = 1, \ldots, k\})$$

-The augmented tangent cone W_{u^*} of T at u^*:

$$W_{u^*} = \{(v, -z) | v = Y\lambda + d^y, d^y \leq 0, z = X\lambda + d^x, d^x \geq 0, \lambda \in G_{\lambda^*}\}$$

Then we say that the model of value efficiency has a solution with $\delta \geq 0$ if $(y, -x) \in W_{u^*}$. Now, let $z^*, \lambda^s, \delta^s, \mu^s$ be a solution of the VE model, with $\epsilon > 0, z^* > 0$ only if $\delta^s > 0$ or $\delta^s = 0$ and $(s^-, s^+) \neq (0, 0)$. In both cases:

$$(v^s, -z^s) \in W_{u^*}$$

$$y^s = Y\lambda^s \geq y,$$

$$x^s = X\lambda^s \leq x$$

$$(y, x) \neq (y^s, x^s)$$

Then we have the following:

$$v(y,-x) < v(y^s,-x^s) \leq v(y^*,-x^*)$$

Then we conclude that $(y,-x)$ is value inefficient. [11]
The idea of Theorem (2.3.1) can be summarized as: we make one LP, the we put the DMU under evaluation in this LP. If the objective value of this LP was positive, then it can be conclude that the DMU under evaluation is not value efficient.

2.3.1 An illustrative example

Assume six DMUs with one input and one output, as shown in following table and diagram (2.6).

DMU	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
Output	1	4	7	9	12	8
Input	3	3	5	7	11	10

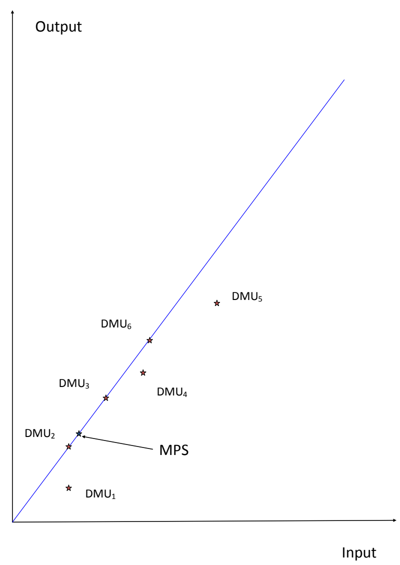

Fig. 2.6: Example diagram

Before formulating VE model, following substitution in model (2.3.2) should be done:

$$w^x = g^x = x_o$$

$$w^y = g^y = y_o$$

and also, $A\lambda \leq b$ substitute by $\mathbf{1}^T\lambda = \mathbf{1}$.
Then, following problem is formulated, and it it is desirable to maximize outputs and minimize inputs:

$$\begin{aligned}
&\text{Maximize} \quad 1\lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 12\lambda_5 + 8\lambda_6 \\
&\text{Minimize} \quad 3\lambda_1 + 3\lambda_2 + 5\lambda_3 + 7\lambda_4 + 11\lambda_5 + 10\lambda_6, \\
&\text{subject to} \\
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\
&\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \geq 0.
\end{aligned} \tag{2.3.3}$$

To solve above model, several MOLP method can be used. As the problem is two criteria, with a good visual method also it is solvable for DM.
Let DM chooses her MPS as (4,5.5) which is shown in diagram as well. As can be seen selected MPS can easily be written as convex combination of DMU_2 and DMU_3. When MPS can be written as convex combination of these two, it means that in model (2.3.3), their optimal values are positive and the other DMU get zero value. By knowing this in VE model λ_2 and λ_3 will be defined unbounded and the other λs are non-negative. Consider for example, DMU_5 is the DMU under evaluation. Now, the value efficiency score will be formulated as:

$$\begin{aligned}
&\text{Minimize} \quad \delta + \epsilon(1^T s^+ + 1^T s^-) \\
&\text{subject to} \\
&\lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 12\lambda_5 + 8\lambda_6 - 12\delta - s^+ = 12, \\
&3\lambda_1 + 3\lambda_2 + 5\lambda_3 + 7\lambda_4 + 11\lambda_5 + 10\lambda_6 - 11\delta + s^- = 11, \\
&\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 = 1, \\
&\lambda_1, \lambda_4, \lambda_5, \lambda_6 \geq 0, \\
&\lambda_2, \lambda_3 \text{Un restricted.}
\end{aligned} \tag{2.3.4}$$

The optimal solution of problem (2.3.4) is $\delta = 0.14$ which means that the DMU under evaluation DMU_5 is vale inefficient. By selected MPS here only DMU_2 and DMU_3 are Value efficient. [11]
Remark: As mentioned before in traditional DEA analysis, if a DMU obtains score 1 it means that the $DMU$is efficient. In combined model where both input and output treated at the same time, efficient DMUs get a score of 0 and inefficient unit a positive score.
The reason of this is that in combine model for one DMU to be efficient , Simultaneously the output should increase while input is increases. We can say that one efficient DMU for example can be value efficient or value in efficient.

2.4 More on Value Efficiency

In value efficiency, MPS plays a key role. If we can consider that DM can figure out one MPS which lies in efficient frontier, then one possibility to achieve the solution of the problem is line search on efficient frontier.

It is important to remember, the efficient frontier is in related with the scale assumption that we made, it means that if the BCC model is used to evaluate technical efficiency, then the same model has to be used for VEA as well.

Example 2.4.1: Consider four DMUs A, B, C, D; which consume one input to produce one output. We used BCC model to evaluate efficiency analysis.

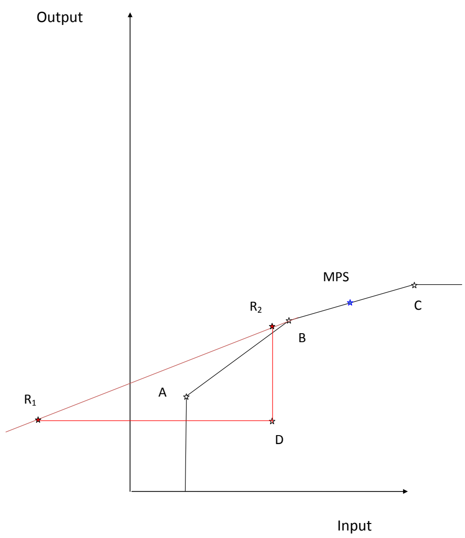

Fig. 2.7: Example diagram

Let the DM assumes that MPS lie on convex combination of B and C. If we assume the output oriented model, reference point for in point D is R_2, which is specified as:

$$R_2 : (1 + \mu)B + (-\mu)C$$

Assume input oriented model. R_1 can be assumed reference point for inefficient point D. As it is shown in figure (2.7), this reference point has negative input. For evaluating inefficient DMU by input oriented model in this example, unit D can never be made equally preferred to MPS be at the current output level, even if it can be possible to produce the current amount of output with negative resource.

To be more precise we can say that, suppose the DM has selected the unit that has large amount of input and output value as MPS. In this situation we can not find any unit which has small amount of input and output which is equally preferred by reducing the use of inputs.(see [17])

3. AN APPROACH TO IMPROVE ESTIMATES OF VALUE EFFICIENCY

3.1 Introduction

In this chapter two refinements are proposed:
1.Some bounds on VE scores are introduced.
2.A way of more accurate computation of VE scores will be introduced with minimal DM involvement.
The base of value efficiency analysis is on pseudoconcave DM's value function. This value function (VF) is not known explicitly.
VEA searches for the VE of DMU, as a part of distance from the a value function contour which passes via the MPS. This contour (value function) can be guessed by tangent of value function at the MPS. By using this method of choosing MPS and tangent cone will makes value efficiency score optimistic approximation of the true one.
As mentioned before the DM has a pseudoconcave value function which is unknown $v(u), u = (y, -x) \in \mathbb{R}^{m+p}$ and is strictly increasing (i.e. strictly increasing in y and strictly decreasing in x) and with value function $v(u)$ which has its maximum in $v(u^*), u^* = (y^*, -x^*) \in T$, at the MPS u^*, where T stands for the feasible set. In practice, the value function is unknown. That is the reason that we approximate the contour by using all possible tangent hyperplanes which passes through MPS. A new 'efficiency frontier' will be define by those hyperplanes and by considering this new frontier, efficiency is defined using a standard DEA technique and the scores which obtains are called value efficiency scores.
To get into problem, consider fig (3.1), which discusses about the basic idea of VE. It is shown five units (A, B, C, D, E) in the figure, which each DMU produces two outputs and consumes one single input. The efficiency of standard DEA is calculated as: $\frac{OB}{OB^1}$. What we want to compute is the ratio: $\frac{OB}{OB^4}$, but as we know the VF is unknown, so we are not able to do it. If we have the possibility to approximate the Value function (which calls indifference contour) by a tangent hyperplane of VF at MPS, we can compute the ratio: $\frac{OB}{OB^3}$. As practically it is not possible, then all possible tangents

of the contour will be considered. The ratio $\frac{OB}{OB^2}$ will be computed as an approximation to the (true) VE score which is the best approximation that can be achieved. As we have seen in chapter two, we can evaluate the VEA as standard DEA by using linear programming.

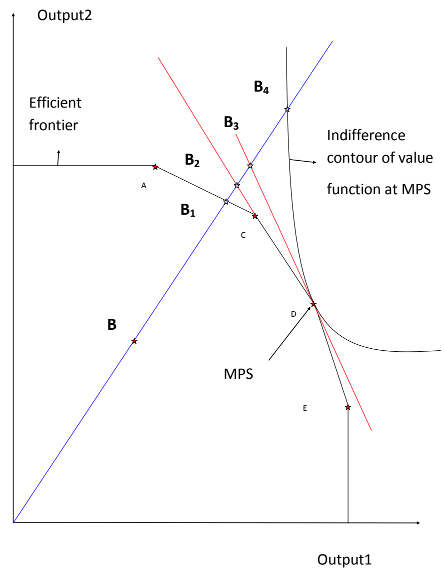

Fig. 3.1: Example diagram

We remind that a DMU_0 with input/output vector $u = (y_0, x_0)$ is value inefficient with respect to strictly increasing pseudoconcave value function $v(u) = v(y, -x)$ (increasing in y and decreasing in x) which its maximum happens at u^*, if the optimum value of the model of (3.1.1) and (3.1.2) means $Z^* = W^*$ is strictly positive in the following problem formulation.
Primal VE model:

$$\begin{aligned}
&\text{Maximum} \quad z = \delta + \epsilon(1^T s^+ + 1^T s^-) \\
&\text{subject to} \\
&Y\lambda - \delta w^y - s^+ = g^y, \\
&X\lambda + \delta w^x + s^- = g^x, \\
&A\lambda + \mu = b, \\
&s^-, s^+ \geq 0, \\
&\epsilon > 0, \\
&\lambda_j \geq 0, \qquad\qquad \text{if} \quad \lambda_j^* = 0, j = 1, \ldots, n, \\
&\mu_j \geq 0, \qquad\qquad \text{if} \quad \mu_j^* = 0, j = 1, \ldots, k.
\end{aligned} \tag{3.1.1}$$

Dual value efficiency model:

$$
\begin{aligned}
&\text{Minimum} \quad w = v^T g^x - \xi^T g^y - \eta^T b \\
&\text{subject to} \\
&-\xi^T Y + v^T X + \eta^T A - \gamma = 0, \\
&\xi^T w^y + v^T w^x = 1, \\
&\xi, v \geq \epsilon, \\
&\epsilon > 0, \\
&\gamma_j \geq 0, && \text{if} \quad \lambda_j^* = 0, j = 1, \ldots, n, \\
&\gamma_j = 0, && \text{if} \quad \lambda_j^* > 0, j = 1, \ldots, n, \\
&\eta_j \geq 0, && \text{if} \quad \mu_j^* = 0, j = 1, \ldots, k, \\
&\eta_j = 0, && \text{if} \quad \mu_j^* > 0, j = 1, \ldots, k,
\end{aligned}
\tag{3.1.2}
$$

Where λ^* and μ^* are corresponding to MPS: $y^* = Y\lambda^*, x^* = X\lambda^*$.(for more information see [19])

3.2 Essential and geometrical consideration

Consider five DMUs where each DMU consumes one input to produce two outputs, also; here constant return to scale and output-oriented DEA model is assumed.
For example, consider table(3.1), which gives numerical value of inputs (man-hour) and outputs (sale-profit) corresponded with each DMU.

Tab. 3.1: Table of input and output data

Units	**Sales**	**Profit**	**Man-hours**
A	37.838	0.929	54.932
B	80.019	2.983	94.596
C	98.931	1.861	68.703
D	90.245	0.987	55.000
E	86.775	0.437	50.157

Firstly, we scale all output vales, by dividing the corresponding output values to their input. (without loss of any generality)
See table (3.2), which is presented in terms of output per man-hour.

Tab. 3.2: Table of scaled data

Units	**Sales**	**Profit**
A	0.689	0.0169
B	0.846	0.0315
C	1.440	0.0271
D	1.641	0.0179
E	1.730	0.087

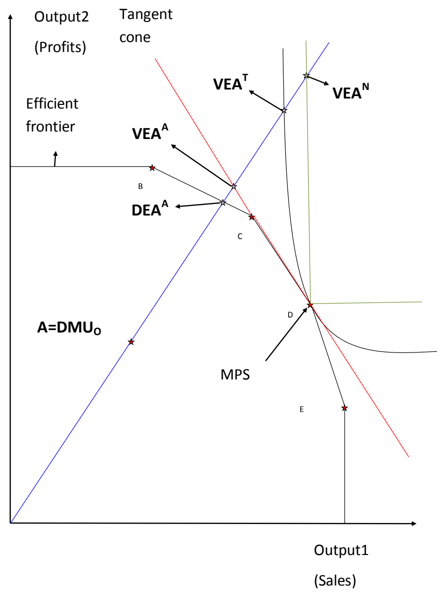

Fig. 3.2: Example diagram

Let the figure(3.2) which horizontal line stands output 1(Sales) and vertical line stands for output 2(profit).
Consider A as DMU under evaluation, it means that $A = DMU_O$ in fig(3.2). The standard efficiency of DMU_O is defined by DEA^A which by definition is $\frac{O-DMU_O}{O-DEA^A}$. This efficiency definition measures how close the DMU is efficient.
To find value efficiency, we need to know DM's MPS which is located on the efficient frontier, and also the evaluation of DMUs which is done by DM with using value function. Needless to say that the solution can be existing or virtual.
The goal is to evaluate how much improvement in input and output the DMU needs to achieve a point at the contour of value function. So, we need to evaluate $\frac{O-DMU_O}{O-VEA^T}$. But as it was mentioned, the value function is not known and no information about VEA^T exists.

The proposed idea which is discussing here is; for each value function $v(u)$, we approximate the $\frac{O-DMU_O}{O-VEA^T}$ for for unit DMU_O. It means $\frac{O-DMU_O}{O-VEA^N}$ for upper bound and $\frac{O-DMU_O}{O-DEA^A}$ for lower bound will be computed. Also, for every point VEA^T which located in the intersection of ray from the origin through point DMU_O and value function contour through MPS, we can prove that: $DEA^A \leq VEA^T \leq VEA^N$.

Lemma 3.2.1: Show $DEA^A \leq VEA^T \leq VEA^N$.

Proof: On contradiction let $VEA^T < DEA^A$. As we chose MPS in a way that $MPS \approx DEA^T$, then MPS which considered $MPS \approx DEA^T$ should be less than DEA^A, $(MPS \approx DEA^T < DEA^A)$. On the other hand, if we assume $VEA^T > VEA^N$ then:

$MPS \approx VEA^T > VEA^N \Rightarrow MPS > VEA^N$. It means that we got $VEA^N < MPS < DEA^T$ which can not be possible because in this situation the bounds will be too loose. For more information see [19].

The idea of finding lower and upper bound to approximate VEA^T is made on the idea of using tangent cone at the MPS. This tangent cone is defined by constraints which are binding on MPS. The cone of binding constraint at MPS, contains solution less preferred than MPS. VEA^A is defined as an intersection of the ray from origin through $A = DMU_O$ and the tangent cone at MPS. According to the point VEA^A, we can define a tighter lower bound on value efficiency which is $\frac{O-DMU_O}{O-VEA^A}$. The defined lower and upper bound is as follow:

$$\left[\frac{O - DMU_O}{O - VEA^A}, \frac{O - DMU_O}{O - VEA^N}\right]$$

3.3 Theoretical considerations

Assume n DMUs which each DMU uses m inputs to produce p outputs, and $X \in \mathbb{R}_+^{m\times n}$ and $Y \in \mathbb{R}_+^{p\times n}$ are matrices consisting non-negative elements which contains observed inputs and outputs measures for DMU, respectively. x_j is used to show the vector of inputs consumed by DMU_j and x_{ij} is the quantity of input i consumed for DMU_j. A similar notation can be considered for output.

Consider $\Lambda = \left\{\lambda | \lambda \in \mathbb{R}^n_+, A\lambda \leq b\right\}$, also consider $T = \{u | u = U\lambda, \lambda \in \Lambda\}$ as a feasible set. Furthermore assume $e_i \in \Lambda, i = 1, \ldots, n$.
It is noted that, combined general DEA model is defined as:

$$\begin{aligned}
&\text{Maximum} \quad z = \delta + \epsilon(1^T s^+ + 1^T s^-) \\
&\text{subject to} \\
&Y\lambda - \delta w^y - s^+ = g^y, \\
&X\lambda + \delta w^x + s^- = g^x, \\
&A\lambda \leq b, \\
&\lambda, s^+, s^- \geq 0, \\
&\epsilon > 0,
\end{aligned} \tag{3.3.1}$$

A DMU in general model is efficient if and only if the optimal value Z^* is equal to 1 and all slack variables s^+, s^- are equal to zero otherwise we say that DMU is inefficient. (see [20])
Let the optimal value of model(3.3.1) be shown by $\delta^e \geq 0$, and we define $0 = \delta^0$.
It is so important to remember that all efficient points can not consider equally "good" and also we can not say that all inefficient points equally are "bad". As mentioned in chapter II, it can happen that some inefficient points can define more preferred than some other efficient ones, this depends on the value of each DMU. In fact, these kind of consideration can assume when the VE is defined.
Also it was mentioned in chapter II that The weighted true value efficiency is defined as follow:

$$E^w_t(u^0) = \delta^t$$

where δ^t is the optimal value of model (2.3.1).
The idea of value efficiency can be shortly reviewed as follow:
To evaluate true VE scores, we should know value function but the value function is not known exactly. We assume DM choose MPS on efficient frontier. It is not enough to figure out what is VF by knowing only MPS. The only thing is we can define the region which is consists of all DMUs that MPS is the best point which is preferred for DM between all DMUs. This region is the augmented tangent cone $W(u^*)$ at the MPS which $W(u^*) \subseteq V = \{u = (y, -u) | v(u) \leq v(u^*)\}$ where V is any pseudoconcave value function which has its maximum at u^*. In fact if we substitute augmented tangent cone $W(u^*)$ instead of V in true VE model we get the model (3.1.1) which was defined in chapter II and named as VEA model and its optimum shown by δ^a.
This approximated VE score can be assumed as lower bound for the true value efficiency score δ^t, because $W(u^*) \subseteq V$,i.e: $\delta^a \leq \delta^t$.
Now we should calculate the upper bound for δ^t. As $v(u)$ is strictly increas-

ing function to compute upper bound:

$$\{u|v(u) \leq v(u^*)\} \cap \{u|u^* < u)\} \neq \emptyset$$

So, we do not need pseudoconcavity assumption and an upper bound is obtained as follow:

$$\begin{aligned}
&\text{Maximum} \quad \delta \\
&\text{subject to} \\
&u - \delta w \geq u_0, \\
&u \in K = \{u|u^* < u\}^C, \\
&w > 0,
\end{aligned} \tag{3.3.2}$$

The solution of this problem is called "naive" value efficiency score and shown by δ^n. Then what we got up to here is: $\delta^a \leq \delta^t \leq \delta^n$.

Lemma 3.3.1: $\delta^e \leq \delta^a$ where δ^e is optimal value of:

$$\begin{aligned}
&\text{Maximum} \quad z = \delta + \epsilon(1^T s^+ + 1^T s^-) \\
&\text{subject to} \\
&Y\lambda - \delta w^y - s^+ = g^y, \\
&X\lambda - \delta w^x - s^- = g^x, \\
&A\lambda \leq b, \\
&s^-, s^+ \geq 0, \\
&\epsilon > 0, \\
&\lambda \geq 0,
\end{aligned} \tag{3.3.3}$$

Proof: As it is clear, the objective function of model (3.3.3) and model (3.1.1) are exactly the same and the constraints of model (3.1.1) includes the constraints of model (3.3.3), as in model (3.3.3) there is more constraints than model (3.1.1), then the optimal value is not improved, so: $\delta^e \leq \delta^a$. (for more information see Bazaraa et.al (1993) [9])

Then the following inequalities is achieved: $\delta^0 \leq \delta^e \leq \delta^a \leq \delta^t \leq \delta^n$.

It is important to know that, there is no need to locate true value efficiency score if δ^a and δ^n are so near together.

Lemma 3.3.2: Let function $f : \mathbb{R}^n \to \mathbb{R}$ be pseudoconcave on convex set S, then we can prove that $\forall x_1, x_2 \in S : f(\lambda x_1 + (1-\lambda)x_2) > min\{f(x_1), f(x_2)\}$ for $\lambda \in [0, 1]$.

Proof: See Bazaraa et.al (1993) [9].

Lemma 3.3.3: Consider v is pseudoconcave and strictly increasing value function which is defined in $\mathbb{R}^{m+p}$ and assume δ^a is a solution of model (3.1.1) which is corresponding to value efficiency analysis of unit u_0, with an assumption that $u^* = MPS$. If $\exists \delta^s, 0 \leq \delta^s \leq \delta^a$ and $u^s = u_0 + \delta^s u_0$ such that $u^* \approx u^s$ then $\exists u^m \in T$ s.t $v(u^m) > v(u^*)$.

Proof: Assume $u \in T$ and $W(u^*)$ is augmented tangent cone of T at u^*. By Halme et.at (1999) [10] it can be written, if $u \in W(u^*) \Rightarrow v(u) < v(u^*)$ and also for every $u^a \in W(u^*)$, u^a has the form of $u^a = u_0 + \delta^a u_0$. On the other hand, by assumption $u^s \approx u^*$ then we can say that $v(u^s) = v(u^*)$. Also as we see in assumption $\delta^s \leq \delta^a$ and v is strictly increasing, it can be conclude that:

$$(u^s = u_0 + \delta^s u_0) \leq (u^a = u_0 + \delta^a u_0) \Rightarrow v(u^s) < v(u^a)$$

As $v(u^s) < v(u^a)$ so $v(u^*) < v(u^a)$, which is in conflict (because if $u^a \in W(u^*)$ then $v(u^a) < v(u^*) = v(u^s)$).

so v has no maximum at u^* in set T then there is $u^m \in T$ such that $v(u^*) < v(u^m)$.

3.4 *An approach for finding the value efficiency score*

The approach which is presented in this section, makes an ability for DM to start doing a simple line search. This search will start from u_0 going through the ray. The starting point is shown by $A = DMU_o$, it means that the search is on the ray $u_0 + \delta u_0, \delta \in [0, \delta^n]$.

3.4.1 *Proposed approach*

1. **Step 0: Find MPS**
   In this step the DM introduces MPS ($u^* \in T$) which is located on efficiency frontier.

2. **Step 1: Find value efficiency of point $\mathbf{u_0}$**
   Denote selected starting point by u_0, otherwise stop.
   Define a ray $r = (1+\delta)u_0, \delta > 0$ and compute the values of δ^e and δ^n.
   If the pseudoconcavity of value function is confirmed, then determine δ^a if not, put the value of δ^e as δ^a.

3. ***Step 2: An indifference point to $\mathbf{u}^*$ on the ray $\mathbf{r}$***
   By varying the point δ, for which $\delta \in [0, \delta^n]$ (it meant that finding the amount of $\delta^e, \ldots, \delta^n$) we try to find the most nearest point to u^*. and we name this point as u, i.e:$u \approx u^*$. The DM makes indifferent between u and u^*,i.e $v(u) \approx v(u^*)$. Let the corresponding parameter value ,which its point is the nearest one to u^*, be δ^I.

4. ***Step 3: Check the consistency***
   For δ^I following cases can happen:
   If $\delta^I \in [\delta^0, \delta^e]$, then go to step 4.
   If $\delta^I \in [\delta^e, \delta^a]$, then go to step 5.
   If $\delta^I \in [\delta^a, \delta^n]$, then use δ^I as true VEA.

5. ***Step 4: No consistent choice***
   If the chosen δ by DM is $\delta^I < \delta^e$, then it means that the point $u_0+\delta^I u_0$ is dominated by $u_0 + \delta^e u_0 \in T$. As the value function (unknown) was considered to be increasing, then we have $v(u^*) = v(u^I) < v(u^e)$. It means that, The point which was chosen by DM and introduced as MPS is a point which its solution is less preferred than feasible solution. In this case two option will introduce:

- The DM can go back to step 2 and correct the evaluation.
-If the DM insist on this δ^I, (s)he can choose a new MPS. The new MPS can be found through new search or solution of model (3.3.1).

6. ***Step 5: Inconsistent choice***
Let the DM has chosen $\delta^e \leq \delta^I < \delta^a$. There are two possibilities:
- The DM has made an inconsistent choice.
- The choice in a case of loose of pseudoconcavity is consistent. These two different choices are illustrated in Fig(3.3). In figure (a), a pseu-

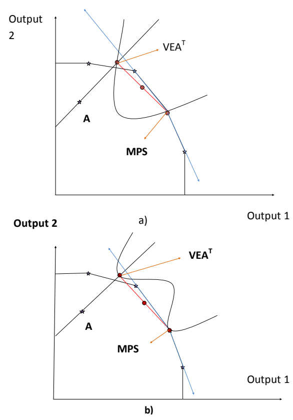

Fig. 3.3: Example diagram of step 5

doconcave function is shown where in figure (b), pseudoconcavity is not met.
Let the convex combination $u(\mu) = \mu u^I + (1-\mu)u^*, \mu \in [0,1]$ where $u^I = (1+\delta^I)u_0$. The end points in fig(3.3) stands for $u^I = VEA^T$ and $u^* = MPS$.
If the DM can show a point (at least one) on the line $u(\mu)$ corresponds to VE function, which is less preferred than MPS then the function is not pseudoconcave (Lemma(3.3.2)).
(fig (a)) stands for pseudoconcave VF, if DM is able to find or put such points in line search in which lemma(3.3.2) does not hold then pseudoconcavity is not met (fig (b)).
In a case we have fig(b), the current solution can be accepted an assumption about pseudoconcavity can ignore if these assumption is not necessity for DM, otherwise we go to step (2).
If no evidence which can not reject pseudoconcavity found, then we consider the current point as u^I and by lemma (3.3.2); there exist at leat one point $u^D \in T$ preferred to u^*. To search this point see Korhonen and Laakso 1986 [21].

3.5 ***Illustrative example***

The data set for application is collected for evaluation of academic research of economics and business administration at Helsinki school. For the application which is used in this section the same DM who was helped for the research in Korhonen et al. (1998)([22]) is chosen. (Professor Tainio, vice vector of economics at Helsinki school, Finland).
Now, we need to describe precisely the output variables.
Outputs are chosen as follow:

1. ***The research quantity***
   -Articles which are published in international journals
   -The number of scientific books which are published
   -Citation
2. ***The activity of research***
   -The number of articles which are refereed in international journals
   -Papers which submitted and accepted in conferences
   -Conference presentations
3. ***Impact of research***
   -The number of citation which are cited by other researchers
   -Number of foreign co-authors which are in the articles that accepted in international journals
   -The number of invited presentations and plan for international conferences
4. ***The number and quality of educating young researchers and their activity***
   -The number of doctoral degree which is produced
   -The number of doctoral student supervised

The rang of standard values are from 0 to 100 which located in table(3.3). The recognition of these standards values derived with the help from AHP (Satty,1980 [23]) which is performed by the member of TUTKE. (see Korhonen et al. 1998 [22] for details).
The **inputs** which are used in this research are:
-The budget which each unit is consumed for research activities.

Tab. 3.3: Table of output and input data

Units	Quality	activity	Impact	students	Budget
A	67	100	48	100	70
B	38	36	32	25	32
C	5	8	0	9	34
D	21	23	2	35	101
E	54	37	9	6	25
F	43	64	33	55	64
G	35	42	18	65	46
H	2	5	1	23	25
I	27	36	19	25	28
J	34	36	0	11	23
K	7	8	0	7	7
L	42	45	35	51	68
M	4	7	2	0	8
N	18	15	27	3	15
O	25	21	25	25	37
P	74	47	82	5	29
Q	27	51	20	0	12
R	76	55	74	3	119

To achieve the first evaluation of DMU the standard DEA is used with combined orientation,(see Joro et.al 1998 [24]); and use VRS assumption (Banker et.al (1984) [25]).
The formulation (3.3.1), is used to evaluate a DMU with combined standard DEA. As mentioned before model (3.3.1)is searching for input/output improvement and evaluates the efficiency score δ^e. To evaluate a DMU with model (3.3.1), we substitute $A\lambda \leq b$ by $1^T\lambda = 1$.(see chapter I for VRS formulation)
To evaluate other efficiency scores we need MPS (step 0), our DM (prof. Tainio) to perform free search on efficient frontier, used Pareto race method (see Korhonen and Wallenius, 1998 [[24]). He chose a point on efficient frontier which has the highest value for him as MPS based on this search method. The obtained MPS is listed in table(3.4).

Tab. 3.4: Most preferred solution

Output/input	values
Quality	71.4
activity	67
Impact	69.2
students	40.8
Budget	44.5

Now, we are performing (step 1) that evaluates VEA score ,δ^a, which obtained from model (3.1.1).
The DM is asked to consider a point on ray r as mentioned in (step 2) to calculate true value efficiency score then put this point as equally as the MPS. These scores are listed in table (3.5). In this study, after first irritation, all the result which reveal were in the range: $\delta^a \leq \delta^t \leq \delta^n$.

Tab. 3.5: Table of efficiency scores

Units	DEA	VEA	true VEA
A	0.000	0.000	0.050
B	0.109	0.109	0.150
C	0.640	0.669	0.750
D	0.603	0.615	0.800
E	0.050	0.050	0.250
F	0.211	0.211	0.350
G	0.000	0.033	0.400
H	0.179	0.245	0.750
I	0.118	0.118	0.200
J	0.114	0.114	0.150
K	0.000	0.000	0.300
L	0.264	0.264	0.400
M	0.059	0.432	0.500
N	0.042	0.105	0.150
O	0.233	0.233	0.400
P	0.000	0.000	0.000
Q	0.000	0.000	0.000
R	0.000	0.000	0.450

I should mention that, the possible error that would happen in this evaluation is about the distance of unit from MPS. It means that an approximation which is very far from MPS can be imprecise. Need less to say that this approximation has no effect on DEA values.

4. CONCLUSIONS

In this thesis, I introduced the main concept of DEA and then developed the idea by introducing Value Efficiency Analysis (VEA). The idea of VEA came to researchers mind as they understood that not all efficient DMUs which were evaluated by traditional DEA model can introduce equally good and nor all inefficient DMUs equally bad. One may prefer one inefficient DMU to the efficient one.

In VEA Decision Maker (DM) plays very important role. To evaluate VEA, pseudoconcave value function is desirable. The idea begins with DM's MPS which lies on efficient frontier and it is desirable that the DMU under evaluation is as close as MPS.

As the psedoconcave value function is not known nor easy to guess then the tangent cone at MPS is introduced to estimate the counter of value function. By use of these assumption, VE model was produced. To evaluate one DMU in order to get if it is value efficient or not, we put the DMU under evaluation on this model if the solution was positive it conclude that the DMU under evaluation is value inefficient. It is important to add that one efficient DMU can be value efficient or value inefficient. The idea even improved in last chapter . In last part, an interactive algorithm introduced and lower and upper bounds on the true scores was found without any DM involvements. The approach is tested on academic research at the Helsinki School of Economics and Business Administration.

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