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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Matching Conditions for the Laplace and Squared Laplace Operator on a Y-graph

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Jean-Claude Kiik

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Matching Conditions for the Laplace and Squared Laplace Operator on a Y-graph

Jean-Claude Kiik

Självständigt arbete i matematik 15 högskolepoäng, Grundnivå

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Abstract

The Laplace operator and squared Laplace operator on a branching graph are studied. Boundary conditions connecting three functions at a vertex are determined. It is shown that in general the boundary conditions are linear relationships between derivatives of the same degree.

Introduction

The background to the study of quantum graphs, a graph considered together with an operator and certain boundary conditions, lies in the theory of partial differential equations. The main interest of the square Laplace operator can be seen to stem from the equations for free oscillations of a plate, see Lifshitz and Landau in [7]. The equation

$$\frac{\rho\partial^2\zeta}{\partial t^2} + \frac{Eh^2}{12(1-\sigma^2)}\Delta^2\zeta = 0$$

is a partial differential equation in the time domain, where ρ is the density of matter, ζ the distance function, t time, E the Young's Modulus (the measure of stiffness of the elastic material), h the height of the plate, σ the Poisson ratio (the negative ratio of transverse to axial strain). Fourier transform can be used on the equation to go from a time domain to a frequency domain. Using a Fourier transform we get the following equation,

$$-\rho k^2 u + \frac{Eh^2}{12(1-\sigma^2)}\Delta^2 u = 0$$

where,

$$u = \int_{-\infty}^{\infty} \zeta \mathrm{e}^{kt} dt.$$

Setting $-\rho = \dfrac{Eh^2}{12(1-\sigma^2)} = 1$, we get the following partial differential equation for u,

$$\Delta^2 u = k^2 u.$$

This equation can be used to describe waves in rods. Our aim is to study the system of three rods sharing one clamp with the three ends free. This system is described by the square Laplacian Δ^2 on a Y-graph with certain boundary conditions at the vertex.

A lot of focus in research is on inverse scattering problems, see [6], which will not be covered in this report. The boundary, or matching, conditions are what will be discussed in the following sections. The first section will state definitions of concepts concerning unbounded operators and quantum graphs. The second section will treat the negative Laplace operator on a Y-graph and the necessary boundary conditions for it to be symmetric and connected at a vertex. The third section will treat the case of a symmetric Y-graph, with respect to the three angles between the branches, and the square of the Laplace operator acting on it. The last section will treat the general case of the square Laplace operator, with different angles between each branch.

Operators and Graphs

A graph Γ consists of a finite set of edges $E = \{e_j\}$ and a set of vertices $V = \{v_i\}$, where each edge lies between two vertices (v_i, v_k). If two edges share a vertex they are said to be connected and if all edges share one vertex v the graph Γ is said to be a star graph. We will focus on a graph with three edges sharing one vertex, which we will call a Y-graph. In order to uniquely define this Y-graph each edge e_j must have a length, making it a metric graph,

Definition 1. *A graph Γ is said to be a* ***metric graph*** *if each edge e_j is identified by an interval of the real line, i.e. $e_j = (0, l_j]$ where $0 < l_j \leq \infty$.*

We therefore extend the so-called Y-graph to be a metric graph with each edge e_j that can be identified with the interval $[0, \infty)$ with all the left end-points identified. The edges emanate outward from the origin and we say that the operators we are interested in are acting on functions $f_i(x)$, where x is a point along an edge. The Y-graph with each branch parametrised by a function f_i, $i = 1, 2, 3$, can be seen in figure 1.
So far we have seen the Y-graph of interest and the functions on each edge, but since we will be treating unbounded operators the domain of the operators needs to be defined. The space $L^2[a, b]$ is defined as

Definition 2. *The space $L^2[a, b]$ is the collection of real (complex) valued square integrable functions with the inner product between f and g defined as*

$$\langle f, g \rangle = \int_a^b f(x)\overline{g(x)}dx$$

and the norm of f defined as

$$\|f\|^2_{L^2[a,b]} = \langle f, f \rangle = \int_a^b |f(x)|^2 dx < \infty.$$

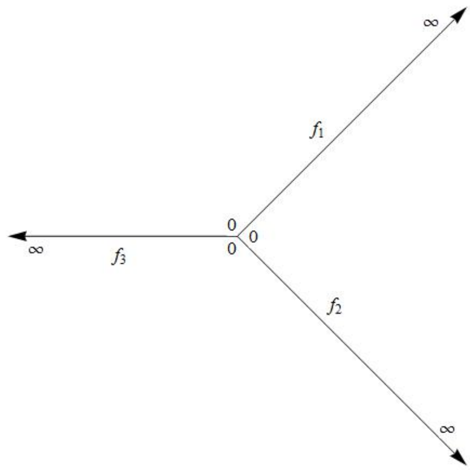

Figure 1: Y-graph equipped with functions on each branch.

We have defined three functions and their domains, one on each edge of the Y-graph, and therefore we consolidate the Hilbert space with,

Definition 3. *The space $L^2(\Gamma)$ on Γ consists of functions that are measurable and square integrable on each edge e_j and such that*

$$\|f\|^2_{L^2(\Gamma)} = \sum_{e_j \in E} \|f\|^2_{L^2(e_j)} < \infty$$

i.e., $L^2(\Gamma)$ is the orthogonal direct sum of spaces $L^2(e_j)$. The scalar product in this setting

$$\langle f, g \rangle = \sum_{j=1}^{3} \int_\Gamma f_j(x) \overline{g_j(x)} dx.$$

The operators acting on the functions will be the two differential operators,

- The operator of negative second differentiation - the Laplace operator: $L_{II} = -\frac{d^2}{dx^2}$.
- The operator of fourth order differentiation: $L_{IV} = L_{II}^2 = \frac{d^4}{dx^4}$.

We also note that our operators are linear, i.e. fulfilling

Definition 4. *Let X, Y be two vector spaces. A function L that maps every vector u from the domain $D(L) \subset X$ into a vector $v = Lu$ of Y is called a **linear operator** from X to Y if L preserves linear relations, that is, if*

$$L(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 L u_1 + \alpha_2 L u_2$$

for all u_1, u_2 from D(L) and all scalars α_1, α_2.

A symmetric linear operator is defined as

Definition 5. *A linear operator that satisfies the relationship*

$$\langle Lf, g\rangle = \langle f, Lg\rangle$$

is said to be ***symmetric*** *if the condition holds for all f and g in the domain $D(L)$.*

The triple of metric graph, operator and conditions at the vertex V of the Y-graph are what we define as a quantum graph.

Definition 6. *A quantum graph is a metric graph Γ equipped with an operator L acting on it, together with matching (vertex) conditions.*

Second Order Linear Differential Operator

The three functions defined on the Y-graph with the Laplace operator acting on them are defined on separate edges and their boundary values at the origin are independent, with an angle of 2α between two of the edges, according to figure 2. Each edge is parametrised by a real parameter x and the corresponding functions are denoted by f_j for $j = 1, 2, 3$. The boundary values $f_j(0)$ and $f_j'(0)$ should be connected by certain linear conditions to be called matching conditions. The conditions should be chosen so that the operator is symmetric as stated in the previous section.

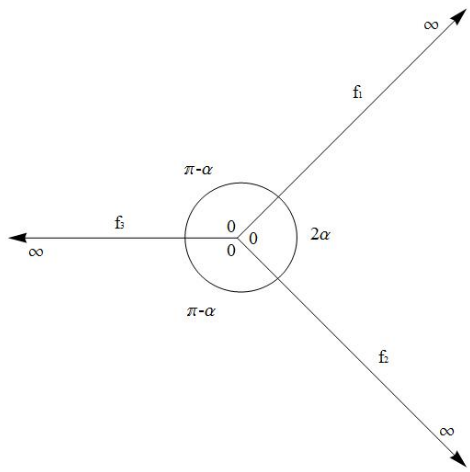

Figure 2: Y-graph with angles 2α and $\pi - \alpha$.

The Laplace operator $L_{II} = -\dfrac{d^2}{dx^2}$ has the domain $\bigoplus_{i=1}^{3} W_2^2[0, \infty)$, consisting of all functions f_j from $L^2[0, \infty)$ such that $f_j'' \in L^2[0, \infty)$. The subscript

II will be dropped for the remainder of this section and L will denote the previous L_{II}. Let us calculate the boundary form of the operator L,

$$\begin{aligned}\langle Lf,g\rangle - \langle f,Lg\rangle &= \sum_{j=1}^{3}\{\int_0^\infty -f_j''\overline{g_j}dx + \int_0^\infty f_j\overline{g_j''}dx\} \\ &= \sum_{j=1}^{3}\{-f_j'\overline{g_j}|_0^\infty + \int_0^\infty f_j'\overline{g_j'}dx + f_j\overline{g_j}|_0^\infty - \int_0^\infty f_j'\overline{g_j'}dx\} \\ &= \sum_{j=1}^{3}\{f_j'(0)\overline{g_j(0)} - f_j(0)\overline{g_j'(0)}\}.\end{aligned}$$

The question of which vertex conditions need be imposed upon a function from the domain of the operator in order for the sum to be identically 0 is the same as making the operator symmetric. Here we assume that both functions f and g satisfy the same conditions. The following two vertex conditions can be considered on the Y-graph equipped with the negative Laplacian,

Example 1. *The* ***Dirichlet*** *condition with* $f_1(0) = f_2(0) = f_3(0) = 0$ *satisfies the stated requirements and yields the boundary form equal to 0. Imposing the Dirichlet condition at the origin vertex on the Y-graph is an example of a decoupling condition, it disconnects the edges entering the vertex. Under this condition the operator would be the direct sum of the operator on each edge. The edges are in this case independent of each other.*

Example 2. *The* ***Robin*** *condition* $f_j'(0) = h_j f_j(0)$*, with* $h_j \in \mathbb{R}$ *also satisfies the requirements and yields a boundary form equal to 0. This is yet another example of a decoupling condition. In a physical application this would have the same result as the previous example, namely that the system would fall apart at the origin due to the edges being independent of each other.*

A vertex condition for physical applications often require the functions to be continuous at the vertex. In this case the edges are not independent. This important vertex condition is referred to as the standard, Neumann or Kirchoff conditions.

Example 3. *The* ***standard*** *conditions are the following*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ f_1'(0) + f_2'(0) + f_3'(0) = 0. \end{cases} \tag{1}$$

They are chosen so that the system is connected at the origin and the boundary form is identically 0, since

$$\begin{aligned}\langle Lf, g\rangle - \langle f, Lg\rangle &= \sum_{j=1}^{3}\{f_j'(0)\overline{g_j(0)} - f_j(0)\overline{g_j'(0)}\} \\ &= \overline{g(0)}\sum_{j=1}^{3} f_j'(0) - f(0)\sum_{j=1}^{3}\overline{g_j'(0)} \equiv 0.\end{aligned}$$

An example of a more general vertex condition than the above can be considered to be the δ-type condition

Example 4. *The δ-**type** condition is considered to be the following,*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ f_1'(0) + f_2'(0) + f_3'(0) = hf(0) = h\delta\,[f] \end{cases}$$

for $h \in \mathbb{R}$. These conditions reduce to the standard conditions when $h = 0$.

All of the above examples in this section can be generalised in the following manner,

$$i(S-I)\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = (S+I)\begin{pmatrix} f_1' \\ f_2' \\ f_3' \end{pmatrix}$$

where I denotes the identity matrix and S is a unitary matrix containing the information about the conditions. From this we can see that for $S = -I$ we get the Dirichlet condition and with $S = I$ we obtain the Neumann condition. For the standard conditions we have

$$S = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

Fourth Order Linear Differential Operator

Symmetric Angle

We now consider the operator of fourth order differentiation $L_{IV} = \dfrac{d^4}{dx^4}$ acting on a function on the metric Y-graph, with a symmetric angle of 2α between two edges, see Figure 3. For $\alpha = 0$ the branches f_1 and f_2 would lie on the same branch and for $\alpha = \pi$ the branches f_1 and f_2 would lie along f_3. Therefore we restrict the angle α and require that $\alpha \neq 0$, $\alpha \neq \pi$. In this section and the following we will be treating the same operator and for

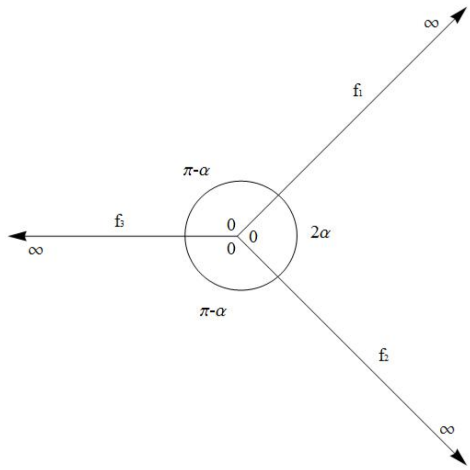

Figure 3: Y-graph with angles 2α and $\pi - \alpha$.

simplicity the subscript IV will be dropped in these two sections, leaving L in place of L_{IV}.
The boundary form of the operator L may be evaluated with the aid of partial integration as follows

$$
\begin{aligned}
&\langle Lf, g\rangle - \langle f, Lg\rangle = \\
&\quad = \sum_{j=1}^{3} \Big(\int_0^\infty f_j^{(IV)} \overline{g_j} dx - \int_0^\infty f_j \overline{g_j^{(IV)}} dx \Big) \\
&\quad = \sum_{j=1}^{3} \Big(f_j^{'''} \overline{g_j} |_0^\infty - f_j^{''} \overline{g_j^{'}} |_0^\infty + \int_0^\infty f_j^{''} \overline{g_j^{''}} dx - f_j \overline{g_j^{'''}} |_0^\infty + f_j^{'} \overline{g_j^{''}} |_0^\infty - \int_0^\infty f_j^{''} \overline{g_j^{''}} dx \Big) \\
&\quad = \sum_{j=1}^{3} \Big(- f_j^{'''}(0) \overline{g_j(0)} + f_j^{''}(0) \overline{g_j^{'}(0)} - f_j^{'}(0) \overline{g_j^{''}(0)} + f_j(0) \overline{g_j^{'''}(0)} \Big).
\end{aligned}
$$

The boundary form above shows us that we need more vertex conditions than in the previous section to ensure that the operator L is symmetric, more precisely in total six are needed. Our aim is to describe conditions that are straight forward generalisations of the standard conditions described in the previous section. The first equation in (1), page 5, means that the function is continuous even at the vertex. We reuse this condition requiring $f_1(0) = f_2(0) = f_3(0)$.
Another additional condition can be obtained if we require that the tangential lines to the graph of f_j lie in the same plane. We should get one condition. Let us discuss how to express this condition using derivatives of f_j. We solve this problem by introducing the coordinate system $\mathbb{R}^3$ by considering our Y-graph lying in the xy-plane in $\mathbb{R}^3$, according to figure 4. Let

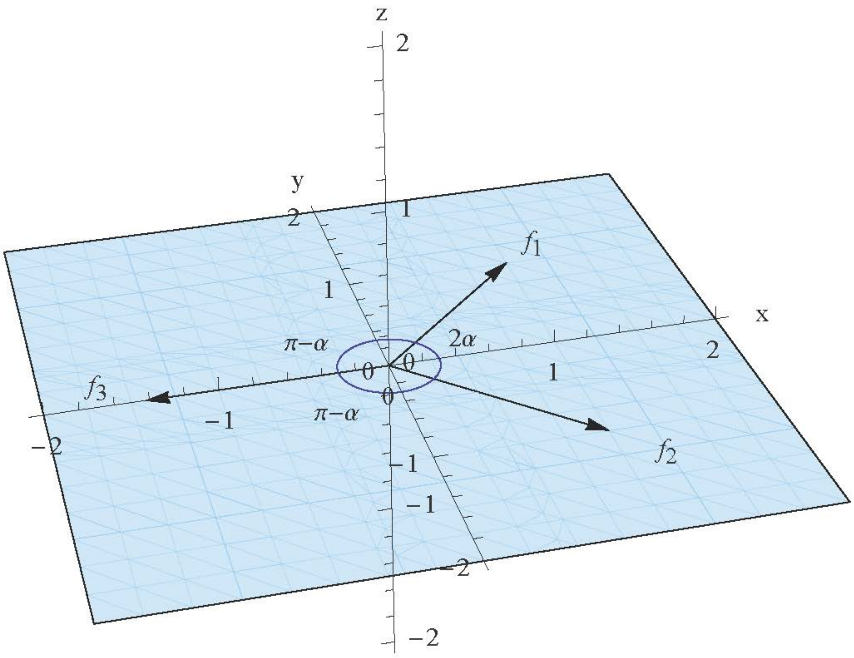

Figure 4: Y-graph in $\mathbb{R}^3$ with angles 2α, $\pi - \alpha$ and $\pi - \alpha$.

$\vec{a}^j$ be the tangent vector to the curves $z = f_j(x_j)$ considered in $\mathbb{R}^3$. Further, let ϕ_j denote the angles between the tangential lines f'_j and the xy-plane, it can be observed that $\tan \phi_j = f'_j(0)$. The vector $\vec{a}^3$ is parallel to the x-axis, therefore has zero second coordinate and can be chosen equal to,

$$\vec{a}^3 = (1, 0, \tan \phi_3) = (1, 0, f'_3(0)).$$

The tangential vectors $\vec{a}^1$ and $\vec{a}^2$ may then be chosen equal to,

$$\vec{a}^1 = (-\cos\alpha, \sin\alpha, \tan\phi_1) = (-\cos\alpha, \sin\alpha, f_1'(0))$$
$$\vec{a}^2 = (-\cos\alpha, -\sin\alpha, \tan\phi_2) = (-\cos\alpha, -\sin\alpha, f_2'(0)).$$

We want the branches of our Y-graph to be dependent and therefore find a matching condition through examining the determinant of the vectors $\vec{a}^j$,

$$\det\begin{pmatrix} -\cos\alpha & \sin\alpha & f_1'(0) \\ -\cos\alpha & -\sin\alpha & f_2'(0) \\ 1 & 0 & f_3'(0) \end{pmatrix} = \sin\alpha(f_1'(0) + f_2'(0) + 2\cos\alpha f_3'(0)) = 0.$$

The restrictions $\alpha \neq 0, \pi$, mean that the value of $\sin\alpha$ is non-zero and can be divided in the last equality above. This gives us the matching condition relating the first derivatives of f_j,

$$f_1'(0) + f_2'(0) + 2\cos\alpha f_3'(0) = 0.$$

So far we have the following two matching conditions,

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) = f(0) \\ f_1'(0) + f_2'(0) + 2\cos\alpha f_3'(0) = 0 \end{cases} \tag{2}$$

To find the remaining matching conditions we use these together with the boundary form at the beginning of this section, but we write $f^{'}, f^{''}, f^{'''}$ instead of $f^{'}(0), f^{''}(0), f^{'''}(0)$ respectively for all derivatives below for simplicity, since it is understood that we are investigating at the vertex. We use the conditions in (2) in the third step below,

$$
\begin{aligned}
&\langle Lf, g\rangle - \langle f, Lg\rangle = \\
&= \sum_{j=1}^{3}\Big(- f_j^{'''}(0)\overline{g_j(0)} + f_j^{''}(0)\overline{g_j^{'}(0)} - f_j^{'}(0)\overline{g_j^{''}(0)} + f_j(0)\overline{g_j^{'''}(0)}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + f_1^{''}\overline{g_1^{'}} + f_2^{''}\overline{g_2^{'}} + f_3^{''}\overline{g_3^{'}} \\
&- f_1^{'}\overline{g_1^{''}} - f_2^{'}\overline{g_2^{''}} - f_3^{'}\overline{g_3^{''}} + f_j(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + f_1^{''}(-\overline{g_2^{'}} - 2\cos\alpha\overline{g_3^{'}}) + f_2^{''}\overline{g_2^{'}} + f_3^{''}\overline{g_3^{'}} \\
&- (-f_2^{'} - 2\cos f_3^{'})\overline{g_1^{''}} - f_2^{'}\overline{g_2^{''}} - f_3^{'}\overline{g_3^{''}} + f(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + (-f_1^{''} + f_2^{''})\overline{g_2^{'}} + (-2\cos\alpha f_1^{''} + f_3^{''})\overline{g_3^{'}} \\
&- f_2^{'}(-\overline{g_1^{''}} + \overline{g_2^{''}}) - f_3^{'}(-2\cos\alpha\overline{g_1^{''}} + \overline{g_3^{''}}) + f(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big).
\end{aligned}
$$

Looking at the last expression above we search for conditions that connect together derivatives of the same order and make the boundary form equal to zero. We choose the following conditions as our final matching conditions,

$$
\begin{cases}
2\cos\alpha f_1^{''}(0) = 2\cos\alpha f_2^{''}(0) = f_3^{''}(0) \\
\sum_{j=1}^{3} f_j^{'''}(0) = 0.
\end{cases}
$$

We now have the following six matching conditions for the operator L on our Y-graph in figure 3,

$$
\begin{cases}
f_1(0) = f_2(0) = f_3(0) = f(0) \\
f_1'(0) + f_2'(0) + 2\cos\alpha f_3'(0) = 0 \\
2\cos\alpha f_1^{''}(0) = 2\cos\alpha f_2^{''}(0) = f_3^{''}(0) \\
\sum_{j=1}^{3} f_j^{'''}(0) = 0.
\end{cases}
\tag{3}
$$

Under these conditions the fourth order operator L is symmetric (and even self-adjoint). Let us consider two special cases.

Example 5. *In the case* $\alpha = \frac{\pi}{2}$, $\cos\alpha = 0$ *and the Y-graph resembles the letter T. The boundary conditions in (2) become,*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) = f(0) \\ f_1'(0) + f_2'(0) = 0 \end{cases}$$

Under these conditions the boundary form of L becomes

$$\begin{aligned} \langle Lf, g\rangle - \langle f, Lg\rangle &= \\ &= -\Big(\sum_{j=1}^{3} f_j'''\Big)\overline{g(0)} + (-f_1'' + f_2'')\overline{g_2'} + f_3''\overline{g_3'} \\ &\quad - f_2'(-\overline{g_1''} + \overline{g_2''}) - f_3'\overline{g_3''} + f(0)\Big(\sum_{j=1}^{3} \overline{g_j'''}\Big). \end{aligned}$$

We observe that two separate sets of matching conditions ensure that this boundary form equals 0, namely

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) = f(0) \\ f_1'(0) + f_2'(0) = 0 \\ f_1''(0) = f_2''(0) \\ f_3''(0) = 0 \\ \sum_{j=1}^{3} f_j'''(0) = 0, \end{cases}$$

and

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) = f(0) \\ f_1'(0) + f_2'(0) = 0 \\ f_3'(0) = 0 \\ f_1''(0) = f_2''(0) \\ \sum_{j=1}^{3} f_j'''(0) = 0. \end{cases}$$

Example 6. *We consider the case when* $\alpha = \frac{\pi}{3}$, *thus* $\cos\alpha = \dfrac{1}{2}$. *In this case the boundary conditions in (3) become*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) = f(0) \\ f_1'(0) + f_2'(0) + f_3'(0) = 0 \\ f_1''(0) = f_2''(0) = f_3''(0) \\ \sum_{j=1}^{3} f_j'''(0) = 0. \end{cases}$$

The first two matching conditions coincide with the standard conditions in (1). This is expected since the Y-graph is symmetric with respect to the angle between each branch when $\alpha = \frac{\pi}{3}$.

General Angle

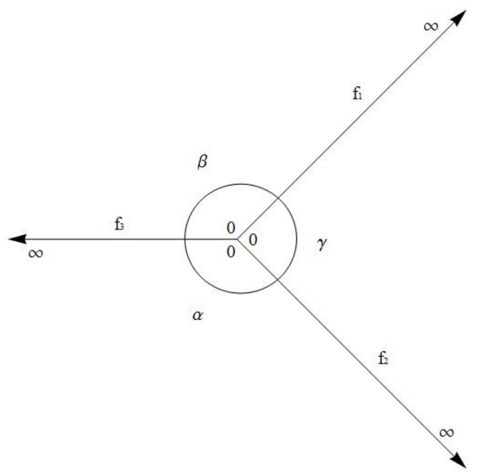

Figure 5: Y-graph with angles α, β and γ.

In this section we examine a case similar to the previous section with the operator L acting on the Y-graph, but with the exception that we let the angle between each branch be different. We denote the angles by α, β and γ according to figure 5 and require that $\alpha, \beta, \gamma \neq 0$. As before the boundary form for the operator L evaluates to

$$\begin{aligned} &\langle Lf, g\rangle - \langle f, Lg\rangle = \\ &\qquad = \sum_{j=1}^{3} \Big(- f_j'''(0)\overline{g_j(0)} + f_j''(0)\overline{g_j'(0)} - f_j'(0)\overline{g_j''(0)} + f_j(0)\overline{g_j'''(0)} \Big). \end{aligned}$$

As earlier we search for six conditions to make the boundary form equal to zero. The first condition of continuity is still applicable $f_1(0) = f_2(0) =$

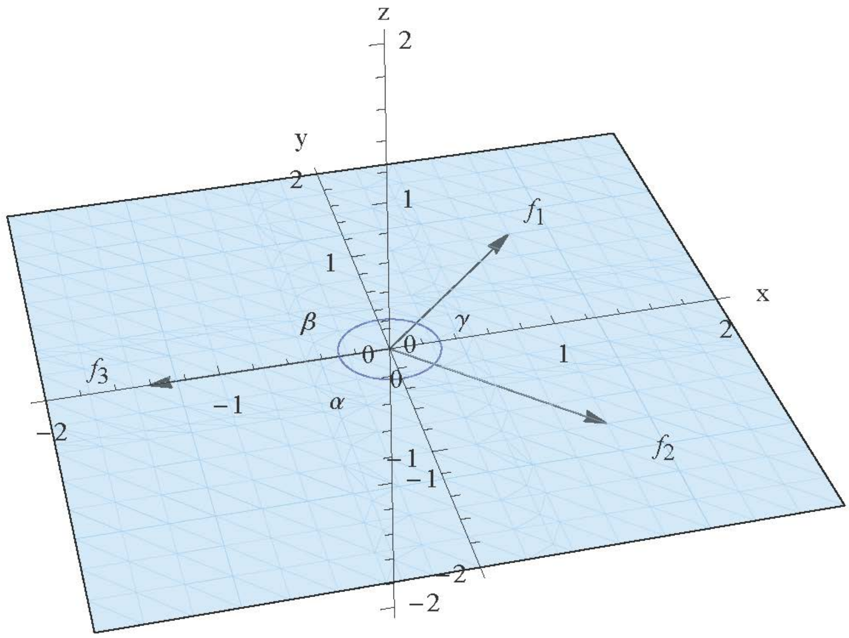

Figure 6: Y-graph in $\mathbb{R}^3$ with angles α, β and γ.

$f_3(0)$ and ensures that the branches are connected at the vertex. We continue in much the same way as in the previous section, by noting that the balance equation means that the tangential lines to the graph of f_j lie in the same plane, from which we should get one condition. We solve this problem again by introducing the coordinate system $\mathbb{R}^3$ and consider our Y-graph lying in the xy-plane in $\mathbb{R}^3$, according to figure 6. Let $\vec{a}^j$ be the tangent vector to the curves $z = f_j(x_j)$ considered in $\mathbb{R}^3$. Further, let ϕ_j denote the angles between the tangential lines f_j' and the xy-plane, it can be observed that $\tan\phi_j = f_j'(0)$. The vector $\vec{a}^3$ is parallel to the x-axis, therefore has zero second coordinate and can be chosen equal to,

$$\vec{a}^3 = (1, 0, \tan\phi_3) = (1, 0, f_3'(0))$$

The tangential vectors $\vec{a}^1$ and $\vec{a}^2$ may then be chosen equal to,

$$\vec{a}^1 = (-\cos(\pi - \beta), \sin(\pi - \beta), \tan\phi_1) = (\cos\beta, \sin\beta, f_1'(0))$$
$$\vec{a}^2 = (-\cos(\pi - \alpha), -\sin(\pi - \alpha), \tan\phi_2) = (\cos\alpha, -\sin\alpha, f_2'(0))$$

We want the branches of our Y-graph to be dependent and therefore find a matching condition through examining the determinant of the vectors $\vec{a}^j$,

$$\det\begin{pmatrix} \cos\beta & \sin\beta & f_1'(0) \\ \cos\alpha & -\sin\alpha & f_2'(0) \\ 1 & 0 & f_3'(0) \end{pmatrix} =$$
$$\sin\alpha f_1'(0) + \sin\beta f_2'(0) - f_3'(0)(\cos\beta\sin\alpha + \sin\beta\cos\alpha = 0.$$

Using the trigonometric reduction $\sin\alpha\cos\beta + \cos\alpha\sin\beta = \sin(\alpha + \beta)$ we get the following matching condition,

$$\sin\alpha f_1^{'}(0) + \sin\beta f_2^{'}(0) - \sin(\alpha + \beta) f_3^{'}(0) = 0.$$

Further, noting that $\sin(\alpha + \beta) = \sin(2\pi - \gamma) = -\sin\gamma$ this condition can be written as

$$\sin\alpha f_1^{'}(0) + \sin\beta f_2^{'}(0) + \sin\gamma f_3^{'}(0) = 0$$

Thus, so far we have the following two matching conditions,

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\alpha f_1^{'}(0) + \sin\beta f_2^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \end{cases}$$

To find the remaining matching conditions we use a rearrangement of the conditions above

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ f_3^{'}(0) = -\dfrac{\sin\alpha f_1^{'}(0) + \sin\beta f_2^{'}(0)}{\sin\gamma} \end{cases}$$

together with the boundary form at the beginning of this section. In order to use the stated rearrangement we require that γ is non-zero and $\gamma \neq \pi$ so that $\sin\gamma \neq 0$. Below we write $f^{'}, f^{''}, f^{'''}$ instead of $f^{'}(0), f^{''}(0), f^{'''}(0)$ respectively for all derivatives for simplicity, since it is understood that we are investigating at the vertex. The boundary form of the operator L becomes,

$$
\begin{aligned}
&\langle Lf, g\rangle - \langle f, Lg\rangle = \\
&= \sum_{j=1}^{3}\Big(- f_j^{'''}(0)\overline{g_j(0)} + f_j^{''}(0)\overline{g_j^{'}(0)} - f_j^{'}(0)\overline{g_j^{''}(0)} + f_j(0)\overline{g_j^{'''}(0)}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + f_1^{''}\overline{g_1^{'}} + f_2^{''}\overline{g_2^{'}} + f_3^{''}\overline{g_3^{'}} \\
&- f_1^{'}\overline{g_1^{''}} - f_2^{'}\overline{g_2^{''}} - f_3^{'}\overline{g_3^{''}} + f(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + f_1^{''}\overline{g_1^{'}} + f_2^{''}\overline{g_2^{'}} + f_3^{''}\Big(- \frac{\sin\alpha\overline{g_1^{'}} + \sin\beta\overline{g_2^{'}}}{\sin\gamma)}\Big) \\
&- f_1^{'}\overline{g_1^{''}} - f_2^{'}\overline{g_2^{''}} - \Big(- \frac{\sin\alpha f_1^{'} + \sin\beta f_2^{'}}{\sin\gamma}\Big)\overline{g_3^{''}} + f(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big) \\
&= -\Big(\sum_{j=1}^{3} f_j^{'''}\Big)\overline{g(0)} + \Big(f_1^{''} - \frac{\sin\alpha f_3^{''}}{\sin\gamma}\Big)\overline{g_1^{'}} + \Big(f_2^{''} - \frac{\sin\beta f_3^{''}}{\sin\gamma}\Big)\overline{g_2^{'}} \\
&- f_1^{'}\Big(\overline{g_1^{''}} - \frac{\sin\alpha\overline{g_3^{''}}}{\sin\gamma}\Big) - f_2^{'}\Big(\overline{g_2^{''}} - \frac{\sin\beta\overline{g_3^{''}}}{\sin\gamma}\Big) + f(0)\Big(\sum_{j=1}^{3}\overline{g_j^{'''}}\Big).
\end{aligned}
$$

Looking at the last expression above we search for conditions that connect together derivatives of the same order and make the boundary form equal to zero. We choose the following conditions as our final matching conditions,

$$
\begin{cases}
f_1^{''}(0) - \dfrac{\sin\alpha f_3^{''}}{\sin\gamma} = 0 \\
f_2^{''}(0) - \dfrac{\sin\beta f_3^{''}}{\sin\gamma} = 0 \\
\sum_{j=1}^{3} f_j^{'''}(0) = 0
\end{cases}
$$

Since we have stated earlier that γ is non-zero and $\gamma \neq \pi$ we rearrange and, in total, have the following six matching conditions for the operator L on the Y-graph in figure 5 and 6,

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\alpha f_1^{'}(0) + \sin\beta f_2^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \\ \sin\gamma f_1^{''}(0) - \sin\alpha f_3^{''} = 0 \\ \sin\gamma f_2^{''}(0) - \sin\beta f_3^{''} = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0. \end{cases}$$

There is an interesting symmetry between the angles of the Y-graph and the first derivatives, namely that the first derivatives of each branch is associated with the angle subtending the other two branches, the angle opposite the branch itself. We continue by considering some special cases.
Let us examine the cases when the angles α and β are equal to π.

Example 7. *In the case* $\alpha = \pi$ *the value of* $\sin\alpha = 0$ *and we get a similar case to Example 5, where two different sets of conditions ensure that the boundary form is zero,*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\beta f_2^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \\ \sin\gamma f_1^{''}(0) = 0 \\ \sin\gamma f_2^{''}(0) - \sin\beta f_3^{''} = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0, \end{cases}$$

and

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\beta f_2^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \\ \sin\gamma f_1^{'}(0) = 0 \\ \sin\gamma f_2^{''}(0) - \sin\beta f_3^{''} = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0. \end{cases}$$

Example 8. *In the case* $\beta = \pi$ *the value of* $\sin\beta = 0$ *and this case is similar to the previous example when* $\alpha = \pi$*, giving the two different sets of matching conditions,*

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\alpha f_1^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \\ \sin\gamma f_2^{'}(0) = 0 \\ \sin\gamma f_1^{''}(0) - \sin\alpha f_3^{''} = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0, \end{cases}$$

and

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \sin\alpha f_1^{'}(0) + \sin\gamma f_3^{'}(0) = 0 \\ \sin\gamma f_1^{''}(0) - \sin\alpha f_3^{''} = 0 \\ \sin\gamma f_2^{''}(0) = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0. \end{cases}$$

Example 9. *Further, let us examine the case when α, β, γ all are equal to $\frac{2\pi}{3}$. In this case*

$$\sin\alpha = \sin\beta = \ sin\gamma = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}.$$

The matching conditions become

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ \frac{\sqrt{3}}{2} f_1^{'}(0) + \frac{\sqrt{3}}{2} f_2^{'}(0) + \frac{\sqrt{3}}{2} f_3^{'}(0) = 0 \\ \frac{\sqrt{3}}{2} f_1^{''}(0) - \frac{\sqrt{3}}{2} f_3^{''} = 0 \\ \frac{\sqrt{3}}{2} f_2^{''}(0) - \frac{\sqrt{3}}{2} f_3^{''} = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0) = 0. \end{cases}$$

Dividing by $\frac{\sqrt{3}}{2}$ gives the following matching conditions

$$\begin{cases} f_1(0) = f_2(0) = f_3(0) \\ f_1^{'}(0) + f_2^{'}(0) + f_3^{'}(0) = 0 \\ f_1^{''}(0) = f_2^{''}(0) = f_3^{''}(0) = 0 \\ \sum\limits_{j=1}^{3} f_j^{'''}(0). \end{cases}$$

These conditions are expected since they concur with the standard conditions in (3) and with the results in Example 6.

Conclusion

A final remark about the results achieved above can be made, namely that in all three cases observed, the standard conditions are a set of linear relationships between derivatives of the same degree. For the Laplace operator noting that the standard conditions are linear relationships is straight forward. The standard conditions for the square Laplace operator in the symmetric case can be realised to have a linear relationship since $2\cos\alpha$ is finite. Linear relationships in the case of a general angle between each branch discussed in the fourth section can be realised since $\sin\alpha, \sin\beta$ and $\sin\gamma$ are all finite. The linearity of the matching conditions seems to extend to an operator of any even order of differentiation, or perhaps to any order of differentiation, but proving this would require further study. The symmetry in the matching conditions in the general case would perhaps also extend to a star graph with any finite, or perhaps infinite, number of branches, but this would also warrant further study.

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