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Cyclic homology and T-equivariant (co)homology of free loop spaces

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Abstract

Cyclic homology is related to many mathematical areas as algebraic topology, K-theory and differential geometry. In this thesis we will investigate some of the relations between cyclic homology and $\mathbb{T}$-equivariant cohomology of free loop spaces. In particular we will prove that $HC_*^-(S^\bullet(X)) \cong H_{\mathbb{T}}^*(LX)$ for simply connected spaces X.

Acknowledgements

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I would also like to thank Thomas Ohlson Timoudas, one of few friends I actually can discuss mathematics with. Thank you for reading (parts of) this thesis and for being a good friend.

Contents

Introduction

Cyclic homology is a homology theory for cyclic objects and has relations to many mathematical branches and illuminates interactions between these branches. For instance, cyclic homology has connections to algebraic topology (in particular $\mathbb{T}$-equivariant (co)homology), differential geometry (cyclic homology generalizes the de Rham cohomology) and algebraic K-theory (in particular Lie algebra homology of matrices).

In 1983, John D.S. Jones proved in [Jon] that, given a simply connected topological space X, the negative cyclic homology of the singular cochain algebra of X, denoted by $HC_*^-(S^\bullet(X))$, is isomorphic to the $\mathbb{T}$-equivariant cohomology of the free loop space of X, denoted by $H_{\mathbb{T}}^*(LX)$.

The purpose of this thesis is to reproduce the proof of this theorem and make it accessible for a broader public by giving giving necessary background, including theory for (co)simplicial and (co)cyclic objects and some theory for equivariant (co)homology. Some of the proofs are completely my own, and others are inspired by other proofs but where I do it in my own way.

The thesis consist of three chapters (and an appendix). The first chapter deals with (co)simplicial and (co)cyclic objects. In sections 1.4. and 1.5. we prove the existence of $\mathbb{T}$-actions on realizations of (co)cyclic spaces, without using any category theory (in contrast to how it is proved usually).

The second chapter is the most technical part. Here we introduce Hochschild and cyclic homology for simplicial and cyclic complexes, respectively and prove some homotopy-like conditions.

In the third we introduce and motivate the notion of equivariant (co)homology, and go into details on how to construct contractible spaces with free G-action. The chapter ends with proving the main theorem and giving some applications of it.

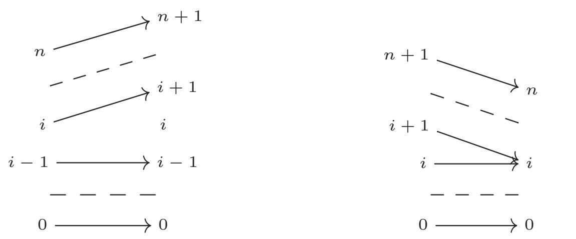
(Co)Simplicial and (Co)Cyclic objects

1.1. (Co)Simplicial Objects

Definition 1.1.1. *The simplicial category* $\mathbf{\Delta}$ is the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$, $n \in \mathbb{Z}_{\geq 0}$, and whose morphisms are generated by

(a) face maps $\delta_i : [n] \to [n+1]$, $0 \leq i \leq n+1$, where δ_i is the unique injective non-decreasing map that misses $i \in [n+1]$ (i.e. $i \notin \operatorname{Im} \delta_i$), and

(b) degeneracy maps $\sigma_i : [n+1] \to [n]$, $0 \leq i \leq n$, where σ_i is the unique surjective non-decreasing map that repeats i.



$\delta_i : [n] \to [n+1]$ $\qquad$ $\sigma_i : [n+1] \to [n]$

One can show that all non-decreasing maps $[m] \to [n]$ are generated by the faces and the degeneracies described above.

Lemma 1.1.2. The faces δ_i and the degeneracies σ_i in $\mathbf{\Delta}$ satisfies the relations

$$\begin{gathered}\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{for} \quad i < j, \\ \sigma_j \sigma_i = \sigma_i \sigma_{j+1} \quad \text{for} \quad i \leq j, \\ \sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & \text{if } i < j \\ \mathrm{id}_{[n]} & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & \text{if } i > j+1. \end{cases}\end{gathered} \tag{1.1}$$

Proof. Each of the equalities can be verified by just applying both sides of that equality on arbitrary elements of their domain and checking that the image coincide. $\square$

Definition 1.1.3. A *simplical object* $X_{\cdot}$ in a category $\mathcal{C}$ is a covariant functor

$$X_{\cdot} : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}.$$

We will set $X_n := X([n])$, $d_i := X(\delta_i) : X_n \to X_{n-1}$ and $s_i := X(\sigma_i) : X_n \to X_{n+1}$, $0 \leq i \leq n$. Here d_i and s_i are called the faces and the degeneracies of $X_.$, respectively. Note that that the faces and degeneracies of $X_.$ will satisfy the opposite relations of (1.1), i.e.

$$\begin{aligned} d_i d_j &= d_{j-1} d_i \quad \text{for} \quad i < j, \\ s_i s_j &= s_{j+1} s_i \quad \text{for} \quad i \leq j, \end{aligned}$$

$$d_i s_j = \begin{cases} s_{j-1} d_i & \text{if } i < j \\ \mathrm{id}_{X_n} & \text{if } i = j \text{ or } i = j+1 \\ s_j d_{i-1} & \text{if } i > j+1. \end{cases} \tag{1.2}$$

A *cosimplicial object* in a categeory $\mathcal{D}$ is a covariant functor

$$Y^{\cdot} : \mathbf{\Delta} \to \mathcal{D}.$$

We will set $Y^n := Y^{\cdot}([n])$, $\delta_i := Y^{\cdot}(\delta_i) : Y^n \to Y^{n+1}$, $0 \leq i \leq n+1$ and $\sigma_j := Y(\sigma_j) : Y^n \to Y^{n-1}$, $0 \leq j \leq n-1$. Here δ_i and σ_i are called the faces and the degeneracies of $Y^{\cdot}$, respectively. Obviously the faces and degeneracies of $Y^{\cdot}$ will satisfy the relations of (1.1).

Lemma 1.1.4. (a) A collection of objects $X_0, X_1, \ldots$ in a category $\mathcal{C}$ with maps $d_i : X_n \to X_{n-1}$ and $s_i : X_{n-1} \to X_n$, that satisfies the relations of (1.2) defines a simplicial object $X_. : \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$, where $X_.([n]) = X_n$, $X_.(\delta_i) = d_i$ and $X_.(\sigma_i) = s_i$.

(b) A collection of objects $Y^0, Y^1, \ldots$ with maps $\delta_i : Y^n \to Y^{n+1}$ and $\sigma_j : Y^{n+1} \to Y^n$, that satisfies the relations of (1.1) defines a cosimplicial object.

Proof. See Proposition 8.1.3 and Corollary 8.1.4 in [Wei]. □

Example 1.1.5. For any group G, let $E_n(G) := G^{n+1}$ and let $d_i : E_n(G) \to E_{n-1}(G)$ and $s_i : E_{n-1}(G) \to E_n(G)$ be given by

$$d_i(g_1, \ldots, g_{n+1}) = \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) & \text{if } i = 1, \ldots, n, \end{cases}$$

$$\text{and} \quad s_i(g_1, \ldots, g_n) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n)$$

One can easily check that d_i and s_i satisfies relations of (1.2), so $E_.(G)$ will therefore, by *Lemma 1.1.4. (a)*, define a simplicial object $E_.(G) : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Grp}$.

Example 1.1.6. Let $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0,\ t_0 + \cdots + t_n = 1\}$ be the geometric n-simplex, and let $\delta_i : \Delta^n \to \Delta^{n+1}$ and $\sigma_i : \Delta^n \to \Delta^{n-1}$ be given by

$$\delta_i(t_0, \ldots, t_n) = (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_n) \quad i = 0, \ldots n+1$$

$$\sigma_i(t_0,\dots,t_n) = (t_0,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n) \quad i=0,\dots n-1.$$

One can easily check that δ_i and σ_i satisfies the relations of (1.1), and hence $\Delta^{\cdot} : \mathbf{\Delta} \to \mathrm{Top}$ defines a cosimplicial object according to *Lemma 1.1.4. (b)*.

Definition 1.1.7. Given a simplicial space $X_{.} : \mathbf{\Delta} \to \mathrm{Top}$, one defines *the geometric realization* $|X_{.}|$ of $X_{.}$ as the quotient

$$(\coprod X_n \times \Delta^n)/\sim$$

where $\sim$ is the equivalence relation generated by the relations $(\varphi^*(x), y) \sim (x, \varphi_*(y))$ for any $x \in X_n$, $y \in \Delta^m$, and any $\varphi \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$, and where $\varphi^* = X_{.}(\varphi)$ and $\varphi_* = \Delta^{\cdot}(\varphi)$ (note that we said that the equivalence relation is generated by the relation $(\varphi^*(x), y) \sim (x, \varphi_*(y))$, and not that the relation itself is an equivalence relation).

Definition 1.1.8. We say that an element $x \in X_p$ is non-degenerated if x cannot be written on the form $s_i x'$. An element $(x,u) \in X_p \times \Delta^p$ is non-degenerated if x is non-degenerated and u is an interior point of Δ^p.

Now consider following lemma:

Lemma 1.1.9. Each point $(x,u) \in X_p \times \Delta^p$, $p \in \mathbb{Z}_{\geq 0}$ is, under the equivalence relation $\sim$, equivalent to a unique non-degenerated element $(y,v) \in X_q \times \Delta^q$.

Proof. See Lemma 14.2 in [Ma2].

Example 1.1.10. For any space K that can be triangulated (i.e. is homeomorphic to a union of a collection of geometric simplexes that only intersects in common faces (see Definition 6.1 in [Arm]), we can order the vertices of that triangulation. Now there is an associated simplicial set $K_{.}$ where

$$K_n = \left\{ (v_{k_0},\dots,v_{k_n}) \;\middle|\; \begin{array}{c} v_{k_0},\dots,v_{k_n} \text{ are the vertices of some simplex of triang}(K) \\ \text{and } k_0 \leq \dots \leq k_n \text{ (with repetition of vertices allowed)} \end{array} \right\}$$

and where

$$d_i(v_{k_0},\dots,v_{k_n}) = (v_{k_0},\dots,\hat{v}_{k_i},\dots,v_{k_n})$$

and

$$s_i(v_{k_0},\dots,v_{k_{n-1}}) = (v_{k_0},\dots,v_{k_i},v_{k_i},\dots,v_{k_{n-1}})$$

(in other words, d_i removes the i'th element, while s_i repeats the i'th element).

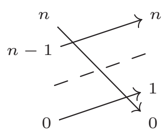
Obviously if $(v_{k_0},\dots,v_{k_n}) \in K_n$ has no repeated vertices then it is non-degenerated (and vice versa) and hence for every inner point $u \in \Delta^n$, we have that $\big((v_{k_0},\dots,v_{k_n}),u\big)$ is non degenerate, and hence not identified with any other point than itself in $\coprod_{i \leq n} K_i \times \Delta^i$ (by the lemma above). Hence for every n-simplex in the triangulation there is a corresponding n-simplex in $|K_{.}|$ (and vice versa since the degenerated elements does not contribute with geometric simplexes), and we get identification in such a way that $K \cong |K_{.}|$.

1.2. (Co)Cyclic Objects

Definition 1.2.1. *The cyclic category* $\mathbf{\Delta}C$ has the same objects as $\mathbf{\Delta}$, and its morphisms are generated by the ordinary face operators $\delta_i : [n] \to [n+1]$, the ordinary degeneracy operators $\sigma_i : [n] \to [n-1]$ and cyclic operators $\tau_n : [n] \to [n]$, subject to the following relations:

$$\begin{array}{rcll} \tau_n\delta_i & = & \delta_{i-1}\tau_{n-1} & \text{for } 1 \leq i \leq n \ , \\ \tau_n\delta_0 & = & \delta_n \ , & \\ \tau_n\sigma_i & = & \sigma_{i-1}\tau_{n+1} & \text{for } 1 \leq i \leq n \ , \\ \tau_n\sigma_0 & = & \sigma_n\tau_{n+1}^2 \ , & \\ \tau_n^{n+1} & = & \mathrm{id}_{[n]} \ . & \end{array} \tag{1.3}$$

One should not think of morphisms of $\mathbf{\Delta}C$ as a morphism of sets, since for sets, we have that $\mathrm{Hom}_{\mathrm{Set}}([n],[0]) = *$, while in the cyclic category $\mathrm{Hom}_{\mathbf{\Delta}C}([n],[0])$ contains $n+1$ morphisms. The relations above is motivated by the fact that the cyclic permutation $(a_0, \dots, a_n) \mapsto (a_1, \dots, a_n, a_0)$ satisfies the relations of (1.3).



Definition 1.2.2. A *cyclic object* in a category $\mathcal{C}$ is a covariant functor

$$X_\cdot : \mathbf{\Delta}C^{\mathrm{op}} \to \mathcal{C}.$$

We set $X_n := X_\cdot([n])$, $d_i := X_\cdot(\delta_i)$, $s_i := X_\cdot(\sigma_i)$ and $t_n := X_\cdot(\tau_n)$.

A *cocyclic object* in a categeory $\mathcal{D}$ is a covariant functor

$$Y^\cdot : \mathbf{\Delta}C \to \mathcal{D}.$$

We set $Y^n := Y^\cdot([n])$, $\delta_i = Y^\cdot(\delta_i)$, $\sigma_i := Y^\cdot(\sigma_i)$ and $\tau_n = Y^\cdot(\tau_n)$.

Lemma 1.2.3. (a) A collection of objects $X_0, X_1, \dots$ in a category $\mathcal{C}$ with maps $d_i : X_n \to X_{n-1}$, $s_i : X_n \to X_{n+1}$ and $t_n : X_n \to X_n$, that satisfies the relations of (1.2) and the opposite relations of (1.3), i.e.

$$\begin{array}{rcll} d_it_n & = & t_{n-1}d_{i-1} & \text{for } 1 \leq i \leq n \ , \\ d_0t_n & = & d_n \ , & \\ s_it_n & = & t_{n+1}s_{i-1} & \text{for } 1 \leq i \leq n \ , \\ s_0t_n & = & t_{n+1}^2s_n \ , & \\ t_n^{n+1} & = & \mathrm{id}_{X_n} \ , & \end{array} \tag{1.4}$$

defines a cyclic object $X_\cdot : \mathbf{\Delta}C \to \mathcal{C}$, where $X_\cdot([n]) = X_n$, $X_\cdot(\delta_i) = d_i$, $X_\cdot(\sigma_i) = s_i$ and $X_\cdot(\tau_n) = t_n$.

(b) A collection of objects $Y^0, Y^1, \dots$ with maps $\delta_i : Y^n \to Y^{n+1}$, $\sigma_i : Y^n \to Y^{n-1}$ and $\tau_n : Y^n \to Y^n$, that satisfies the relations of (1.1) and (1.3) defines a cocyclic object $Y^{\cdot} : \mathbf{\Delta} C \to \mathcal{C}$, where $Y^{\cdot}([n]) = Y^n$, $Y^{\cdot}(\delta_i) = \delta_i$, $Y^{\cdot}(\sigma_i) = s_i$ and $Y^{\cdot}(\tau_n) = \tau_n$.

Proof. See Proposition 9.6.4 in [Wei]. $\square$

Lemma 1.2.4. Every morphism θ in $\mathbf{\Delta} C$ can be decomposed as $\theta = \delta_{i_1} \cdots \delta_{i_a} \sigma_{j_1} \cdots \sigma_{j_b} \tau_n^k$, where $i_a \leq \cdots \leq i_1$ and $j_1 < \cdots < j_b$.

Proof. This follows directly from the relations in (1.1) and (1.3). $\square$

We present a cyclic space that will be of main importance in this thesis.

Example 1.2.5. For a fixed $n \in \mathbb{Z}_{\geq 0}$, we construct the cyclic set $\lambda^n_{\cdot}$, where $\lambda^n_{\cdot}([m]) = \mathrm{Hom}_{\mathbf{\Delta} C}([m],[n])$, and d_i, s_i and t_i are given by

$$d_i \varphi = \varphi \circ \delta_i, \qquad s_i \varphi = \varphi \circ \sigma_i, \qquad t_n \varphi = \varphi \circ \tau_n$$

(easy to check that the relations in (1.2) and (1.4) are satisfied).

Example 1.2.6. Given an algebra A, we construct a cyclic module $A_{\cdot}$ where $A_n = A^{\otimes(n+1)}$ and

$$d_i(a_0, \dots, a_n) = \begin{cases} (a_0, \dots, a_i a_{i+1}, \dots, a_n) & 0 \leq i \leq n-2 \\ (-1)^{|a_n|(|a_0| + \cdots + |a_{n-1}|)}(a_n a_0, a_1, \dots, a_{n-1}) & i = n \end{cases}$$

$$s_i(a_0, \dots, a_n) = (a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n)$$
$$t_n(a_0, \dots, a_n) = (-1)^{n + |a_n|(|a_0| + \cdots + |a_{n-1}|)}(a_n, a_0, a_1, \dots, a_{n-1})$$

In this thesis we will be in particular interested in algebras on the form $S^*(X)$ with multiplication given by the cup product (see [Hat]).

1.3. $\mathbb{T}$-action on realizations of cyclic spaces

In this thesis we will regard the circle as the topological group $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, and will prove that there is a $\mathbb{T}$-action on realizations $|X_{\cdot}|$ of cyclic spaces $X_{\cdot}$. This is done by introducing a new type of realization $\wr X_{\cdot} \wr$ which is equipped with an $\mathbb{T}$-action, and then showing that $\wr X_{\cdot} \wr$ is homeomorphic to $|X_{\cdot}|$ (and therefore there is an $\mathbb{T}$-action on $|X_{\cdot}|$ as well).
The equivalence of these two realizations is often shown using category theory. In this section the equivalence will be shown without using any category theory, which is the own work of the author.

We start by presenting a cocyclic set that is of importance in this section.

Example 1.3.1. Let $\Lambda^{\cdot}$, be the cocyclic space $\Lambda^{\cdot}([n]) = \mathbb{T} \times \Delta^n$, where the face and degeneracy maps are given by the products of the identity of $\mathbb{T}$ with the usual face and degeneracy maps of $\Delta^{\cdot}$ (see *Example 1.1.6.*), and where τ_n is given by

$$\tau_n(z, u_0, \ldots, u_n) = (ze^{-i2\pi u_0}, u_1, \ldots, u_n, u_0).$$

One verifies that $\Lambda^{\cdot}$ is a cocyclic space by verifying that the relations of (1.1) and (1.3) holds for δ_i, σ_i and τ_n

Definition 1.3.2. We introduce a new kind of geometric realization of cyclic spaces given by the quotient

$$\wr X_. \wr = \left(\coprod X_n \times \Lambda^n\right) / \approx$$

where $\approx$ is generated by all relations $(\theta^*(x), (z, \boldsymbol{u})) \approx (x, \theta_*(z, \boldsymbol{u}))$ where $x \in X_n$, $(z, \boldsymbol{u}) \in \Lambda^m$ (here z is the $\mathbb{T}$-coordinate, an $\boldsymbol{u}$ is the Δ^m-coordinate) and $\theta \in \mathrm{Hom}_{\boldsymbol{\Delta} C}([m], [n])$.

This realization has a induced $\mathbb{T}$-action on $\wr X_. \wr$ given by $w.(x, (z, \boldsymbol{u})) = (x, (wz, \boldsymbol{u}))$, (where $w \in \mathbb{T}$ and $(x, (z, \boldsymbol{u})) \in \wr X \wr$). This action is well-defined as the action commutes with the faces, the degeneracies and the cyclic operators.

Definition 1.3.3. Given a cyclic space $X_.$, we say that a pair of elements $(x_1, (z_1, \boldsymbol{u}_1))$, $(x_2, (z_2, \boldsymbol{u}_2)) \in \coprod X_n \times \Lambda^n$ are *directly equivalent (or d.e.) via* θ if there is some $\theta \in \mathrm{Mor}\, \boldsymbol{\Delta} C$ such that $x_1 = \theta^* x_2$ and $(z_2, \boldsymbol{u}_2) = \theta_*(z_1, \boldsymbol{u}_1)$. Note that d.e. elements are $\approx$-equivalent, but that the reverse is not true in general.

We say that map $\Psi : \coprod X_n \times \Lambda^n \to \coprod X_n \times \Lambda^n$ preserves d.e. if whenever $(x_1, (z_1, \boldsymbol{u}_1))$ and $(x_2, (z_2, \boldsymbol{u}_2))$ are d.e., then $\Psi(x_1, (z_1, \boldsymbol{u}_1))$ and $\Psi(x_2, (z_2, \boldsymbol{u}_2))$ are d.e..

Theorem 1.3.4. There is a homeomorphism of spaces $\wr X_. \wr \cong |X_.|$.

Proof. The idea of the proof can be explained in three main ideas.

1. We construct a map $\Psi : \coprod X_n \times \Lambda^n \to \coprod X_n \times (\{1\} \times \mathrm{int}(\Delta^n))$ such that
   - i) $(x, (z, \boldsymbol{u})) \approx \Psi(x, (z, \boldsymbol{u}))$
   - ii) Ψ preserves d.e.
   - iii) Ψ restricted to $\coprod X_n \times (\{1\} \times \mathrm{int}(\Delta^n))$ is the identity map

   In a similar manner we construct a map $\Psi' : \coprod X_n \times \Delta^n \to \coprod X_n \times \mathrm{int}(\Delta^n)$ that satisfies i).

2. Since $(x, (z, \boldsymbol{u})) \approx \Psi(x, (z, \boldsymbol{u})) \in X_m \times (\{1\} \times \mathrm{int}(\Delta^m)) \cong X_m \times \mathrm{int}(\Delta^m)$ (for some $m \in \mathbb{Z}_{\geq 0}$), we can identify $\wr X_. \wr$ with the quotient

$$(\coprod X_n \times \mathrm{int}(\Delta^n)) / \approx$$

where $(x, \boldsymbol{u}) \approx (x', \boldsymbol{u}')$ iff $(x, (1, \boldsymbol{u})) \approx (x', (1, \boldsymbol{u}'))$.

In a similar manner it follows that $|X_.|$ can be identified with

$$(\coprod X_n \times \operatorname{int}(\Delta^n))/\sim$$

3. Now as Ψ preserves d.e., if $(x_1, (z_1, \boldsymbol{u}_1))$ and $(x_2, (z_2, \boldsymbol{u}_2))$ are d.e., then $\Psi(x_1, (z_1, \boldsymbol{u}_1)) = (x'_1, (1, \boldsymbol{u}'_1))$ and $\Psi(x_2, (z_2, \boldsymbol{u}_2)) = (x'_2, (1, \boldsymbol{u}'_2))$ are d.e..
   As $\boldsymbol{u}'_1$ and $\boldsymbol{u}'_2$ are inner points (i.e., none of their coordinates are zero), we must have that $(x'_1, (1, \boldsymbol{u}'_1))$ and $(x'_2, (1, \boldsymbol{u}'_2))$ are d.e. via morphisms in $\boldsymbol{\Delta}$, since applying τ_n will result on a $\mathbb{T}$-coordinate different from 1 (this is not true in general if $\boldsymbol{u}'_1$ and $\boldsymbol{u}'_2$ are boundary points, which is easy to check). This means in particular that $(x'_1, \boldsymbol{u}'_1) \sim (x'_2, \boldsymbol{u}'_2)$.

4. Now assume that $(x, (1, \boldsymbol{u})) \approx (x', (1, \boldsymbol{u}')$. Then there exists a chain of direct equivalences

$$(x, (1, \boldsymbol{u})) \overset{\text{d.e.}}{\approx} (x_1, (z_1, \boldsymbol{u}_1)) \overset{\text{d.e.}}{\approx} \cdots \overset{\text{d.e.}}{\approx} (x_n, (z_n, \boldsymbol{u}_n)) \overset{\text{d.e.}}{\approx} (x', (1, \boldsymbol{u}'))$$

   Applying Ψ on this chain yields the following chain of direct equivalences

$$(x, (1, \boldsymbol{u})) = \Psi(x, (1, \boldsymbol{u})) \overset{\text{d.e.}}{\approx} (x_1, (z_1, \boldsymbol{u}_1)) \overset{\text{d.e.}}{\approx} \cdots$$
$$\cdots \overset{\text{d.e.}}{\approx} (x_n, (z_n, \boldsymbol{u}_n)) \overset{\text{d.e.}}{\approx} \Psi(x', (1, \boldsymbol{u}')) = (x', (1, \boldsymbol{u}))$$

   By 3. above it follows that $\Psi(x_k, (z_k, \boldsymbol{u}_k))$ and $\Psi(x_{k+1}, (z_{k+1}, \boldsymbol{u}_{k+1}))$ are d.e. via morphisms in $\boldsymbol{\Delta}$. In particular that means $(x, (1, \boldsymbol{u}))$ and $(x', (1, \boldsymbol{u}'))$ are equivalent via morphisms in $\boldsymbol{\Delta}$, and hence we know that two elements of $\coprod X_n \times \operatorname{int}(\Delta^n)$ are identified under $\approx$ only if they are identified under $\sim$. On the other hand, if two elements are identified under $\sim$ then they are also identified under $\approx$ (as $\operatorname{Mor} \boldsymbol{\Delta} \subset \operatorname{Mor} \boldsymbol{\Delta} C$). Hence $\sim$ and $\approx$ are the same equivalence relations on $\coprod X_n \times \operatorname{int}(\Delta^n)$, and therefore it follows from 2. that $\wr X_. \wr \cong |X_.|$.

Now in order to complete the proof we have to define Ψ which needs some preparing lemmas.

Lemma 1.3.5. Assume that $\boldsymbol{u} \in \Delta^n$ is a point with zero-coordinates at positions $i_1 < \cdots < i_\ell$. Then $u = \delta_{i_\ell} \cdots \delta_{i_1} \boldsymbol{u}'$ for some inner point $\boldsymbol{u}' \in \Delta^{n-\ell}$. Let $\Psi_1 : \coprod X_n \times \Lambda^n \to \coprod X_n \times \Lambda^n$ be the map sending $(x, (z, \boldsymbol{u}))$ to the $\approx$-equivalent element $(d_{i_1} \cdots d_{i_\ell} x, (z, \boldsymbol{u}')$. Ψ_1 preserves d.e..

Proof. We just need to prove the cases when $(x_1, (z_1, \boldsymbol{u}_1))$ and $(x_2, (z_2, \boldsymbol{u}_2))$ are d.e. via τ_n, δ_i and σ_i, since every morphism in $\boldsymbol{\Delta} C$ is a composition of these.

We prove the case of direct association via τ_n. Assume we have only a zero in position i in $(t_n x, (z, \boldsymbol{u}))$, i.e. $(t_n x, (z, u)) = (t_n x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n))$.

Then the d.e. element $(x, \tau_n(z,u)) = (x, (e^{i2\pi(\tilde{u}+u_0)}, u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n, u_0))$ has a zero in position $i-1$.

Now $\Psi_1(t_n x, (z, \boldsymbol{u})) = (d_i t_n x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n))$ and $\Psi_1(x, \tau_n(z, \boldsymbol{u})) = (d_{i-1}x, (e^{-i2\pi\tilde{u}+u_0}, u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n, u_0))$ are obviously d.e. via τ_{n-1}.

If there would be several zeros, then we repeat the process above. For δ_i and σ_i we prove the assertion in an analogous way as above. $\square$

Lemma 1.3.6. Given a cyclic space $X_\cdot$, for any element $(x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n)) \in X_n \times \Lambda^n$, there is some $k < n+1$ such that $(t_n^{-k}x, \tau_n^k(e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$ is on the form $(y, (e^{-i2\pi\tilde{v}}, v_0, \ldots, v_n))$ where $\tilde{v} \leq v_n$. Obviously, these two elements are equivalent under $\approx$ (even d.e. via τ_n^k)

Proof. We can w.l.o.g. assume that $\tilde{u} \in [0,1)$. We introduce a partial order on $\mathbb{R}/\mathbb{Z}$ by considering the order of the representatives in $[0,1)$ (so, for instance, $0.5 \underset{\mathbb{R}/\mathbb{Z}}{>} 2$ as $0.5 > 0$ in $[0,1)$).

Assume to get a contradiction that there is some $\tilde{u} \in [0,1)$ and $(u_0, \ldots, u_n) \in \Delta^n$ such that

$$\begin{aligned}
\tilde{u} &> u_n \quad \text{in } \mathbb{R} && (\star)\\
\tilde{u} + u_0 &> u_0 \quad \text{in } \mathbb{R}/\mathbb{Z} && (\star\star)\\
\tilde{u} + u_0 + u_1 &> u_1 \quad \text{in } \mathbb{R}/\mathbb{Z} && (\star\star\star)\\
&\vdots\\
\tilde{u} + u_0 + \ldots + u_{n-1} &> u_{n-1} \quad \text{in } \mathbb{R}/\mathbb{Z} && (\star\cdots\star)
\end{aligned}$$

From $(\star\star)$ we get that $\tilde{u} + u_0 < 1$ in $\mathbb{R}$ (otherwise $\tilde{u} + u_0 = 1 + \varepsilon$, where $\varepsilon < u_0$, which gives $\tilde{u} + u_0 \underset{\mathbb{R}/\mathbb{Z}}{=} \varepsilon \underset{\mathbb{R}/\mathbb{Z}}{<} u_0$, contradicting $(\star\star)$). This is equivalent to

$$\tilde{u} < 1 - u_0 \quad \text{in } \mathbb{R}.$$

Now we put $\hat{u} := \tilde{u} + u_0$, which by above has the property $\hat{u} \in [0,1)$. Hence $(\star\star\star)$ is reduced to an inequality of the form $\hat{u} + u_1 > u_1$, where $\hat{u} \in [0,1)$, and hence we can repeat the argument above to get

$$\hat{u} < 1 - u_1 \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad \tilde{u} < 1 - u_0 - u_1 \quad \text{in } \mathbb{R}.$$

Repeating this enough many times will yield the inequality

$$\tilde{u} < 1 - u_0 - u_1 - \ldots - u_{n-1} = u_n \quad \text{in } \mathbb{R},$$

which contradicts ($\star$).
From the contradiction we conclude that either $\tilde{u} \leq u_n$ or that there is some $j \in \mathbb{Z}$; $0 \leq j \leq n-1$, such that

$$\tilde{u} + u_0 + \cdots + u_j \leq u_j \ .$$

Hence

$$(t_n^{-j-1}x, (e^{-i2\pi(\tilde{u}+u_0+\cdots+u_j)}, u_{j+1}, \ldots, u_n, u_0, \ldots, u_j))$$

is an element on the form $(y, (e^{-i2\pi\tilde{v}}, v_0, \ldots, v_n))$ where $\tilde{v} \leq v_n$ and which is identified with $(x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$ under $\approx$. This completes the proof. $\square$

Lemma 1.3.7. Let $\Psi_2 : \coprod X_n \times \Lambda^n \to \coprod X_n \times \Lambda^n$ be a map sending an element $(x, (z, u))$ to the $\approx$-equivalent element $(t_n^{-k}x, \tau_n^k(z, u)) = (y, (e^{-i2\pi\tilde{v}}, v_0, \ldots, v_n))$, where $\tilde{v} \leq v_n$ and k is as small as possible (such map exists by previous lemma). Ψ_2 preserves d.e.

Proof. Let $(x, \theta_*(z, u))$ and $(\theta^* x, (z, u))$ be d.e. via $\theta \in \mathrm{Hom}_{\boldsymbol{\Delta}C}([m], [n])$.
Now $\Psi_2(x, \theta_*(z, u)) = (t_n^{-k}x, \tau_n^k\theta_*(z, u))$ and $\Psi_2(\theta^* x, (z, u)) = (t_m^{-j}\theta^* x, \tau_m^j(z, u))$ are obviously d.e. via $\tau_n^k\theta\tau_m^{-j}$. $\square$

Lemma 1.3.8. Let $L \subseteq \coprod X_n \times \Lambda^n$ be the set of all elements of the form $(x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$, where $\tilde{u} \leq u_n$ and where $(u_0, \ldots, u_n) \in \Delta^n$ is an inner point. Now define $\Psi_3 : L \to \coprod X_n \times \Lambda^n$, which takes an element $(x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$ to its $\approx$-equivalent element $(t_{n+1} \circ s_n(x), (1, \tilde{u}, u_0, \ldots, u_{n-1}, u_n - \tilde{u}))$.
If $(x_1, (z_1, u_1))$ and $(x_2, (z_2, u_2))$ are d.e., then $\Psi_3(x_1, (z_1, u_1))$ and $\Psi_3(x_2, (z_2, u_2))$ are also d.e..

Proof. We have by *Lemma 1.2.4.* that every morphism φ can be decomposed as $\varphi = \delta_{i_1} \cdots \delta_{i_a}\sigma_{j_1} \cdots \sigma_{j_b}\tau_n^k$, where $i_a \leq \cdots \leq i_1$ and $j_1 < \cdots < j_b$. Now if two elements of L are d.e. via some θ, then θ is on the form $\sigma_{j_1} \cdots \sigma_{j_b}\tau_n^k$, i.e. without any faces, since faces takes the $\boldsymbol{u}$-coordinate to boundary points.

Case 1: *θ is on the form $\sigma_{j_1} \cdots \sigma_{j_b}$.*
We check that $\Psi_3(s_i x, (e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$ and $\Psi_3(x, \sigma_i(e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n))$ are d.e. in the same manner as in *Lemma 1.3.5.*.

Case 2: *θ is on the form $\sigma_{j_1} \cdots \sigma_{j_b}\tau_n^k$.*
If this is the case then θ has to be on the form

$$\boldsymbol{\sigma} \cdot \sigma_{n-k}\sigma_{n-k+1} \cdots \sigma_{n-1}\tau_n^k \ ,$$

where $\boldsymbol{\sigma}$ is a composition of degeneracies.
The reason why θ has to be on this form is because after applying τ_n^k on $(e^{-i2\pi\tilde{u}}, u_0, \ldots, u_n)$ we get $(e^{-i2\pi(\tilde{u}+u_0+\cdots+u_{k-1})}, u_k, \ldots, u_n, u_0, \ldots, u_{k-1})$. Now the only way to get the last coordinate of $\boldsymbol{u}$ to be greater than or equal to $\tilde{u} + u_0 + \cdots + u_{k-1}$ is to at least add the last $k+1$ $\boldsymbol{u}$-coordinates.

Now by *Case 1*, it is enough to check that Ψ_3 preserves direct equivalence for elements that are d.e. via $\sigma_{n-k}\sigma_{n-k+1}\cdots\sigma_{n-1}\tau_n^k$. This is done again in the same manner as in *Lemma 1.3.5.*. $\square$

Continuation of the proof of Theorem 1.3.5. Now we can define Ψ as the composition $\Psi := \Psi_1\Psi_3\Psi_2\Psi_1 : \coprod X_n \times \Lambda^n \to \coprod X_n \times (\{1\} \times \mathrm{int}(\Delta^n))$. As Ψ_1, Ψ_2, Ψ_3 satisfies properties 1 i) and ii), we conclude that Ψ does that as well. It's also easy to check that Ψ satisfies condition 1 iii).

We also set $\Psi' := \Psi_1$ (which satisfies 1 i)). This ends the proof. $\square$

Corollary 1.3.9. There is an $\mathbb{T}$-action on $|X_\cdot|$.

Proof. Since $|X_\cdot| \cong \wr X_\cdot \wr$ (by *Theorem 1.3.5.*), it is sufficient to prove that there is an $\mathbb{T}$-action on $\wr X_\cdot \wr$. Let $f : \mathbb{T} \times \wr X_\cdot \wr \to \wr X_\cdot \wr$ be given by

$$w \times (x, (z, u_0, \ldots, u_m)) \longmapsto (x, (wz, u_0, \ldots, u_n)) \ .$$

One can easily check that f commutes with the face, degeneracy and cyclic maps of $\Lambda^\cdot$, which makes f well-defined, and will therefore define a $\mathbb{T}$-action on $\wr X_\cdot \wr$. $\square$

1.4. Realization of cosimplicial and cocyclic spaces

In this section we define realizations of cosimplicial spaces, and prove that there is an $\mathbb{T}$-action on realizations of cocyclic spaces.

We will deal with spaces of continuous maps $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$, equipped with the so-called compact-open topology. Given a compact subset $K \subseteq X$ and an open set $U \subseteq Y$, let $V(K, U) := \{f \in \mathrm{Hom}_{\mathrm{Top}}(X, Y) \mid f(K) \subseteq U\}$. The subsets $V(K, U)$ defines a sub-basis for the compact-open topology on $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$. More details may be found in the appendix of [Hat].

Definition 1.4.1. The geometric realization $|Y^\cdot|$ of a cosimplicial space $Y^\cdot$ is the subspace of $\prod \mathrm{Hom}_{\mathrm{Top}}(\Delta^n, Y^n)$ consisting of all sequences $\alpha_n : \Delta^n \to Y^n$ such that the diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha_n"] \arrow[d, "\varphi_*"'] & Y^n \arrow[d, "\varphi_*"] \\ \Delta^m \arrow[r, "\alpha_m"'] & Y^m \end{tikzcd}$$

commutes for all $\varphi \in \mathrm{Hom}_{\mathbf{\Delta}}([n], [m])$.

Theorem 1.4.2. There is a $\mathbb{T}$-action on realizations of cocyclic spaces.

Proof. The arguments here are similar to the arguments given in the previous section. We will define a new kind of geometric realization $\wr Y^{\cdot} \wr$ of cocyclic spaces $Y^{\cdot}$, where $\wr Y^{\cdot} \wr$ is the subspace of $\prod \mathrm{Hom}_{\mathrm{Top}}(\Lambda^n, Y^n)$ consisting of all sequences $\tilde{\alpha}_n : \Lambda^n \to Y^n$ such that the diagram

$$\begin{tikzcd} \Lambda^n \arrow[r, "\tilde{\alpha}_n"] \arrow[d, "\theta_*"'] & Y^n \arrow[d, "\theta_*"] \\ \Lambda^m \arrow[r, "\tilde{\alpha}_m"'] & Y^m \end{tikzcd}$$

commutes for all $\theta \in \mathrm{Hom}_{\mathbf{\Delta}C}([n], [m])$.

<u>Claim:</u> *A sequence* $\{\tilde{\alpha}_n\} \in \wr Y^{\cdot} \wr$ *is completely determined by its values on elements on the form* $(1, u_0, \ldots, u_n)$ *(i.e. with the* $\mathbb{T}$*-coordinate fixed at 1).*

Proof. For any element $(e^{-2\pi\tilde{u}}, u_0, \ldots, u_n) \in \Lambda^n$ there is some $k \in \mathbb{Z}$ such that $\tau_n^k(e^{-2\pi\tilde{u}}, u_0, \ldots, u_n) = (e^{-2\pi\tilde{v}}, v_0, \ldots, v_n)$, where $\tilde{v} \leq v_n$ (see the proof of *Lemma 1.3.6.*). Hence

$$(e^{-2\pi\tilde{u}}, u_0, \ldots, u_n) = \tau_n^{-k} \circ \sigma_n \circ \tau_{n+1}(1, \tilde{v}, v_0, \ldots, v_{n-1}, v_n - \tilde{v}) \ .$$

Since $\{\tilde{\alpha}_n\}$ has to satisfy the commuting property above, it follows that

$$\tilde{\alpha}_n(e^{-2\pi\tilde{u}}, u_0, \ldots, u_n) = \tau_n^{-k} \circ \sigma_n \circ \tau_{n+1}(\tilde{\alpha}_{n+1})(1, \tilde{v}, v_0, \ldots, v_{n-1}, v_n - \tilde{v}) \ . \qquad \blacktriangle$$

Now there is an obvious map $\Psi : |Y^{\cdot}| \to \wr Y^{\cdot} \wr$ where $\{\alpha_n\}$ is sent to $\{\tilde{\alpha}_n\}$ where

$$\tilde{\alpha}_n\big|_{\{1\}\times\Delta^n} = \alpha_n$$

(this determines $\{\tilde{\alpha}_n\}$ completely by the claim above).

Now let $\Phi : \wr Y^{\cdot} \wr \to |Y^{\cdot}|$ be the map sending $\{\tilde{\alpha}_n\}$ to $\{\alpha_n\}$ where

$$\alpha_n = \tilde{\alpha}_n\big|_{\{1\}\times\Delta^n} \ .$$

One can easily check that Ψ and Φ are continuous and each others inverses giving that

$$|Y^{\cdot}| \cong \wr Y^{\cdot} \wr \ .$$

Now by the homeomorphism above, it is enough to show that there is an $\mathbb{T}$-action on $\wr Y^{\cdot} \wr$ in order to complete the proof.

Define a $\mathbb{T}$-action on $\wr Y^{\cdot} \wr$, where for any $w \in \mathbb{T}$ we set $w.\{\tilde{\alpha}_n\} = \{w.\tilde{\alpha}_n\}$ where

$$w.\tilde{\alpha}_n(z, \boldsymbol{u}) = \tilde{\alpha}_n(wz, \boldsymbol{u})$$

for any $(z, \boldsymbol{u}) \in \Lambda^n$. This completes the solution. $\square$

1.5. Realization of $\lambda^n[m] = \mathrm{Hom}_{\mathbf{\Delta}C}([m],[n])$.

For any $n \in \mathbb{Z}_{\geq 0}$ there is a cyclic space $\lambda^n_\cdot : \mathbf{\Delta}C^{\mathrm{op}} \to \mathrm{Top}$ where $\lambda^n[m] = \mathrm{Hom}_{\mathbf{\Delta}C}([m],[n])$ is endowed by the discrete topology and with $d_i(\varphi) = \varphi \circ \delta_i$, $s_i(\varphi) = \varphi \circ \sigma_i$ and $t_m(\varphi) = \varphi \circ \tau_m$.

In this section we will prove that $|\lambda^n_\cdot| = \mathbb{T} \times \Delta^n$ which is a result we will be in need of later on. In order to do that we need following lemma.

Lemma 1.5.1. Let $X_\cdot$ be a simplicial space equipped with a G-action $G \times X_n \to X_n$, such that the G-action commutes with the faces and the degeneracies. Then there is an induced G-action on $|X_\cdot|$ and

$$|X_\cdot/G| \cong |X_\cdot|/G \ .$$

Proof. The induced G-action on $|X_\cdot|$ is given by $g(x,u) = (gx,u)$ (well-defined as the G-action commutes with the simplicial morphisms).

Since the G-action on $X_\cdot$ commutes with the simplicial morphisms, it doesn't matter if we first identify $(x, \varphi_*(u))$ with $(\varphi^*(x), u)$ (i.e. realizing), and then identifying $(x, \varphi_*(u))$ with $(gx, \varphi_*(u))$ (and $(\varphi^*(x), u)$ with $(g.\varphi^*(x), u)$ (i.e. quoting out the G-action) or doing it in the reversed order. This explains why $|X_\cdot/G| \cong |X_\cdot|/G$ holds. □

Theorem 1.5.2. For every fixed $n \in \mathbb{Z}_{\geq 0}$ the realization of the cyclic space $\lambda^n_\cdot$ is the topological space $\Lambda^n = \mathbb{T} \times \Delta^n$.

Proof. We start by ordering the vertices of Δ^n as $v_0, \dots, v_n$ where $v_i = (0, \dots, 0, 1, 0 \dots, 0)$ (1 in the $(i+1)$-th place). Now triangulate $\mathbb{R} \times \Delta^n$ in the following way.

The vertex set of the triangulation is the set of all pairs (k, v_i) where $k \in \mathbb{Z}$ and v_i is a vertex of Δ_n. We order these vertices such that $(k, v_i) < (\ell, v_j)$ if either $k < \ell$ or $k = \ell$ and $i < j$.

The geometric q-simplices of the triangulation are of two types. The first type consist of all q-simplices that are spanned by vertices on the form

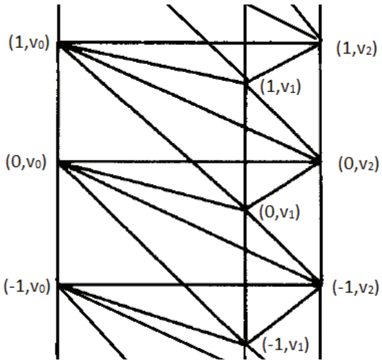
$$(k, v_{r_s}), (k, v_{r_{s+1}}), \dots, (k, v_{r_q}), (k+1, v_{r_0}), (k+1, v_{r_1}), \dots, (k+1, v_{r_{s-1}})$$

where $r_0 < \cdots < r_q$ (strict inequalities).

The second type consist of all q-simplices that are spanned by vertices on the form

$$(k, v_{r_s}), (k, v_{r_{s+1}}), \dots, (k, v_{r_{q-1}}), (k+1, v_{r_0}), (k+1, v_{r_1}), \dots, (k+1, v_{r_s})$$

where $r_0 < \cdots < r_{q-1}$ (strict inequalities). The triangulation of $\mathbb{R} \times \Delta^2$ is visualized here below



Now let Σ^n be the simplicial set generated by this triangulation (see *Example 1.1.10.*). We define an operation $\beta_q : \Sigma^n[q] \to \Sigma^n[q]$ where the vertices of $\beta_q\sigma$ are the same as the vertices of σ except that the last vertex of σ, say (k, v_s), is replaced by $(k-1, v_s)$ (which becomes the first vertex of $\beta_q\sigma$).

One can easily check that $d_i\beta_q = \beta_{q-1}d_{i-1}$, $d_0\beta_q = d_q$, $s_i\beta_q = \beta_{q+1}s_{i-1}$ and $s_0\beta_q = \beta_{q+1}^2 s_q$. That means that β_q satisfies all relations of (1.4) but the last one. Actually β_q^{q+1} translates a q-simplex by -1.

Obviously, in our triangulation of $\mathbb{R} \times \Delta^n$, we get geometrical simplices of all dimensions between 0 and $n+1$, but not of higher dimensions. Every geometric q-simplex where $q < n+1$ is a face of some geometric $(n+1)$-simplex

$$\big((k, v_r), (k, v_{r+1}), \ldots, (k, v_n), (k+1, v_0) \ldots (k+1, v_r)\big)$$

(note that the geometric $(n+1)$-simplices of $\mathbb{R} \times \Delta^n$ has to be of the second type). The $(n+1)$-simplex above is actually the simplex $\beta_{n+1}^{-k(n+2)} s_n \beta_n^{n-r} \iota_n$, where $\iota = 0 \times \Delta^n$. Hence, as every non-degenerated $(n+1)$-simplex can be generated by just using the operations β_q and s_i on ι, and as every q-simplex is a face of some $(n+1)$-simplex, it follows that every non-degenerated q-simplex of Σ^n (i.e. every geometric q-simplex of the triangulation) can be generated using β_q, s_i, d_i. Hence every simplex of Σ^n can bee generated using β_q, s_i, d_i.

Now compare this to the cyclic set $\lambda^n_{\cdot}$, which is generated by using d_i, s_i and t_q by $\mathrm{id}_{[n]} \in \lambda^n[n]$ (obviously if $\varphi \in \lambda^n_q$, then $\varphi = \varphi^*(\mathrm{id}_{[n]})$). Hence λ^n is obtained from Σ^n by identifying β_q^{q+1} with the identity. Define a $\mathbb{Z}$-action $\mathbb{Z} \times \Sigma^n[q] \to \Sigma^n[q]$ given by $n.\sigma = \beta_q^{n(q+1)}(\sigma)$.

This $\mathbb{Z}$-action commutes with the faces and the degeneracies, and $n \in \mathbb{Z}$ acts on a simplex by translating it $-n$ steps in the $\mathbb{Z}$-direction. The induced $\mathbb{Z}$-action on the realization $|\Sigma^n_{\cdot}| = \mathbb{R} \times \Delta^n$ translates every point by $-n$ in the $\mathbb{R}$-direction.

Obviously, quoting out the action of $\mathbb{Z}$ on $\Sigma^n_{\cdot}$ is equivalent to identifying β_q^{q+1} with the identity, which is, by previous reasoning, the cyclic space $\lambda^n_\cdot$. Hence

$$|\lambda^n_\cdot| = |\Sigma^n_\cdot/\mathbb{Z}| = |\Sigma^n_\cdot|/\mathbb{Z} = \mathbb{T} \times \Delta^n$$

(the third equality follows from the previous lemma). $\square$

Hochschild and Cyclic homology

2.1. Hochschild homology

Often Hochschild homology is presented as a homology theory for algebras. However in [Jon], J.D.S. Jones defines Hochschild homology for simplicial complexes over a ring R, which we will do in this thesis as well.

In order to define Hochschild homology we need the notion of double complexes of R-modules (and the total complexes of these).

Definition 2.1.1. A double complex of R-modules is a collection of R-modules $C_{n,m}$ indexed by $\mathbb{Z} \times \mathbb{Z}$ together with vertical differentials $d_v : C_{n,m} \to C_{n-1,m}$ and horizontal differentials $d_h : C_{n,m} \to C_{n,m-1}$ such that $d_v^2 = d_h^2 = d_v d_h + d_h d_v = 0$.

The total complex of C_{**}, denoted by $\operatorname{Tot} C_{**}$ is a chain complex with

$$\operatorname{Tot}_n C_{**} = \prod_{n=p+q} C_{p,q}$$

and with a differential $d := d_v + d_h$ (sometimes called the total differential).

Definition 2.1.2. Given simplicial chain complex of R-modules, $E_. : \mathbf{\Delta}^{\mathrm{op}} \to R{-}\mathrm{Ch}$, we can define a double complex $C_{**}(E_.)$, where $C_{p,q}(E_.) = E_.[p]_q$. The vertical differential $b_v : E_.[p]_q \to E_.[p]_{q-1}$ is the ordinary differential of $E_.[p]$ and the horizontal differential is given by $(-1)^q b_h$, where $b_h : E_.[p]_q \to E_.[p-1]_q$ is given by $b_h = \sum(-1)^i d_i$.

The total complex of a of $C_{**}(E_.)$ is called the *Hochschild complex of* $E_.$ and is denoted by $C_*(E_.)$, and the total differential $b = b_v + (-1)^q b_h$ is called the *Hochschild boundary*. The homology of $C_*(E_.)$ is called the *Hochschild homology of* $E_.$ is denoted by $HH_*(E_.)$.

Example 2.1.3. This example will explore how the Hochschild homology as a homology theory for algebras is related to Hochschild homology as a homology theory for simplicial complexes. Given a d.g. algebra A, let $A_. = A, A^{\otimes 2}, \dots$ be the simplicial complex described in *Example 1.2.6.*. Then the Hochschild homology, viewed as a homology theory for simplicial complexes, of $A_.$ will coincide with the Hochschild homology, viewed as a homology theory for algebras, of A (see § 5.3 in [Lo1]).

Example 2.1.4. A very common homology theory for topological spaces is the singular homology theory. Given a topological space U, the singular chain complex of U, denoted by $S_*(U)$, is a chain complex with $S_n(U)$ being the free R-module generated by all continuous maps $\alpha : \Delta^n \to U$, and with differential $\partial\alpha = \sum(-1)^i \alpha \circ \delta_i$, where δ_i are the faces of the cosimplicial space $\Delta^{\cdot}$ (see *Example 1.1.6.*). The homology of $S_*(U)$ is denoted by $H_*(U)$ and is called the singular homology of U (singular homology are treated in more detail in [Hat]). One can show that S_* is a functor from the category of topological spaces to the category of chain complexes of R-modules (given that we have fixed some ring R).

Now to any simplicial space $X_.$ there is an associated simplicial chain complex $S_*(X_.) = S_*(X_0), S_*(X_1), \ldots$ of singular chain complexes (with faces and degenracies equal to $S_*(d_i)$ and $S_*(s_i)$ respectively). Hence the Hochschild complex of $S_*(X_.)$ is the total complex of

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
S_2(X_0) \arrow[d, "\partial"] & S_2(X_1) \arrow[l, "b_h"'] \arrow[d, "\partial"] & S_2(X_2) \arrow[l, "b_h"'] \arrow[d, "\partial"] & \cdots \arrow[l] \\
S_1(X_0) \arrow[d, "\partial"] & S_1(X_1) \arrow[l, "-b_h"'] \arrow[d, "\partial"] & S_1(X_2) \arrow[l, "-b_h"'] \arrow[d, "\partial"] & \cdots \arrow[l] \\
S_0(X_0) & S_0(X_1) \arrow[l, "b_h"'] & S_0(X_2) \arrow[l, "b_h"'] & \cdots \arrow[l]
\end{tikzcd}$$

We will denote the total homology of this double complex by $HH(S_*(X_.))$. The goal of this section is to establish an isomorphism $HH(S_*(X_.)) \cong H_*(|X_.|)$, given that $X_.$ satisfies some conditions. The author could not find a proof for this isomorphism in the literature, so a proof will be offered, by using other theorems that will be stated without proof.

Definition 2.1.5. A *bisimplicial object* $X_{..}$ in a category $\mathcal{C}$ is a bigraded sequence of objects $X_{m,n}$ in $\mathcal{C}$ together with horizontal faces and horizontal degenracies ($d_i^h : X_{m,n} \to X_{m-1,n}$ and $s_i^h : X_{m,n} \to X_{m+1,n}$) that satisfies the simplicial relations in (1.2), as well as vertical faces and vertical degeneracies ($d_i^v : X_{m,n} \to X_{m,n-1}$ and $s_i^v : X_{m,n} \to X_{m,n+1}$) that also satisfies the simplicial relations in (1.2).
This is equivalent to a covariant functor $X_{..} : \mathbf{\Delta}^{\mathrm{op}} \times \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{C}$.

Lemma 2.1.6. For any CW-complex X, $S_*(X)$ is a simplicial space and there is a homotopy equivalence $\varphi : |S_*(X)| \to X$ given by

$$S_n(X) \times \Delta^n \ni (f, p) \longmapsto f(p) \in X$$

Proof. The second theorem of § 16.2 in [Ma2] together with Theorem 4.5 (Whitehead theorem) in [Hat] will give the desired result. $\square$

Lemma 2.1.7. Given a bisimplicial R-module $C_{..}$, let $\mathrm{diag}(C_{..})$ be the chain complex with $\mathrm{diag}_n(C_{..}) = C_{n,n}$ in degree n and with differential $d = \sum_{i=0}^{n}(-1)^i d_i$ where $d_i = d_i^h d_i^v$. For any simplicial space $X_.$, we have that

$$H(\mathrm{Tot}(C_{**}(S_\bullet(X_.)))) = H(\mathrm{diag}(C_{**}(S_\bullet(X_.))))$$

Note that the left-hand side is the Hochschild homology of $S_*(X_.)$. Note also that $C_{**}(S_\bullet(X_.)$ can be viewed as a bisimplicial set, since $S_*(X_k)$ it self is a simplicial set (see *Lemma 2.1.6* or look forward for *Definition/Lemma 2.3.2.* where the simplicial structure is explicitly given).

Proof. Theorem 8.5.1 (Eilenberg-Zilber theorem) and Theorem 8.3.8 in [Wei] will together give the desired result. □

Definition/Lemma 2.1.8. A bisimplicial space $X_{..}$ can be realized either by
(a) realizing the simplicial space $A_.$, where $A_k = |X_{k,.}|$, or
(b) realizing the simplicial space $B_.$, where $B_k = |X_{.,k}|$, or
(c) realizing the simplicial set $D_.$, where $D_k = X_{k,k}$
All these realizations are homeomorphic.

Proof. See the lemma at page 86 in [Qui]. □

Definition 2.1.9. A simplicial space is called *good* if for every $n \in \mathbb{Z}_{\geq 1}$ and every $0 \leq i \leq n$, the inclusion $s_i(A_{n-1}) \hookrightarrow A_n$ is a cofibration.

Lemma 2.1.10. If $f : X_. \to Y_.$ is a simplicial map between good simplicial spaces, such that $f_n : X_n \xrightarrow{\sim} Y_n$ is a homotopy equivalence for each $n \in \mathbb{Z}_{\geq 0}$, then $|X_.| \simeq |Y_.|$.

Proof. See Proposition A.1.(ii),(iv) in [Seg]. □

Theorem 2.1.11. Let $X_.$ be a good simplicial CW-complex. Then $H(|X_.|) = HH(S_*(X_.))$.

Proof. Obviously $C_{**}(S_\bullet(X_.))$ can be regarded as a bisimplicial set $C_{..}$ (since $S_*(X_n)$ itself is a simplicial space), which can be realized in the sense of *Definition/Lemma 2.1.8. (a)*. That means that $|C_{..}|$ is the realization of the simplicial space $A_.$, where $A_n = |S_*(X_n)|$.

Now let $\varphi_n : A_n = |S_*(X_n)| \to X_n$ be the simplicial map described in Lemma 2.1.6. We have by Lemma 2.1.6. that $A_n = |S_*(X_n)| \simeq X_n$. Hence, as A_n and X_n are both CW-complexes and homotopic, we have by Lemma 2.1.10. that their realizations are also homotopic, i.e. that $|C_{..}| = |A_.| \simeq |X_.|$, and hence

$$H(|C_{..}|) = H(|X_.|) \tag{2.1}$$

Now we realize $C_{..}$ in the sense of Definition/Lemma 2.1.8.(c) instead. Then we have that $|C_{\cdot\cdot}|$ is the realization of the simplicial set $D_{\cdot}$ where $D_n = \mathrm{diag}_n(C_{\cdot\cdot}) = S_n(X_n)$.

Hence

$$\begin{aligned} HH_*(S_\bullet(X_{\cdot})) &= H_*(\mathrm{Tot}(C_{\bullet\bullet}(S_\star(X))) = H_*(\mathrm{diag}(C_{\bullet\bullet}(S_\star(X_{\cdot}))) \\ &= H_*(D_{\cdot}) = H_*(|D_{\cdot}|) = H_*(|C_{\cdot\cdot}|) = H_*(|X_{\cdot}|) \end{aligned}$$

The second equality follows from Lemma 2.1.5., the forth equality follows from the equivalence of simplicial and singular homology, the last equality follows from (2.1). $\square$

2.2. Cyclic homology

In this section we introduce cyclic homology.

Definition 2.2.1. Given a cyclic chain complex $E_{\cdot} : \mathbf{\Delta} C \to R\text{-Ch}$, define

$$h_p := t_{p+1}s_p : E_{\cdot}[p] \to E_{\cdot}[p+1] \quad \text{and} \quad N_p := \sum_{i=0}^{p} (-1)^{ip} t_p^i : E_{\cdot}[p] \to E_{\cdot}[p].$$

The operator

$$B_p = (-1)^q(1 - (-1)^{p+1}t_{p+1})h_p N_p : E_{\cdot}[p] \to E_{\cdot}[p+1]$$

is called *Connes' boundary* and satisfies

$$B^2 = 0, \quad Bb = -bB$$

where $b = b_v + (-1)^q b_h$ is the the Hochschild boundary (the equalities are proved in 1.3 and 1.4 in [LQ]).

Definition 2.2.2. Given cyclic chain complex $E_{\cdot} : \mathbf{\Delta}^{\mathrm{op}} \to R - Ch$, let $R[u]$ be a polynomial ring with $|u| = -2$, there is an associated double complex over $R[u]$ given by

$$C^-(E_{\cdot}) = R[u] \otimes C(E_{\cdot})$$

where $C(E_{\cdot})$ is the Hochschild complex, and the differential is $\partial = b + uB$ (where b is the Hochschild boundary and B is Connes' boundary).

If $C(E)$ is positively graded then $C^-(E) = R[u] \otimes C(E)$ is the following complex

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\cdots \longleftarrow & & u^2C_2(E) & \xleftarrow{uB} & uC_1(E) & \xleftarrow{uB} & C_0(E) \\
 & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
\cdots \longleftarrow & & u^2C_1(E) & \xleftarrow{uB} & uC_0(E) & & \\
 & & {\scriptstyle b}\downarrow & & & & \\
\cdots \longleftarrow & & u^2C_0(E) & & & &
\end{array}$$

The homology of the total complex of $C^-(E)$ is called *the negative cyclic homology* of E and is denoted by $HC^-(E)$.

2.3. Shuffle product and Connes' Boundary

In this section we introduce shuffle products in order to prove some conditions for Connes' boundary on simplical complexes $S_*(X_.)$ and $S^*(Y^\cdot)$.

We will grade the singular cochain complex $S^*(U)$ negatively, i.e. $S^{-n}(W) = \mathrm{Hom}_{\mathbb{Z}}(S_n(W), R)$ in order to make the singular cochain differential of degree -1. We will also use following sign convention for singular cochains:

$$\widehat{\partial}(c)(x) = (-1)^{|c|+1}c(\partial x) \tag{2.2}$$

where $\widehat{\partial}$ is the singular cochain differential and ∂ is the singular chain differential.

A type of product called the shuffle product ∇ will be of main importance in this section. In order to define this product we need some preparing definitions.

Definition 2.3.1. A (p, q)-shuffle is a permutation $\nu \in S_{p+q}$ such that

$$\nu(1) < \nu(2) < \cdots < \nu(p) \quad \text{and} \quad \nu(p+1) < \nu(p+2) < \cdots \nu(p+q)\ .$$

Definition/Lemma 2.3.2. Given a topological space U, the singular chain complex $S_*(U)$ is a simplicial set, with faces $F_i(\alpha) = \alpha \circ \delta_i$ and degeneracies $D_i(\alpha) = \alpha \circ \sigma_i$, where δ_i and σ_i are the faces and the degeneracies of the cosimplicial set $\Delta^\cdot$. We denote the faces and the degeneracies by capital letters in order to not mix them with the ordinary faces and degeneracies when we are dealing with singular chains complexes of simplicial spaces (which is in fact a bisimplicial set).

Definition 2.3.3. If $\alpha \in S_n(U)$ and $\beta \in S_n(V)$, let $\alpha \times \beta \in S_n(U \times V)$ be given by

$$(\alpha \times \beta)(u) = \alpha(u) \times \beta(u)$$

for any $u \in \Delta^n$.

Now we are ready to define the shuffle product.

Definition 2.3.4. For any topological spaces U and V, the *shuffle product* $\nabla : S_p(U) \otimes S_q(V) \to S_{p+q}(U \times V)$, is given by the sum

$$\sum_{\nu=(p,q)\text{-shuffle}} \mathrm{sgn}(\nu) D_{\nu(p+q)} \cdots D_{\nu(p+1)}\alpha \times D_{\nu(p)} \cdots D_{\nu(1)}\beta\ .$$

Theorem 2.3.5. (a) (The Eilenberg-Zilber theorem) The shuffle product $\nabla : S_*(U) \otimes S_\bullet(V) \to S_{*+\bullet}(U \times V)$ is a natural chain equivalence and has a natural homotopy inverse, called the Alexander-Whitney map

$$AW : S_n(U \times V) \to (S_*(U) \otimes S_\bullet(V))_n\ .$$

(b) The shuffle product satisfies following properties

$$\begin{array}{cc} (i) & \partial(\alpha\nabla\beta) = \partial(\alpha)\nabla\beta + (-1)^{|\alpha|}\alpha\nabla\partial(\beta) \\ (ii) & \alpha\nabla\beta = (-1)^{|\alpha||\beta|}\beta\nabla\alpha \\ (iii) & \alpha\nabla(\beta\nabla\gamma) = (\alpha\nabla\beta)\nabla\gamma \end{array}$$

Proof. For (a) see Corollary 1.4., Ch. VI in [Bre] and for (b) see Theorem 5.2. in [EM]. □

Remark 2.3.6. If α is of odd homological degree, then it follows that $\alpha\nabla\alpha = 0$ by (ii) in the lemma above.

Lemma 2.3.7. Let $X_{.}$ be a cyclic space and let $\Lambda^{\cdot}$ be cocyclic space defined in *Example 1.3.1.* Then for any $\alpha \in S_q(X_n)$ and any $\beta \in S_p(\Lambda^m)$ and any $\varphi \in \mathrm{Hom}_{\mathbf{\Delta}C}([m],[n])$ we have that

$$\pi_{m*}(\varphi^*\alpha\nabla\beta) = \pi_{n*}(\alpha\nabla\varphi_*\beta)$$

where π_k is the projection map $X_k \times \Lambda^k \to \wr X_{.}\wr = |X_{.}|$.

Proof. Just check that for any $u \in \Delta^{p+q}$ and any (p,q)-shuffle ν that

$$\begin{aligned} & D_{\nu(p+q)}\cdots D_{\nu(q+1)}\varphi^*\alpha(u) \times D_{\nu(q)}\cdots D_{\nu(1)}\beta(u) \\ & \qquad \approx D_{\nu(p+q)}\cdots D_{\nu(q+1)}\alpha(u) \times D_{\nu(q)}\cdots D_{\nu(1)}\varphi_*\beta(u) \end{aligned}$$

where $\approx$ is the equivalence relation described in *Definition 1.3.2.*. □

In next definition we need the notion of slant product

$$/ : S^{-n}(U \times V) \times S_k(U) \to S^{-n+k}(V) \ .$$

Let $\alpha \in S^{-n}(U \times V)$ and $b \in S_k(U)$. Then the slant product of α and b, denoted α/b, is given by the equality $\alpha/b(c) = \alpha(b\nabla c)$ for any $c \in S_{n-k}(V)$. This definition of slant product is due to [Jon], but is not standard terminology (there are authors who defines slant product in a different way).

Definition 2.3.8. Let W be a space with a circle action $f : \mathbb{T} \times W \to W$, and let

$$I : S^{-n}(W) \to S^{-n+1}(W), \quad J : S_n(W) \to S_{n+1}(W)$$

be given by the formulas $I(x) = (-1)^{|x|}f^*(x)/z$ and $J(x) = (-1)^{|x|}f_*(z\nabla x)$ where z is the fundamental 1-cycle in $S_1(z)$ (i.e. the cycle that generates $H_1(\mathbb{T}) = \mathbb{Z}$).

Lemma 2.3.9. I and J satisfies

$$\widehat{\partial}I = -I\widehat{\partial}, \quad \partial J = -J\partial$$

and

$$I^2 = J^2 = 0 \ .$$

Proof. We prove the properties only for I, since the properties for J can be proved in a similar manner. We have that

$$(\delta \circ I(x))[\alpha] = -f^*(x)/z[d\alpha] = -f^*(x)[z\nabla d\alpha] = f^*(x)[d(z\nabla\alpha) - \underbrace{dz}_{=0}\nabla\alpha]$$

$$= f^*(x)[d(z\nabla\alpha)] = (-1)^{|x|}\delta f^*(x)[z\nabla\alpha] = (-1)^{|x|}f^*(\delta x)[z\nabla\alpha] = -(I \circ \delta(x))[\alpha]$$

In order to prove that $I^2 = 0$, let $\mu : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ be the multiplication map. Obviously μ can be regarded as a $\mathbb{T}$-action on $\mathbb{T}$. Since group actions $G \times W \to W$ has to satisfy $g_2.(g_1.w) = (g_2 g_1).w$, it follows that

$$f(g_2, f(g_1, w)) = f(\mu(g_2, g_1), w) \ . \tag{2.3}$$

Now

$$\begin{aligned} I(I(x))[\alpha] &= (-1)^{|x|+1}I(x)[f_*(z\nabla\alpha)] = -x[f_*(z\nabla f_*(z\nabla\alpha))] \\ &\overset{(2.2)}{=} -x[f_*(\mu_*(\underbrace{z\nabla z}_{=0})\nabla\alpha)] = 0 \end{aligned}$$

($z\nabla z = 0$ by *Remark 2.2.2.*). $\square$

Proposition 2.3.10. Given a cyclic set $X_.$ (endowed by the discrete topology) there is a natural chain map $\varphi : C_*(S_\bullet(X_.)) \to S_*(|X|)$ and a natural map $h : C_*(S_\bullet(X_.)) \to S_{*+2}(|X_.|)$ such that $\partial h - hb = J\varphi - \varphi B$, where ∂ is the singular chain differential.

Proof. We start by describing the chain map $\varphi : C_*(S_\bullet(X_.)) \to S_*(|X_.|)$.

Let $\pi_n : X[n] \times \Lambda^n \to \wr X \wr = |X|$ be the obvious projection map. If $\alpha \in S_q(X_n)$ then $\varphi(\alpha) = \pi_{n*}(\alpha\nabla\kappa_n)$ where κ_n is the fundamental singular n-simplex (i.e. $\kappa_n : \Delta^n \to \mathbb{T} \times \Delta^n$ is given by $u \mapsto (1, u)$).

Now let $d_i : X_n \to X_{n-1}$ be a the i'th face map. We have that for any $u \in \Delta^{q+n-1}$ that

$$(d_{i*}\alpha\nabla\kappa_{n-1})(u) \approx (\alpha\nabla\delta_{i*}\kappa_{n-1})(u)$$

(see *Lemma 2.3.7.*). Hence

$$\pi_{(n-1)*}(d_{i*}\alpha\nabla\kappa_{n-1}) = \pi_{n*}(\alpha\nabla\delta_{i*}\kappa_{n-1}) \tag{2.4}$$

One can also easily check that $\partial\kappa_n = \sum(-1)^i\delta_{i*}\kappa_{n-1}$ where ∂ is the singular chain differential. Hence we have that

$$\begin{aligned} \varphi b(\alpha) &= \pi_{n*}(\partial\alpha\nabla\kappa_n) + (-1)^n\pi_{(n-1)*}\left(\sum(-1)^i d_{i*}\alpha\nabla\kappa_{n-1}\right) \\ &= \pi_{n*}(\partial\alpha\nabla\kappa_n) + (-1)^n\pi_{n*}\left(\alpha\nabla\sum(-1)^i\delta_{i*}\kappa_{n-1}\right) \\ &= \pi_{n*}(\partial\alpha\nabla\kappa_n) + (-1)^n\pi_{n*}(\alpha\nabla\partial\kappa_n) \\ &= \partial\pi_{n*}(\alpha\nabla\kappa) = \partial\varphi(\alpha) \end{aligned} \tag{2.5}$$

showing that φ is a chain map. The naturality of φ follows from the naturality of ∇ and π_{k*} (w.r.t. cyclic maps).

Now we prove the existence of a natural map $h : C_*(S_\bullet(X_.)) \to S_{*+2}(|X_.|)$ that satisfy $\partial h - hb = J\varphi - \varphi B$, for any cyclic discrete space $X_.$.

Since $X_.$ is a cyclic discrete set, it follows that for any q and n that $S_q(X_n)$ is identified with X_n (since the image of each map in $S_q(X_n)$ is limited to a single point of X_n), so we will denote a map in $S_q(X_n)$ by its image in X_n.

We will prove the assertion by induction on the degree of $x \in C_*(S_\bullet(X_.))$. We define h for singular 0-simplices, and then extend it by linearity (possible as $S_0(X)$ is free). If $x \in C_0(S_\bullet(X_.)) = S_0(X_0)$ is a singular 0-simplex, consider the element $Q = hb(x) + J\varphi(x) - \varphi B(x)$.

Obviously $hb(x) = 0$ as $b(x) \in C_{-1}(S_\bullet(X_.)) = 0$. Now explicit computation gives that $J\varphi(x) - \varphi B(x)$ is a constant map $\Delta^1 \to |X_.|$ where every point of Δ^1 is sent to $\pi_{0*}(x \times 1)$. Obviously constant maps in $S_1(|X_.|)$ are boundaries, so we conclude that $Q = \partial a$ for some $a \in S_2(|X_.|)$. Now let $h(x) = a$ and hence we get that

$$\partial h(x) - hb(x) = J\varphi(x) - \varphi B(x) \ ,$$

ending the base case.

Now suppose that we have defined h for all elements of degrees strictly less than n, then we will define it for elements of degree n. We have that φ and ∇ are natural. From the naturality of ∇ it follows that J is also natural with respect to cyclic maps. Obviously f commutes with B (as f is a map of cyclic spaces). Moreover we want h to be natural as well, which gives that every term in the equality

$$\partial h - hb = \varphi J - \varphi B$$

is natural with respect to cyclic maps. Hence for any $x \in S_p(X_q) \subset C_n(S_\bullet(X_.))$ define a cyclic map $f : \lambda^q_. \to X_.$ given by $\mathrm{id}_{[q]} \mapsto x$ (this determines f completely as $\lambda^q_.$ is generated by using d_i, s_i and t_ℓ by $\mathrm{id}_{[q]}$). If we manage to construct h that satisfies the required equality for $\mathrm{id}_{[q]} \in S_p(\lambda^q_q)$ (recall that for a discrete set A we can identify $S_p(A)$ with A), then we have managed to do that for $x = f_*(\mathrm{id}_{[q]}) \in S_p(X_q)$.

Again let $Q = hb(\mathrm{id}_{[q]}) + J\varphi(\mathrm{id}_{[q]}) - \varphi B(\mathrm{id}_{[q]})$. Differentiating Q gives

$$\partial Q = \partial h(b(\mathrm{id}_{[q]})) + \underbrace{\partial J\varphi(\mathrm{id}_{[q]})}_{=-J\varphi(b(\mathrm{id}_{[q]}))} - \underbrace{\partial \varphi B(\mathrm{id}_{[q]})}_{=-\varphi B(b(\mathrm{id}_{[q]}))} \tag{2.6}$$

By the inductive hypothesis we have that

$$\partial h(b(\mathrm{id}_{[q]})) = h(\underbrace{b^2(\mathrm{id}_q)}_{=0}) + J\varphi(b(\mathrm{id}_{[q]})) - \varphi B(b(\mathrm{id}_{[q]})) \ .$$

Substituting $\partial h(b(\mathrm{id}_{[q]}))$ with the right-hand side above in (2.6) will prove that $\partial Q = 0$, making Q into a cycle of $S_{n+1}(|\lambda^q_\cdot|) = S_{n+1}(\mathbb{T} \times \Delta^q)$. Since $H_{n+1}(\mathbb{T} \times \Delta^q) = 0$, it follows that Q is a boundary, i.e $Q = \partial a$ for some $a \in S_{n+2}(|\lambda^q_\cdot|)$. Now let $h(\mathrm{id}_{[n]}) = a$ and hence $Q = \partial h(\mathrm{id}_{[q]})$ and the required equality is satisfied. □

Theorem 2.3.11. Given a cocyclic space $Y^\cdot$ there is a natural chain map $\psi : C_*(S^\bullet(Y^\cdot)) \to S^*(|Y^\cdot|)$ and a natural map $j : C_*(S^\bullet(Y^\cdot)) \to S^{*+2}(|Y^\cdot|)$ such that $\widehat{\partial} j - jb = I\psi - \psi B$, where $\widehat{\partial}$ is the singular cochain differential.

The proof of this theorem is very technical, but as there are some misprints in [Jon], a proof will be offered.

Proof. We start by describing the natural chain map $\psi : C_*(S^\bullet(Y^\cdot)) \to S^*(|Y^\cdot|)$.

Recall the definition of $\wr Y^\cdot \wr$ in § 1.4 and recall that $\wr Y^\cdot \wr$ is homeomorphic to $|Y^\cdot|$. Define a map $\rho_n : \Lambda^k \times \wr Y^\cdot \wr \to Y^k$ given by $(u, \{\alpha_n\}) \mapsto \alpha_k(u)$. By the definition of $\wr Y \wr$ it follows that the diagram

$$\begin{tikzcd} \Lambda^n \times \wr Y^\cdot \wr \arrow[r, "\rho_n"] \arrow[d, "\theta\times 1"'] & Y^n \arrow[d, "\theta"] \\ \Lambda^m \times \wr Y^\cdot \wr \arrow[r, "\rho_m"'] & Y^m \end{tikzcd}$$

commutes.

For $x \in S^q(Y(n))$, let $\psi(x) = \rho_n^*(x)/\kappa_n$, where κ_n is the fundamental n-simplex in Λ^n. By the commuting diagram above it follows that $\psi(\delta_i^*(x)) = \rho_{n-1}^*(\delta_i^* x)/\kappa_{n-1} = \rho_n^*(x)/\delta_{i*}\kappa_{n-1}$.
Moreover, as $\partial\kappa_n = \sum(-1)^i \delta_{i*}\kappa_{n-1}$, and as $\widehat{\partial}(x/a) = \widehat{\partial}(x)/a + (-1)^{|x|} x/\partial a$ (use *Theorem 2.3.5.(b)(i)* and the sign convention (2.2) to verify this), it follows in a similar manner as in (2.5) that ψ is a chain map.

Now we define $j : C_*(Y^\cdot) \to S^{*+2}(|Y^\cdot|)$. For $x \in S^q(Y^n)$ let

$$j(x) = \rho_n^*(x)/h(\iota_n)$$

where $h : C_*(\lambda^n_\cdot) \to S_*(|\lambda^n_\cdot|)$ is the map defined in the previous proposition and $\iota_n \in S_0(\lambda^n_\cdot)$ is the constant map $* \mapsto \mathrm{id}_{[n]}$. Now we have that

$$\widehat{\partial} j(x) = \rho_n^*(\widehat{\partial} x)/h(\iota_n) + (-1)^q \rho_n^*(x)/\partial(h(\iota_n))$$

$$jb(x) = \rho_n^*(\widehat{\partial} x)/h(\iota_n) + (-1)^q \rho_{n-1}^*(b_h x)/h(\iota_{n-1})$$

and hence

$$\widehat{\partial} j(x) - jb(x) = (-1)^q \rho_n^*(x)/\partial h(\iota_n)) - (-1)^q \rho_{n-1}^*(b_h x)/h(\iota_{n-1}) \tag{2.7}$$

Simplifying $\rho^*_{n-1}(b_h x)/h(\iota_{n-1})$:
In our case $b_h = \sum(-1)^i \delta_i^*$. By the commuting diagram above we have that

$$\rho^*_{n-1}(b_h x)/h(\iota_{n-1}) = \sum(-1)^i \rho^*_n(x)/\delta_{i*}h(\iota_{n-1})$$

Now define a cyclic map $f_i : \lambda^{n-1}_{\cdot} \to \lambda^n_{\cdot}$ given by $\mathrm{id}_{[n-1]} \to d_i(\mathrm{id}_{[n]})$ f_i completely as every element $\lambda^{n-1}_{\cdot}$ is generated by using morphisms in $\mathbf{\Delta}C$ on id_{n-1}). Now obviously f_i induces a map on the realizations $\tilde{f}_i : \Lambda^{n-1} \to \Lambda^n$ which is the face map δ_i. Now since h is natural with respect to cyclic maps it follows that $\delta_{i*}h(\iota_{n-1}) = \tilde{f}_{i*}h(\iota_{n-1}) = h(f_{i*}(\iota_{n-1})) = h(d_{i*}\iota_n)$. This gives

$$\rho^*_{n-1}(b_h x)/h(\mathrm{id}_{[n-1]}) = \rho^*_n(x)/hb_h(\iota_n))$$

Now since $\iota_n \in S_0(\lambda^n_n)$, it follows that $\partial\iota_n = 0$ and hence $hb_h(\iota_n)) = h(b_h\iota_n + \partial\iota_n) = hb(\iota_n)$. This gives that

$$\rho^*_{n-1}(b_h x)/h(\mathrm{id}_{[n-1]}) = \rho^*_n(x)/hb(\iota_n)) \qquad \blacktriangle$$

Substituting $\rho^*_{n-1}(b_h x)/h(\iota_{n-1})$ with $\rho^*_n(x)/hb(\iota_n))$ in (2.7) gives

$$\widehat{\partial}j(x) - jb(x) = (-1)^q \rho^*_n(x)/\partial h(\iota_n)) - (-1)^q \rho^*_{n-1}(x)/hb(\iota_n)$$

$$= (-1)^q \rho^*_n(x)/\big(\partial h(\iota) - hb(\iota)\big) \overset{\substack{\text{Prop.}\\ 2.3.10.}}{=} (-1)^q \rho^*_n(x)/J\varphi(\iota_n) - (-1)^q \rho^*_n(x)/\varphi B(\iota_n)$$

By looking at explicit formulas for each of the maps one can check that $(-1)^q \rho^*_n(x)/J\varphi(\iota_n) = I\psi(x)$ and $(-1)^q \rho^*_n(x)/\varphi B(\iota_n) = \psi B(x)$. This completes the proof. $\square$

We end this chapter with following definition.

Definition 2.3.12. A cocyclic space $Y^{\cdot}$ is said to *converge* if the map $\psi : C_*(S^\bullet(Y^{\cdot}) \to S^*(|Y^{\cdot}|)$ induces an isomorphism on homology.

Connections to Equivariant (co)homology and free loop spaces

In this chapter we establish some of the connections between cyclic homology and equivariant cohomology of free loop spaces. Spectral sequence arguments are used in many occasions in § 3.3 and § 3.4, so the reader who is unfamiliar with spectral sequences may read the appendix in preparation to these sections.

3.1. Equivariant (co)homology

In this section we will consider connected topological groups G that acts on topological spaces X, where both G and X are of CW-complex homotopy type.

When G acts freely on X, then the quotient X/G is as nice as X is. E.g. if X is a manifold and G acts freely on X, then X/G is a manifold as well.

The idea of G-equivariant homology, H_*^G, is that whenever G acts freely on X, then $H_*^G(X)$ computes the homology groups $H_*(X/G)$.

If G acts ***non***-freely on a topological space X, then X/G may be pathological and not nice as X is.

Example 3.1.1. The circle $\mathbb{T}$ acts on the sphere S^2 by rotating it. As the $\mathbb{T}$-action is trivial on the poles, the $\mathbb{T}$-action is not free. Quoting out the $\mathbb{T}$-action of S^2 gives a closed line segment which is ***not*** a manifold, even though S^2 is.

If we puncture the sphere at the poles, the $\mathbb{T}$-action becomes free and quoting out the $\mathbb{T}$-action gives an open line segment which is an open manifold precisely as the (double-)punctured sphere.

In order to overcome the problem of pathology when quoting out non-free G-actions of X, we are considering something similar to X, but not really X, namely a homotopy equivalent space $X \times EG$ where EG is a contractible space with a free G-action. The diagonal G-action on $X \times EG$ is obviously free. This gives rise to following more general definition of G-equivariant homology.

Definition 3.1.2. Given a space X with some G-action $G \times X \to X$ where G is connected of CW-complex homotopy type, then the equivariant homology is given by

$$H_*^G(X) := H_*((X \times EG)/G)$$

where EG is a contractible space with a free G-action. The equivariant cohomology is defined to be $H^*_G(X) := H^*((X \times EG)/G)$. The space $(X \times EG)/G$ is called the *Borel construction* of the G-space X.

This definition will make sense only if EG exists, which we will prove in next section.
Now after generalizing the definition of equivariant (co)homology, it is natural to ask whether this definition is really generalizing the original definition (restricted to spaces on which G acts freely). In other words, if G already acts freely on X, is there an isomorphism of homology $H(X/G) \cong H((X \times EG)/G)$?
Another question one may ask is if H^G_* is independent of choice of space EG. These questions are answered in the following theorem.

Theorem 3.1.3. (a) If G acts freely on X, then $H(X/G) = H((X \times EG)/G)$.

(b) The G-equivariant (co)homology is independent of choice of contractible space EG with free G-action.

Sketch of proof of (a) We make the sketch of the proof in case X is connected. The more general case will then follow easily. Let $p : (X \times EG)/G \to X/G$ be given by $(x, e) \mapsto x$. One can show that p is a fibration (see [Dol]). For every $x \in X/G$ we have that

$$p^{-1}(x) = \{(gx, e) \mid g \in G, e \in EG\} \overset{(gx,e)=(x,g^{-1}e)}{=} \{(x, e) \mid e \in EG\} \cong EG$$

Note that the last homeomorphism holds for all $x \in X/G$ iff the G-action is free, because otherwise there is some $g \neq \mathrm{id}_G$ and $x \in X$ such that $gx = x$ and hence $(x, e) = (x, ge) \in (X \times EG)/G$, while e and ge are not identified in EG (as G acts freely on EG).

Hence there is a fiber bundle $EG \to (X \times EG)/G \to X/G$ which gives rise to a long exact sequence of homotopy groups

$$\cdots \to \pi_n(EG) \to \pi_n((X \times EG)/G) \xrightarrow{p_*} \pi_n(X/G) \to \pi_{n-1}(EG) \to \cdots$$

(see Theorem 4.41. in [Hat]). Since EG is contractible we get that that p induces an isomorphism $p_* : \pi_*((X \times EG)/G) \to \pi_*(X/G)$. Since X and EG are homotopy equivalent to CW-complexes it follows from the Whiteheads' theorem that p is a homotopy equivalence.

Proof of (b) Let EG and $\widetilde{EG}$ be contractible spaces with free G-action. From (a) we get

$$\begin{aligned} H_*((X \times EG)/G) &\overset{(a)}{=} H_*(((X \times EG) \times \widetilde{EG})/G) \\ = H_*(((X \times \widetilde{EG}) \times EG)/G) &\overset{(a)}{=} H_*((X \times \widetilde{EG})/G) \end{aligned}$$

□

3.2. Simplicial homotopies and EG

From now on we will denote the quotient $(X \times EG)/G$ by $X \times_G EG$.

Definition 3.2.1. Let $A_.$ and $B_.$ be two simplicial objects in a category $\mathcal{C}$. Two simplicial maps $f, g : A_. \to B_.$ are called *simplicially homotopic* if there are morphisms $h_i : A_n \to B_{n+1},\ 0 \le i \le n$, in $\mathcal{C}$ such that $d_0 h_0 = f$ and $d_{n+1} h_n = g$ and

$$d_i h_j = \begin{cases} h_{j-1} d_i & \text{if } i < j \\ d_i h_{i-1} & \text{if } i = j \neq 0 \\ h_j d_{i-1} & \text{if } i > j+1 \end{cases}$$
$$s_i h_j = \begin{cases} h_{j+1} s_i & \text{if } i \le j \\ h_j s_{i-1} & \text{if } i > j \end{cases} \tag{3.1}$$

We call $\{h_j\}$ a simplicial homotopy from f to g and write $f \simeq g$.

Proposition 3.2.2. Let $A_.$ and $B_.$ be simplical spaces, and let $f, g : A_. \to B_.$ be simplicial maps. If there is a simplicial homotopy $\{h_j\}$ from f to g, then there is a homotopy $h : I \times |A_.| \to |B_.|$ from $|f|$ to $|g|$.

Proof. See Corollary 11.10. in [Ma2]. □

In the previous section we proved some conditions for EG, but didn't prove the existence of EG. Given a topological group G, recall from *Example 1.1.5.* the simplicial space $E_.G$ where $E_nG = G^{n+1}$ and where the the faces and the degeneracies are given by

$$d_i(g_1, \ldots, g_{n+1}) = \begin{cases} (g_2, \ldots, g_{n+1}) & \text{if } i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) & \text{if } i = 1, \ldots, n \end{cases}$$

and

$$s_i(g_1, \ldots, g_{n+1}) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_{n+1}) .$$

The next theorem will display the connection between EG and $E_.G$.

Theorem 3.2.3. Let G be a topological group with homotopy equivalent to a CW-complex. The realization $EG = |E_.G|$ of the simplicial space $E_.G$ is a contractible space with a free G-action.

Proof. We start by proving that G acts freely on EG. We start by defining a G-action on E_nG for $n \ge 0$. Let $g \cdot (g_0, \ldots, g_n) := (g_0, \ldots, g_{n-1}, g_n g^{-1})$. The action commutes with the faces and the degeneracies, so it induces a G-action on the realization EG. Now given any point $((g_1, \ldots, g_{n+1}), \boldsymbol{u}) \in EG$, we may by *Lemma 1.1.9.* assume that $(g_1, \ldots, g_{n+1})$ is non-degenerated and $\boldsymbol{u} \in \Delta^n$ is an inner point. From the definition of s_i, $(g_1, \ldots, g_{n+1})$ is non-degenerated iff $g_1, \ldots, g_n$ are all different from the identity (but there is no such restriction on g_{n+1}). Now applying any $g \neq \text{id}$ on this point gives

$((g_1, \ldots, g_n, g_{n+1}g^{-1}), \boldsymbol{u})$. Since $g_1, \ldots, g_n$ are still different from the identity, we conclude that $((g_1, \ldots, g_n, g_{n+1}g^{-1}), \boldsymbol{u})$ is a non-degenerated point, and can therefore not be in the same equivalence class as $((g_1, \ldots, g_{n+1}), \boldsymbol{u})$ because that contradicts the uniqueness of the non-degenerated points in each $\sim$-equivalence class (see *Lemma 1.1.9.*). This proves that G acts freely on EG.

Now we prove that EG is contractible. Let $A_.$ be a constant simplicial one-point space (i.e. $A_n = \{*\}$ for all n, and the simplicial maps acts by triviality). Let $\iota : E_nG \to A_n$ be the constant simplicial map, and let $\rho : A_n \to E_nG$ be given by $* \mapsto (1, \ldots, 1)$. Obviously $\iota\rho = \mathrm{id}_{A_.}$, and hence $|\iota||\rho| = \mathrm{id}_{|A_.|}$. If we can show that $\rho\iota \simeq \mathrm{id}_{E_.G}$ then it follows from *Proposition 3.2.2.* that that $|\rho||\iota| = \mathrm{id}_{EG}$ and hence EG is homotopic to $|A_.| = \{*\}$, which is equivalent to the contractibility of EG.

Let $h_i : E_nG \to E_{n+1}G$ be given by

$$\begin{aligned}
h_0(g_1, \ldots, g_{n+1}) &= (g_1 \cdots g_{n+1}, 1, \ldots, 1)\\
h_1(g_1, \ldots, g_{n+1}) &= (g_1, g_2 \cdots g_{n+1}, 1, \ldots, 1)\\
h_2(g_1, \ldots, g_{n+1}) &= (g_1, g_2, g_3 \cdots g_{n+1}, 1, \ldots, 1)\\
&\vdots\\
h_n(g_1, \ldots, g_{n+1}) &= (g_1, \ldots, g_{n+1}, 1)
\end{aligned}$$

One can easily check that $d_0h_0 = \rho\iota$, $d_{n+1}h_n = \mathrm{id}_{E_.G}$ and that h_i satisfies the relations of (3.1). This ends the proof. $\square$

The last thing we do in this section is to construct $EG \times_G W$ as the realization of some simplicial space, given that W is a space with a G-action.

Theorem 3.2.4. Let W be a space on which G acts on. The space $EG \times_G W$ is the realization of the simplicial space $[n] \mapsto G^n \times W$ and where

$$d_i(g_1, \ldots, g_n, w) = \begin{cases} (g_2, \ldots, g_n, w) & \text{if } i = 0 \\ (g_1, \ldots, g_ig_{i+1}, \ldots, g_n, w) & \text{if } i = 1, \ldots, n-1 \\ (g_1, \ldots, g_{n-1}, g_nw) & \end{cases}$$

and

$$s_i(g_1, \ldots, g_n, w) = (g_1, \ldots, g_i, 1, g_{i+1}, \ldots, g_n, w)\ .$$

Proof. The space $EG \times W$ is obviously the realization of the simplicial space $E_.G \times W$ (where the the faces and the degeneracies acts on $E_.G$ as usual and by triviality on W). From *Lemma 1.5.1.* we have that $EG \times_G W = |E_.G \times_G W|$. Recall that the G-action on E_nG is given by $g \cdot (g_1, \ldots, g_{n+1}) = (g_1, \ldots, g_{n+1}g^{-1})$.
Now let $\alpha : E_nG \times_G W \to G^n \times W$ be given by $(g_1, \ldots, g_n, g_{n+1}, w) \mapsto (g_1, \ldots, g_n, g_{n+1}^{-1}w)$. One can easily check that α is well defined and bijective. If we endow $G^n \times W$ with the faces and the degeneracies given in the theorem, α becomes a simplicial bijective map, i.e. a simplicial isomorphism. Hence α induces a homeomorphism on the realizations. $\square$

3.3. Connections to $\mathbb{T}$-equivariant (co)homology

In this section we prove that $HC^-_{-*}(S^\bullet(Y^\cdot)) = H^*_{\mathbb{T}}(|Y^\cdot|)$ as $R[u]$ modules ($|u| = -2$).

Definition 3.3.1. Suppose $A_.$ is a simplicial object in an abelian category $\mathcal{A}$. To each such object there is an associated chain complex $C_*(A_.)$ where $C_n(A_.) = A_n$ and the differential is the sum $\partial = \sum(-1)^i d_i$.
The normalized chain complex of this simplicial object is the quotient $N_*(A_.) = C_*(A_.)/D_*(A_.)$ where $D_*(_.A) \subseteq C_*(A_.)$ is the subcomplex generated by all degeneracies (i.e. $D_n(A_.) = \sum s_i(A_{n-1})$).

Lemma 3.3.2. Given a simplicial object $A_.$ in an abelian category $\mathcal{A}$, then $H(C_*(A_.)) = H(N_*(A_.))$.

Proof. See *Theorem 8.3.8* in [Wei] □

The singular chain groups $S_n(X)$ is a good example of a simplicial object in an abelian category (the category of R-modules), and we will denote $N_*(S_\bullet(X))$ by simply $N_*(X)$.

Lemma 3.3.3. Given a simplicial space $X_.$, $HH(S_*(X_.)) = HH(N_*(X_.))$.

Proof. If $C_{**}(S_\bullet(X_.))$ and $C_{**}(N_\bullet(X_.))$ are filtrated by columns the projection map $C_{**}(S_\bullet(X_.)) \to C_{**}(N_\bullet(X_.))$ becomes a map of filtration. As the projection map is a quasi-isomorphism on each column (see previous lemma) it follows that their total homology are isomorphic by *Theorem A.2.2.*. □

Definition 3.3.4. Given a topological space W with a circle action $f : \mathbb{T}\times W \to W$ there are associated double complexes $U^-(W) = R[u]\otimes S_*(W)$ and $U^+(W) = R[u^{-1}]\otimes S_*(W)$ ($|u| = -2$), with total differentials on the form $\partial_U = \partial + uJ$ (for definition of J, see *Definition 2.3.8.*). In particular this means that $U^-(W)$ is a second quadrant double complex while $U^+(W)$ is a first quadrant double complex. $U^+(W)$ is visualised in the diagram below

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \iddots \\
S_2(W) & u^{-1}\otimes S_1(W) \arrow[l, "uJ"'] \arrow[d, "\partial"'] & u^{-2}\otimes S_0(W) \arrow[l, "uJ"'] \\
S_1(W) & u^{-1}\otimes S_0(W) \arrow[l, "uJ"'] \\
S_0(W)
\end{tikzcd}$$

We also define the double complex $V^-(W) = R[u] \otimes S^*(W)$ with total differential $\partial_V = \widehat{\partial} + uI$ (for definition of I see *Definition 2.3.8*) which is third quadrant double complex.

Proposition 3.3.5. Let W be a space with a circle action $f : \mathbb{T} \times W \to W$.
(a) There is an isomorphism of homology $H_*(U^+(W)) \cong H_*^{\mathbb{T}}(W)$.
(b) There is an isomorphism of (co)homology $H_*(V^-(W)) \cong H^*_{\mathbb{T}}(W)$.

Proof. (a) Let $\pi_2 : S_*(V \times W) \to S_*(W)$ be given by $\alpha \times \beta \mapsto \beta$. Let M_* be the Hochcshild complex of the simplicial complex $[n] \to N_*(\mathbb{T}^n \times W)$ where faces and degeneracies are induced from the faces and degeneracies in *Theorem 3.2.4*. By *Theorem 2.1.9.* and *Lemma 3.3.2* it follows that $H(M_*) = H(E\mathbb{T} \times_{\mathbb{T}} W) = H^{\mathbb{T}}(W)$ (in order to apply *Theorem 2.1.9.* we need to to have that $s_i(\mathbb{T}^n \times W) \hookrightarrow \mathbb{T}^{n+1} \times W$ is a cofibration, but this follows as it is a CW-subcomplex inclusion which are always cofibrations (see *Cor 1.4., Ch VII* in [Bre])).

Note that the Hochschild complex can be defined for any collection of complexes $C_1, C_2, \ldots$ equipped with faces that satisfy $d_i d_j = d_{j-1} d_i$ whenever $i < j$ (the degenracies has no role in the definition of Hochschid homology). Such a collection of complexes is called a semi-simplicial complex.

Now define a semi-simplicial complex $[n] \mapsto A_n = N_*(\mathbb{T})^{\otimes n} \otimes N_*(W)$ where the faces of $A_1 = N_*(\mathbb{T}) \otimes N_*(W)$ are given by

$$d_i(\alpha \otimes \beta) = \begin{cases} \pi_2(\alpha \nabla \beta) & i = 0 \\ f_*(\alpha \nabla \beta) & i = 1 \ . \end{cases}$$

For $A_i,\ i > 1$ the faces are given by

$$d_i(\alpha_{k_1} \otimes \cdots \otimes \alpha_{k_n} \otimes \beta) = \begin{cases} \pi_2(\alpha_{k_1} \nabla \alpha_{k_2}) \otimes \alpha_{k_3} \otimes \cdots \otimes \alpha_{k_n} \otimes \beta & i = 0 \\ \alpha_{k_1} \otimes \cdots \otimes \mu_*(\alpha_{k_i} \nabla \alpha_{k_{i+1}}) \otimes \cdots \otimes \alpha_{k_n} \otimes \beta & i = 1, \ldots, n-1 \\ \alpha_{k_1} \otimes \cdots \otimes \alpha_{k_{n-1}} \otimes f_*(\alpha_{k_n} \nabla \beta) & i = n \end{cases}$$

One can easily check that if $i < j$ then $d_i d_j = d_{j-1} d_i$ (so the above construction is really a semi-simplicial complex).

Now define a map $\eta : C_{**}(A_.) \to M_*$ given by

$$N_*(\mathbb{T})^{\otimes n} \otimes N_*(W) \ni \alpha_{k_1} \otimes \cdots \otimes \alpha_{k_n} \otimes \beta \longmapsto \alpha_{k_1} \nabla \cdots \nabla \alpha_{k_n} \nabla \beta \in N_*(T^n \times W)$$

One can easily check that η commutes with the horizontal and vertical differentials (so it is really a map of complexes).

If we filter $C_{**}(A_.)$ as $0 \subset A_0 \subset A_0 \oplus A_1 \subset \cdots \subset C_{**}(A_.)$ and M_* by columns, η becomes a map of filtered complexes. Now as $\eta : N_*(\mathbb{T})^{\otimes n} \otimes N_*(W) \xrightarrow{\sim} N_*(\mathbb{T}^n \times W)$

is a chain equivalence follows easily from *Lemma 3.3.2.* and from the fact that ∇ is a chain equivalence), it follows that the total homology of $C_{**}(A_{\cdot})$ and M_* are isomorphic.

Now let $E(z) \hookrightarrow N_*(\mathbb{T})$ be the exterior subalgebra generated by a 1-cycle $z \in N_1(\mathbb{T})$ that generates $H_1(N_*(\mathbb{T})) = R$. Now we can construct a simplicial complex $[n] \mapsto B_n = E_1(z)^{\otimes n} \otimes N_*(W)$ where the faces are inherited from $A_{\cdot}$ and the degeneracies s_i are given by inserting $1 \in R = E_0(z)$ in position i.

Obviously the inclusion $E(z) \hookrightarrow N_*(\mathbb{T})$ is a chain equivalence, so by spectral sequence argument we conclude that $C_{**}(A_{\cdot}) \simeq C_{**}(B_{\cdot})$.

Now let L_{**} be the double complex obtained by normalizing $C_{**}(B_{\cdot})$ with respect to the degeneracies of $B_{\cdot}$. That is $L_{p,*} = B_p / \sum s_i(B_{p-1})$. Filtering L_{**} and $C_{**}(B_{\cdot})$ by rows we get by *Lemma 3.3.2.* and spectral sequence arguments that the projection map $\mathrm{Tot}(C_{**}(B_{\cdot})) \to \mathrm{Tot}(L_{**})$ is a chain equivalence.

Finally we have that $L_{**} = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} E_1(z)^{\otimes n} \otimes N_*(W)$ and we have for every $n \geq 2$ that the induced horizontal differential $(-1)^{n+p} \sum (-1)^i d_i = (-1)^p d_n$. This equality follows as $d_0 = \cdots = d_{n-1} = 0$, which follows from the fact that $z \nabla z = 0$ (see *Remark 2.3.6.*).

On $E_1(z) \otimes N_*(W)$ we have that the induced zeroth face d_0 is trivial as $d_0(z \otimes \beta) = \pi_2(z \nabla \beta)$ is a degenerated element and hence trivial in $N_*(W)$ (this is why we are working with $N_*(W)$ instead of $S_*(W)$). Hence L_{**} is on the form

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
N_2(W) \arrow[d, "\partial"'] & E_1(z) \otimes N_1(W) \arrow[l, "-d_1"'] \arrow[d, "\partial"'] & E(z)^{\otimes 2} \otimes N_0(W) \arrow[l, "\mathrm{id} \otimes d_2"'] \\
N_1(W) \arrow[d, "\partial"'] & E_1(z) \otimes N_0(W) \arrow[l, "d_1"'] & \\
N_0(W) & &
\end{tikzcd}$$

As $N_*(W) \simeq S_*(W)$ and as $(-1)^{|\beta|} \mathrm{id}^{\otimes n-1} \otimes d_n(z \otimes \cdots \otimes z \otimes \beta) = z \otimes \cdots \otimes z \otimes J(\beta)$, the map $U^+(W) \to L_{**}$ given by $u^{-k} \otimes \alpha \mapsto z^{\otimes k} \otimes \alpha$ is a map of double complexes that induces a chain equivalence on columns, which by spectral sequence arguments yields $\mathrm{Tot}\, L_{**} \simeq U^+(W)$. Since $H_*(\mathrm{Tot}\, L_{**}) = H_*^{\mathbb{T}}(W)$, the theorem follows.

(b) Just dualize the arguments above. $\square$

The next step is to prove that when a cocyclic space $Y^{\cdot}$ converges (recall the definition of convergence in *Definition 2.3.12.*) then $HC^-(S^*(Y^{\cdot})) = H_*(V^-|Y^{\cdot}|)$ and thereby $HC^-(S^*(Y^{\cdot})) = H^*_{\mathbb{T}}(|Y^{\cdot}|)$ (by the proposition above).

Lemma 3.3.6. (a) Given cyclic set $X_.$, there is a natural $R[u]$-module map $\zeta : C^-(S_*(X_.)) \to U^-(|X_.|)$ such that the induced map

$$C(S_*(X_.)) = C^-(S_*(X_.))/uC^-(S_*(X_.)) \to U^-(|X_.|)/uU^-(|X_.|) = S_*(|X_.|)$$

is the chain map φ defined in *Proposition 2.3.10.*.

(b) Given cocyclic space $Y^\cdot$, there is a natural $R[u]$-module map $\xi : C^-(S^*(Y^\cdot)) \to V^-(|Y^\cdot|)$ such that the induced map

$$C(S^*(Y^\cdot)) = C^-(S^*(Y^\cdot))/uC^-(S^*(Y^\cdot)) \to U^-(|Y^\cdot|)/uU^-(|Y^\cdot|) = S_*(|Y^\cdot|)$$

is the chain map ψ defined in *Theorem 2.3.11.*.

Proof. (a) Recall that the total differential of $C^-(S_*(X))$ is $\partial^- = b + B$ and that the total differential of $U^-(|X_.|)$ is $\partial_U = \partial + J$. That means ζ has to satisfy following equality

$$\zeta \circ (b + B) = (\partial + J) \circ \zeta$$

Now we make an ansatz where $\zeta = \sum \zeta_n u^n$ and $\zeta_n : C(S_*(X_.)) \to S_{*+2n}(|X_.|)$ is R-linear. Hence the equality above is equivalent to

$$\zeta_n b + \zeta_{n-1} B = \partial \zeta_n + J \zeta_{n-1} \tag{3.2}$$

for every $n > 0$.

We construct ζ_n inductively by setting $\zeta_0 = \varphi$ and $\zeta_1 = h$ (maps defined in Proposition 2.3.10.) which are proved to satisfy (3.2).

Suppose that we have constructed ζ_n for $n \geq 2$ and that satisfy (3.2). Now we construct ζ_{n+1} inductively on the degree d of the argument. By naturality it is sufficient to construct ζ_{n+1} for $\mathrm{id}_{[q]} \in S_p(\lambda^q_q)$, $p + q = d$ (this is motivated in the proof of *Proposition 2.3.10.*).

For $d = 0$, let $w = \zeta_n B(\mathrm{id}_{[0]}) - J\zeta_n(\mathrm{id}_{[0]})$. By using the equality (3.2) and that $b(\mathrm{id}_{[0]}) = 0$, one gets easily that w is a cycle of degree $2n + 1 \geq 3$ of $S_*(|\lambda^0_.|)$ and hence also a boundary (as $H_{2n+1}(|\lambda^0_.|) = H_{2n+1}(\mathbb{T}) = 0$). This means that $w = \partial a$ for some $a \in S_{2n+2}(|\lambda^0_.|)$, and we let $\zeta_{n+1}(\mathrm{id}_{[0]}) = a$, and hence (3.2) is satisfied.

Now assume that we have defined ζ_{n+1} for elements of degree $< m$. Now if $\mathrm{id}_{[q]} \in S_p(\lambda^q_.)$ where $p + q = m$, we can in a similar manner as above show that $w = \zeta_{n+1} b(\mathrm{id}_{[q]}) + \zeta_n B(\mathrm{id}_{[q]}) - J\zeta_n(\mathrm{id}_{[q]})$ is a boundary and the arguments continues as above. This ends the inductive step.

(b) As in the proof of (a) we construct a map $\xi : C_*^-(S^\bullet(Y^\cdot)) \to S^*(|Y^\cdot|)$ such that $\xi = \sum \xi_n u^n$ and $\xi_n : C_n(S^\bullet(Y^\cdot)) \to S^{2n}(|Y^\cdot|)$ is R-linear. The ξ_i has to satisfy

$$\xi_n b + \xi_{n-1} B = \widehat{\partial}\xi_n + I\xi_{n-1} \tag{3.3}$$

Given some $x \in S^{-q}(Y^m)$ let $\xi_n(x) = \rho_m^*(x)/\zeta(\mathrm{id}_{[m]})$ where $\mathrm{id}_{[m]} \in S_m(\lambda_m^m)$. Now the equality in (3.3) follows from the equality in (3.2) in the same way as *Theorem 2.3.11.* follows from *Proposition 2.3.10.*. □

Theorem 3.3.7. Let $Y^\cdot$ be a cocylic space. If $Y^\cdot$ converges, then

$$HC_*^-(S^\bullet(Y^\cdot)) = H_{\mathbb{T}}^*(|Y^\cdot|)$$

Proof. We show that $HC_*^-(S^\bullet(Y^\cdot)) = H_*(V^-(|Y^\cdot|))$. Filtering $C^-(S^\bullet(Y^\cdot))$ and $V^-(|Y^\cdot|)$ by columns ξ becomes a map of filtered complexes, and induces $\psi : u^k C(S^*(Y^\cdot) \to u^k S^*(|Y^\cdot|)$ as a map between the $(-k)$'th columns. As we are assuming that $Y^\cdot$ converges ψ is a quasi-isomorphism, and hence the equality of homology follows from spectral sequence arguments (see *Theorem A.2.2.*). □

3.4. Free loop spaces

Given that a space X is simply connected, we prove that $HC_*^-(S^\bullet(X)) = H_{\mathbb{T}}^*(LX)$. Before going in more depth about this, suppose X is a topological space and $E_\cdot$ is a cyclic space. From these two objects one can construct a cocylic space $X^{E_\cdot}$ where $X^E[n] = \mathrm{Hom}_{\mathrm{Top}}(E_\cdot[n], X)$, and $\delta_i\alpha = \alpha \circ d_i$, $\sigma_i\alpha = \alpha \circ s_i$ and $\tau_n\alpha = \alpha \circ t_n$, and d_i, s_i and t_n are the faces, the degeneracies and the cyclic operators of $E_\cdot$, respectively.

Lemma 3.4.1. Let $E_\cdot$ be a cyclic set and X a topological space. There is a homeomorphism between the realization $|X^E|$ of the cocyclic space X^E and the space $\mathrm{Hom}_{\mathrm{Top}}(|E_\cdot|, X)$.

Proof. We define a map $\Phi : |X^E| \to \mathrm{Hom}_{\mathrm{Top}}(|E|, X)$, sending $\{\alpha_n : \Delta^n \to X^E[n]\} \in |X^E|$ to $\alpha \in \mathrm{Hom}_{\mathrm{Top}}(|E_\cdot|, X)$ where α is given by $\alpha(e_n, \boldsymbol{u}_n) = [\alpha_n(\boldsymbol{u}_n)](e_n)$.

To verify that Φ is a homeomorphism (continuous with a continuous inverse) is left as an easy exercise for the reader. □

Now we define free loop spaces.

Definition 3.4.2. The space

$$LX = \mathrm{Hom}_{\mathrm{Top}}(\mathbb{T}, X),$$

equipped with the compact-open topology, is called *the free loop space* of X.

Recall from *Lemma 1.2.4.* that every morphism $\beta \in \mathrm{Hom}_{\mathbf{\Delta}C}([n],[m])$ can be decomposed as $\alpha \circ \tau_n^k$ where $\alpha \in \mathrm{Hom}_{\mathbf{\Delta}}([n],[m])$. This means in particular that

$$\mathrm{Hom}_{\mathbf{\Delta}C}([n],[m]) \cong \mathrm{Hom}_{\mathbf{\Delta}}([n],[m]) \times \mathbb{Z}_{n+1}$$

(the isomorphism is explicitly given by $\beta = \alpha \circ \tau_n^k \mapsto (\alpha, k)$).

Recall from *Example 1.2.5.* the cyclic space $\lambda^0_{.}$ where $\lambda^0_n = \mathrm{Hom}_{\mathbf{\Delta}C}([n],[0])$. Obviously $\mathrm{Hom}_{\mathbf{\Delta}}([n],[0]) = *$, and hence $\lambda^0_n = * \times \mathbb{Z}_{n+1} \cong \mathbb{Z}_{n+1}$.

Now given a space X, we can construct a cocyclic space $X^{\lambda^0_{.}}$. As a space $X^{\lambda^0_{.}}[n] = X^{n+1}$ (since $\#(\lambda^0_{.}[n]) = n+1$).

Lemma 3.4.3. If X is simply connected, then $X^{\lambda^0_{.}}$ converges.

Proof. See § 5 in [BS]. $\square$

Before proving the main theorem of this thesis recall from *Example 1.2.6.* that any algebra A, defines a cyclic module $A_{.} = A, A^{\otimes 2}, A^{\otimes 3}, \ldots$. The negative cyclic homology of $A_{.}$ will be denoted by $HC_*^-(A)$. The singular cochain complex of a space X, $S^*(X)$, can be viewed as a differential graded algebra with with multiplication given by the cup product

$$\cup : S^p(X) \times S^q(X) \to S^{p+q}(X)$$

where $\cup$ is the composition

$$S^p(X) \times S^q(X) \xrightarrow{AW^*} S^{p+q}(X \times X) \xrightarrow{d^*} S^{p+q}(X)$$

where AW^* is the dual of the Alexander-Whitney map (see Theorem 2.3.5,) and d^* is the dual of the diagonal map $d : X \to X \times X$, $d(x) = (x, x)$.

Theorem 3.4.4. If X is simply connected, there is an isomorphism

$$HC_*^-(S^\bullet(X)) \cong H_{\mathbb{T}}^*(LX) .$$

Proof. We have by *Lemma 3.4.1., 3.4.3.* and *Theorem 3.3.7.* that $HC_*^-(S^\bullet(X^{\lambda^0_{.}})) \cong H_{\mathbb{T}}^*(\mathrm{Hom}(|\lambda^0_{.}|, X))$. By *Theorem 1.5.2.*, $|\lambda^0_{.}| = \mathbb{T}$ and hence $HC_*^-(S^\bullet(X^{\lambda^0_{.}})) \cong H_{\mathbb{T}}^*(LX)$.

Thus it is enough to show that $HC_*^-(S^\bullet(X^{\lambda^0_{.}})) \cong HC_*^-(S^\bullet(X))$ in order to complete the proof.

Filter $C_*^-(S^\bullet(X^{\lambda^0_{.}}))$ and $C_*^-(S^\bullet(X))$ by columns, and let $\theta : C_*^-(S^\bullet(X)) \to C_*^-(S^\bullet(X^{\lambda^0_{.}}))$ be the map induced by the dual of the Alexander-Whitney map, $AW^* : S^p(X) \otimes S^q(X) \xrightarrow{\sim} S^{p+q}(X \times X)$. One can easily verify that AW^* is a map of cyclic complexes, which makes θ into a map of double complexes.

As the Alexander-Whitney map is a chain equivalence, θ induce quasi-isomorphisms $S^*(X)^{\otimes n} \xrightarrow{\sim} S^*(X^n)$ on the columns of the complexes, which by spectral sequence arguments gives that the the total homology of the both double complexes are isomorphic. $\square$

3.5. Applications

In 1999 M. Chas and D. Sullivan founded a new mathematical branch called string topology (see [CS]), which is the study of homology theories on free loop spaces. String homology, which is the $\mathbb{T}$-equivariant homology of free loop spaces, is one of the homology theories that are of importance.

Hence it is not surprising that the isomorphism $HC_*^-(S^\bullet(X)) \cong H_{\mathbb{T}}^*(LX)$ for simply connected spaces X has applications in this branch, especially if one wants to study string homology of spheres $S^n,\ n \geq 2$ and complex projective spaces $\mathbb{C}P^m,\ m \geq 1$ (as they are simply connected).

Examples where this isomorphism is applied may be found in [VP] where M. Vigué-Poirrier offers isomorphisms that may be of importance in the study and computation of $H_{\mathbb{T}}^*(LX;\mathbb{K})$ for simply connected spaces X and where $\mathrm{char}(\mathbb{K}) = 0$.

For fields with $\mathrm{char}(\mathbb{K}) > 0$ there are theoretical and computational contributions by M. El-Haouari and B. Ndombol in [EN1] and [EN2].

Even if it is hard to compute $\mathbb{T}$-equivariant cohomology, it is not impossible, and has been done for several spaces.

For instance $H(LS^{2k+1};\mathbb{Q}) = \mathbb{Q}[u] \oplus \mathbb{Q}[y]$ where $|u| = 2$ and $|y| = 2k$, computed in [Bas] (cyclic homology was however not involved in the computation).

For $\mathrm{char}(\mathbb{K}) > 0$ there is an isomorphism of algebras

$$H_{\mathbb{T}}^*(LS^{2q+1};\mathbb{K}) = \frac{\mathbb{K}[u] \otimes \Gamma(sx, y)}{< u \otimes \gamma^n(sx), \gamma^n(sx) \otimes y >, \quad n \nmid p}$$

(proved in [EN2]). In contrast to the previous example, this computation involved study of negative cyclic homology.

Appendix A - Spectral sequence of a filtration

This is a brief introduction to spectral sequences where we just present results needed in the thesis. The subject is treated in more detail in many books, e.g. [Wei].

A.1. Introduction

Definition A.1.1. A *homological spectral sequence* starting with E^a in an abelian category $\mathcal{A}$ consist of following data:

1. A family $\{E^r_{p,q}\}$ of objects of $\mathcal{A}$ defined for $p, q \in \mathbb{Z}$ and $r \geq a$.
2. Maps $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ that are differentials in the sense that $d^r d^r = 0$.
3. $E^{r+1}_{p,q}$ is isomorphic to the homology of E^r_{**} at the spot $E^r_{p,q}$.

Definition A.1.2. A *filtration* of a chain complex C is a chain of subcomplexes $\cdots \subseteq F_{p-1}C \subseteq F_pC \subseteq \cdots$ of C. We say that the filtration is *exhaustive* if $C = \cup F_pC$ and *complete* if $\varprojlim C/F_pC = C$.

A map $f : C \to D$ of filtered complexes is called a map of filtered complexes if $f(F_pC) \subseteq F_pD$

Theorem A.1.3. A filtration F of a chain complex C naturally determines a spectral sequence starting with $E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}$ and $E^1_{p,q} = H_{p+q}(E^0_{p,*})$

Proof. See *Theorem 5.4.1* in [Wei] □

Theorem A.1.4. (Eilenberg-Moore Comparison Theorem) Let $f : C \to D$ be a map of filtered complexes of modules, where both C and D are complete and exhaustive. Suppose that for some $r \geq 0$ the induced map $f^r : E^r_{p,q}(C) \to E^r_{p,q}(D)$ is an isomorphism for all $p, q \in \mathbb{Z}$. Then $f_* : H_*(B) \to H_*(C)$ is an isomorphism.

Proof. See *Theorem 5.5.11* in [Wei]. □

A.2. Spectral sequence of a double complex

Definition A.2.1. Let C be a double complex. We may filter $\mathrm{Tot}(C)$ by the columns of C, by letting

$$F_nC = \begin{cases} C_{p,q} & \text{if } p \leq n \\ 0 & \text{if } p > n \end{cases} \qquad \qquad \begin{array}{cccc|ccc} \cdots & * & * & 0 & 0 & \cdots \\ \cdots & * & * & 0 & 0 & \cdots \\ \cdots & * & * & 0 & 0 & \cdots \\ \cdots & * & * & 0 & 0 & \cdots \end{array}$$

This gives rise to a spectral sequence where $E^1_{p,q} = H_q(C_{p,*})$.

Theorem A.2.2. Let C_{**} and D_{**} be double complexes such that $C_{p,q} = D_{p,q} = 0$ in the forth quadrant (i.e. when $p > 0$ and $q < 0$). Assume that $f : \mathrm{Tot}(C_{**}) \to \mathrm{Tot}(D_{**})$ is a map of filtered complexes where the filtration is taken with respect to columns. If the induced map $f_p : C_{p,*} \to D_{p,*}$ is a quasi-isomorphism for all $p \in \mathbb{Z}$, then $H(\mathrm{Tot}(C_{**})) = H(\mathrm{Tot}(D_{**}))$.

Proof. We have that $f^1_{p,q} : E^1_{p,q}(C) \to E^1_{p,q}(D)$ is an isomorphism since $f^1_{p,q} = H_q(f_p)$, where the right-hand side is an isomorphism by assumption. As $\mathrm{Tot}(C_{**})$ and $\mathrm{Tot}(D_{**})$ are exhaustive and complete, we may apply *Theorem A.1.4.*, and that completes the proof. □

One can also filter a double complex by rows, which is the same as exchanging rows and columns and then filtering by columns. This results in the following theorem.

Theorem A.2.3. Let C_{**} and D_{**} be double complexes such that $C_{p,q} = D_{p,q} = 0$ in the second quadrant (i.e. when $p < 0$ and $q > 0$). Assume that $f : \mathrm{Tot}(C_{**}) \to \mathrm{Tot}(D_{**})$ is a map of filtered complexes where the filtration is taken with respect to rows. If the induced map $f_p : C_{*,p} \to D_{*,p}$ is a quasi-isomorphism for all $p \in \mathbb{Z}$. Then $H(\mathrm{Tot}(C_{**})) = H(\mathrm{Tot}(D_{**}))$. □

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