SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Universal bounds on the eigenvalues of compact finite quantum graphs

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2014 - No 20

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Pavel Kurasov

2014

Abstract

In this Master Thesis we search for and find universal upper and lower bounds for all the eigenvalues of a quantum graph with delta-conditions. Only graphs where all strengths of the matching conditions are non-negative will be considered.

The spectrum of a quantum graph can be calculated using the Rayleigh quotient, which involves quadratic forms. As the quadratic form's domain depends on certain properties of the graph it is possible to derive upper and lower bounds on the eigenvalues of quantum graphs with delta-conditions by looking at how changes in the underlying graph affects the domain. We present a number of alterations preserving the total strength of the conditions and the total length of the graph, which will always shift the eigenvalues in a known direction. Combining these results with the lower bound given by A. Friedlander for quantum graphs with standard conditions we derive new universal upper and lower bounds for graphs with delta type matching conditions.

Acknowledgements

First of all I would like to express my deep gratitude to my supervisor, Professor Pavel Kurasov. His expertise and knowledge of the subject has of course been invaluable, but even more important has been his enthusiasm, encouragement and dedication, without which the thesis would never have become what it is.

The people working at the Department of Mathematics at Stockholm University have created a wonderful study environment during my years. I couldn't have hoped for a more welcoming and academically developing environment.

I would also like to thank my parents for their constant support and care, for a wonderful upbringing which have played a crucial role in taking me where I am today.

A special thanks to my colleague and best friend Isak Trygg Kupersmidt, with whom I have shared my studies and work, both during the writing of this thesis and the courses leading here. I am sure that we together have achieved much more than any of us could have done alone.

Contents

Introduction

Quantum graphs are metric graphs with ordinary differential operators acting on all the edges. The matching conditions at the end points of the edges connect together the values of the functions and are such that the values at all end-points belonging to the same vertex are connected. Such a set up has properties of both ordinary differential operators and partial differential operators. Problems where for example wave propagation in thin structures is studied can in many cases be approximated by the study of a corresponding quantum graph. For a survey of such applications and problems, and when the approximation is suitable see [8].

Many interesting questions and applications of quantum graphs are concerned with the spectrum of the graph. The spectrum of a finite compact quantum graph is given by the eigenvalues of the corresponding differential operator. In this text mainly the Laplace operator will be considered, that is the differential expression will be

$$L(u) = -u''(x)$$

and the spectrum is given by all $\lambda \in \mathbb{C}$ such that there is a non-trivial solution to the differential equations

$$L(u) = \lambda u(x)$$

satisfying the matching conditions. The spectrum of an operator contains a lot of information about the operator and is an important tool for understanding the operator.

The goal of this text is to find universal bounds, under some restrictions, on the spectrum of quantum graphs. That means values a_n, b_n such that the n:th eigenvalue, λ_n, of each quantum graph must lie between them, i.e. $a_n \leq \lambda_n \leq b_n$. We also search for the graphs where the equalities are realized. The matching conditions considered are so called δ-conditions, which always

have a real parameter α_v for each vertex v called the strength of the condition the sum of these strengths will be called the total strength of the quantum graph.

A change in total length or the total strength move the spectrum in a known direction. The interesting question is therefore to find bounds given a total length and a total strength. The reason for doing so is at least twofold. Firstly, in the process of finding these bounds and the corresponding extremal graphs a better understanding of how the geometry of the graph and the distribution of the strengths affect the spectrum is found. Secondly the spectrum is normally very difficult to calculate, and in all but a few simple graphs, impossible to calculate explicitly. If there are universal bounds depending on a few simple parameters, estimates of the spectrum are easily calculated and can in turn facilitate other arguments and investigations.

The first chapter contains an introduction to and a definition of quantum graphs, and some important tools used in the later chapters are presented. In chapter 2 a number of alterations in geometry and the strengths distribution and how they affect the spectrum are presented. In chapter 3 these results are combined with an earlier lower bound for quantum graphs with standard conditions to obtain a lower bound on the spectrum. One of the theorems in chapter 2 directly gives the graph with largest possible eigenvalues. The spectrum of this graph is, though, not as easily calculable, but this is done in chapter 4 to achieve upper bounds on the spectrum. Lastly the results are summarized and an outlook on possible improvements and extensions of the results are presented.

Chapter 1

Introduction to quantum graphs

A quantum graph is a self-adjoint operator and it can be described as a **metric graph** equipped with a **differential operator** on the edges with a set of **matching conditions** at the vertices. This chapter will be devoted to define and present these three parts of a quantum graph one by one. After that the most straightforward technique to calculate the spectrum of a quantum graph is presented and a number of such calculations exemplified. Lastly the quadratic form for a quantum graph and the Rayleigh quotient formula for characterization of the spectrum are presented. For a deeper and more thorough introduction to quantum graphs see [11] and [2]. A good short survey article for quantum graphs is [9].

1.1 Definitions

1.1.1 Metric graphs

A definition of a metric graph starts with a set of edges. Let E be a set of compact intervals on the real line $\mathbb{R}$. All results in this text will cover compact finite graphs, but it is possible to also consider semi-infinite edges. Let each edge $e_n \in E$ be parametrized in the following way:

$$e_n = [x_{2n-1}, x_{2n}]$$

where $x_{2n-1} < x_{2n}$. From this the vertices can be defined as disjoint subsets of the set of all endpoints of the edges. Let $\mathcal{V}$ be the set of all endpoints i.e. $\mathcal{V} = \{x_n\}_{n=1}^{2N}$ where N is the number of edges. The set of all vertices can then be defined as a partition of $\mathcal{V}$ into M disjoint subsets, i.e.

$$\mathcal{V} = v_1 \cup v_2 \cup \ldots \cup v_M$$
$$v_i \cap v_j = \emptyset \quad \text{if } i \neq j.$$

The set of all vertices v_i will be denoted V. The number of endpoints connected at a vertex is called the degree of the vertex. We define an equivalence relation, the points x, y are said to be equivalent if either there exists an edge e_n such that $x, y \in e_n$ and $x = y$ or there exists a vertex v_i such that $x, y \in v_i$. With this equivalence it is possible and quite intuitive to define the metric graph Γ as the quotient set of all the edges with this equivalence relation,

$$\Gamma = E/_{x \sim y}.$$

In this text we will mostly focus on connected graphs, but in some intermediate steps we will also consider graphs formed by several disconnected components. A graph is said to be connected if any two points $x, y \in \Gamma$ can be connected with a continuous path. If a graph is disconnected it is often of interest to talk about the number of connected components.

Each edge e_n has the length which will be denoted by $l_n = x_{2n} - x_{2n-1}$ and particularly in this text an important parameter of a quantum graph is the total length $\mathcal{L}(\Gamma) = \sum_{n=1}^{N} l_n$. It is simple to define a metric on a connected graph where the distance $\rho(x, y)$ is simply the length of the shortest path between the two points. Since each edge is simply a subset of the real line it is straightforward to consider the Lebesgue measure dx on a graph.

We can now define two important function spaces on a quantum graph.

Definition. The Hilbert space $L_2(\Gamma)$ consists of all complex valued functions that are measurable and square integrable on each edge e_n. In other words, it is the direct sum of the spaces $L_2(e_n)$,

$$L_2(\Gamma) = \bigoplus_{n=1}^{N} L_2(e_n).$$

The scalar product is given by

$$\langle u, v \rangle = \int_\Gamma u \overline{v} dx.$$

Definition. The Sobolev space $W_2^2(\Gamma \setminus \mathcal{V})$ is the direct sum of the corresponding Sobolev spaces on all edges, $W_2^2(\Gamma \setminus \mathcal{V}) = \bigoplus_{n=1}^N W_2^2(e_n)$ i.e. the space of all functions which are square integrable and have square integrable second derivatives on each edge.

At this point it is important for the reader to highlight that the points of a quantum graph are all points on the edges and the vertices, not just the vertices which is custom for combinatorial graphs. It is, however, not necessarily reasonable to talk about the value of $f(v_i)$ since f need not attain the same value at all endpoints connected at the vertex. Since we have equated endpoints belonging to the same vertex it is therefore not initially well-defined what $f(x_j)$ means. To circumvent this problem we define the functions values at the endpoint as a limit, i.e. $f(x_j) = \lim_{x \to x_j} f(x)$ where the limit is taken from inside the interval. At the endpoints one often consider the normal derivatives

$$\partial(x_j) = \begin{cases} \lim_{x \to x_j} & x_j = x_{2n-1} \text{ for some edge } e_n \\ -\lim_{x \to x_j} & x_j = x_{2n} \text{ for some edge } e_n \end{cases}$$

that is, the derivative is always considered outwards from the vertex. This is helpful since the normal derivative is independent of the direction of the parametrization of the edge.

1.1.2 The differential operator

The second component of a quantum graph is a differential operator acting on the edges. This is what makes quantum graphs suitable for the study of wave propagation. There are three different kinds of differential operators generally considered:

the **Laplace operator**

$$L(f(x)) = -\frac{d^2}{dx^2} f(x),$$

the **Schrödinger operator**

$$L_q(f(x)) = \left(-\frac{d^2}{dx^2} + q(x) \right) f(x),$$

and the **magnetic Schrödinger operator**

$$L_{q,a}(f(x)) = \left(\left(i\frac{d}{dx} + a(x) \right)^2 + q(x) \right) f(x).$$

The magnetic Schrödinger operator describes quantum particles moving under the influence of a magnetic potential a and an electric potential q. The Laplace operator is the same as the Schrödinger operator with an electric potential identically equal to zero and the Schrödinger operator is the same as the magnetic Schrödinger with the magnetic potential identically equal to zero. So the Laplace operator describes a quantum particle moving freely and the Schrödinger operator describes quantum particles moving under the influence of just an electric potential. In what follows we will only consider the **Laplace** operator L even though much should be possible to extend to the other two types of operators as well.

An operator is not well defined without a domain, and the domain of the operators play crucial role in our analysis. Two operators of interest are the **minimal** and the **maximal** operators given by the smallest and biggest reasonable domains one can associate with the differential expression. The minimal operator is defined on smooth functions, i.e. infinitely differentiable functions, with support separated from the endpoints. The maximal operator is defined on the domain of all functions in the Hilbert space $L_2(\Gamma)$ which are mapped to functions still in $L_2(\Gamma)$. Formally the domains of the operators can be described as

$$\mathrm{Dom}(L^{min}(\Gamma)) = C_0^\infty(\Gamma \setminus V)$$

and

$$\mathrm{Dom}(L^{max}(\Gamma)) = \bigoplus_{n=1}^{N} W_2^2(e_n)$$

where the Sobolev space $W_2^2(e_n)$ is the space of all square integrable functions which has square integrable second derivatives.

The minimal operator can quite easily be shown to be symmetric by integra-

tion by parts. Let $u, v \in \mathrm{Dom}(L^{min}(\Gamma)) = C_0^\infty(\Gamma \setminus V)$.

$$\begin{aligned}
\langle L^{min}u, v\rangle &= \sum_{n=1}^{N} \int_{e_n} -u''(x) \cdot \overline{v(x)} dx \\
&= \sum_{n=1}^{N} \int_{e_n} -u'(x)\overline{v'(x)} dx + \Big(u'(x_{2n-1})\overline{v(x_{2n-1})} - u'(x_{2n})\overline{v(x_{2n})}\Big) \\
&= \sum_{n=1}^{N} \int_{e_n} -u'(x)\overline{v'(x)} dx \\
&= \sum_{n=1}^{N} \int_{e_n} u(x)\overline{(-v''(x))} dx + \Big(u(x_{2n-1})\overline{v'(x_{2n-1})} - u(x_{2n})\overline{v'(x_{2n})}\Big) \\
&= \sum_{n=1}^{N} \int_{e_n} u(x)\overline{(-v''(x))} dx \\
&= \langle u, L^{min}v\rangle.
\end{aligned}$$

The boundary terms disappear since the functions u, v and their derivatives are equal to zero at the endpoints.

The maximal operator is not symmetric, following a similar calculation as for the minimal operator gives, for $u, v \in \mathrm{Dom}(L^{max}) = \bigoplus_{n=1}^{N} W_2^2(e_n)$,

$$\begin{aligned}
&\langle L^{max}u, v\rangle - \langle u, L^{max}v\rangle = \\
&= \sum_{n=1}^{N} \int_{e_n} -u''(x) \cdot \overline{v(x)} dx - \int_{e_n} u(x) \cdot \overline{-v''(x)} dx \\
&= \sum_{n=1}^{N} \int_{e_n} -u'(x)\overline{v'(x)} dx - \int_{e_n} u'(x)\overline{-v'(x)} dx + \\
&+ \Big(u'(x_{2n-1})\overline{v(x_{2n-1})} - u'(x_{2n})\overline{v(x_{2n})}\Big) - \Big(u(x_{2n-1})\overline{v'(x_{2n-1})} - u(x_{2n})\overline{v'(x_{2n})}\Big) \\
&= \sum_{x_j \in \mathcal{V}} \partial u(x_j) \cdot \overline{v(x_j)} - u(x_j) \cdot \overline{\partial v(x_j)} \qquad (1.1)
\end{aligned}$$

which is not in general zero, so the maximal operator is not necessarily symmetric.

Notice that neither the minimal nor the maximal operator does in any way reflect how the edges are connected. The maximal operator can be shown to be the adjoint of the minimal operator, so the minimal operator is not self-adjoint.

The domain of any self-adjoint operator associated with $L = -\frac{d^2}{dx^2}$ in $L_2(\Gamma)$ should definitely contain the domain of the corresponding minimal operator

and be contained in the domain of the corresponding maximal operator. Any self-adjoint operator, corresponding to the same differential expression on the edges, can in fact be characterized by restricting the domain of the maximal operator in a suitable way. At each vertex there should be as many conditions as the degree of the vertex, and theses conditions should be such that the boundary terms in (1.1) cancel. It is first now that the connectivity of the graph is reflected in the operator since the conditions should only connect the values at the end-points belonging to the same vertex. These conditions are what is called the matching conditions of the quantum graph.

1.1.3 Matching conditions

The role of the matching conditions is two-fold, they are necessary to make the operator self-adjoint and they reflect how the edges are connected. The choice of matching conditions often reflect a physical interpretation, as we will see later. In many cases it is of interest to distinguish between matching and boundary conditions, where the boundary conditions are the conditions at all vertices of degree one and the matching conditions are the conditions at inner vertices, i.e vertices of degree two or more. This distinction is mostly relevant for studies of inverse problems, dynamic control and other such problems where the boundary vertices play a different role than the inner vertices. In this text such a distinction is not of interest and therefore all conditions will simply be called matching conditions.

As was stated before we need as many conditions as we have endpoints at the vertices to assert the self-adjointness of the operator. For this purpose it is relevant to rewrite the sum of the boundary terms in (1.1) as

$$\langle L^{max}u, v\rangle - \langle u, L^{max}v\rangle = \sum_{m=1}^{M}\left(\sum_{x_j\in v_m}\partial u(x_j)\cdot\overline{v(x_j)} - u(x_j)\overline{\partial v(x_j)}\right)$$

so that the sum is taken over each vertex separately. From this representation it follows that it is enough to consider the vertices separately since the matching conditions for each vertex should only connect the values of the function at the end-points belonging to the vertex. It is in fact possible to parametrize all possible matching conditions in several different ways suitable in different cases. See [11] or [2] for a detailed description. We will in this text only consider the three most common types of matching conditions. Some basic facts about these matching conditions, for example the self-adjointness of the corresponding operator, will merely be stated here. For proofs and deeper investigation again please consult [11] or [2].

Dirichlet conditions

The simplest condition at a vertex v is perhaps the Dirichlet matching conditions. It states that

$$u(x_j) = 0, \quad \forall x_j \in v.$$

Notable with this condition is that the condition is actually a condition at each separate end-point at the vertex independent of the other end-points at the vertex. If a quantum graph has Dirichlet matching conditions at all vertices then that quantum graph is essentially a collection of independent intervals with separate conditions on all endpoints. The operator with Dirichlet conditions at all vertices is not the same as the minimal operator, note the difference between compact support inside the interval and just the condition of attaining the value 0 at the end-point. The obtained operator from such a quantum graph is self-adjoint.

Standard conditions

The standard conditions are also known as Neumann conditions. The standard conditions at a vertex v state that

$$\begin{cases} u(x_i) = u(x_j) \quad \forall x_i, x_j \in v \\ \sum_{x_k \in v} \partial u(x_k) = 0. \end{cases}$$

Here $u(v_m)$ is well defined, since the conditions require that u attains the same value at all end-points connected at the vertex. The first condition means that the function $u(x)$ is continuous at the vertex v_m. If the vertex is of degree one this simply translates into the usual Neumann condition at the end-point, $u'(x_j) = 0$, and this is why these matching conditions are sometimes called Neumann conditions.

Consider a point x inside an edge. At that point any function $u(x)$ in the domain $W_2^2(\Gamma \setminus V)$ must be continuous and so must its derivative. This is exactly what the standard conditions require at a vertex of degree two. So instead of that whole edge we might as well consider the same graph but with that edge divided into two edges connected at a vertex with standard conditions. The standard conditions are the only case when this is true. That is why it is natural to call it the standard conditions, these are the conditions that must hold at all points besides the vertices.

The fact that vertices of degree two with standard conditions can be inserted or removed without affecting the spectrum is crucial for our analysis further on.

δ conditions

δ-conditions is a broader class of conditions that include the standard conditions. δ-conditions of strength α_v at a vertex v are given by

$$\begin{cases} u(x_i) = u(x_j) \quad \forall x_i, x_j \in v \\ \sum_{x_k \in v} \partial u(x_k) = \alpha_v \cdot u(v). \end{cases}$$

The value α_v is called the strength of the δ-condition and is assumed to be a real number. If the constant α_v is equal to zero then these conditions coincide with the standard matching conditions. The function is still required to be continuous at the vertex but the derivative is not necessarily continuous at a vertex of degree two, in fact if $\alpha_v \neq 0$ the derivative is only continuous if the value of the function at the vertex is zero. The interpretation of δ-conditions is that there is a point-potential situated at the vertex. Such a potential is often called a δ-potential and that's why the conditions are called δ-matching conditions.

This text will investigate certain bounds on the eigenvalues of quantum graphs with δ-conditions, while standard and to some extent Dirichlet conditions will play a supporting role. Throughout the whole text, if nothing else is mentioned, the strengths α_v will always be assumed to be non-negative.

1.2 Calculating the spectrum

The spectrum is perhaps the most important, interesting and well studied characteristic of quantum graphs. The eigenvalues of an self-adjoint operator H on a Hilbert space, over $\mathbb{C}$, are given by all $\lambda \in \mathbb{C}$ for which the equation

$$H(u) = \lambda u.$$

has a non-trivial solution u. The pair λ and u are called eigenvalue and eigenfunction, when the operator is acting on a space of functions, respectively. In many cases there are a number of linearly independent solutions u and in that case that number is called the multiplicity of the eigenvalue.

This is all very similar to the definition of eigenvalues and eigenvectors for linear operators in finite dimensional spaces.

The spectrum of an operator H is somewhat more complicated and given by the set of all $\lambda \in \mathbb{C}$ such that the operator $(H - \lambda I)$ does not have an inverse that is a bounded linear operator. If λ is an eigenvalue then the operator $(H - \lambda I)$ is not one-to-one and can therefore not have an inverse. It is, however, not necessarily so that the all λ:s in the spectrum are eigenvalues for the operator. If this is the case, and $\{\lambda_n\}$ has no finite accumulation point, one says that the spectrum of the operator is pure discrete.

The spectrum of finite compact quantum graphs is in fact pure discrete. This is something we will use in this text without a proof. The question is handled in other texts such as [11] and [2].

It is in general very difficult to calculate the spectrum of a quantum graph explicitly and many different techniques for calculating it exist. When the differential operator is the Laplace operator, i.e. when there is no magnetic or electric potential involved, calculating the spectrum is at least in theory rather straightforward. We will see later that all the eigenvalues of a quantum graphs with δ-conditions of non-negative strength are non-negative. We can therefore assume $\lambda = k^2$, for some $k \in \mathbb{R}$. As a consequence all the solutions to the differential equation

$$\begin{aligned} L(u(x)) &= \lambda u(x) \\ -\frac{d^2u(x)}{dx^2} &= \lambda u(x) \end{aligned}$$

are given by $u(x) = A_i \sin(kx) + B_i \cos(kx)$ for $\lambda \neq 0$, where A_i and B_i depend on the edge e_i. If $\lambda = 0$ is an eigenvalue then the solution is instead given by a linear function $u(x) = a_i x + b_i$, the second derivative of a linear function is always 0. The question of calculating the spectrum is reduced to finding all k for which it is possible to choose all A_i and B_i such that $u(x)$ is a function from the domain and then look for a eigenfunction for $\lambda = 0$ separately. Since all such $u(x)$ will be in $W_2^2(\Gamma \setminus \mathcal{V})$ the only requirements left to satisfy are the matching conditions. The following simple examples illustrates the procedure described.

Example 1. *Calculate the spectrum of the quantum graph consisting of a single interval of length $\mathcal{L}$ and with standard conditions at each end-point.*

Let the interval be parametrized from 0 to $\mathcal{L}$.

Figure 1.1: The interval of length $\mathcal{L}$.

We first look for a possible solution for $\lambda = 0$, $u(x) = ax+b$ for some $a, b \in \mathbb{R}$. Since both vertices are of degree one the matching condition is simply

$$u'(x) = 0$$

at the endpoints. Since $u'(x) = a$ this implies that $a = 0$ so the eigenfunction is in fact a constant function. The first eigenvalue and eigenfunction is given by

$$\lambda_0 = 0, \quad u_0 = b.$$

Next we consider $\lambda \neq 0$. As has been concluded earlier $u(x)$ must be of the form

$$u(x) = A\sin(kx) + B\cos(kx)$$

for some A, B and k. The derivative of $u(x)$ must therefore be

$$u'(x) = kA\cos(kx) - kB\sin(kx).$$

The matching condition at the first vertex $x = 0$, gives

$$\begin{aligned} u'(x) &= 0 \\ &\Leftrightarrow \\ kA\cos(0) - kB\sin(0) &= 0 \\ &\Leftrightarrow \\ kA &= 0 \\ &\Leftrightarrow \\ A &= 0 \end{aligned}$$

since $k \neq 0$, so $u(x) = B\cos(kx)$. At the other vertex the matching condition becomes

$$u'(\mathcal{L}) = B\sin(k\mathcal{L}) = 0.$$

Since we are looking only for non-trivial solutions $u(x)$ it follows that $B \neq 0$, so for the matching condition to be fulfilled it must be true that $\sin(k\mathcal{L}) = 0$. This is true for $k = \frac{\pi}{\mathcal{L}} + \frac{n\pi}{\mathcal{L}}$. The spectrum is therefore given by

$$\begin{aligned} \lambda_0 &= 0 \\ \lambda_n &= \left(\frac{\pi}{\mathcal{L}} + \frac{n\pi}{\mathcal{L}}\right)^2, \quad n = 0, 1, 2 \dots \end{aligned}$$

Example 2. *Calculate the spectrum of the quantum graph consisting of a single interval of length $\mathcal{L}$ and with a δ-condition with strength $\alpha \neq 0$ at one end-point and a standard condition at the other.*

Parametrize the interval from 0 to $\mathcal{L}$ with the standard condition at 0.

Figure 1.2: The interval of length $\mathcal{L}$ with one standard condition and one δ-condition.

We first look for the possible eigenvalue $\lambda = 0$ with $u(x) = ax + b$. The standard matching condition again requires $u'(0) = 0$, and since $u'(x) = a$ this implies that $u(x) = b$. However, at the other vertex there is a δ-condition of strength $\alpha \neq 0$. This condition requires

$$u'(\mathcal{L}) = \alpha u(\mathcal{L})$$

but since $u'(x) = 0$ and $\alpha \neq 0$ this would imply that $u(\mathcal{L}) = b = 0$ so that $u(x) = 0$. This is though a trivial solution and therefore not an eigenfunction. Therefore $\lambda = 0$ is not an eigenvalue if $\alpha \neq 0$.

For $\lambda > 0$, where $\lambda = k^2$, any solution to the equation $-\frac{\partial^2}{\partial x^2}u = \lambda u$ on the interval is again of the form

$$u(x) = A\sin(kx) + B\cos(kx),$$

and

$$u'(x) = kA\cos(kx) - kB\sin(kx).$$

The standard condition states that $u'(0) = 0$ which implies that $A = 0$, so therefore the solution must be of the form $u(x) = B\cos(kx)$.

The δ-condition states that $\partial u(k\mathcal{L}) = \alpha u(k\mathcal{L})$. Since the edge is incoming at $x = \mathcal{L}$ the normal derivative is

$$\partial_n u(\mathcal{L}) = -u'(\mathcal{L}) = kB\sin(k\mathcal{L}).$$

The matching condition becomes

$$kB\sin(k\mathcal{L}) = \alpha B\cos(k\mathcal{L})$$

which is only non-trivially solvable when $\cos(k\mathcal{L})$ is not zero. Then it is possible to divide both sides by the factor $B\cos(k\mathcal{L})$ and the equation becomes

$$k\tan(k\mathcal{L}) = \alpha.$$

In conclusion the spectrum is given by $\lambda_n = k_n^2$ where k_n is the n:th nonzero solution to the equation $k\tan(k\mathcal{L}) = \alpha$.

Notice that $\lambda_n(\Gamma_1) < \lambda_n(\Gamma_2)$ where Γ_1 is the interval with only standard conditions and Γ_2 is the interval with one δ-condition with positive strength. Also note that for Γ_2, if α increases so does each $\lambda_n(\Gamma_2)$. This is in fact something that holds generally for all quantum graphs with δ-conditions, and we will prove this later in the text.

The second thing to observe is that when $\mathcal{L}$ increases λ_n decreases for both graphs. This is also something that translates to a general fact for all quantum graphs with δ-conditions. These two properties are why we will be looking for universal bounds on graphs given a total length and total strength of the graph. If we do not make these restrictions there are no bounds to found since no bounds exist.

The following two examples and the previous will be used explicitly in the derivation of the universal bounds on the spectrum.

Example 3. *Calculate the spectrum of the star graph with n edges of the same length* $\frac{\mathcal{L}}{n}$*, and with standard matching conditions at the vertices.*

The edges are parametrized from the outer vertices to the inner. Se figure 1.3 for an example.

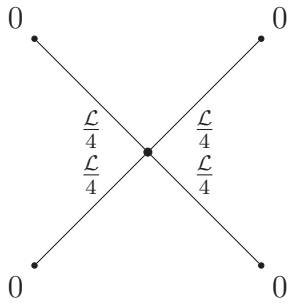


Figure 1.3: Example of a star graph with four edges.

For $\lambda = 0$ the eigenfunction must be of the form $u(x) = a_i x + b_i$, where a_i, b_i depend on the edge e_i. At the outer vertices, the vertices of degree one, the

conditions requires that $u'(0) = 0$ on all edges. However, since $u'(x) = a_i$ for all x this means that $u(x)$ is in fact a constant function $u(x) = b_i$ on each edge. At the middle vertex the continuity condition requires

$$u_i(\frac{\mathcal{L}}{n}) = u_j(\frac{\mathcal{L}}{n}), \quad \forall i, j \in \{1, 2, \ldots, n\}.$$

and since $u_i(x) = b_i$ this means that the constant functions on all edges must be equal, $b_i = b_j$ for all i, j. The derivative condition is trivially satisfied since

$$\sum_i u'(\frac{\mathcal{L}}{n}) = \sum_i 0 = 0.$$

Since the function is determined on all edges up to the multiplication of a constant, $\lambda = 0$ is an eigenvalue of multiplicity 1.

Consider $\lambda \neq 0$, then the solutions are given by $u(x) = A_i \sin(kx) + B_i \cos(kx)$ on all edges e_i. Since the standard conditions state that the normal derivative must be zero at the outer vertices and

$$u'(0) = kA_i \cos(0) - kB_i \sin(0) = kA_i$$

it follows that $A_i = 0$ for $k \neq 0$ so $u(x) = B_i \cos(kx)$. The continuity condition gives that

$$B_i \cos\left(k\frac{\mathcal{L}}{n}\right) = B_j \cos\left(k\frac{\mathcal{L}}{n}\right)$$

so $B_i = B_j = B$ for all i, j if $\cos\left(k\frac{\mathcal{L}}{n}\right) \neq 0$. Assume $\cos\left(k\frac{\mathcal{L}}{n}\right) \neq 0$, then the standard condition at the middle vertex states

$$\sum_i kB \sin\left(k\frac{\mathcal{L}}{n}\right) = nkB \sin\left(k\frac{\mathcal{L}}{n}\right) = 0$$

so for $B \neq 0$ the solutions are given by

$$k_m = m\frac{n\pi}{\mathcal{L}}, \quad m = 1, 2, \ldots$$

The corresponding eigenvalues $\lambda = k_m^2$ has multiplicity one since all B_i:s are decided up to multiplication with a constant.

If $\cos\left(k\frac{\mathcal{L}}{n}\right)$ is zero, i.e. $k = \left(\frac{n\pi}{2\mathcal{L}}\right) + m\frac{\pi}{\mathcal{L}}$, then B_i and B_j can attain different values without violating the continuity condition. The conditions on the

derivatives at the middle vertex states that

$$\sum_i kB_i \sin\left(k\frac{\mathcal{L}}{n}\right) = 0$$
$$\Rightarrow$$
$$\sum_i B_i \sin\left(k\frac{\mathcal{L}}{n}\right) = 0$$

which is always solvable since $\sin\left(k\frac{\mathcal{L}}{n}\right) = \pm 1$. The equation has which has $n-1$ linearly independent solutions since there are n values B_i that can be chosen. To summarize the spectrum is given by

$$\lambda_0 = 0$$
$$\lambda_{m\cdot n} = \left(\frac{m\cdot n\pi}{\mathcal{L}}\right)^2, \quad m = 1, 2, \dots$$
$$\lambda_{m\cdot n+1} = \lambda_{m\cdot n+2} \dots = \lambda_{(m+1)\cdot n-2} = \lambda_{(m+1)\cdot n-1} = \left(\left(\frac{n\pi}{2\mathcal{L}}\right) + m\frac{\pi}{\mathcal{L}}\right)^2, \quad m = 0, 1, 2, \dots$$

Example 4. *Calculate the spectrum of the following star graph with n edges. The edges are all of the same length* $\frac{\mathcal{L}}{n}$ *and the matching conditions are standard conditions at all vertices but the middle vertex where it is a* δ*-condition with strength* $\alpha > 0$*.*

The solution is similar to that of example 3. Let the edges are parametrized in the same way. For $\lambda = 0$ and the corresponding possible eigenfunction $u(x) = a_i x + b_i$ the standard conditions at the vertices of degree one still implies that $u(x) = b_i$ while the continuity condition at the middle vertex requires that $b_i = b_j$ for all i, j. However, the derivative condition at the middle vertex v_0 requires that

$$\sum_i u'(\frac{\mathcal{L}}{n}) = \alpha u(v_0),$$

but $u'(x) = 0$ for all x so this is only possible if $u(x) = 0$, but this is a trivial solution and therefore not an eigenfunction. As a consequence, $\lambda = 0$ is not an eigenvalue.

Let $\lambda > 0$, it follows that the solutions are given by $u(x) = B_i \cos(kx)$ since standard conditions still apply at the outer vertices. At the middle there are two cases: $\cos\left(k\frac{\mathcal{L}}{n}\right) = 0$ and $\cos\left(k\frac{\mathcal{L}}{n}\right) \neq 0$. If $\cos\left(k\frac{\mathcal{L}}{n}\right) = 0$ the solutions are unchanged since $\alpha\cdot 0 = 0$. If $\cos\left(k\frac{\mathcal{L}}{n}\right) \neq 0$ the solutions differ. The continuity

condition still implies that $B_i = B_j = B$, but the derivative condition now becomes

$$\begin{aligned}\sum_i kB\sin\left(k\frac{\mathcal{L}}{n}\right) &= \alpha B\cos\left(k\frac{\mathcal{L}}{n}\right)\\ nk\sin\left(k\frac{\mathcal{L}}{n}\right) &= \alpha\cos\left(k\frac{\mathcal{L}}{n}\right)\\ nk\tan\left(k\frac{\mathcal{L}}{n}\right) &= \alpha.\end{aligned}$$

This equation will always have a solution on the interval $\left(0, \frac{n\pi}{2\mathcal{L}}\right)$ since $\tan(x)$ goes from zero to infinity on the interval $\left(0, \frac{\pi}{2}\right)$. In fact there will be exactly one solution in each interval

$$\left(\left(\frac{n\pi}{2\mathcal{L}}\right) + m\frac{\pi}{\mathcal{L}}, \left(\frac{n\pi}{2\mathcal{L}}\right) + (m+1)\frac{\pi}{\mathcal{L}}\right) \quad m = 0, 1, 2, \ldots$$

so each of the eigenvalues λ_{nm} are given by the solution of the equation

$$nk\tan\left(k\frac{\mathcal{L}}{n}\right) = \alpha \tag{1.2}$$

on the corresponding interval. To summarize

$$\lambda_{m\cdot n} = k_m^2 \quad m = 0, 1, 2, \ldots$$

$$\lambda_{m\cdot n+1} = \lambda_{m\cdot n+2} \ldots = \lambda_{(m+1)\cdot n-2} = \lambda_{(m+1)\cdot n-1} = \left(\left(\frac{n\pi}{2\mathcal{L}}\right) + m\frac{\pi}{\mathcal{L}}\right)^2, \quad m = 0, 1, 2, \ldots$$

where k_m are the ordered positive solutions to equation 1.2.

The two preceding examples give rise to the first observation about universal bounds formulated in the following corollary.

Corrolary 1. *There is no upper bound on λ_n for $n \geq 1$, nor on the difference $\lambda_1 - \lambda_0$, given only a total length $\mathcal{L}$ and a total strength α.*

Proof. Consider the star graph in Example 4. As the number of edges, n, goes to infinity, so does $\lambda_1 = \left(\frac{n\pi}{2\mathcal{L}}\right)^2$. The same is, however, not true for the lowest eigenvalue λ_0. To see why the lowest eigenvalue is bounded when the number of edges tend to infinity consider the limit

$$\lim_{n\to\infty} nk\tan\left(\frac{k\pi\mathcal{L}}{n}\right) = k^2\pi\mathcal{L}$$

and notice that for each α the solution to $k^2\pi\mathcal{L} = \alpha$ is bounded. This implies that the difference $\lambda_1 - \lambda_0$, which is called the spectral gap and is of special importance, is in general not bounded given just a total length and total strength. □

As a consequence of this there should be some further restriction on the quantum graph, besides the total length and total strength, to find a meaningful upper bound. In this text we will later use the set of edges for this. In other words the upper bounds will be found given a total strength and a set of edges.

The last example we will consider, which will also be used in the derivation later, is the single loop graph with one vertex with a δ-condition. But first a useful tool for simplifying the calculation of the spectrum of graphs with some kind of symmetry will be presented.

Proposition 1. *Let A and B be two commuting self-adjoint operators. Then the eigenfunctions of A and B can be chosen to be equal.*

Proof. For our purposes we prove this proposition for a bounded operator B with discrete spectrum. Then the operator B is defined on the whole Hilbert space $\mathcal{H}$. Moreover we assume that B has only two non-equal eigenvalues, μ_0, μ_1 but the argument extends to any number of eigenvalues. This is quite a big restriction of the theorem, but it is sufficient for all purposes in this text and the proof is greatly simplified.

Any function u in the Hilbert space $\mathcal{H}$ can be written as a sum of eigenfunctions of B, $u = u_0 + u_1$ and $Bu = \mu_0 u_0 + \mu_1 u_1$. We can write $\mathcal{H}$ as an orthogonal sum of the two eigenspaces,

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \qquad (*)$$

and we define the two projectors P_0, P_1 to be the projectors onto the two subspaces. B can be written in the block-operator form with respect to the orthogonal decomposition $(*)$ as

$$B = \left(\begin{array}{c|c} \mu_0 & 0 \\ \hline 0 & \mu_1 \end{array}\right).$$

The first step is to prove that if $u \in \mathrm{Dom}(A)$ then $P_0 u$ and $P_1 u$ lie in $\mathrm{Dom}(A)$. By assumption $AB = BA$ and thus the domains of AB and BA coincides.

It follows that Bu must lie in $\mathrm{Dom}(A)$. If $u \in \mathrm{Dom}(A)$, then we can write it as

$$u = u_0 + u_1$$

and we know that

$$Bu = \mu_0 u_0 + \mu_1 u_1.$$

Any linear combination of functions in $\mathrm{Dom}(A)$ also lies in $\mathrm{Dom}(A)$, so if we can write u_0 and u_1 as a linear combinations of for example u and Bu we know that u_0 and u_1 lies in $\mathrm{Dom}(A)$. Such linear combinations do exist. Assume that both μ_0 and μ_1 are non zero, then

$$u_0 = \frac{u - \frac{Bu}{\mu_1}}{1 - \frac{\mu_0}{\mu_1}},$$

$$u_1 = \frac{u - \frac{Bu}{\mu_0}}{1 - \frac{\mu_1}{\mu_0}}$$

are such linear combinations. Since $\mu_0 \neq \mu_1$ both of them can not be zero, let μ_0 be the non zero eigenvalue. Then the sought for linear combinations are

$$u_0 = \frac{Bu}{\mu_0}$$

and

$$u_1 = u - \frac{Bu}{\mu_0}.$$

As a consequence A can also be written in the block-operator form with respect to the orthogonal decomposition $(*)$

$$A = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right)$$

where α is an operator from $P_0\mathrm{Dom}(A)$ to $\mathcal{H}_0$, β an operator from $P_1\mathrm{Dom}(A)$ to $\mathcal{H}_0$ and so on.

Since $AB = BA$ it must hold that

$$\left(\begin{array}{c|c} \mu_0\alpha & \mu_1\beta \\ \hline \mu_0\gamma & \mu_1\delta \end{array}\right) = \left(\begin{array}{c|c} \mu_0\alpha & \mu_0\beta \\ \hline \mu_1\gamma & \mu_1\delta \end{array}\right)$$

which implies that $\beta = \gamma = 0$ since $\mu_0 \neq \mu_1$. So

$$A = \left(\begin{array}{c|c} \alpha & 0 \\ \hline 0 & \delta \end{array}\right)$$

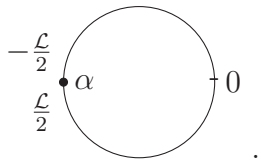
which means that the eigenvalues and eigenfunctions can be found for α and δ independently such that all eigenfunctions for α lie in $\mathcal{H}_0$ and all eigenfunctions for δ in $\mathcal{H}_1$. Hence the eigenfunctions of A can be chosen such that they are eigenfunctions of B as well.

□

We can now use this proposition in the next example.

Example 5. *Calculate the spectrum of the single loop graph S of length $\mathcal{L}$, and a single vertex which has a δ-condition of strength $\alpha \neq 0$.*

Parametrize the single loop graph S in the following way:



.

For the possible eigenvalue $\lambda = 0$ the corresponding eigenfunction must be of the form $u(x) = ax + b$. The continuity condition at the vertex requires that

$$\begin{aligned} u(\frac{\mathcal{L}}{2}) &= u(-\frac{\mathcal{L}}{2}) \\ \frac{a\mathcal{L}}{2} + b &= \frac{-a\mathcal{L}}{2} + b \\ \frac{a\mathcal{L}}{2} &= \frac{-a\mathcal{L}}{2} \end{aligned}$$

which is only true if $a = 0$. This in turn implies that $u(x)$ is a constant function $u(x) = b$ so $u'(x) = 0$. The derivative condition

$$\begin{aligned} u'(\frac{-\mathcal{L}}{2}) - u'(\frac{\mathcal{L}}{2}) &= \alpha u(v) \\ 0 - 0 &= \alpha b \end{aligned}$$

which is only true if $\alpha = 0$ or $b = 0$, both which is impossible since $\alpha > 0$ and $b = 0$ would make $u(x) = 0$ and therefore not a nontrivial function. We conclude that $\lambda = 0$ is in fact not an eigenvalue.

Let R_Γ be the reflecting operator which takes $u(x)$ to $u(-x)$. Then

$$RL = LR$$

since

$$(LR(u))(x) = -u''(-x) = (RL(u))(x).$$

We want to show that the domains of LR and RL coincide. Let $u(x)$ be a function in the domain of L, i.e. a function from $W(\Gamma \setminus V)$ satisfying the matching conditions. With the given parametrization Ru must satisfy the matching conditions if u does, the same equations are obtained. As a consequence we can apply Proposition 1.

The eigenvalues of the reflecting operator are easily calculated since

$$\left(R^2(u)\right)(x) = u(-(-x)) = u(x) = (I(u))(x)$$

where I is the identity operator. The only possible eigenvalues of R are therefore ± 1 since the only eigenvalue of I is 1. The eigenfunctions corresponding to the eigenvalue 1 must satisfy the equation

$$u(-x) = 1 \cdot u(x),$$

which is exactly the definition of an even function. The eigenfunctions corresponding to the eigenvalue -1 must instead satisfy the equation

$$u(-x) = -1 \cdot u(x),$$

which is the definition of an odd function. As consequence the eigenfunctions of R must be either even or odd. By Proposition 1 the eigenfunctions of R and L can be chosen such that they coincide. It follows that all the eigenfunctions of L can be chosen such that they are either even or odd functions.

The odd eigenfunctions must be of the form $A\sin(kx)$ and all even functions of the form $B\cos(kx)$. Every odd continuous function must be zero at $x = \pm\frac{\mathcal{L}}{2}$. The derivative conditions are automatically satisfied for odd functions. $u(\frac{\mathcal{L}}{2}) = 0$ implies that

$$k = \frac{2n\pi}{\mathcal{L}}, \quad n \in \mathbb{N}.$$

At the vertex v every even eigenfunction will automatically satisfy the continuity condition. The derivative condition gives the following condition for the corresponding eigenvalues:

$$2Bk\sin\left(\frac{k\mathcal{L}}{2}\right) = \alpha B\cos\left(\frac{k\mathcal{L}}{2}\right) \quad \Leftrightarrow \quad k\tan\left(\frac{k\mathcal{L}}{2}\right) = \frac{\alpha}{2}. \tag{1.3}$$

The function $k \tan\left(\frac{k\mathcal{L}}{2}\right)$ is piece-wise increasing and has a singularity for each $k = \frac{2n\pi}{\mathcal{L}}$. In other words there will be exactly one eigenvalue corresponding to an even eigenfunction between every consecutive pair of eigenvalues corresponding to odd eigenfunctions and vice versa. Since $0 \cdot \tan\left(\frac{0 \cdot \mathcal{L}}{2}\right) = 0$ it follows that there must exist some solution to the equation in the interval $[0, \frac{2\pi}{\mathcal{L}})$, so λ_0 corresponds to an even solution. To summarize

$$\lambda_{2n} = k_n^2, \quad n = 0, 1, 2, \ldots$$

where k_n is the n:th solution to equation (1.3) and

$$\lambda_{2n+1} = \left(\frac{2n\pi}{\mathcal{L}}\right)^2, \quad n = 0, 1, 2, \ldots$$

1.3 Rayleigh quotient and quadratic forms

Central tools in many of the proofs presented later will be the quadratic form of an operator and the Rayleigh quotient. The quadratic form of an operator is defined by

$$\langle Lu, u \rangle.$$

The **Rayleigh quotient** $R(u)$ is defined by

$$R(u) = \frac{\langle Lu, u \rangle}{\langle u, u \rangle}.$$

The eigenvalues and eigenfunctions of an operator can be obtained by minimizing the Rayleigh quotient. Before the characterization of the eigenvalues by the Rayleigh quotient is presented the quadratic form will be described in more detail.

For the space $L_2(\Gamma)$ the inner product is given by $\langle u, v \rangle = \int_\Gamma u(x)\overline{v(x)}dx$, so

$$\langle Lu, u \rangle = \int_\Gamma -u''(x)\overline{u(x)}dx.$$

By partial integration it is possible to rewrite that expression so that it does only contain the first derivative.

Theorem 1. *The quadratic form of the Laplace operator on a quantum graph Γ with δ-conditions is given by*

$$\langle Lu, u\rangle_\Gamma = \int_\Gamma |u'|^2 dx + \sum_{v\in V} \alpha_v |u(v)|^2$$

where V is the set of vertices of Γ. And $\int_\Gamma$ means that the integral is taken over all the edges in Γ.

Proof. Let the e_i denote the edges and V the set of vertices of Γ. Then

$$\begin{aligned}
\langle \mathcal{L}u, u\rangle_\Gamma &= \int_\Gamma -u''(x)\overline{u(x)}dx \\
&= \int_\Gamma |u'|^2 dx + \sum_i u'(x_{2i-1})\overline{u(x_{2i-1})} - u'(x_{2i})\overline{u(x_{2i})} \\
&= \int_\Gamma |u'|^2 dx + \sum_i \partial u(x_{2i-1})\overline{u(x_{2i-1})} + \partial u(x_{2i})\overline{u(x_{2i})} \\
&= \int_\Gamma |u'|^2 dx + \sum_{v\in V} [\overline{u(v)} \underbrace{\sum_{x_k\in v} \partial u(x_k)}_{=\alpha_v u(v)}] \\
&= \int_\Gamma |u'|^2 dx + \sum_{v\in V} \alpha_v |u(v)|^2.
\end{aligned}$$

□

The expression

$$\int_\Gamma |u'|^2 dx + \sum_{v\in V} \alpha_v |u(v)|^2$$

is well-defined for all functions $u(x)$ which are continuous at all vertices and have a square integrable first derivative, i.e. all continuous functions in $W_2^1(\Gamma \setminus \mathcal{V})$. The domain of L is though restricted to functions in $W_2^2(\Gamma \setminus \mathcal{V})$ satisfying the matching conditions at all vertices.

This extended definition of the quadratic form with a larger domain is the one we will use in the rest of the text and is what will be meant by the expression $\langle Lu, u\rangle$. This bigger domain for the quadratic form on a graph Γ will be written as $\mathrm{Dom}_Q(\Gamma)$.

Theorem 2. *The spectrum of a quantum graph is given by minimizing the Rayleigh quotient in the following way.*

$$\lambda_0(\Gamma) = \min_{u\in Dom_Q(\Gamma)} \frac{\langle Lu, u\rangle}{\langle u, v\rangle} \tag{1.4}$$

and for $n \geq 1$ λ_n is given by any of the two following expressions

$$\lambda_n = \min_{\substack{u \in Dom_Q(\Gamma) \\ u \perp u_i, i < n}} \frac{\langle Lu, u \rangle}{\langle u, u \rangle} \tag{1.5}$$

$$\lambda_n = \max_{\substack{A_n \subset Dom_Q(\Gamma) \\ \dim(A_n) = n}} \left(\min_{\substack{u \perp A_n \\ u \in Dom_Q(\Gamma)}} \frac{\langle Lu, u \rangle}{\langle u, u \rangle} \right) \tag{1.6}$$

where u_i denotes the i:th eigenfunction. For each n, a function which minimizes the expression is an eigenfunction corresponding to that eigenvalue.

Proof. The proof is rather straightforward once certain properties of the Laplace operator on a finite compact metric graph are taken into account:

(i) The spectrum $\{\lambda_n\}_{n \in \mathbb{N}}$ is discrete and bounded from below, indexed such that $\lambda_i \leq \lambda_{i+1}$ and has a unique accumulation point at $+\infty$.

(ii) The eigenfunctions u_n can all be chosen orthogonal, i.e. $u_i \perp u_j$, for $i \neq j$. The eigenfunctions form an orthonormal basis for the Hilbert space $L_2(\Gamma)$. Note that this space is larger then the domain for the operator L.

In particular, any function $u(x)$ in the domain of the operator can be written as $u(x) = \sum_{n=0}^{\infty} \langle u, u_n \rangle u_n$ where u_n is the n:th normalized eigenfunction.

Using the decomposition in (ii) and the fact that $\lambda_0 \leq \lambda_i$ for all i the formula

for λ_0 can be proved.

$$\begin{aligned}
\frac{\langle Lu, u\rangle}{\langle u, u\rangle} &= \frac{\langle (\sum_{i=0}^{\infty} \lambda_i \langle u, u_i\rangle u_i), u\rangle}{\langle u, u\rangle} \\
&= \sum_{i=0}^{\infty} \frac{\langle \lambda_i \langle u, u_i\rangle u_i, u\rangle}{\langle u, u\rangle} \\
&= \sum_{i=0}^{\infty} \lambda_i \frac{\langle u, u_i\rangle \langle u_i, u\rangle}{\langle u, u\rangle} \\
&= \sum_{i=0}^{\infty} \lambda_i \frac{|\langle u, u_i\rangle|^2}{\langle u, u\rangle} \\
&\geq \lambda_0 \sum_{i=0}^{\infty} \frac{|\langle u, u_i\rangle|^2}{\langle u, u\rangle} \qquad (1.7) \\
&= \lambda_0 \frac{\langle u, u\rangle}{\langle u, u\rangle} \\
&= \lambda_0
\end{aligned}$$

To see that any function u that minimizes the expression is in fact an eigenfunction to λ_0 we first conclude that any eigenfunction u_0 does minimize the expression.

$$\frac{\langle Lu_0, u_0\rangle}{\langle u_0, u_0\rangle} = \frac{\langle \lambda_0 u_0, u_0\rangle}{\langle u_0, u_0\rangle} = \lambda_0 \frac{\langle u_0, u_0\rangle}{\langle u_0, u_0\rangle} = \lambda_0$$

To see that a function which minimizes the expression must in fact be an eigenfunction it is enough to consider the expansion and note that if any of the terms $\lambda_i \langle u, u_i\rangle u_i$ are non-zero for a $\lambda_i \neq \lambda_0$ then the inequality in (1.7) will be strict. And if all the terms $\lambda_i \langle u, u_i\rangle u_i$ for $\lambda_i \neq \lambda_i$ are zero, then u is by definition an eigenfunction for λ_0.

For λ_n we will only prove formula (1.6) since that is the one we will be using. Formula (1.5) can be proven in a similar manner.

Consider a subset A of $\mathrm{Dom}_Q(\Gamma)$ of dimension n. By an argument of dimension the following intersection

$$A^{\perp} \cap \mathrm{span}\{u_0, u_1, \ldots u_n\}$$

where $A^{\perp}$ is the orthogonal complement to A, must be non-empty. Choose a function $u \in A^{\perp} \cap \mathrm{span}\{u_0, u_1, \ldots u_n\}$. Since it is in the span of the first $n+1$ eigenfunctions it can be written as a linear combination of them, i.e.

$$u = \sum_{i=0}^{n} \langle u, u_i\rangle u_i.$$

Now, by looking at the Rayleigh quotient of this function we see that it is bounded from above by λ_n,

$$\begin{aligned}\frac{\langle Lu, u\rangle}{\langle u, u\rangle} &= \frac{\langle (\sum_{i=0}^{n} L\langle u, u_i\rangle u_i), u\rangle}{\langle u, u\rangle} \\ &= \frac{\langle (\sum_{i=0}^{n} \lambda_i\langle u, u_i\rangle u_i), u\rangle}{\langle u, u\rangle} \\ &= \sum_{i=0}^{n} \lambda_i \frac{|\langle u, u_i\rangle|^2}{\langle u, u\rangle} \\ &\leq \lambda_n \sum_{i=0}^{n} \frac{|\langle u, u_i\rangle|^2}{\langle u, u\rangle} \\ &= \lambda_n \frac{\langle u, u\rangle}{\langle u, u\rangle} \\ &= \lambda_n.\end{aligned}$$

Some middle steps have been excluded in the above calculations since they are very similar to those in the derivation for the formula for λ_0.

To see that equality is given exactly for the eigenfunctions first note that if the subset A is chosen to be $\text{span}\{u_0, u_1, \ldots, u_{n-1}\}$ then $R(u)$ is in fact λ_n. The value of $R(v)$ must be greater for any function v such that $\langle u, u_i\rangle u_i$ is not zero for some $i > n$ where $\lambda_i > \lambda_n$. If v is not such a function then by definition it is an eigenfunction of λ. If A is not chosen to be the span of the first $n-1$ eigenfunctions then it must be possible, since the eigenfunctions are orthogonal, to choose some function v in the intersection which would generate a lower $R(v)$ which would contradict the maximizing statement.

□

We now possess a powerful tool for investigating the eigenfunctions by the quadratic form of the operator.

The scalar product $\langle u, u\rangle$ is for a quantum graph given by

$$\langle u, u\rangle = \int_\Gamma u(x)\overline{u(x)}dx = \int_\Gamma |u(x)|^2 dx.$$

As before $\int_\Gamma$ means that the integral is taken over all the edges in Γ. Putting this expression together with the derived expression for the quadratic form

we get the following explicit expression for the Rayleigh quotient:

$$R(u) = \frac{\langle Lu, u\rangle}{\langle u, u\rangle} = \frac{\int_\Gamma |u'(x)|^2\,dx + \sum_v \alpha_v |u(v)|^2}{\int_\Gamma |u(x)|^2\,dx}. \tag{1.8}$$

Note that the Rayleigh quotient will always be non-negative if all the strengths are non-negative and as a consequence all eigenvalues of a quantum graph with δ-conditions of non-negative strengths are themselves non-negative. This is a fact we used in the examples.

Chapter 2

Altering the graph

One technique for getting estimates on the spectrum of a quantum graph Γ is to first investigate how certain alterations of the underlying metric graph and of the matching conditions change the spectrum and then use this knowledge to estimate the eigenvalues of $\mathcal{L}(\Gamma)$ by alter Γ into some other graph Γ' for which the eigenvalues are easy to calculate. In this chapter a number of such alterations and its effects are presented. Some of these results are new and some are taken from other texts. Then in chapter 3 and 4 these results will be used to find universal upper and lower bounds for the spectrum.

2.1 Alteration of the total length and the total strength

Any finite compact quantum graph Γ with δ-conditions, with $\alpha_v \geq 0$, have a total length and a total strength. These two values contain information about the spectrum. It is of interest to know how the spectrum is affected by changes in $\mathcal{L}(\Gamma)$ and $\alpha(\Gamma)$. This question has been studied in more detail in [3] and [4]. While we in this text are focused on what alterations of a quantum graph will always move the spectrum in one direction these two texts also cover more difficult cases where an alteration might move the spectrum in both directions depending on the underlying graph and characterizes when the spectrum will move either direction.

Theorem 3. *Given a quantum graph Γ with δ-conditions, with all $\alpha_v \geq 0$, construct a new graph Γ' with the same matching conditions but inequalitygth*

of one edge e_0 multiplied with a constant $c > 1$. The eigenvalues of the new graph is bounded from above by the corresponding eigenvalues of the old graph, $\lambda_n(\Gamma) \geq \lambda_n(\Gamma')$ for all $n = 0, 1, 2, \ldots$

Proof. Consider the formula for the eigenvalues given by the Rayleigh quotient. Take the function u which minimizes the expression for Γ and construct a corresponding function $f(x)$ in $\mathrm{Dom}_Q(\Gamma')$ by simply setting $f(x) = u(x/c)$ for the new edge e_0' and $f(x) = u(x)$ for the rest. We compare the expressions which enter in the Rayleigh quotient corresponding to the edge e_0,

$$\int_{e_0} |u(x)|^2 dx < \int_{e_0'} |f(x)|^2 dx = c \int_{e_0} |u(x)|^2 dx,$$

$$\int_{e_0} |u'(x)|^2 dx > \int_{e_0'} |f'(x)|^2 dx = \frac{1}{c} \int_{e_0} |u'(x)|^2 dx,$$

and since the values at the vertices are unchanged,

$$\sum_v \alpha_v u(v) = \sum_v \alpha_v f(v).$$

As a consequence of this we have

$$\begin{aligned}
R(u) &= \frac{\sum_{e \in E} \int_e |u'(x)|^2 dx + \sum_{v \in V} \alpha_v u(v)}{\sum_{e \in E} \int_e |u(x)|^2 dx} \\
&= \frac{\sum_{e \in E, e \neq e_0} \int_e |u'(x)|^2 dx + \int_{e_0} |u'(x)|^2 dx + \sum_{v \in V} \alpha_v u(v)}{\sum_{e \in E, e \neq e_0} \int_e |u(x)|^2 dx + \int_{e_0} |u(x)|^2 dx} \\
&\geq \frac{\sum_{e \in E, e \neq e_0} \int_e |u'(x)|^2 dx + \frac{1}{c} \int_{e_0} |u'(x)|^2 dx + \sum_{v \in V} \alpha_v u(v)}{\sum_{e \in E, e \neq e_0} \int_e |u(x)|^2 dx + c \int_{e_0} |u(x)|^2 dx} \\
&= \frac{\sum_{e \in E, e \neq e_0} \int_e |f'(x)|^2 dx + \int_{e_0'} |f'(x)|^2 dx + \sum_{v \in V} \alpha_v f(v)}{\sum_{e \in E, e \neq e_0} \int_e |f(x)|^2 dx + \int_{e_0'} |f(x)|^2 dx} \\
&= R(f)
\end{aligned}$$

and as described earlier this is enough to prove the theorem. Note also that if $u|_{e_0} \not\equiv 0$ then the inequality is strict. Notice that the function $f(x)$ does

not lie in the domain of the operator corresponding to the new quantum graph, but it lies in the domain of the quadratic form which is enough. This illustrates very nicely the technique we will use many times in this text. □

A deeper investigation of what happens when the edge lengths are varied can be found in [4], if α is not required to be non-negative the question is quite a lot more complicated and it is not always true that the spectrum decreases when the edge length increases.

This tells us, in a sense, that if the total length $\mathcal{L}(\Gamma)$ increases, the spectrum $\lambda_n(\Gamma)$ decreases. Of course it is not the case that all graphs with a larger total length will have a larger λ_n then all graphs with a shorter smaller length, but it is valid as a general principle. A quantum graph with larger total strength usually have a lower λ_n then a quantum graph with shorter total length. Also, by varying the length of Γ it is possible to let $\lambda_n(\Gamma)$ attain any positive value, something we will see later is not possible if the total length is held fixed.

A similar relationship holds for the total strength $\alpha(\Gamma)$. It is, however, not as strict and it is actually possible to increase all α_v for a graph without affecting the spectrum in some very special cases. However, before we turn to that question we make the following crucial observation.

Observation 1. *Let Γ and Γ' be two quantum graphs with the same underlying metric graph but with different strengths α_v in the matching conditions. Then the domains of the quadratic forms for Γ and Γ' coincide.*

Proof. The quadratic form of a quantum graph with δ-conditions is given by

$$\langle Lu, u\rangle = \int_\Gamma |u'(x)|^2\,dx + \sum_{v\in V} \alpha_v |u(v)|^2$$

and its domain is $\{u \in L^2(\Gamma) : u' \in L^2(\Gamma),\, u \in C(\Gamma)\}$. Since α_v does not enter in the domain it follows that the two domains are the same. □

This property is very important since this implies that especially the function which is the actual minimum in the Rayleigh quotient for $\lambda_n(\Gamma)$ will also be in the domain of the quadratic form for Γ'. Can we show that the Rayleigh quotient with that function is lower for Γ' we have shown that $\lambda_n(\Gamma')$ also must be lower. This method is surprisingly useful and central to many of the proofs in this text.

Theorem 4. *The eigenvalues of a quantum graph Γ depend positively on the strength α_v at all vertices. That is, if Γ and Γ' are two quantum graphs such that the underlying metric graph is the same, but the matching conditions are such that $\alpha_v \geq \alpha'_v$ for all vertices v, then*

$$\lambda_n(\Gamma) \geq \lambda_n(\Gamma').$$

Proof. By the Rayleigh quotient the eigenvalues are given by

$$\lambda_0(\Gamma) = \min_{u \in \mathrm{Dom}_Q(\Gamma)} \frac{\int_\Gamma |u'(x)|^2 \, dx + \sum_v \alpha_v |u(v)|^2}{\int_\Gamma |u(x)|^2 \, dx}$$

and

$$\lambda_n(\Gamma) = \max_{\substack{A_n \subset \mathrm{Dom}_Q(\Gamma) \\ \dim(A_n) = n}} \left(\min_{\substack{u \perp A_n \\ u \in \mathrm{Dom}_Q(\Gamma)}} \frac{\int_\Gamma |u'(x)|^2 \, dx + \sum_v \alpha_v |u(v)|^2}{\int_\Gamma |u(x)|^2 \, dx} \right)$$

for Γ and correspondingly

$$\lambda_0(\Gamma') = \min_{u \in \mathrm{Dom}_Q(\Gamma')} \frac{\int_{\Gamma'} |u'(x)|^2 \, dx + \sum_v \alpha'_v |u(v)|^2}{\int_\Gamma' |u(x)|^2 \, dx}$$

and

$$\lambda_n(\Gamma') = \max_{\substack{A_n \subset \mathrm{Dom}_Q(\Gamma') \\ \dim(A_n) = n}} \left(\min_{\substack{u \perp A_n \\ u \in \mathrm{Dom}_Q(\Gamma')}} \frac{\int_{\Gamma'} |u'(x)|^2 \, dx + \sum_v \alpha'_v |u(v)|^2}{\int_{\Gamma'} |u(x)|^2 \, dx} \right)$$

for Γ'.

Since the underlying metric graphs are identical, $\mathrm{Dom}_Q(\Gamma') = \mathrm{Dom}_Q(\Gamma)$. Since we also have $\alpha'_v \leq \alpha_v$ the later two expressions will always be less or equal to the first two and the result follows.

□

2.2 Alterations with fixed total length and total strength

Any interesting universal bounds will have to be restricted to a total length and total strength since increasing or decreasing these can make any eigenvalue arbitrarily small or large. The next question to ask is how the geometry of the underlying metric graph and the distribution of the strengths in the δ-conditions affect the spectrum given a fixed total length and total strength. A number of such statements will now be presented and used later on for the derivation of the universal bounds.

Theorem 5. *Let v be a vertex of the quantum graph Γ with the matching conditions*

$$\begin{cases} \psi \text{ is continuous at } v \\ \sum_{x_k \in v} \partial\psi(x_k) = \alpha_v \psi(v) \end{cases}$$

and let Γ' be the graph obtained from Γ by separating the vertex v into two vertices v' and v'' and endowing them with vertex conditions

$$\begin{cases} \psi \text{ is continuous at } v' \\ \sum_{x_k \in v'} \partial\psi(x_k) = \alpha_{v'} \psi(v') \end{cases} \quad \text{and} \quad \begin{cases} \psi \text{ is continuous at } v'' \\ \sum_{x_k \in v''} \partial\psi(x_k) = \alpha_{v''} \psi(v'') \end{cases}$$

such that $\alpha_{v'} + \alpha_{v''} = \alpha_v$ then

$$\lambda_n(\Gamma) \geq \lambda_n(\Gamma').$$

Note that this theorem is independent of the matching conditions at all other vertices.

Proof. Any $\psi \in \mathrm{Dom}_Q(\Gamma)$ will also be in $\mathrm{Dom}_Q(\Gamma')$ since if ψ is continuous at v then it is also continuous at v' and v''. The reverse, though, does not hold since in general $\psi(v') \neq \psi(v'')$.

Let $\lambda_n(\Gamma)$ be denoted by λ_n, and $\lambda_n(\Gamma')$ be denoted by λ'_n. Then

$$\lambda_n = \max_{\substack{A_n \subset \mathrm{Dom}_Q(\Gamma) \\ \dim(A_n) = n}} \left(\min_{\substack{u \perp A_n \\ ||u|| = 1 \\ u \in \mathrm{Dom}_Q(\Gamma)}} \langle Lu, u \rangle \right)$$

and

$$\lambda_n' = \max_{\substack{A_n' \subset \mathrm{Dom}_Q(\Gamma') \\ \dim(A_n') = n}} \left(\min_{\substack{u \perp A_n' \\ ||u|| = 1 \\ u \in \mathrm{Dom}_Q(\Gamma')}} \langle Lu, u \rangle \right).$$

Putting these together we get that:

$$\lambda_n' = \max_{\substack{A_n' \subset \mathrm{Dom}_Q(\Gamma') \\ \dim(A_n') = n}} \left(\min_{\substack{u \perp A_n' \\ ||u|| = 1 \\ u \in \mathrm{Dom}_Q(\Gamma')}} \langle Lu, u \rangle \right) \underset{(1)}{\leq} \max_{\substack{A_n' \subset \mathrm{Dom}_Q(\Gamma') \\ \dim(A_n') = n}} \left(\min_{\substack{u \perp A_n' \\ ||u|| = 1 \\ u \in \mathrm{Dom}_Q(\Gamma)}} \langle Lu, u \rangle \right)$$

$$\underset{(2)}{=} \max_{\substack{A_n \subset \mathrm{Dom}_Q(\Gamma) \\ \dim(A_n') = n}} \left(\min_{\substack{u \perp A_n \\ ||u|| = 1 \\ u \in \mathrm{Dom}_Q(\Gamma)}} \langle Lu, u \rangle \right) = \lambda_n.$$

(1) follows as $\mathrm{Dom}_Q\Gamma \subset \mathrm{Dom}_Q\Gamma'$, and adding additional conditions on the elements we can choose to minimize with will increase the value of the expression. (2) follows as A_n' is chosen to maximize the expression. As we have already restricted us to the smaller domain $\mathrm{Dom}_Q(\Gamma)$ when minimizing, choosing A_n such that it contains an element not in $\mathrm{Dom}_Q(\Gamma)$ will only increase the number of elements available to the min-function. As a consequence restricting A_n to $\mathrm{Dom}_Q\Gamma$ will not effect the value of the expression. □

Obviously this theorem goes in the other direction as well. If two vertices of a quantum graph are glued together in this way, this will increase the spectrum.

Corrolary 2. *Let Γ be quantum graph with a set of edges $E(\Gamma)$ and a total strength $\alpha(\Gamma)$. The spectrum of Γ is bounded from above by the spectrum of the corresponding flowergraph F_E, i.e.*

$$\lambda_n(\Gamma) \leq \lambda_n(F_E).$$

The corresponding flower graph is the graph obtained by gluing together both endpoints of all edges at a single vertex and assigning the whole total strength $\alpha(\Gamma)$ *to the matching condition there.*

Proof. As stated by Theorem 5, if two of the vertices of Γ are glued together then the spectrum increases. Of course it is possible to glue together vertices one by one and thereby increasing the spectrum until the corresponding flower graph F is obtained. See figure 2.1 for an example.

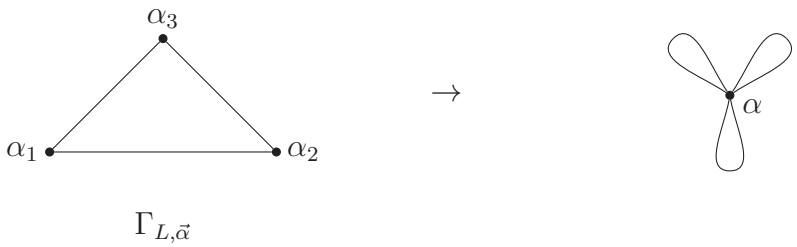


Figure 2.1: An example of a graph and its corresponding flowergraph.

□

As a consequence of this corollary, if the spectrum of the obtained flower graph is easily calculated, for example if all edges of Γ are of the same length, this allows for a quick upper bound on the spectrum of Γ. Also, by better understanding the flower graph it is possible to get upper bounds on other graphs as well. We will come back to this question in chapter 4.

Another question of interest is how the distribution of the strengths α_v affect the spectrum. In general this depends on the underlying metric graph, see [3] for a good investigation of this, but there is enough we can say about λ_0 for us to obatin a universal bound later.

Theorem 6. *Every quantum graph* Γ *has one vertex* v' *such that putting a* δ*-condition of strength*

$$\alpha = \textstyle\sum_{v \in V} \alpha_v$$

at v'*, and standard conditions at all other vertices, gives a new quantum graph* Γ' *such that:*

$$\lambda_0(\Gamma) \geq \lambda_0(\Gamma').$$

Proof. Let $f \in \mathrm{Dom}_Q(\Gamma)$ be a function that minimizes

$$\frac{\langle Lf, f\rangle}{\langle f, f\rangle}.$$

In other words, let f be a eigenfunction of Γ corresponding to the eigenvalue λ_0. Let v' be the vertex where $|f|$ attains the smallest value.

Let Γ' be the graph created from Γ be putting $\alpha_v = 0$ for all $v \neq v'$ and $\alpha_{v'} = \alpha$. Let us denote the corresponding operator by L'. By Theorem 1 $\mathrm{Dom}_Q(\Gamma) = \mathrm{Dom}_Q(\Gamma')$, so $f \in \mathrm{Dom}_Q(\Gamma')$. We use the Rayleigh quotient of the two quantum graphs to show that $\lambda_0(\Gamma) \geq \lambda_0(\Gamma')$

$$\lambda_0(\Gamma) = \frac{\int_\Gamma |f'(x)|^2\,dx + \sum_v \alpha_v |f(v)|^2}{\int_\Gamma |f(x)|^2\,dx} \geq \frac{\int_\Gamma |f'(x)|^2\,dx + \alpha |f(v')|^2}{\int_\Gamma |f(x)|^2\,dx} \geq \lambda_0(\Gamma').$$

□

Chapter 3

Lower bounds on the eigenvalues

3.1 Standard conditions

The theorem presented below is proven by Friedlander in [6]. The methodology and proof technique he uses is very different from the rather geometric approach used in this text. A summary of his proof will be given and then we will see how the majority of his results can be proven using geometrical tools and that in some cases improvements can be made. This proof is only valid for standard conditions but we will use it to prove the same bounds for graphs with δ-conditions with positive α:s. The theorem by Friedlander is not possible to extend to any good estimate for λ_0 for δ-conditions since for a quantum graph with standard conditions λ_0 is always 0 which is not true for graphs with $\alpha > 0$.

Proposition 2 (By Friedlander)**.** *The eigenvalues of a quantum graph Γ, of length $\mathcal{L}$ and with standard conditions at all vertices is bounded by*

$$\lambda_n(\Gamma) \geq \left(\frac{(n+1)\pi}{2\mathcal{L}}\right)^2, \; n = 1, 2, 3... \tag{3.1}$$

with equality only when Γ is the star graph with $n+1$ edges of the same length if $n \geq 2$, and equality for $n = 1$ only if Γ is the single interval of length $\mathcal{L}$.

The proof is quite technical and it is not necessary to understand to proof to understand any of the other results presented.

Proof. We will denote the star graph mentioned in the theorem by $\Gamma = H^{n+1}$. By Theorem 5 it is enough to consider connected trees. This is since any graph can be turned into a tree by a number of vertex cuts, and each cut will only relax the continuity conditions for the functions and thereby increase the domain for the quadratic form. Now this is a crucial step since Friedlander here loses almost all information of how the graph is connected, we will see later how this information can be used to produce better bounds, at least for λ_0.

Let Γ be a connected tree and fix an arbitrary λ_n, $n \geq 2$, to minimize. Let $\psi_0(x) = \text{const}$, $\psi_1(x), \dots, \psi_n$ be the n first eigenfunctions of Γ. For any collection of points $x_0, x_1, \dots, x_m$, $m < n$, it is possible to find a non-zero linear combination ψ of $\psi_0, \dots, \psi_n$ that vanishes at all those points. Since $\lambda_0 \leq \lambda_1 \leq \ldots \leq \lambda_n$, ψ is a linear combination of ψ_i and

$$\lambda_i = \frac{\int_\Gamma |\psi_i'(x)|^2 \, dx}{\int_\Gamma |\psi_i(x)|^2 \, dx}$$

it follows that

$$\int_\Gamma |\psi'(x)|^2 \, dx \leq \lambda_n \int_\Gamma |\psi(x)|^2 \, dx.$$

since as in the proof of Theorem 2

$$\begin{aligned} \frac{\int_\Gamma |\psi'(x)|^2 \, dx}{\int_\Gamma |\psi(x)|^2 \, dx} &= \frac{\langle L\psi, \psi \rangle}{\langle \psi, \psi \rangle} \\ &= \frac{\langle (\sum_{i=0}^n \lambda_i \langle \psi, u_i \rangle u_i), \psi \rangle}{\langle \psi, \psi \rangle} \\ &\leq \lambda_n \frac{\langle \psi, \psi \rangle}{\langle \psi, \psi \rangle} \\ &\leq \lambda_n. \end{aligned}$$

The collection of graphs given by the closure of $\Gamma \setminus \{x_0, \dots, x_m\}$ will be denoted by $\Gamma(x_0, \dots, x_m)$. The next lemma about metric trees allows us to choose $x_0, \dots, x_m$ in a suitable way.

Lemma 1. *Let G be a connected metric tree and let $n \geq 2$. Then there exist points $x_0, \dots, x_m$, $m < n$, such that the length of each connected component of $\Gamma(x_0, \dots, x_m)$ is less or equal to $\mathcal{L}(G)/(n+1)$.*

Now let $x_0, \ldots, x_m$ be points chosen such as described in Lemma 1 and ψ the corresponding linear combination of the first eigenfunctions. For at least one of the connected components of $\Gamma(x_0, \ldots, x_m)$, denoted Γ_1, ψ is not identically zero. By the same reasoning as earlier

$$\int_{\Gamma_1} |\psi'(x)|^2 \, dx \leq \lambda_n \int_{\Gamma_1} |\psi(x)|^2 \, dx.$$

By construction $\psi(x)$ must be zero at one of its leaves v, i.e. one of its vertices of degree one, so there $\psi(x)$ satisfies the Dirichlet boundary condition. The next step is to give a lower bound on all W_2^1 functions that satisfy the Dirichlet condition at that leaf. This step is done somewhat more general in the original proof, but is simplified here. All W_2^1 non-zero functions f which satisfy the Dirichlet condition at a v satisfy the following inequality

$$\int_{\Gamma_1} |f'(x)|^2 \, dx \geq \frac{\pi^2}{4\mathcal{L}(\Gamma_1)^2} \int_{\Gamma_1} |f(x)|^2 \, dx \tag{3.2}$$

with equality if and only if Γ_1 is the interval and $f(x)$ is proportional to $\sin(\pi s/(2\mathcal{L}(\Gamma))$ where s is the distance from v. This is proven by use of a so called symmetrization technique.

First, it can be assumed that $f \geq 0$ since replacing f with $|f|$ does not change either side of (3.2)). For $t \geq 0$, let $m_f(t)$ be the measure of the set $\{x \in \Gamma_1 : f(x) < t\}$. This is a lower semi-continuous function that increases from 0 to $M = max f(x)$. It is possible to uniquely define a continuous non-decreasing function $f^*(s)$ on the interval $[0, \mathcal{L}(\Gamma_1)]$ such that $f^*(0) = 0$ and $m_{f^*}(t) = m_f(t)$. It follows that

$$\int_{\Gamma_1} |f(x)|^2 \, dx = \int_0^M t^2 \, dm_f(t) = \int_0^{\mathcal{L}(\Gamma_1)} |f^*(s)|^2 \, ds.$$

To simplify things, recall that any eigenfunction of the graph will be of the form $u(x) = A_i \sin(kx) + B_i \cos(kx)$ and any such function on a finite graph will have a finite number of critical points. Let t be a regular value of $f(x)$. The number of pre-images of t under $f(x)$ is finite. Let that number be denoted with $n(t)$. The co-area formula, see [1] implies that

$$\int_{\Gamma_1} |f'(x)|^2 \, dx = \int_0^M \sum_{x : f(x) = t} |f'(x)| \, dt.$$

To see why the co-area formula should hold consider first a monotone increasing function $\phi(x)$ on an interval $[0, b]$. Let $M(b)$ be the maximum of

$\phi(x)$ depending on b. Then

$$\frac{d}{db}\int_0^b |\phi'(x)|^2 dx = |\phi'^{\prime}(b)|^2.$$

Consider now instead

$$\frac{d}{db}\int_0^{M(b)} |\phi'(x(t))| dt = M'(b)|\phi'(b)|$$

where $x(t)$ is the inverse function of ϕ. Since ϕ is monotone increasing $M'(b) = \phi'(b)$ and ϕ is monotone increasing $\phi'(b)|\phi'(b)| = |\phi'(b)|^2$. This proves the co-area formula for a monotone increasing function. We concluded before that $f(x)$ must have a finite number of critical points, and therefore the graph can be divided into components such that the function ϕ is monotone on each such component. Since we only consider the absolute value of the derivative the argument directly translates to the case of a monotone decreasing function. Summing over all such monotone components give the result.

By algebraic manipulation the chain of inequalities

$$\sum_{x:f(x)=t} |f'(x)| \geq n(t)^2 \left(\sum_{x:f(x)=t} \frac{1}{|f'(x)|}\right)^{-1} \geq \left(\sum_{x:f(x)=t} \frac{1}{|f'(x)|}\right)^{-1} = \frac{1}{m_f'(t)} \tag{3.3}$$

can be derived, which in turn implies

$$\int_{\Gamma_1} |f'(x)|^2 \, dx \geq \int_0^M \frac{1}{m_f'(t)} \, dt.$$

The same argument can be applied to f^*, but since f^* takes every regular value once all inequalities becomes equalities. Together with earlier results this gives

$$\int_{\Gamma_1} |f'(x)|^2 \, dx \geq \int_0^M \frac{1}{m_f'(t)} \, dt = \int_{\Gamma_1} |(f^*)'(s)|^2 \, ds.$$

Now, $f^*(x)$ is a function on the interval $[0, \mathcal{L}(\Gamma_1)]$, with the Dirichlet condition at 0 and standard condition at $\mathcal{L}(\Gamma_1)$, since $m_f'(\mathcal{L}(\Gamma_1)) = 0$. The first eigenvalue of the corresponding quantum graph can be shown to be $\frac{\pi^2}{4\mathcal{L}(\Gamma_1)^2}$ so

$$\int_{\Gamma_1} |(f^*)'(s)|^2 \, ds \geq \frac{\pi^2}{4\mathcal{L}(\Gamma_1)^2} \int_0^{\mathcal{L}(\Gamma_1)} |f^*(s)|^2 \, ds \tag{3.4}$$

and this actually proves the inequality (3.2). It is then relatively straightforward to check that the only graph and function for which equality holds is the first eigenfunction on the interval. The eigenfunctions can always be choosen so that $\int_\Gamma |\psi_i(x)|^2\,dx = 1$. If they are chosen such, then

$$\int_\Gamma |\psi_0'(x)|^2\,dx \leq \int_\Gamma |\psi_1'(x)|^2\,dx \leq \ldots \leq \int_\Gamma |\psi_n'(x)|^2\,dx$$

and especially

$$\int_\Gamma |\psi_n'(x)|^2\,dx \geq \int_\Gamma |\psi'(x)|^2\,dx$$

since ψ is a linear combination of $\psi_0, \ldots, \psi_n$. To finish the inequality (3.1) it is enough to notice that since $x_0, \ldots, x_n$ were chosen in accordance with Lemma 1 $\mathcal{L}(\Gamma_1) \leq \frac{\mathcal{L}(\Gamma)}{n+1}$ so for any non-zero component Γ_i of Γ

$$\int_{\Gamma_1} |(f^*)'(s)|^2\,ds \geq \left(\frac{\pi(n+1)}{2\mathcal{L}(\Gamma)}\right)^2 \int_0^{\mathcal{L}(\Gamma_1)} |f^*(s)|^2\,ds$$

which of course implies the inequality for the whole graph.

The proof of the fact that the equality only holds for H^{n+1} is rather technical. First one can conclude that all non-zero components of $\Gamma(x_0, \ldots, x_n)$ must be of maximal length, i.e. $\frac{\mathcal{L}(\Gamma)}{n+1}$, otherwise (3.4) would imply a higher eigenvalue. Then it is possible to conclude that the eigenfunction must in fact be non-zero on all edges. To complete the proof it is then proven that the only way to glue these $n+1$ edges is like in the star graph. □

3.2 General δ-conditions

It is in fact possible to extend the result of Friedlander to graphs with general δ-conditions, with $\alpha \geq 0$. The lower bound is actually the same for $\lambda_n \geq 1$ but differs for λ_0.

Theorem 7. *For $n \geq 1$ the n:th eigenvalue λ_n of a quantum graph Γ with δ-conditions of non-negative strengths, of length $\mathcal{L}$ and total strength α is bounded from below by the n:th eigenvalue of the regular star graph with $n+1$ edges and standard conditions at all vertices. The equality is realized only for the regular star graph with a δ-condition of strength α at the middle vertex and standard conditions at all other.*

The lowest eigenvalue λ_0 is bounded from below by the lowest eigenvalue of the interval of length $\mathcal{L}$ with the standard condition at one vertex and a δ-condition of strength α at the other. In other words

$$\lambda_n(\Gamma) \geq \left(\frac{(n+1)}{2}\right)^2 \left(\frac{\pi}{\mathcal{L}}\right)^2, \quad n \geq 1$$

and

$$\lambda_0(\Gamma) \geq k_0^2$$

where k_0 is the first non-negative solution to the equation $k_0 \tan(k_0\mathcal{L}) = \alpha$.

Proof for λ_n, $n \geq 1$. Let Γ be a quantum graph with total length $\mathcal{L}$ and total strength α. Then by Theorem 4

$$\lambda_n(\Gamma) \geq \lambda_n(\Gamma')$$

where Γ' is the graph where all $\alpha_v = 0$. Γ' is a graph with standard conditions and therefore the spectrum is bounded from below by Friedlander's result,

$$\lambda_n(\Gamma) \geq \left(\frac{(n+1)}{2}\right)^2 \left(\frac{\pi}{\mathcal{L}}\right)^2.$$

This is already a lower bound on all graphs with δ-conditions. The important question is whether the bound is sharp, i.e. if there exists a graph with length $\mathcal{L}$ and total strength α for which the inequality becomes an equality. By Example 4

$$\lambda_n\left(H_\alpha^{n+1}\right) = \left(\frac{(n+1)}{2}\right)^2 \left(\frac{\pi}{\mathcal{L}}\right)^2$$

where H_α^{n+1} is the star graph with $n+1$ edges and a strength α at the middle vertex. Notice how α does not enter in the right-hand side. The strength at the middle vertex can be chosen such that $\alpha(H_\alpha^{n+1}) = \alpha(\Gamma)$ so the inequality is in fact sharp. □

Proof for λ_0. The lower bound on λ_0 does in fact depend on the total strength α. Therefore Friedlander's result does not help here and a completely new proof has to be constructed. This is done mainly with help of the results in chapter 2 and the following lemma taken from [10]. In fact many of the ideas for the proof for λ_0 comes from [10] where a similar bound is shown for quantum graphs with standard conditions.

Lemma 2 (Kurasov, Naboko)**.** *Let Γ be a quantum graph of total length $\mathcal{L}$ and total strength α. Let Γ^2 be the graph constructed from Γ by duplicating each edge of Γ and doubling all α_v.*

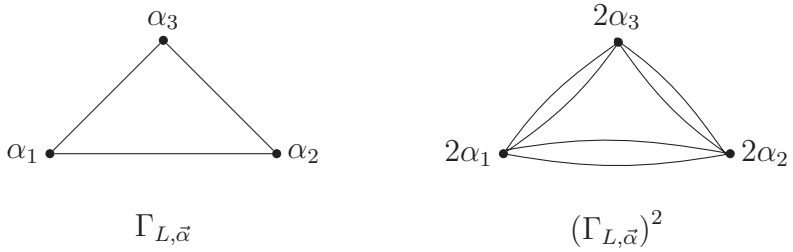


Figure 3.1: An example of Γ and Γ^2.

Then

$$\lambda_n(\Gamma) \geq \lambda_n\left(\Gamma^2\right).$$

Proof of Lemma 2. Let ψ_n be the n:th eigenfunction of Γ and λ_n the corresponding eigenvalue. Let $\psi_n^{(2)}$ be the function on Γ^2 which attains the same value as ψ_n on both the original edge and its duplicate. Since

$$-\frac{d^2}{dx^2}\psi_n(x) = \lambda_n\psi_n(x)$$

and all new edges on Γ^2 are duplicates of the original graph it follows that

$$-\frac{d^2}{dx^2}\psi_n^{(2)}(x) = \lambda_n\psi_n^{(2)}(x).$$

The continuity condition is obviously satisfied. On Γ the function ψ_n satisfies the matching conditions

$$\sum_{x_k\in v}\partial\psi_n(x_k) = \alpha_v\psi(v)$$

since Γ^2 is a duplication of all the edges the left hand side will be doubled and therefore ψ_n^2 will satisfy the matching condition with $2\alpha_v$. From this it follows that any eigenvalue of Γ is an eigenvalue of Γ^2, but not necessarily the other way around. □

It is now possible to construct a chain of inequalities which gives the lower bound for λ_0. Let Γ be a quantum graph with δ-conditions. By Lemma 2

$$\lambda_0(\Gamma) \geq \lambda_0\left(\Gamma^2\right).$$

For the next inequality, note that Γ^2 is in fact of even degree, i.e. all vertices are of even degree. Therefore by a famous theorem by Euler [5] there exists an Eulerian cycle, i.e. a cycle which passes every edge exactly once. Take such a cycle and at each vertex it passes split it up so that the ingoing and outgoing edge of the circle form a separate vertex. By Theorem 5 each such cut will decrease λ_0. When all vertices have been split up what is left will be a single loop graph with total length $2\mathcal{L}$ and a number of vertices with strengths α_v such that the sum of them is 2α, let it be denoted with $S_{2\vec{\alpha}}$.

By Theorem 6 there is one vertex such that moving all the strength to that vertex will decrease the value of λ_0. Since the underlying metric graph is the single loop it does not matter which vertex this is, the resulting graph will always be the single loop graph with one vertex with strength 2α, denoted $S_{2\alpha}$. By Example 5

$$\lambda_0(S_{2\alpha}) = k_0^2$$

where k_0 is the first non-negative solution to the equation $k\tan(k\mathcal{L}) = \alpha$.

To see that inequality is in fact sharp we look at the graph consisting of a single interval of length $\mathcal{L}$ and with a strength α at one vertex and standard condition at the other. Example 2 tells us that λ_0 of the interval is in fact given by the first solution to $k\tan(k\mathcal{L}) = \alpha$. □

An interesting thing to note about this is that if the underlying graph Γ is such that all vertices are of even degree then there already exists an Eulerian cycle so there is no need to double the edges. This gives a better bound for λ_0 for such a graph.

Corrolary 3. *Let Γ be a quantum graph such that all vertices are of even degree, the total length is $\mathcal{L}$ and the total strength is α. Then the lowest eigenvalue λ_0 is bounded from below by the lowest eigenvalue of the interval of length $\frac{\mathcal{L}}{2}$ with the standard condition at one vertex and a δ-condition of strength α at the other. In other words*

$$\lambda_0(\Gamma) \geq k_0^2$$

where k_0 is the first non-negative solution to the equation $k_0\tan(k_0\frac{L}{2}) = \alpha$.

Proof. The proof is identical to that for λ_0 for graphs that not are even besides that the step where the edges are duplicated is skipped which results in the larger bound. □

This is one reason to why the more geometric proof technique used in this text might be preferred to that of Friedlander. More information about the geometry of the underlying graph might be used to achieve better estimates. It should be possible to directly derive the result of Friedlander using similar techniques which would allow for the same improved estimates for all λ_n. It has though been tried without success.

Chapter 4

Upper bounds on the eigenvalues

The corollary to Example 4 immediately tells us that the total length and the total strength of a quantum graph are not alone enough to give an upper bound on the spectrum. For any total length and total strength it is possible to construct a graph with arbitrarily large λ_1, and therefore of course arbitrarily large any λ_n, $n \geq 1$. At least some other property of the graph must be taken into account to obtain an actual bound. This something can be the set of edges. Theorem 2 tells us that given a set of edges, E, and a total strength α, the highest possible λ_n is achieved by the flower graph F_E consisting of all those edges connected to with both endpoints to one single vertex with a matching condition with strength α. In other words, can we characterize the spectrum of the flower graph in a good way we have characterized an upper bound for corresponding graphs. In this chapter the spectrum of the flower graph and its implications for the upper bound will be investigated.

4.1 The regular flower graph

One flower graph for which the spectrum is rather easily calculated is the regular flower graph, i.e. the flower graph were all edges are of the same length. First we calculate the spectrum of this graph and then we extend the reasoning to include first flower graphs were the edge lengths are rationally independent and then to all flower graphs.

Example 6. *The spectrum of a regular flower graph with n edges.*

A regular flower graph is a flower graph were all edges are of the same length. Assume each edge is of length $2l$. Parametrize the edges from $-l$ to l. With this parametrization it can be seen that the graph is symmetric with respect to the reflection operator $Ru(x) = u(-x)$ we used for the circle, and therefore Proposition 1 can be applied. As a consequence the eigenfunctions can be chosen such that they are either odd or even.

For the odd eigenfunctions, $u(x) = A_i \sin(kx)$, the continuity condition implies that

$$A_i \sin(kl) = A_i \sin(-kl) = 0$$

while the condition on the derivatives is always fulfilled. The multiplicity is n since that is the number of linearly independent ways to choose the constants A_i. The corresponding solutions are $k = \frac{m\pi}{l}$.

The even eigenfunctions $u_i(x) = B_i \cos(kx)$ are very similar to those of the regular star graph. If $\cos(kl) = 0$, the continuity condition is fulfilled independently of the value of B_i. However, the derivative condition states that

$$2\sum_{i=1}^{n} kB_i \sin(kl) = 0$$

where the 2 comes from the fact that each edge have two incoming ends contributing the same amount to the sum by symmetry. The corresponding eigenvalues,

$$\lambda = \left(\frac{\pi}{2l} + \frac{m\pi}{l}\right)^2$$

have multiplicity $n-1$ since that is the number of linearly independent ways B_i can be chosen in. If $\cos(kl) \neq 0$ then the continuity condition implies that $B_i = B_j = B$ and the derivative condition

$$2\sum_{i=1}^{n} kB_i \sin(kl) = \alpha B_i \cos(kl)$$

gives that the eigenvalues are given by $\lambda = k^2$ where k are given by the solutions to the equation

$$2nk \tan(kl) = \alpha, \tag{4.1}$$

and since all B_i:s are decided up to one constant the multiplicity of this eigenvalue is 1.

By the same reasoning as for the star graph it is easily seen that the first eigenvalue corresponds to the first solution of equation (4.1). Then the following will be the even eigenvalues and then come the odd. The odd come before the next solution to the (4.1) since the left-hand side is increasing and $\tan(m\pi) = 0$.

This example gives us the first upper bounds for some graphs.

Theorem 8. *Let Γ be a quantum graph such that the length of each edge is a multiple of a constant $2l$ giving a total length of $n \cdot 2l$ for some integer n. Let α be the total strength. The eigenvalues are bounded from above by the corresponding eigenvalues of the regular flower graph with n edges of length $2l$ with the strength α of its matching condition, in other words:*

$$\lambda_{m\cdot 2\cdot n} \leq k_m^2, \quad m = 0, 1, 2, \dots$$

where k_m are the ordered non-negative solutions to (6),

$$\lambda_{m\cdot 2\cdot n+1}, \dots, \lambda_{m\cdot 2\cdot n+n-1} \leq \left(\frac{\pi}{2} + \frac{m\pi}{l}\right)^2, \quad m = 0, 1, 2, \dots$$

and

$$\lambda_{m\cdot 2\cdot n+n}, \dots, \lambda_{((m+1)\cdot 2\cdot n-1} \leq \left(\frac{(m+1)\pi}{l}\right)^2, \quad m = 0, 1, 2, \dots$$

Proof. Since the length of all edges is a multiple of $2l$ it is possible to divide each edge into intervals of length $2l$. At the end points of each such interval it is possible to put vertices of degree two if there is none. We now apply Theorem 2 to see that the spectrum is bounded from above by the spectrum of the flower graph with n edges of length $2l$. The actual bounds follow from Example 6. □

4.2 Pairwise rationally independent edges

Assume F is a flower graph with edges of lengths $2l_1 \leq 2l_2 \leq \dots \leq 2l_n$ such that $l_i \neq ql_j$ for any $q \in \mathbb{Q}$ and $i \neq j$, i.e. the lengths are pairwise rationally independent. The eigenfunctions can by the same reasoning as for the regular star graph be chosen to be either symmetric or antisymmetric.

For the antisymmetric solutions $u(x) = A_i \sin(kx)$ the continuity condition implies that $A_i \sin(kl_i) = 0$, the derivative condition in that case is satisfied by antisymmetry. This implies that $k = m\frac{\pi}{l_i}$, $m \in \mathbb{N}$. Since the lengths are rationally independent it is not possible that $k = m_i\frac{\pi}{l_i} = m_j\frac{\pi}{l_j}$ since that would imply that l_i, l_j were in fact rationally dependent. Each odd eigenfunction therefore must be of the form $u(x) = A_i \sin(kl_i)$ for one such k on the edge e_i and identically equal to zero on the other edges.

The symmetric solutions are somewhat more complicated. Since the edges are pairwise rationally independent it is not possible that

$$B_i \cos(kl_i) = B_j \cos(kl_j) = 0$$

for $j \neq i$. The continuity condition therefore implies that if there is a solution with zero in the middle, then there is only one edge on which the function is nonzero. The derivative condition states that

$$2kB_i \sin(kl_i) = \alpha B_i \cos(kl_j) = 0$$

but there is no k such that $\cos(kl_i) = \sin(kl_i) = 0$. If the function is zero at one end-point then by continuity at the vertex it must be zero at all, since this is not possible it follows that $B_i \cos(kl_i) \neq 0$ for all i.

The continuity condition states that $B_i \cos(kl_i) = B_j \cos(kl_j)$ for all i, j, and since $\cos(kl_i) \neq 0$ this is always solvable.

The derivative condition states

$$2\sum_{s=1}^{n} kB_s \sin(kl_s) = \alpha B_i \cos(kl_i)$$

and since $B_i \cos(kl_i)$ is nonzero

$$2\sum_{s=1}^{n} \frac{kB_s \sin(kl_s)}{B_i \cos(kl_i)} = \alpha. \tag{4.2}$$

Since $B_i \cos(kl_i) = B_j \cos(kl_j)$ for each term in the sum $B_i \cos(kl_i)$ can be substituted by the corresponding $B_s \cos(kl_s)$ giving

$$2\sum_{s=1}^{n} \frac{kB_s \sin(kl_s)}{B_i \cos(kl_i)} = 2\sum_{s=1}^{n} \frac{kB_s \sin(kl_s)}{B_s \cos(kl_s)} = 2k\sum_{s=1}^{n} \tan(kl_s). \tag{4.3}$$

The right hand side is an increasing function with a singularity at each

$$k = \frac{\pi}{2l_i} + \frac{m\pi}{l_i}, \quad m = 0, 1, 2, \ldots$$

As a consequence, if all these k were sorted there would be one solution to the equation (4.2) between every two such k. This describes all the eigenvalues of the quantum graph.

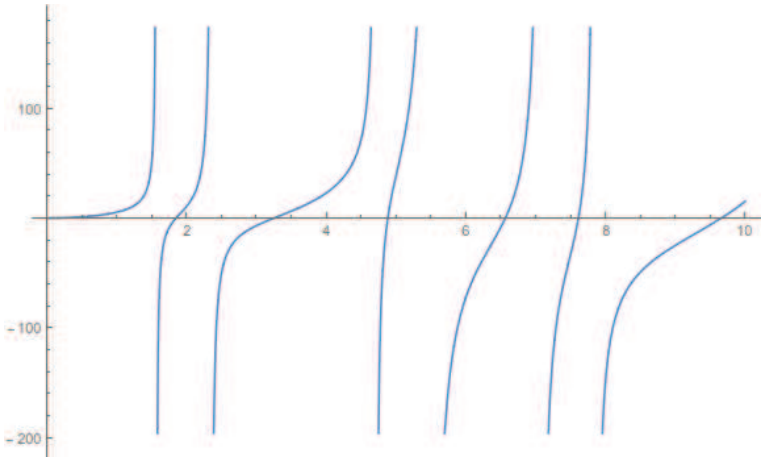


Figure 4.1: An example of the function (4.3) corresponding to a flower graph with three edges of length $\frac{2}{7}$, $\frac{2}{3}$ and 1, $2k\left(\tan\left(k\frac{2}{7}\right)+\tan\left(k\frac{2}{3}\right)+\tan\left(k\right)\right)$.

4.3 Rationally dependent edges

Now consider a flower graph such that the edges of length $2l_i$ are such that there exists $q \in \mathbb{Q}$ such that $l_i = ql_j$ for some i, j such that $i \neq j$. We are now looking at graphs were at least some edge lengths are rationally dependent.

The antisymmetric solutions are barely affected by this. There might now be some $k = m\frac{\pi}{l_i}$ such that $A_i \sin(kl_i) = A_j \sin(kl_j) = 0$. This means that there are some eigenfunctions which are nonzero on more then one edges. However, this will only mean that instead of having two different eigenvalues there will be one with multiplicity two. So there is still one eigenvalue, counting multiplicities, given by

$$k = m\frac{\pi}{l_i}$$

for each l_i and for each $m \in \mathbb{N}$.

Consider the symmetric eigenfunctions. In the same manner if two edges are rationally independent there will be some k:s such that $\cos(kl_i) = \cos(kl_j) =$

0. This will of course remove one singularity, and therefore one eigenvalue, for each such k. However, for this k it is possible to choose B_i, B_j such that

$$2k\left(B_i \sin(kl_i) + B_j \sin(kl_j)\right) = 0$$

which gives a new eigenvalue at the singularity. The symmetric solutions which are nonzero at the middle vertex are not affected by the rational dependence.

Note that if $\alpha \to \infty$ then the k:s corresponding to the solutions which are not zero at the middle will tend to next singularity to the right. This allows us to formulate the following general upper bound for any quantum graph with δ-conditions based solely on the length of the edges.

Theorem 9. *Let Γ be a quantum graph with δ-conditions of any positive strength and with n edges of lengths $2l_1, 2l_2, \ldots, 2l_n$. Then the eigenvalues are bounded from above by the corresponding eigenvalues of the same graph with Dirichlet conditions, in other words:*

$$\lambda = \left(m\frac{\pi}{l_i}\right)^2, \quad m \in \mathbb{N}, 1 \leq i \leq n \tag{4.4}$$

and

$$\lambda = \left(\frac{\pi}{2l_i} + \frac{m\pi}{l_i}\right)^2, \quad m \in \mathbb{N}, 1 \leq i \leq n. \tag{4.5}$$

Proof. The spectrum of Γ is by Corollary 2 bounded by the spectrum of the corresponding flower graph.

The λ for (4.4) are given by the asymmetric eigenfunctions, which does not change depending on whether the lengths are rationally dependent or not. Also the strengths of the δ-conditions does not affect the values of these eigenvalues.

The λ:s in (4.5) are all given by the singularities of (4.3). There is a solution k to (4.2) between any two singularities of the function (4.3). If all α_v increases, so does the spectrum. The spectrum when all $\alpha_v \to \infty$ gives an upper bound on the spectrum of Γ. As $\alpha \to \infty$ the solutions k between the singularities will come arbitrarily close to but not pass the singularity to the right. The symmetric solutions given by rationally dependent edges where the symmetric solutions are 0 at the middle vertex are unaffected by this. When all strengths $\alpha_v \to \infty$ the eigenfunctions converges to the eigenfunctions of the same graph but with Dirichlet conditions. □

Summary

We have now completed our search for universal bounds for compact finite quantum graphs with δ-conditions of positive strength. We have found lower bounds for all eigenvalues given a total length $\mathcal{L}$ and a total strength α and we have found universal upper bounds for all eigenvalues given a set of edges E and a total strength α. The bounds we have found are sharp, so it is not possible to find any better bounds without taking into account some other parameter. Not so surprisingly since so few parameters of the quantum graph enter in the calculation of the bounds the gap between the lower and upper bound is quite large. Consider for example a graph with n edges of the same length $2l$, so the total length is $n2l$ and the total strength $\alpha = 0$. For such a graph the bounds on the eigenvalues are

$$\begin{aligned}
0 \leq &\lambda_0 \leq 0 \\
\left(\frac{\pi}{n2l}\right)^2 \leq &\lambda_1 \leq \left(\frac{\pi}{2l}\right)^2 \\
\left(\frac{3\pi}{n4l}\right)^2 \leq &\lambda_2 \leq \left(\frac{\pi}{2l}\right)^2 \\
&\vdots \\
\left(\frac{(n+1)\pi}{n4l}\right)^2 \leq &\lambda_n \leq \left(\frac{\pi}{l}\right)^2 \\
&\vdots
\end{aligned}$$

so the gap is quite wide.

The techniques and theorems presented and used in this text should be possible to extend to achieve better bounds if more parameters of the graph are included in the analysis. We have seen that the more connected the graph is the higher the eigenvalues are. One parameter which should be possible to use to get better bounds is the edge connectivity of the graph, i.e. the

least number of edges that have to be removed to separate the graph into two disconnected components. It should be possible to find for example the graph of connectivity 2 with the lowest or highest spectrum.

For the upper bound we used the set of edges E of the graph Γ and not just the total length. Another possible extension is to find the graph with a given set of edges with the lowest λ_n and use it to find a lower bound.

Another thing to investigate is how one can use these and maybe other alteration theorems to achieve bounds on more concrete graphs. One thing that would be very helpful would be if it was possible to find characterizations on how the spectrum is affected when edges are added. In proving the lower bound for λ_0 we had to double all the edges to obtain a graph which we knew would have an Euler cycle, in general much fewer additions of edges should be needed to obtain an even degree graph. If this can be done while maintaining control of the spectrum better bounds should be possible to find.

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