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The theory of dynamical system and control applied to macroeconomics

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The theory of dynamical system and control applied to macroeconomics

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Abstract

The basics of difference equations, mathematical control theory and the theory of dynamical systems is presented and applied to macroeconomics. The theory of dynamical systems and mathematical control theory is applied to the problem of setting the interest rate at a level that minimizes a measure of social costs. The theory of difference equations is used to briefly discuss the widely debated theory of capitalism as laid out by economist Thomas Piketty.

The economic world is a misty region. The first explorers used unaided vision. Mathematics is the lantern by which what was before dimly visible now looms up in firm, bold outlines. The old phantasmagoria disappear. We see better. We also see further.

- Irving Fisher

Innehåll

Introduction

For anyone who wishes to formulate a set of comprehensible and coherent thoughts, hoping to elucidate some aspect of reality, mathematics is an indispensable tool. The indispensability lies in the ability of mathematics to express concepts in an exact and stringent fashion, minimizing ambiguity. Many conclusions of economics have far-reaching political consequences and thus incentives to distort facts and theory are strong for parties with vested interests. In addition, not only outright distortion but, merely uttering ideas about the economy in an obscure fashion is problematic. Modern economics is a science, and as such, it strives to present hypotheses about reality that are testable. Obscurity in the formulation of economic hypotheses reduces the testability of these hypotheses which impairs attempts of doing science.

Nothing is more stringent or has the ability to elucidate reality quite like mathematics. Mathematics has proven to be an priceless tool for almost all sciences, aiding and guiding the minds of countless of scientists in their inquiry of countless phenomenon, from Albert Einstein to Milton Friedman, from relativity theory to permanent income theory. In economics, the virtue of mathematics is that it imposes logical discipline; on any one wishes to unravel the mysteries of the economic world and on the public discussion of economic concepts. Without mathematics it would be too easy to formulate and propagate incoherent theories while at the same time concealing these incoherences behind mazy rethorics.

The purpose of this text is to offer a comprehensive introduction to one of the most important areas of mathematics used in modern economics - mathematical control theory. Mathematical control theory is about the ability of a planer to steer a dynamical system - a system that evolves over time - in some desirable direction. This could be solving the consumers problem of determining the consumption behavior that maximizes utility over a period of time, or solving the problem facing a central bank who wishes to set the interest rate at a level that minimizes the detrimental effects of inflation and unemployment.

Introducing the methods of mathematical control theory applied to modern macroeconomics is not an easy task. When applied to modern macroeconomics, mathematical control theory utilizes every branch of mathematics that most master students, or even Ph.D students, in economics have encountered throughout their education; Single- and Multivariable Calculus, Foundations of Real Analysis, Linear Algebra, Ordinary Differential equations, Ordinary Difference equation etc. To cover even a single one of these topics an entire book would be insufficient, hundreds of books would be needed. Thus, in the first chapter some of these topics will be covered - albeit to a very limited extent. In subsequent chapters mathematical control theory is introduced as a means of controlling **discrete-time, time-invariant, linear dynamical systems**. After reading this thesis, I hope the reader has learned as much about the applications of mathematical control theory to modern macroeconomics as I did writing it. Let's begin!

Kapitel 1

Foundations

1.1 Banach's fixed point theorem

In this section the Banach fixed point theorem is presented along with the relevant definitions and lemmas necessary for proving it. The theorem will be an indispensable tool solving a major equation in later chapters, namely the **Bellman equation**.

The following definitions and theorems are standard. I first learned of them when reading Walter Rudin's classical textbook **Principles of mathematical analysis** (1976) [1]

1.1.1 Definition A set X, containing elements which we call **points**, is a **metric space** if with any two points $p, q \in X$ there is associated a real number $d(p, q)$, called the **distance** from p to q, such that

(i) $d(p, q) > 0$ if $p \neq q$; $(p, p) = 0$
(ii) $d(p, q) = d(q, p)$
(iii) $d(p, q) \leq d(p, r) + d(r, q)$, for any $r \in X$

A function with the above properties is called a **distance function**.

1.1.2 Theorem The euclidean spaces, $\mathbb{R}^n$, are metric spaces.

Proof Using $d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$ all of the conditions in definition 1.1

are satisfied.

1.1.3 Definition A **sequence** is a function f defined on the set of non-negative integers $J = \{n \in \mathbb{Z} : n \geq 0\}$. When $f(n) = x_n$ for $n \in J$ it's customary to denote the sequence f by the symbol $\{x_n\}$ or, if handling multiple sequences, $x_i(n),\ i \in J$. The values of f, that is, the elements of x_n, are called the **terms** of the sequence. If A is a set and if $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a **sequence in** A.

1.1.4 Definition A sequence $\{x_n\}$ in a metric space X is said to **converge** if there is a point $x \in X$ with the following property: For every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies that $d(x_n, x) \leq \varepsilon$. In this case, x is said to be the limit of $\{x_n\}$. We write

$$\lim_{n\to\infty} x_n = x.$$

If $\{x_n\}$ does not converge, it is said to **diverge**.
The following are some interesting examples of sequences

(i) If $x_n = 1 + (-\frac{1}{2})^n$, the sequence $\{x_n\}$ converges to 1

(ii) The Fibonacci sequence, $\{F_n\}$, with $F_n = F_{n-1}+F_{n-2},\ \ x_1 = x_2 = 1$, $n = 3, 4, 5, ...$, diverges

(iii) With F_n as in (ii) the sequence $\{x_n\}$, with $x_n = \frac{F_n}{F_{n-1}}$ converges and $\lim_{n\to\infty} x_n = \frac{1+\sqrt{5}}{2}$.

1.1.5 Definition A sequence $\{x_n\}$ in a metric space X is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there is an integer N such that $d(x_n, x_m) < \varepsilon$ if $n \geq N$ and $m \geq N$

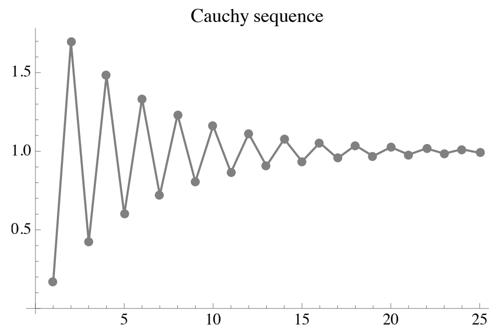
1.1.6 Lemma In any metric space X, every convergent sequence is a Cauchy sequence.

Proof If $\lim_{n\to\infty} x_n = x$ and if $\frac{\varepsilon}{2} > 0$, then there is an integer N such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$. Hence

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon$$

when $n \geq N$ and $m \geq N$. Thus $\{x_n\}$ is a Cauchy sequence.

As an example, if $x_n = 1 + (-\frac{1}{2})^n$ then $\{x_n\}$ is a Cauchy sequence with $\lim_{n\to\infty} x_n = 1$. This sequece is illustrated in the metric space $\mathbb{R}^2$ in figure 1.



Figur 1.1: Graph of the sequence $x_n = 1 + (-\frac{1}{2})^n$

1.1.7 Definition A metric space in which every Cauchy sequence is convergent is said to be **complete**

1.1.8 Definition Let X be a metric space, $x \in X$ and f maps X into X. Then f is said to be **continuous** at x if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$d(f(y), f(x)) < \varepsilon$$

for all $y \in X$ for which $d(y, x)\delta$. If f is continuous att very point of X then f is said to be continuous on X.

1.1.9 Definition Let X be a metric space. I if f maps X in to X and if there is a number $\beta < 1$ such that

$$d(f(x), f(y) \leq \beta d(x, y)$$

for all $x, y \in X$, then f is said to be a **contraction** of X into X. β is called the *modulus* of the contraction.

The following theorem is of the utmost importance for the purpose of solving one of the central equations of this thesis. The theorem is called **Banach's fixed point theorem** after the Polish mathematician Stefan Banach. Worth noting is that this is not the most general fixed point theorem but it is sufficient for the purpose at hand.

1.1.10 Theorem If X is a complete metric space, and if f is a contraction of X in to X, then there exists one and only one $x \in X$ such that $f(x) = x$

Proof Pick any $x_0 \in X$, and define $\{x_n\}$ recursively, by setting

$$x_{n+1} = f(x_n), \quad (n{=}0,\ 1,\ 2,...)$$

Choose $c < 1$ so that

$$d(f(x), f(y)) \leq cd(x, y)$$

for all $x, y \in X$. For $n \geq 1$ we have

$$d(x_{n+1}, x_n) = d(f(x_n), f(x_{n-1})) \leq c\ d(x_n, x_{n-1})$$

. Inductively it follows that

$$d(x_{n+1}, x_n) \leq c^n d(x_1, x_0), \quad (n{=}0,\ 1,\ 2,...)$$

If $n \leq m$, it follows that

$$d(x_n, x_m) \leq \sum_{i=n+1}^{m} d(x_i, x_{i-1}) \leq (c^n{+}c^{n+1}{+}...{+}c^{m-1})d(x_1, x_0) \leq \frac{c^n}{1-c} d(x_1, x_0).$$

This in turn implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, $\lim_{n\to\infty} x_n = x$ for some $x \in X$. The uniqueness follows from the fact that if $f(x) = x$ and $f(y) = y$ then

$$d(f(x), f(y)) = d(x, y) \leq cd(x, y)$$

which is only possible if $d(x, y) = 0$, that is if $x = y$. Moreover, since f is a contraction and therefore f is continous we have

$$f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$$

□

The purpose of this section has been fulfilled; the proof of the Banach fixed point theorem has been presented. Rudin's book has been a priceless resource when putting together this material. The relevance of the theorem might not be apparent yet, however it will be an indispensable tool for proving the existence and uniqueness of solutions to an important set of equations in the dynamic programming theory - the branch of mathematical control theory that will be the central piece of this thesis.

The next section of this chapter will be an excursion in to the field of **Difference equations** in general and **linear difference equations** in particular. Difference equations, like the more familiar **differential equations**, describe the evolution of systems over time. The difference between difference equations and differential equations is that the former treats time as discrete whilst the latter treats time as continuous. People in the real wold, and especially when they participate in the economy, seem to treat time as discrete, we plan in terms of what we are going to consume *today* or *tomorrow*, how much we are going to save of our *monthly* income etc. Difference equations are suitable for modeling economic systems both because of their discrete-time outlook and their dynamic nature.

1.2 Difference equations

Everything changes and nothing stands still.
- Heraclitus

In the reality in general and in the economy in particular, only change is constant. One of the mathematical tools used to model systems that evolve over time is the difference equation. If economists hope to make accurate predictions of phenomenon in the real world, using the mathematical tools that incorporate time is essential.

The purpose of this sections is not to give formal definitions and theorems of general difference equations, rather it's a quick review of how a limited set of difference equations are presented in the literature and of some of their main properties. Many of the concepts and the definition can be found in [2].

1.2.1 Definition Let $\{x_t\}$, t=0, 1, 2,... be a sequence in $\mathbb{R}$ and let f be a function defined for t=0, 1, 2,... and all values of the elements in the sequence. An n**-th order difference equation** is an equation on the form

$$x_{t+n} = f(x_{t+n-1}, x_{t+n-2}, ..., x_{t+1}, x_t, t) \tag{1.1}$$

The general solution of (1.1) is a function $x_t = g(t : C_1, ..., C_n)$ that depends on n arbitrary constants $C_1, ..., C_n$, and satisfies (1.1). It has the property that every solution of (1) can obtained by assigning appropriate values to these constant.

Example: The equation $F_{n+1} = F_n + F_{n-1}$ associated with the Fibonacci sequence is a second order difference equation.

1.2.2 Definition The difference equation (1.1) is said to be **linear** if it can be written on the form

$$x_{t+n} = a_1(t)x_{t+n-1} + a_2(t)x_{t+n-2} + ... + a_{t+n-2}(t)x_2 + a_{t+n-1}(t)x_1$$

where $a_i(t),\ i = 1, 2, ..., n$ are arbitrary functions of time. If in addition, $a_i(t) = a$ for all t, where $a \in \mathbb{R}$, the equation is said to be **autonomous**.

1.2.3 Definition A system of **first order difference equations** in the n sequences $x_1(t), ..., x_n(t)$ can be expressed as

$$\begin{aligned} x_1(t+1) &= f(x_n(t), ..., x_2t, x_1(t), t) \\ x_2(t+1) &= f(x_n(t), ..., x_2t, x_1(t), t) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_3(t+1) &= f(x_n(t), ..., x_2t, x_1(t), t) \end{aligned}$$

1.2.3 Definition If f is linear in the n sequences $x_1(t), ..., x_n(t)$ the system is said to be **linear**.

Note that, because we are handling multiple sequences, the usual notation x_t is replaced by $x_i(t)$.

A dense way of expressing linear systems of difference equations is by using a matrix representation of them. Apart from being economical from a notational point of view it's also a natural way of interconnect the mathematics of difference equations with the mathematics of linear algebra.

The matrix representation of a linear system is

$$x(t+1) = A(t)x(t) + b(t)$$

where $x(t) := \begin{pmatrix} x_1(t) \\ \cdot \\ \cdot \\ x_n(t) \end{pmatrix}$ and

$$A(t) = \begin{pmatrix} a_{11}(t) & . & . & a1n(t) \\ \cdot & . & . & \cdot \\ \cdot & & & \\ a_{n1}(t) & . & . & a_{nn}(t) \end{pmatrix}, \quad b(t) = \begin{pmatrix} b_1(t) \\ \cdot \\ \cdot \\ b_n(t) \end{pmatrix}.$$

Hence A is an $n \times n$ matrix, $x(t)$ is an $n \times 1$ vector and $b(t)$ is an $n \times 1$ vector.

If in addition $A(t) = A$, that is, if all the elements of the matrix $A(t)$ are constants $a_{ij} \in \mathbb{R}$ the system reduces to

$$x(t+1) = Ax(t) + b(t) \tag{1.2}$$

We now prove some important results concerning systems on the form (1.2).

1.2.4 Theorem The solution of a system on the form (1.2) is

$$x(t) = A^t x(0) + \sum_{k=1}^{t} A^{t-k} b(k-1)$$

Proof Inserting $t = 0, 1...$, we get

$$x(1) = Ax(0) + b(0)$$

$$x(2) = Ax(1) + b(1) = A^2x(0) + Ab(0) + b(1)$$

$$x(3) = Ax(2) + b(2) = A^3x(0) + A^2b(0) + Ab(1) + b(2)$$

$$...$$

$$x(t) = A^t x(0) + A^{t-1}b(0) + ... + b(t-1) = A^t x(0) + \sum_{k=1}^{t} A^{t-k} b(k-1)$$

□

1.2.5 Corollary If $x(t+1) = Ax(t)$ then $x(t) = A^t x(0) \quad t = 0, 1...$

1.2.6 Definition A matrix, A, is said to be **stable** if it has the property that $A^t \to 0$ when $t \to \infty$.

1.2.7 Lemma If the matrix A has eigenvalues of moduli strictly less than 1 then A is stable.

Proof From the diagonalization of A we have

$$A^t = P \operatorname{diag}(\lambda_1^t, ..., \lambda_2^t)P^{-1}.$$

For $|\lambda_i| < 1$ we have $\lambda_i^t \to 0$ as $t \to \infty$. The lemma follows.

1.2.8 Lemma If the matrix A has eigenvalues of moduli strictly less than 1 then all the solutions to system $x(t+1) = Ax(t) + b$ are convergent and $\lim_{t\to\infty} x(t) = (A^{t-1} + A^{t-2} + ... + A)b$

Proof Because A has eigenvalues of moduli strictly less than 1 1.2.7 tells us that $A^t \to 0$ when $t \to \infty$. From theorem 1.2.4 we know that the solution to the system with x_0 given is

$$x(t) = A^t x(0) + \sum_{k=1}^{t} A^{t-k} b(k-1)$$

thus when $t \to \infty$ $x(t) \to (A^{t-1} + A^{t-2} + ... + A)b$
□

1.2.9 Theorem If the matrix A has eigenvalues of moduli strictly less than 1 and if $b(t) = b$ then all the solutions to system $x(t+1) = Ax(t) + b$ are convergent and $\lim_{t\to\infty} x(t) = (I_n - A)^{-1}b$ where I_n is the identity matrix of dimension n, $t = 0, 1...$

Proof According to Lemma 1.2.4 the solution to the system for a given $x(0)$ is

$$x(t) = A^t x(0) + (A^{t-1} + A^{t-2} ... + A + I)b$$

. We have that

$$(A^{t-1} + A^{t-2} + ... + A)(I - A) =$$
$$= (A^{t-1} + A^{t-2} ... + A) - (A^{t-1} + A^{t-2} ... + A)A = I - A^t.$$

But

$$(A^{t-1} + A^{t-2} + ... + A)(I - A) = I - A^t \Leftrightarrow (A^{t-1} + A^{t-2} + ... + A) = (I - A^t)(I - A)^{-1}.$$

(The existence of $(I-A)^{-1}$ is guaranteed because since the eigenvalues of A have moduli strictly less than one $|I-A| \neq 0$). From lemma 1.2.7 we have that $A^t \to 0$ when $t \to \infty$, so

$$(A^{t-1} + A^{t-2} + ... + A) \to (I-A)^{-1} \text{ as } t \to \infty$$

In conslusion,

$$A^t x(0) + (A^{t-1} + A^{t-2} ... + A + I)b \to (I-A)^{-1}b \text{ as } t \to \infty$$

The limit $(I-A)^{-1}b$ is called the **steady-state** of the difference equation. □

The basics of difference equations have been put in place. The rest of this section is devoted to applying this knowledge to one of the latest and most interesting controversies of modern macroeconomics - the famous capital growth theory of rockstar-economist Thomas Piketty and the problems with it as laid out by economists Per Krusell and Anthony Smith. The basis for this analysis can be found in [7].

1.3 1.3 Difference equations in action. Piketty vs Krusell and Smith.

Some notation Since this part will be a comparison of the Pikitty growth model and the standard textbook growth model (in the style of Solow, Cass and Koopman) the reader must be made familiar with the standard notations of theories of economic growth. Much of the controversy revolves around the use of either net- or gross variables so the distinction between them is crucial.

Both models have a common accounting framework that can be represented by the three following equations:

$$c_t + i_t = y_t \tag{1.3}$$

$$k_{t+1} = (1-\delta)k_t + i_t \tag{1.4}$$

$$y_t = F(k_t, z_t l) \tag{1.5}$$

where c_t denotes consumption, i_t denotes investments, k_t denotes the capital stock, δ denotes the depreciation rate of capital and y_t denotes gross income/output/production. The function F with arguments k_t and $x_t l$ is called the production function, which is a function of the capital stock k_t and the amount of labour l in the economy. z_t is a process that describes how technological progress affects the productivity of labour. All variables are gross variables at time t. Moreover, net counterparts of the above gross variables will be marked by a tilde, so for example net income will be denoted $\tilde{y}_t$.

In addition to the above mentioned variables the models also use some important parameters, δ as above mentioned is the depreciation rate of capital and g, the growth rate of the economy along a balanced growth path.

The two models arrive at very different conclusions when it comes to what happens in the long run to the capital-income ratio $\frac{k}{y}$. The Long-run-value of a variable is the name economists have of what mathematicians call the steady-state value. Employing the methods of finding the solutions to difference equations and their steady state values we will compare the theoretical implications of the two models and briefly discuss how reasonable the models seem in light of their implications.

The following definition of a special kind of production function will also be relevant.

Definition 1.2.5 For a function F in $\mathbb{R}^n$ is said to fulfill the Inada conditions off it has the following properties.

(i) $F(x) = 0$ if x is the zero-vector in $\mathbb{R}^n$
(ii) F is continously differentiable.
(iii) The function is strictly increasing. $\frac{\partial F}{\partial x_i} > 0$ $i = 1, 2, ..., n$ where x_i are the elements of the n-vector x
(iv) The function is concave in x
(v) $\lim_{x \to 0} \frac{\partial F}{\partial x_i} = \infty, \quad i = 1, 2, ..., n$
(vi) $\lim_{x \to \infty} \frac{\partial F}{\partial x_i} = 0$ $i = 1, 2, ..., n$

1.3.1 The textbook model of economic growth

In this model, labeled the textbook-model by Krusell and Smith, the following assumptions are made (a.) The production function $F(k, \cdot)$, satisfies the Inada conditions. In addition, it's assumed to be homogenous of degree 1, that is

$$\alpha_t F(k_t, z_t l) = F(\alpha_t k_t, \alpha_t z_t l)$$

(b.) Investment, i_t, is a constant fraction $s > 0$ of output. That is, $i_t = sy_t$

From (1.5) and (b) and we get $i_t = sF(k_t, z_t l)$, substituting this into (1.4) along with $i_t = sy_t$ with we get

$$k_{t+1} = (1-\delta)k_t + sF(k_t, z_t l). \tag{1.6}$$

Suppose the labour augmenting technological growth evolves according to $z_t = (1+g)^t$ where $g > 0$ is the rate of technological growth. After dividing both sides of (6) by z_t we get

$$\frac{k_{t+1}}{z_t} = (1-\delta)\frac{k_t}{z_t} + s\frac{F(k_t, z_t l)}{z_t}.$$

Defining $\hat{x}_t = \frac{x_t}{z_t}$ for all t we arrive at

$$(1+g)\hat{k}_{t+1} = (1-\delta)\hat{k}_t + sF(\hat{k}_t, l \Leftrightarrow$$

$$\hat{k}_{t+1} = \frac{1-\delta}{1+g}\hat{k}_t + \frac{s}{1+g}F(\hat{k}_t, l). \tag{1.7}$$

Does this equation have a steady state? And in that case, what is the steady state? We know that $g > 0$, $\delta > 0$ so $|\frac{1-\delta}{1+g}| < 1$ so from theorem 1.2.4 it is at least possible that there is a steady state. Now, in a steady state we have $k_{t+1} = k_t$ so let's examine if the the function

$$g(\hat{k}_t) := \frac{1-\delta}{1+g}\hat{k}_t + \frac{s}{1+g}F(\hat{k}_t, l) = \hat{k}_{t+1}$$

has a fixed point. Since F is concave by assumption, so is g. Since $\lim_{\hat{k}\to 0} g'(\hat{k}) = \infty$ and $\lim_{\hat{k}\to\infty} g'(\hat{k}) = \frac{1-\delta}{1+g} < 1$ this guarantees that g will be steep enough at the origin and flat enough for larger k so as to cross the 45 deg line where $\hat{k}_{t+1} = k_t$. So there exists a fixed point, which we'll call $\hat{k}^*$. Is it stable? To determine this we look at $\Delta\hat{k} = \hat{k}_{t+1} - \hat{k}_t$ by subtracting both sides of (7) by $\hat{k}_t$. We get

$$\Delta\hat{k} = \hat{k}_{t+1} - \hat{k}_t = \frac{1-\delta}{1+g}\hat{k}_t + \frac{s}{1+g}F(\hat{k}_t, l) - k_t = \frac{s}{1+g}F(\hat{k}_t, l) - \frac{g+\delta}{1+g}\hat{k}_t$$

and thus $\Delta\hat{k} = 0$ gives

$$0 = \frac{s}{1+g}F(\hat{k}_t, l) - \frac{g+\delta}{1+g}\hat{k}_t.$$

For $\hat{k}_t = \hat{k}^*$ we must have

$$\frac{s}{1+g}F(\hat{k}_t, l) = \frac{g+\delta}{1+g}\hat{k}_t$$

Since the slope of F is decreasing and the slope of $\frac{g+\delta}{1+g}\hat{k}_t$ is constant we must have that for $\hat{k}_t > \hat{k}^* \Rightarrow \frac{s}{1+g}F(\hat{k}_t, l) < \frac{g+\delta}{1+g}\hat{k}_t$, which implies $\Delta\hat{k} < 0$ so k decreases. Conversely $\hat{k}^* < \hat{k}_t$ implies $\Delta\hat{k} > 0$ and thus $\hat{k}_t$ converges to the steady state $\hat{k}^*$. What is the steady state value $\hat{k}^*$? $\Delta\hat{k} = 0$ gives

$$\frac{s}{1+g}F(\hat{k}_t, l) = \frac{g+\delta}{1+g}\hat{k}_t \Leftrightarrow \hat{k}_t = \frac{s}{g+\delta}F(\hat{k}_t, l)$$

Remembering that the entity of interest if $\frac{\hat{k}_t}{\hat{y}_t}$ and using (5) we get

$$\frac{\hat{k}_t}{\hat{y}_t} = \frac{s}{g+\delta}$$

1.3.2 Piketty's Second law of Capitalism $\mathbf{\frac{k}{y} = \frac{s}{g}}$

Piketty's model uses net variables instead of the textbook gross variables. The difference might seem subtle at first but will prove to be crucial as to the predictions of the model. The main assumptions behind this model are

(a.) The production function $\tilde{F}(k, \cdot) = F(k, \cdot) - \delta k$, is positive and increasing in k and satisfies an Inada condition; namely (vi)

$$\tilde{F}'_k(k, \cdot) \to 0 \text{ when } k \to \infty$$

. it's also assumed to be homogenous of degree 1, that is

$$\alpha_t F(k_t, z_t l) = F(\alpha_t k_t, \alpha_t z_t l)$$

(b.) *Net investment*, defined as $\tilde{i_t} = i_t - \delta k_t$, is a constant fraction $\tilde{s} > 0$ of *net output*. That is, $i_t - \delta k_t = \tilde{s}(y_t - \delta k_t)$

The main differences are of course the translation in to net variables and the relaxed assumptions on the nature of the production function F. These assumption, along with (1.3), (1.4), and (1.5) yield the following. Substituting the definition of net investments in to (1.4) and using $\tilde{y_t} = \tilde{F}(k, z_t l)$ we get

$$k_{t+1} = (1-\delta)k_t + i_t = k_t + \tilde{i_t} = k_t + \tilde{s}\tilde{F}(k_t, z_t l)$$

$$\Leftrightarrow$$

$$k_{t+1} = k_t + \tilde{s}\tilde{F}(k_t, z_t l) \tag{1.8}$$

Some very interesting observations can be made at this stage, observations that give us a hint of what predictions we can expect of this model. Rearranging (1.8) we see that

$$\Delta k = k_{t+1} - k_t = \tilde{s}\tilde{F}(k_t, z_t l).$$

Since we assumed $\tilde{s} > 0$ and that the production function $\tilde{F}(k_t, z_t l)$ is positive and increasing in k, this implies that $\Delta k > 0$ for all t. In other words, the capital stock grows in every time period, no matter if the economy at large grows, shrinks or what not, which is very counter-intuitive. We proceed, as we did with the textbook model, to see what happens to the capital per efficiency unit as time passes. Dividing by z_t, we get

$$(1+g)\hat{k}_{t+1} = \hat{k}_t + \tilde{s}\tilde{F}(\hat{k}_t, l).$$

Dividing by $1+g$ gives the non-linear difference equation

$$\hat{k}_{t+1} = \frac{1}{1+g}\hat{k}_t + \frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l). \tag{1.9}$$

Does this equation have a steady state? By a similar argument as before the concavity of $\tilde{F}$ (implied by (a)) implies the concavity of $h(\hat{k}_t := \frac{1}{1+g}\hat{k}_t + \frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l)$. The limiting behavior of h' in turn implies that there is a fixed point, call it $\hat{k}^*$. Is the fixed point stable? Again, we examine $\Delta\hat{k}_t = \hat{k}_{t+1} - \hat{k}_t$. Subtracting both sides of (1.9) by $\hat{k}_t$ we obtain

$$\Delta\hat{k} = \hat{k}_{t+1} - \hat{k}_t = \frac{1}{1+g}\hat{k}_t + \frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l) - \hat{k}_t = \frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l) - \frac{g}{1+g}\hat{k}_t.$$

In the fixed point we must have $\frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l) = \frac{g}{1+g}\hat{k}_t$. Again, by (a) we have that the slope of $\tilde{F}$ is decreasing as $k \to \infty$ and the slope of $\frac{g}{1+g}\hat{k}_t$. Moreover we have that the slope of $\tilde{F}$ as $\hat{k} \to 0$ is infinite. Thus for $k^* > k$ we have $\frac{g}{1+g}\hat{k}_t > \frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l)$ which implies $\Delta\hat{k} < 0$. Conversely for $\hat{k} < \hat{k}^*$ we have $\Delta\hat{k} > 0$ and thus $\hat{k}$ converges to the fixed point $\hat{k}^*$. What is the steady state value of $\hat{k}$? Using $\Delta\hat{k} = 0$ we obtain $\frac{\tilde{s}}{1+g}\tilde{F}(\hat{k}_t, l) = \frac{g}{1+g}\hat{k}_t \Leftrightarrow$

$$\hat{k}_t = \frac{\tilde{s}}{g}\tilde{F}(\hat{k}_t, l)$$

and thus the capital-income ratio is

$$\frac{\hat{k}_t}{\hat{\tilde{y}}_t} = \frac{\tilde{s}}{g}$$

1.4 Comparing the models

So far we haven't seen why reason for why any of the two models would be better or worse than the other. Part of the reason for why the Piketty-model is less reasonable is what it implies for the savings behavior. What we save is essentially what we produce less what we consume. So let's look at the steady state consumption level. Remember that consumption $c_t = (1-\tilde{s})\tilde{F}(k_t, z_t l)$ So we have

$$\frac{c_t}{y_t} = \frac{F(k_t, z_t l) - i_t}{F(k_t, z_t l)} = \frac{(1-\tilde{s})\tilde{F}(k_t, z_t l)}{F(k_t, z_t l)}$$

From the steady state, $\frac{\hat{k}}{\hat{\tilde{y}}} = \frac{\tilde{s}}{g}$, we get

$$\hat{k}g = \tilde{s}\hat{\tilde{y}} \Leftrightarrow \hat{k}g = \tilde{s}(\hat{F}(k, l) - \delta\hat{k}) \Leftrightarrow$$

$$F(\hat{k}, l) = \frac{g + \tilde{s}\delta}{\tilde{s}}\hat{k} \Leftrightarrow \frac{\hat{F}(\hat{k}, l) - \delta\hat{k}}{F(\hat{k}, l)} = \frac{g}{g + \tilde{s}\delta}$$

With $\frac{\tilde{F}}{F} = \frac{g}{g+\tilde{s}\delta}$ we get

$$\frac{c_t}{y_t} = (1-\tilde{s})\frac{g}{g+\tilde{s}\delta} \tag{1.10}$$

The blow to Piketty's second law of capitalism is the following; look at (1.10) as $g \to 0$ (as it will, according to Piketty's predictions), what will happen to the share of output going to consumption? According to (1.10) it will go to 0! Hence, postulating that net saving $\tilde{s}$ will be constant as growth slows down, as Piketty does, implies the share of output that is devoted to consumption shrinks to 0.

With the textbook model, everything is different. According to textbook model we get

$$\frac{c_t}{y_t} = \frac{F(k_t, z_t l) - i_t}{F(k_t, z_t l)} = \frac{F(k_t, z_t l) - sF(k_t, z_t l)}{F(k_t, z_t l)} = 1 - s$$

Hence, according to the textbook model, the share of output going to consumption is constant and equal to $1-s$. This seems intuitively more accurate and it fits better with the observed data on the matter [7].

Kapitel 2

The Bellman Equation

2.1 Problem formulation

In economics, the core of many models is different entities trying to optimize their behavior, that is, to maximize or minimize some objective, subject to some constraints. A consumer maximizing utility subject to a budget constraint, a firm minimizing costs for a given production volume or a central bank minimizing the social cost of the detrimental effects of inflation and unemployment. The branches of mathematical optimization and optimal control theory are, naturally, the right tools to use when trying to model optimizing behavior. The purpose of this chapter is to present some of the more fundamental conclusions and theorems of **discrete time dynamic programming**, a branch of optimal control theory.

To do this, the concept of a **discrete-time system with output** must be introduced. Even though the term *system* has been used frequently in this thesis so far, no formal definition has been made. To avoid speaking about undefined concepts we now define a system. This definition of a system is called the **internal** definition. The definitions are from [10] and [12] with some notational changes to adapt them to the framework of this paper.

2.1.1 Definition A **system** or **machine** $\Sigma = (\mathcal{T}, \mathcal{X}, \mathcal{U}, \phi)$ consists of:

- A time set $\mathcal{T}$
- A nonempty set $\mathcal{X}$ called the **state space** of Σ
- A nonempty set $\mathcal{U}$ called the control-value or **input-value space** Σ

- A map $\phi : \mathcal{D}_\phi \to \mathcal{X}$ called the **transition map** of Σ which is defined on a subset $\mathcal{D}_\phi$ of

$$\{(\tau, \sigma, x, \omega) | \sigma, \tau \in \mathcal{T}, \sigma \leq \tau, x \in \mathcal{X}, \omega \in \mathcal{U}^{[\sigma,\tau)}\}$$

  Such that the following properties hold:

- **Nontriviality** For each state $x \in \mathcal{X}$, there is at least one pair $\sigma < \tau$ in $\mathcal{T}$ and some $\omega \in \mathcal{U}^{[\sigma,\tau)}$ such that ω is **admissible for** x, that is, so that $(\tau, \sigma, x, \omega) \in \mathcal{D}_\phi$.

- **Restriction** If $\omega \in \mathcal{U}^{[\sigma,\mu)}$ is admissible for x, then for each $\tau \in [\sigma, \mu)$ the restriction $w_1 := \omega|_{[\sigma,\tau)}$ of ω to the subinterval $[\sigma, \tau)$ is also admissible for x and the restriction $w_2 := \omega|_{[\tau,\mu)}$ is admissible for $\phi(\tau, \omega, x, \omega_1)$.

- **Semigroup** if σ, τ, μ are three elements of $\mathcal{T}$ so that $\sigma < \tau < \mu$, if $\omega_1 \in \mathcal{U}^{[\sigma,\tau)}$ and $\omega_2 \in \mathcal{U}^{[\tau,\mu)}$ and if x is a state so that

$$\phi(\tau, \sigma, x, \omega_1) = x_1 \text{ and } \phi(\mu, \tau, x_1, \omega_2) = x_2,$$

  then $\omega = \omega_1\omega_2$ is also admitssible for x and $\phi(\mu, \sigma, x, \omega) = x_2$.

- **Identity** For each $\sigma \in \mathcal{T}$ and each $x \in \mathcal{X}$, the empty sequence $\circ \in \mathcal{U}^{[\sigma,\sigma)}$

2.1.2 Definition A system **with outputs**, $(\mathcal{T}, \mathcal{X}, \mathcal{U}, \mathcal{Y}, h, \phi)$. is given by a system Σ together with

- A set $\mathcal{Y}$ called the **measurement map** or **output-values space**

- A map $h : \mathcal{T} \times \mathcal{X} \to \mathcal{Y}$ called the **measurement map**.

2.1.3 Definition A **discrete-time** system with (or without) outputs is one for which $\mathcal{T} = \mathbb{Z}$.

The above definitions are quite abstract and some examples are necessary to de-mystify the definition of a system. In plain english one could summarize the above definitions as follows. A system is a description of how some state of affairs (elements of the state space in the above definition) evolves over time (the elements in the time set). The change of the state of affairs over time is governed by some ”rule” (the transition map). As the state off affairs evolves over time, the system generates some, in principle, partly observable changes - outputs. The inputs, or controls, are some external forces operating on the system that changes the evolution of the state of affairs. How the inputs, or controls, affect the system is also described in the transition map.

Sometimes the "rule" that governs the system is not known, that is, the transition map is at least partly unknown but the outputs of the system on the other hand are observed. In that case, an external description of the system is more intuitive. One can picture the system as a "blackbox" that accepts inputs and generates outputs. What happens in the "blackbox" might be partially obscured and therefore the observer of the system must examine the systems "input/output-behaviour", that is, how different outputs change outputs. The following image illustrates the "blackbox"-description of a system.

Now over to discussing the types of systems that will be encountered in the rest of this thesis, namely **discrete-time** systems, defined in 2.1.3. For instance, let $x_t \in \mathbb{R}^n$ be a vector of state-variables, $u_t \in \mathbb{R}^n$ be a vector of control variables (or input variables), $f = f(t, x_t, u_t)$ be a function of time t, the state at time t, x_t and the control u_t that describes the evolution of the system. Moreover, let $y_t = r(t, u_t, x_t)$ be outputs, described by a function r. Then the discrete-time system with outputs at hand is

$$x_{t+1} = f(t, x_t, u_t) \tag{2.1}$$
$$y_t = r(t, x_t, u_t)$$

With a system like this in place, the formulation of the optimal control problems (OCP) that are of interest in this thesis, namely the **dynamic programming problem** (DPP), is straight forward. We seek to solve

$$\max\{\sum_{t=0}^{T} r(t, x_t, u_t)\} \tag{2.2}$$

subject to

$$x_{t+1} = f(t, x_t, u_t).$$

Where T is the number of time periods we wish to consider.

As an example, consider the following discrete time system with outputs.

$$x_{t+1} = Ax_t^{\alpha} - u_t$$

$$y_t = \sqrt{u_t}$$

In this case

$$f(t, x_t, u_t) = Ax_t^\alpha - u_t, \;\; 0 < \alpha < 1, \quad A > 0$$

and $y_t = r(t, x_t, u_t) = \sqrt{u_t}$. If we let x_t denote the amount of capital that a individual holds at time t, u_t denote the consumption of the individual at time t and let Ax_t^α be the amount of wealth that the individual can produce at time t with her capital, the above is a simple model of the individual's capital over time.

The individual strives to maximize utility over a time period, from time $t = 0$ to some time $t = T$ in the future. The individual faces the following OCP

$$\max \sum_{t=0}^{T} \sqrt{u_t} \quad \text{subject to} \quad x_{t+1} = Ax_t^\alpha - u_t \tag{2.3}$$

This thesis is all about solving problems similar to the above.

2.2 Dynamic programming equation and the Bellman's principle

One of the most important mathematical concept of this thesis is the **Bellman equation** also called the **Dynamic programming equation**. This equation, which is the result of the **Bellman's optimality principle**, simplifies the problem (2.2) drastically. The optimality principle reads as follows.

Bellmans principle of optimality: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. The principle is summed up in the following theorem.

2.2.1 Definition For each $s = 0, 1..., T-1, T$, let $V_s(x)$ denote the optimal value function

$$V(x_s) = \max_{u_s,...,u_T} \sum_{t=0}^{T} r(t, x_t, u_t)$$

for the optimal control problem

$$\max \sum_{t=0}^{T} r(t, x_t, u_t) \text{ subject to } \; x_{t+1} = f(t, x_t, u_t), \quad u_t \in U$$

with x_0 given.

2.2.2 Theorem The the sequence of value functions satisfies the equation

$$V(x_s) = \max_{u \in \mathcal{U}} [r(s, x_s, u_s) + V(x_{s+1})] \tag{2.4}$$

.

$$V(x_T) = \max_{u \in \mathcal{U}} r(T, x_T, u_T).$$

Proof: Suppose that x_s is the state at time $t = s$. The optimal control at this instant in time, u_s^*, must satisfy the equation

$$\max_{u_s \in \mathcal{U}} [r(s, x_s, u_s) + V(x_{t+1})].$$

To see why we note that at time s the objective function

$$\sum_{t=0}^{T} r(t, x_t, u_t)$$

can be written as

$$r(0, x_0, u_0) + r(1, x_1, u_1) + ... + r(s, x_s, u_s) + \sum_{t=s+1}^{T} r(t, x_t, u_t).$$

Since we are at time $t = s$ all the prior controls have already been executed and the cost up to time $t = s$, which we denote by S

$$S = r(0, x_0, u_0) + r(1, x_1, u_1) + ... + r(s-1, x_{s-1}, u_{s-1})$$

is a sunk cost (that is, a cost that has has already been paid and does not enter in to the consideration of future control-choices). To maximize the objective function from $t = s$ up to the final time $t = T$ then becomes the problem of maximizing

$$S + r(s, x_s, u_s) + \sum_{t=s+1}^{T} r(t, x_t, u_t).$$

But finding u_s that solves

$$\max_{u \in \mathcal{U}} [S + f(s, x_s, u_s) + \sum_{t=s+1}^{T} r(t, x_t, u_t)]$$

is the same as finding the u_s that solves (17), by the definition of $V(x_{s+1})$:

$$V(x_{t+1}) = \max_{\{u\}_{s+1}^{T}} [\sum_{t=s+1}^{T} r(t, x_t, u_t)].$$

Because s arbitrary this equation holds for all time periods $0 \leq t < T$ which proves the theorem. $\square$

Suppose that, as often is the case in economic applications, we want to solve the above OCP with two special modifications:

- The time horizon extends beyond all finite limits and in to infinity
- The instantaneous value function has the form $r(t, x_t, u_t) = \beta^t r(x_t, u_t)$ with $0 < \beta < 1$

This type of problem arises when we, for instance, consider the individual who wants to maximize utility over an infinite time horizon where future consumption is less valuable (measured in utility) than present consumption. The natural name for this problem is of course **infinite horizon discounted dynamic programming problem**. Mathematically it can be presented as

$$\max \sum_{t=0}^{\infty} \beta^t r(x_t, u_t) \quad \text{subject to} \quad x_{t+1} = f(x_t, u_t). \tag{2.5}$$

Note that neither r nor f are explicit functions of time, because of this the problem is often referred to as an **autonomous** problem. The Bellman equation (or dynamic programing equation) for this problem is instead.

$$V(x_s) = \max[r(x_s, u_s) + \beta V(x_{s+1})][2], [3]. \tag{2.6}$$

Reasoning heuristically we arrive at something very similar to the finite horizon undiscounted problem. Since the problem is autonomous, the starting time is irrelevant in the sense that all we care about it *"how much"* time has passed since the starting time rather than *"what was the starting time"*. Compare it to a 100m runner; one doesn't care if the 100m run started at 3 o'clock or 5 o'clock, what is interesting is the time that has passed *since* the run started. With this in mind, we can, with no loss of generality assume that we start at time $t = 0$. At this time the optimal value function must, according to Bellman's principle, be such that the control at time $t = 0$, u_0^* solves

$$\max_{u_0 \in \mathcal{U}} [\beta^0 r(x_0, u_0) + \beta V(x_1)].$$

This holds for all instants of time and thus the Bellman equation (16) holds.

So what do we need the Bellman equation for? It's seems like it's just a way of reformulation one hard problem in to another hard problem. Where was the gain from finding the Bellman equation? The answer is the following: Suppose that we knew the value function. Inserting it in to the Bellman equation we can solve for the sequence of optimal controls by the usual "fist order-condition"-method. Some methods for finding the value function are listed bellow.

2.3 Computing the value function

Two types of computational methods for solving the dynamic programming equation will be examined.

2.3.1 Guessing value function form

This is an "undetermined coefficients"-approach where the form of the solution of $V(x)$ is guessed and then verified. This method yields a solution that is not necessarily unique and information on the uniqueness is not always obtainable [3].

2.3.2 Value function iteration

By constructing a sequence of value functions and associated control functions starting from $V_0 = 0$ the value function is obtained as the limit of the (conditionally) convergent sequence

$$V_{t+1} = \max_{u_t \in \mathcal{U}} \{ r(x_t, u_t) + \beta V_t(x_{t+1})$$

where $x_{t+1} = f(x_t, u_t), \quad t = 1, 2, ...$ is the state transition law. [3]

The second method, **value function iteration**, works, with guaranteed uniqueness, under reasonable conditions on r and g. To prove this we need the Banach fixed point theorem, derived in chapter 1, and a theorem due to American mathematician David Blackwell [6]. Blackwell's theorem consists of a number of conditions that, if fulfilled by a *mapping*, sufficiently proves that the mapping in question is a contraction. We know, thru Banach's theorem, that every contraction on a complete metric has a unique fixed point, and thus to prove that the Bellman equation has a unique solution $V(x)$ we need only need to prove that the map T, where $TV = \max[r(x,u)+\beta V(f(x,u)]$, is a contraction mapping. Note that the fixed point is the v such that $TV = V$.

We begin by stating Blackwells conditions. The proof is from [3]

2.3.1 Theorem Let T be an operator on a metric space of functions, X, with the metric $d(x,y) = \sup_{0\leq t\leq T} |x(t) - y(t)|$. If the operator has the following two properties

- Monotonicity: For any $x, y \in X$, $x \geq y$ implies $T(x) \geq T(y)$.
- Discounting: Let c denote a function that is constant at the real values c for all points in the domain of definition of the function in X. For any positive real c and every $x \in X$, $T(x+c) \leq T(x) + \beta c$ for some $0 \leq \beta < 1$.

Then T is a contraction mapping with modulus β.

Proof For all $x, y \in X$, $x \leq y + d(x,y)$. Applying the two properties of monotonicity and discounting, this gives

$$T(x) \leq T(y + d(x,y) \leq T(y) + \beta d(x,y).$$

Exchanging the roles of x and y and using the samel logic implies

$$T(y) \leq T(x) + \beta d(x,y).$$

Combining these two inequalities gives $|T(x) - T(y)| \leq \beta d(x,y)$ or

$$d(T(x), T(y) \leq \beta d(x,y).$$

□

2.3.2 Theorem Let r be a real valued, continuous, concave, and bounded function. Let the set $S = \{x', x, u : x' \leq f(x,u), u \in \mathbb{R}^n\}$ be convex and compact.
We definite the operator

$$TV = \max_{u\in\mathbb{R}^n}[r(x,u) + \beta V(f(x,u)],\ x' \leq f(x,u),\ x \in X$$

Let Y be the (complete) metric space of continuous bounded functions that maps X into the real line. The operator T maps a continuous bounded function $V(x)$ into a continuous bounded function TV (proof omitted). Then the operator T is a contraction.

Proof We verify Blackwell's conditions. Suppose $V(x) \geq W(x)$ for all $x \in X$, then:

$$TV = \max_{u \in \mathbb{R}^n} \{r(x,u) + \beta V(x')\} \geq \max_{u \in \mathbb{R}^n} \{r(x,u) + \beta W(x')\} = TW, \quad x' \leq f(x,u).$$

Thus T is monotone. Next, notice that for any positive constant c,

$$T(V+c) = \max_{u \in \mathbb{R}^n} \{r(x,u) + \beta[V(x')+c]\} = \max_{u \in \mathbb{R}^n} \{r(x,u) + \beta V(x') + \beta c\} = TV + \beta c$$

$x' \leq f(x,u)$.

Thus T discounts. T satisfies both Blackwell's conditions and is therefore a contraction on a complete metric space. It follows from Banach's theorem that the Bellman equation has a unique fixed point V such that $TV = V$. Moreover this unique fixed point is the limit of the sequence $\{T^n(V_0)\}$ for some initial value (V_0) when $n \to \infty$

The value function iteration method is in essence a numerical method for finding V. This entails that it's hard to find closed form solutions by hand. There are of course exceptions that are pretty easy to solve by hand. Bellow follows one of those examples due to Brock and Mirman (1972) [14].

2.4 Solved problem: (Brock and Mirman, 1972)

Let $u(c) = \ln(c)$ be the utility function and $f(k) = Ak^{\alpha}, \; 0 < \alpha < 1$ be the production function. Further let the depreciation rate of capital be $\delta = 1$
We wish to solve the OCP

$$\max \sum \beta^t \ln(c_t)$$

subject to the equation describing the evolution of capital as

$$k_{t+1} = f(k_t) - c_t$$

with k_0 given.

The first method for solving this problem is given by the method of value function iteration. Using this method we begin by starting from $V_0 = 0$ and continuing until V_j has converged where

$$V_{j+1} = \max_c \{\ln(c) + \beta V_j(k')\}.$$

k' denotes the value of k in the next period, regardless of what the current period is.

So from $V_0(k) = 0$ we solve the one period problem:

$$\max_c \{\ln(c) + \beta V_0(k)\} = \max_c \{\ln(c)\}$$

Subject to $k' = Ak^\alpha - c$. This is obviously solved by choosing $c = Ak^\alpha \Rightarrow k' = 0$. This then gives us

$$V_1(k) = \ln(Ak^\alpha) = \ln A + \alpha \ln k$$

We continue by solving for $V_2(k)$, given by

$$V_2(k) = \max_c \{\ln(c) + \beta(\ln A + \alpha \ln k')\}$$

Differentiating with respect to c we get the first order condition

$$\frac{1}{c} - \frac{\beta\alpha}{Ak^\alpha + c} = 0$$

$$\Leftrightarrow c = \frac{Ak^\alpha}{1+\beta\alpha}$$

$$\Rightarrow k' = \frac{\beta\alpha}{1+\beta\alpha} Ak^\alpha$$

So

$$V_2(k) = \ln \frac{A}{1+\alpha\beta} + \beta \ln A + \alpha\beta \ln \frac{\alpha\beta A}{1+\alpha\beta}$$

Continuing like this, using the knowledge of geometric series, we find that c, k and V converges to

$$c = (1-\beta\alpha)Ak^\alpha$$

$$k' = \beta\alpha Ak^\alpha$$

$$V(k) = \frac{1}{(1-\beta)}\{\ln A(1-\alpha\beta) + \frac{\beta\alpha}{1-\alpha\beta}\ln(A\beta\alpha)\} + \frac{\alpha}{1-\beta\alpha}\ln k$$

□

We now turn to a special kind of dynamic programming problem. One where the the output function $r(x_t, u_t)$ is quadratic in x and u and the transition law function $f(x_t, u_t)$ is linear. This problem is called the **linear quadratic optimal control**-problem or, **LQ control**-problem for short.

Kapitel 3

Optimal Linear Quadratic Control

3.1 Deterministic LQ-control problem

The above mentioned methods for solving the Bellman equation might be suitable for problems not involving to many inputs. However, when the models applied grow larger so does the computational power needed to solve the dynamic programming problems encountered.

Luckily, many problems in macroeconomics are of - or can at least be made into - a certain sort of dynamic programming problem for which there are simple but powerful methods available to solve. One of these simple but powerful methods is *linear quadratic dynamic programing*, where the return function is quadratic and the transition function is linear. Problems that can be solved using linear quadratic dynamic programming are called linear quadratic optimal regulator problems, or LQ-problems for short. [3]

A special case of the deterministic version of the LQ problem can be described mathematically as follows; find a sequence $\{u_t\}_{t=0}^{\infty}$ that solves the OCP

$$\max_{\{u_t\}_{t=0}^{\infty}} [-\sum_{t=0}^{\infty}\{x_t^T Q x_t + u_t^T R u_t\}] \tag{3.1}$$

subject to

$$x_{t+1} = Ax_t + Bu_t,\ x_0 \text{ given}$$

Where Q is a positive semidefinite symmetric matrix and R is a positive definite symmetric matrix. Moreover the system transition law is presented in the state-space form, with the state-variable vector of size $n \times 1$, the state-transition matrix A of size $n \times n$, the control vector u_t of size $k \times 1$ and the scaling matrix B of size $n \times k$. The Bellman equation for this problem is

$$V(x) = \max_u \{-x_t^T Q x_t - u_t^T R u_t + V(x')\} \tag{3.2}$$

where $x' = Ax + Bu$. Guessing the form of the value function to be $V(x) = -x^T P x$ and using the state space equation

$$x' = Ax + Bu$$

we get

$$-x^T P x = \max_u \{-x_t^T Q x_t - u_t^T R u_t + (Ax + Bu)^T P (Ax + Bu)\} \tag{3.3}$$

The first order necessary condition for (3.3) is

$$(R + B^T P B) u = -B^T P A x$$

$$\Leftrightarrow u = -(R + B^T P B)^{-1} B^T P A x$$

. If we define

$$F = -(R + B^T P B)^{-1} B^T P A$$

we get $u = Fx$ as the state feedback control. Substituting this u in to (3.3) we get the **algebraic matrix Riccati equation (ARE)**

$$P = Q + A^T P A - A^T P B (R + B^T P B)^{-1} B^T P A.$$

[3]

By solving this equation for P we obtain the optimal state feedback control, hence, solving this equation is the very core of the LQ-problem. The equation is very difficult to solve by hand, luckily, there are computers that, in many cases, can solve it for us.

3.1.1 Definition The pair of matrices (A,B) is said to be **stabilizable** if there exists a matrix F for which $(A - BF)$ is a stable matrix.

3.1.2 Theorem If (A,B) is stabilizable and R is positive definite an symmetric, Q is positive semidefinite and symmetric, then under the optimal rule $(A - BF)$ is a stable matrix and the following holds. Consider the sequence of matrices $\{P_j\}_0^\infty$ given by the formula

$$P_{j+1} = Q + A' P_j A - A' P_j B (R + B' P_j B)^{-1} B' P_j A, \quad P_0 = 0.$$

The solution, P, to the ARE

$$P = Q + A^{'}PA - A^{'}PB(R + B^{'}PB)^{-1}B^{'}PA$$

is given by $P = \lim_{j\to\infty} P_j$

Proving this is beyond the scope of this paper, however a proof can be found in theorem 5.1, chapter 3. in [15]

There is a more general formulation of the LQ-regulator problem where the loss function also accepts cross-terms between state-variables and control variables. This, generalized problem, can be expressed as finding $\{u_t\}_t^\infty = 0\}$ that solves the OCP

$$\max_{\{u_t\}_{t=0}^\infty} [-\sum_{t=0}^{\infty}\{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\}] \tag{3.4}$$

subject to

$$x_{t+1} = Ax_t + Bu_t,\ x_0 \text{ given}$$

where the matrix S is of dimensions $1 \times k$. To derive the corresponding Riccati equation to this problem an algebraic approach will be used. To proceed, we will need the following two lemma

3.1.4 Lemma Let matrix P be symmetric. For every control u_t that forces the system $x_{t+1} = Ax_t + Bu_t \to 0$ as $t \to \infty$ we have that

$$\sum_{t=0}^{\infty} x_t^T(A^TPA + P)x_t + 2u_t^TB^TPAx_t + u_t^TB^TPBu_t = -x_0^TPx_0 \tag{3.5}$$

Proof We will use the equality

$$x_{t+1}^TPx_{t+1} = x_t^T(A^TPA - P)x_t + 2u_t^TB^TPAx_t + u_t^TB^TPBu_t$$

which follows from simple insertion of $x_{t+1} = Ax_t + Bu_t$. We have that

$$\sum_{t=0}^{N-1} x_t^T(A^TPA-P)x_t+2u_t^TB^TPAx_t+u_t^TB^TPBu_t = \sum_{t=0}^{N-1} x_{t+1}^TPx_{t+1}-x_t^TPx_t =$$

$$= x_1^TPx_1 - x_0^TPx_t + x_2^TPx_2 - x_1^TPx_1 + ... + x_N^TPx_N - x_{N-1}^TPx_{N-1}$$

$$\Leftrightarrow -x_0^T P x_0 + x_N^T P x_N = \sum_{t=0}^{N-1} x_t^T (A^T PA - P) x_t + 2u_t^T B^T PA x_t + u_t^T B^T PB u_t$$

If we let $N \to \infty$ then $t \to \infty$ and so $x_t \to 0$ from which (3.5) follows $\square$

3.1.5 Theorem If symmetric matrix P satisfies the discrete-time algebraic Riccati equation (DARE)

$$P = Q + A^T PA - (A^T PB + S)(R + B^T PB)^{-1}(S^T + B^T PA)$$

then $u_t = -(R + B^T PB)^{-1}(S^T + B^T PA)$ solves the problem (3.4).

Proof We have

$$x_0^T P x_0 + \sum_{t=0}^{\infty} x_t^T (-Q + A^T PA - (A^T PB + S)(R + B^T PB)^{-1}(S^T + B^T PA)) x_t +$$

$$2u_t^T B^T PA x_t + u_t^T B^T PB u_t = 0$$

according to lemma 3.1.3. This in turn implies that

$$\sum_{t=0}^{\infty} \{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\} = \sum_{t=0}^{\infty} \{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\} +$$

$$x_0^T P x_0 + \sum_{t=0}^{\infty} x_t^T (-Q + A^T PA - (A^T PB + S)(R + B^T PB)^{-1}(S^T + B^T PA)) x_t +$$

$$2u_t^T B^T PA x_t + u_t^T B^T PB u_t$$

After canceling out the $x_t^T Q x_t$-term and rearangeing we get

$$\sum_{t=0}^{\infty} \{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\} = x_0^T P x_0 +$$

$$+ \sum_{t=0}^{\infty} \{x_t^T ((A^T PB + S)(R + B^T PB)^{-1}(B^T PA + S^T)) x_t + u_t^T (R + B^T PB) u_t +$$

$$+ 2u_t^T (S^T + B^T PA) x_t\}$$

Now we use the "completing the square"-method; $(x+y)^T A(x+y) = x^T Ax + 2x^T Ay + y^T Ay$ and obtain

$$\sum_{t=0}^{\infty} \{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\} = x_0^T P x_0 + \sum_{t=0}^{\infty} (u_t + (R + B^T PB)^{-1}(B^T PA + S^T) x_t)^T \cdot$$

$$(R + B^T PB)(u_t + (R + B^T PB)^{-1}(B^T PA + S^T)x_t)$$

It's obvious that this attains it's minimum when

$$u_t + (R + B^T PB)^{-1}(B^T PA + S^T)x_t = 0 \Leftrightarrow$$

$$u_t = -(R + B^T PB)^{-1}(B^T PA + S^T)x_t$$

□

We immediately conclude that (3.1) is the special case of the above where $S = 0$

3.2 3.2 Stochastic LQ optimal control problem

Introducing an element of probability is, as we shall see below, no problem for the LQ-framework. The reason for the smooth transition from solving non-stochastic LQ-problems to solving their stochastic counterparts is at the center of the discussion of this section. This smooth transition is due to the very useful principle of **Certainty equivalence** which says that the optimal feedback law derived in the non-stochastic LQ-problem is unaffected by the introduction of certain stochastic elements. This means that to solve for the optimal feedback law in the stochastic LQ-problem we only have to solve the non-stochastic problem. Noteworthy is the fact that this easy transition from solving the deterministic LQ-problem to solving the stochastic version is only possible under assumptions that sometimes limit the scope of the solution method, nevertheless the certainty equivalence principle has proven to be very useful. The following proof is from Ljungqvist and Sargent (2004) [3]

3.2.1 Theorem Consider the OCP of maximizing

$$-E_0 \sum_{t=0}^{\infty} \beta^t \{x_t^T Q x_t + u_t^T R u_t\}, \quad 0 < \beta < 1$$

subject to the stochastic dynamical system transition law

$$x_{t+1} = Ax_t + Bu_t + \epsilon_{t+1}, \quad x_0 \text{ given}$$

.

Given that the matrices R, Q, A, B fulfill the same assumptions as in the non-stochastic LQ-problem, and that the $n \times 1$ vector of stochastic shock variables

ϵ_{t+1} are i.i normal distributed with mean zero and the $n \times n$ identity matrix as covariance matrix, $E\epsilon^T\epsilon$, the optimal feedback law for this problem will be the same as in the non-stochastic version, namely

$$F = -(R + B^T PB)^{-1} B^T PA$$

$$u_t = Fx_t$$

Proof The Bellman equation of the above problem is

$$V(x_t) = -\max_{u_t}[x_t^T Qx_t + u_t^T Ru_t + \beta EV(Ax_t + Bu_t + \epsilon_t)]$$

We continue by using the method of guessing the form of the value function to be

$$V(x_t) = x_t^T Px_t + d.$$

Substituting this guess in to the Bellman equation and get

$$V(x_t) = x_t^T Px_t + d = -\max_{u_t}\{x_t^T Qx_t + u_t^T Ru_t + \beta E[(Ax_t + Bu_t + \epsilon)^T P(Ax_t + Bu_t + \epsilon_t)] + \beta d\}$$

Proceeding we get

$$V(x_t) = -\max_{u_t}\left[x_t^T Qx_t + u_t^T Ru_t + \beta E\begin{pmatrix} x^T A^T PAx+ \\ + \\ x^T A^T PBu + x^T A^T P\epsilon+ \\ + \\ u^T B^T PAx + u^T B^T PBu + u^T B^T P\epsilon+ \\ + \\ +\epsilon^T PAx + \epsilon^T PBu + \epsilon^T P\epsilon \end{pmatrix} + \beta d\right]$$

The assumption that $E[\epsilon_{t+1}] = 0$ reduces this equation to

$$V(x_t) = x_t' Px_t + d - \max_{u_t}\left[x_t' Qx_t + u_t' Ru_t + \beta E\begin{pmatrix} x^T A^T PAx + x^T A^T PBu+ \\ u^T B^T PAx + u^T B^T PBu \end{pmatrix} + \beta d\right]$$

The first order condition of this is of course

$$2Ru + 2\beta B' PAx + 2\beta B' PBu = 0 \Leftrightarrow$$

$$u = -(R + B^{'} PB)^{-1} B^{'} PAx.$$

So if the optimal feedback law is not changed, then what is the corresponding Riccati equation and the matrix d? By substituting the optimal feedback law in to the expression for the value function we arrive at the two expressions

$$d = \frac{\beta}{1-\beta} tr(PE[\epsilon^T \epsilon])$$

$$P = Q + \beta A' PA - \beta^2 A^T PB(R + \beta B^T PB)^{-1} B^T PA$$

Note that the ARE of this problem is identical to the ARE go the deterministic discounted LQ-problem. Under the assumptions made the solution of P can of course be obtained by iteration of the equation

$$P_{j+1} = Q + \beta A^T P_j A - \beta^2 A^T P_j B(R + \beta B^T P_j B)^{-1} B^T P_j A$$

3.3 3.3 Setting the interest rate

In this section the methods described above will be applied to solve the Central Bank's (CB) problem of setting the interest rate to minimize social cost. The relevant variables in this section are the difference between the inflation rate and the target inflation rate π_t, output gap y_t and the *change* in nominal interest rate i_t.

The goal of the policy maker (in this case, the CB) is to minimize the social cost of inflation and output-gap by using the change of the interest rate, i_t, as a control variable. We assume that the CB wants a smooth"use of the interest rate tool and therefore also consider drastic changes in the interest rate costly. To summarize, the CB has the following single-period cost function

$$L_t = \bar{\pi}_t^2 + \lambda y_t^2 + \nu(i_t - i_{t-1})^2$$

The coefficients $\lambda \geq 0$ and $\nu \geq 0$ denote the relative weights put on inflation rate deviation from target, output-gap, and changes in the nominal interest rate respectively. In the case where $\lambda = \nu = 0$ the central bank is said to follow a *strict inflation targeting policy*, otherwise the central bank is said to follow a *flexible* inflation targeting rule. For instance, the Swedish central bank, Sveriges Riksbank, follows a flexible inflation policy which in practice means that not only deviations of inflation fram target but also deviations in output from target is counted as a loss.

The model of the economy that will be used is the model advanced by Svensson and Rudebusch (1998) [4]. In their paper they examine a set of policy rules that are consistent with inflation targeting in a small macroeconomic model of the US economy. In this text, their results will mainly be used to illustrate an area of application of the linear quadratic regulator techniques presented in the previous sections. The material in this section has benefited tremendously from a mail correspondence between myself and Prof. Svensson. That said all the mistakes and potential representations errors are of course my own.

In the Svensson-Rudebusch model the inflation rate in period $t+1$ depends on the inflation rate in the four previous periods, the output gap in period t and on an exogenous stochastic shock variable $\eta_{\pi:t+1}$ of the kind discussed in the above section on stochastic LQ-problems. Further the output gap in period $t+1$ depends on the output gap in the two previous periods and on the difference between the average nominal interest rate and the average inflation rate in the past four periods. In addition, the output gap also depends on a stochastic shock variable $\eta_{y:t+1}$, with the same characteristics as $\eta_{\pi:t+1}$. Hence the two equations, together with coefficients estimates made by Svensson and Rudebusch based on US data, can be summarized as

$$\pi_{t+1} = 0.7\pi_t + 0.1\pi_{t-1} + 0.28\pi_{t-2} + 0.12\pi_{t-3} + 0.14y_t + \eta_{\pi:t+1} \tag{3.6}$$

$$y_{t+1} = 1.16y_t - 0.25y_{t-1} - 0.1(\bar{i}_t - \bar{\pi}_t) + \eta_{y:t+1}. \tag{3.7}$$

.

Remember that we have $\bar{\pi}_t = \frac{1}{4}\sum_{j=0}^{3}\pi_{t-j}$ and $\bar{i}_t = \frac{1}{4}\sum_{j=0}^{3} i_{t-j}$ as the average inflation and nominal interest rate respectively. From the above equations we conclude that firstly, inflation is persistent and depends on past inflation. Secondly, that output gap is not as persistent and that it depends negatively on the real interest rate ($r_t = i_t - \pi_t$).

The above model fits perfectly in to the stochastic LQ-problem framework that was established above. We can rewrite the model in state space-form and thus the problem translates to

$$\max_{u_t}[-E\sum_{t=0}^{\infty}\{x_t^T Q x_t + u_t^T R u_t + 2x_t^T S u_t\}], \quad 0 < \beta < 1$$

subject to

$$x_{t+1} = Ax_t + Bi_t + \eta_{t+1}$$

where

$$x_t = \begin{pmatrix} \pi_t \\ \pi_{t-1} \\ \pi_{t-2} \\ \pi_{t-3} \\ y_t \\ y_{t-1} \\ i_{t-1} \\ i_{t-2} \\ i_{t-3} \end{pmatrix}, A = \begin{pmatrix} 0.7 & 0.1 & 0.28 & 0.12 & 0.14 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.025 & 0.025 & 0.025 & 0.025 & 1.16 & -0.25 & -0.025 & -0.025 & -0.025 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.025 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \eta_t = \begin{pmatrix} \eta_{\pi:t} \\ 0 \\ 0 \\ 0 \\ \eta_{y:t} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ Q = \begin{pmatrix} 1/16 & 1/16 & 1/16 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 1/16 & 1/16 & 1/16 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 1/16 & 1/16 & 1/16 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 1/16 & 1/16 & 1/16 & 1/16 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\nu \\ 0 \\ 0 \end{pmatrix} \text{ and } R = \nu$$

According the theory presented in the previous section, the optimal feedback rule will take the form $u_t = Fx_t$ where $F = -(Q + B^T PB)^{-1}(B^T PA + S^T)$ where the matrix P is the limit of the sequence

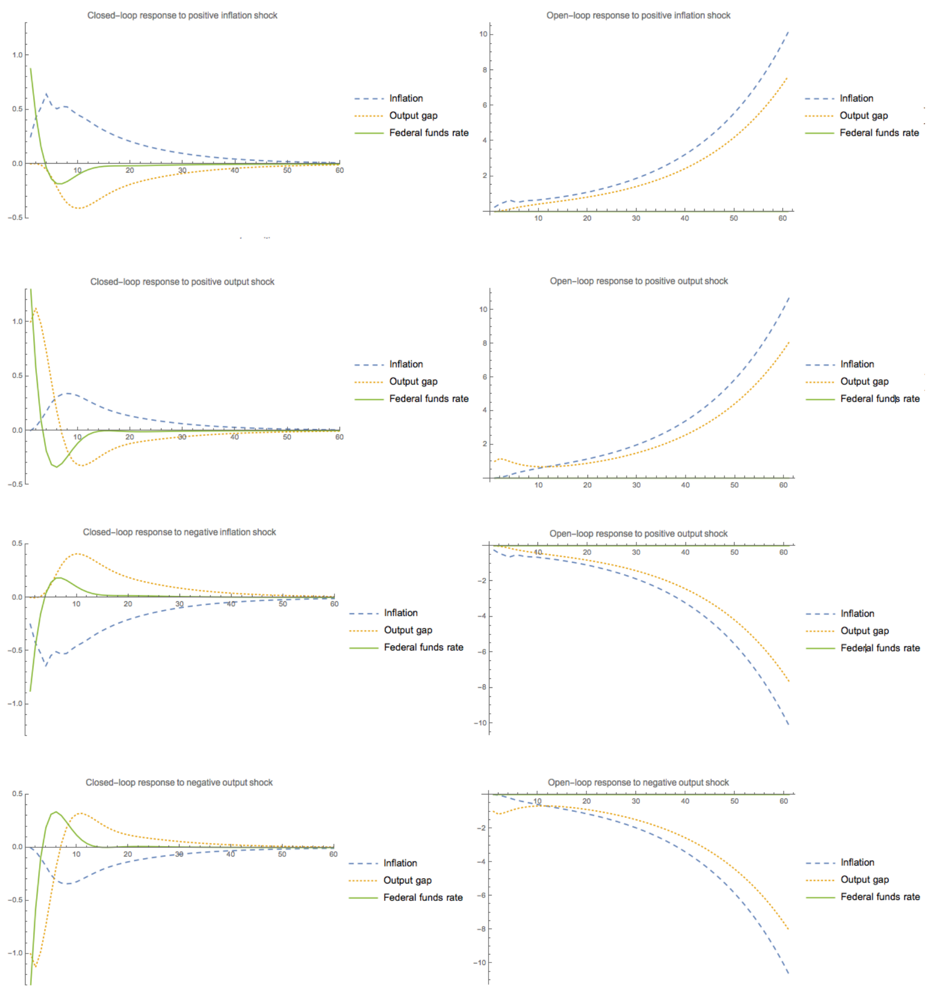
$$P_{j+1} = R + A^T P_j A - A^T P_j B(Q + \beta B^T P_j B)^{-1}(B^T P_j A + S^T)$$

Because this is a 9-dimensional system the Riccati-equation cannot be solved by hand, computer based methods are used. The intricacies of these methods are presented in the appendix.

For instance, let $\lambda = 1$ and $\nu = 0.5$ then, after 39 steps, the sequence $\{P_j\}$ corresponding to these parameters has converged with a 15 decimal precision. The optimal feedback matrix is given by

$$F = -(0.8780, 0.3032, 0.3799, 0.1299, 1.3294, -0.3312, 0.4679, -0.0652, -0.0331).$$

Bellow follows a graphical comparison between the open-loop and closed-loop responses to shocks in inflation and output.



Kapitel 4

Matlab script appendix

```
% Appendix to 3.1 Setting the interest rate
% Purpose: In this script, the procedure for obtaining the solution to the
% Riccati equation by the itereation method is presented.
%I begin by defining the relevant matrices, A,B,R,Q.
A=[0.7 -0.1 0.28 0.12 0.14 0 0 0 0;
   1 0 0 0 0 0 0 0 0
   0 1 0 0 0 0 0 0 0
   0 0 1 0 0 0 0 0 0
   0.025 0.025 0.025 0.025 1.16 -0.25 -0.025 -0.025 -0.025
    0 0 0 0 1 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 1 0 0
    0 0 0 0 0 0 0 1 0];
B=[0; 0; 0; 0; -0.025; 0; 1; 0; 0];
 Q=[1/16 1/16 1/16 1/16 0 0 0 0 0;
   1/16 1/16 1/16 1/16 0 0 0 0 0
   1/16 1/16 1/16 1/16 0 0 0 0 0
   1/16 1/16 1/16 1/16 0 0 0 0 0
   0 0 0 0 1 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0.5 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0];
R=0.5;
S=[0; 0; 0; 0; 0; 0; -0.5; 0; 0];
rank(ctrb(A,B));        % checking the rank of the controlability matrix.
                        % Since the rank=9 as is the dimension of the
                        % system, the system is controlable.
% I begin the Riccati equation iteration by selecting beta and an initial
% value for P, P=0. The iteration continues until the difference between
```

```
% the largest elements of two consecutive P's in the sequence
% of matrices is at most 0.0000000000001.
beta=1;
dif=1;
i=1;
 P0=[0 0 0 0 0 0 0 0 0;
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0
    0 0 0 0 0 0 0 0 0];
while dif>0.0000000000001
    F0=-inv(R+B'*P0*B)*(S'+B'*P0*A);
    nextP=Q+A'*P0*A+(A'*P0*B+S)*F0;
    Pdif=nextP-P0;
    dif=max(abs(Pdif));
    dif=max(dif');
    i=i+1;
    P0=nextP;
end;
% The solution to the Riccati equation is given by P and thus, the feedback
% matrix is given by F.
P0;
F0;
```

.

Kapitel 5

Mathematica script appendix

I begin by defining the relevant matrices, A, B, Q, R, S, with the help of the matrices Cx and Ci. I also define the parameters lambda and nu, defined in the text. The system ssm is also created.

$A = \{\{0.7, -0.1, 0.28, 0.12, 0.14, 0, 0, 0, 0\},$

$\{1, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0\},$

$\{0, 0, 1, 0, 0, 0, 0, 0, 0\},$

$\{0.025, 0.025, 0.025, 0.025, 1.16, -0.25, -0.025, -0.025,$

$-0.025\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 1, 0, 0\},$

$\{0, 0, 0, 0, 0, 0, 0, 1, 0\}\};$

$\text{MatrixForm}[A];$

$B = \{\{0\}, \{0\}, \{0\}, \{0\}, \{-0.025\}, \{0\}, \{1\}, \{0\}, \{0\}\};$

$\text{MatrixForm}[B];$

$\text{lambda} = 1;$

$\text{ny} = 0.5;$

$K = \{\{1, 0, 0\}, \{0, \text{lambda}, 0\}, \{0, 0, \text{ny}\}\};$

$\mathrm{Cx} = \{\{0.25, 0.25, 0.25, 0.25, 0, 0, 0, 0, 0\}, \{0,$

$0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, -1, 0, 0\}\};$

$\mathrm{MatrixForm}[\mathrm{Cx}];$

$\mathrm{Ci} = \{\{0\}, \{0\}, \{1\}\};$

$\mathrm{MatrixForm}[\mathrm{Ci}];$

$\mathrm{ssm} = \mathrm{StateSpaceModel}[\{A, B, \mathrm{Cx}, \mathrm{Ci}\}, \mathrm{SamplingPeriod} \to 0.5];$

$Q = \mathrm{Transpose}[\mathrm{Cx}].K.\mathrm{Cx};$

$\mathrm{MatrixForm}[Q];$

$R = \mathrm{Transpose}[\mathrm{Ci}].K.\mathrm{Ci};$

$S = \mathrm{Transpose}[\mathrm{Cx}].K.\mathrm{Ci};$

$\mathrm{MatrixForm}[S];$

I use the LQRegulatorgains to obtain the optimal gain matrix F. Note that F in this script is -F in the PDF.

$F = \mathrm{LQRegulatorGains}[\mathrm{ssm}, \{Q, R, S\}];$

I connect the optimal gain matrix to the system.

$\mathrm{ctrlssm} = \mathrm{SystemsModelStateFeedbackConnect}[\mathrm{ssm}, F];$

A set of impulse response diagrams are created for initial states corresponding to shocks to inflation and output respectively.

$\mathrm{cloop1} = \mathrm{OutputResponse}[\{\mathrm{ctrlssm}, \{1, 0, 0, 0, 0, 0, 0, 0, 0\}\},$

$0, \{t, 0, 60\}];$

$\mathrm{cloop2} = \mathrm{OutputResponse}[\{\mathrm{ctrlssm}, \{0, 0, 0, 0, 1, 0, 0, 0, 0\}\},$

$0, \{t, 0, 60\}];$

$\mathrm{oloop1} = \mathrm{OutputResponse}[\{\mathrm{ssm}, \{1, 0, 0, 0, 0, 0, 0, 0, 0\}\},$

$0, \{t, 0, 30\}];$

$\mathrm{oloop2} = \mathrm{OutputResponse}[\{\mathrm{ssm}, \{0, 0, 0, 0, 1, 0, 0, 0, 0\}\},$

```
0, {t, 0, 30}];
cloop3 = OutputResponse[{ctrlssm, {−1, 0, 0, 0, 0, 0, 0, 0, 0}},
0, {t, 0, 60}];
cloop4 = OutputResponse[{ctrlssm, {0, 0, 0, 0, −1, 0, 0, 0, 0}},
0, {t, 0, 60}];
oloop3 = OutputResponse[{ssm, {−1, 0, 0, 0, 0, 0, 0, 0, 0}},
0, {t, 0, 30}];
oloop4 = OutputResponse[{ssm, {0, 0, 0, 0, −1, 0, 0, 0, 0}},
0, {t, 0, 30}];

{ListPlot[oloop1, PlotRange → All,
PlotStyle → {Dashing[Medium], Dotted, Joined},
Joined → True,
PlotLegends → {“Inflation”, “Output gap”,
“Federal funds rate”},
PlotLabel → “Open-loop response to positive inflation shock”]
ListPlot[cloop1, PlotRange → {{0, 60}, {−0.5, 1.3}},
PlotStyle → {Dashing[Medium], Dotted, Joined},
Joined → True,
PlotLegends → {“Inflation”, “Output gap”,
“Federal funds rate”},
PlotLabel →
“Closed-loop response to positive inflation shock”]};

{ListPlot[oloop2, PlotRange → All,
```

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel → "Open-loop response to positive output shock"]

ListPlot[cloop2, PlotRange → {{0, 60}, {−0.5, 1.3}},

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel → "Closed-loop response to positive output shock"]};

{ListPlot[oloop3, PlotRange → All,

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel → "Open-loop response to positive output shock",

PlotLabel → "Open-loop response to negative inflation shock"]

ListPlot[cloop3, PlotRange → {{0, 60}, {−1.3, 0.5}},

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel →

"Closed-loop response to negative inflation shock"]};

{ListPlot[oloop4, PlotRange → All,

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel → "Open-loop response to negative output shock"]

ListPlot[cloop4, PlotRange → {{0, 60}, {−1.3, 0.5}},

PlotStyle → {Dashing[Medium], Dotted, Joined},

Joined → True,

PlotLegends → {"Inflation", "Output gap",

"Federal funds rate"},

PlotLabel → "Closed-loop response to negative output shock"]};

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