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Ehrhart polynomials of lattice triangles

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Abstract

The Ehrhart polynomial of a lattice polytope counts the number of lattice points on the boundary and the number of lattice points strictly in the interior of dilations of the polytope. In this thesis, we show that there are infinitely many Ehrhart polynomials of lattice polygons which are not the Ehrhart polynomial of any lattice triangle.

Let (b, i) be a given pair of non-negative integers. We give conditions on (b, i) for there to be a lattice triangle with b boundary points and i interior points. For $b + 2(i - 1) = p$, where p is prime, the condition is particularly simple. This gives us a class of (b, i) for which we know there are no lattice triangles. In addition, we conjecture the non-existence of lattice triangles for other large classes of (b, i).

In the course of our work, we develop tools to study the patterns of for which (b, i) there is a lattice triangle.

Acknowledgements

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Contents

1 Introduction

The Ehrhart polynomial of a lattice polytope in $\mathbb{R}^n$ counts the number of lattice points contained inside or on the boundary of dilations of the polytope. In $\mathbb{R}^2$, there is a known relation between the number of boundary points b, the number of interior points i and the area a of a lattice polygon; a result known as Pick's theorem. There is also an inequality limiting how many boundary points a polygon can have given its number of interior points, known as Scott's inequality.

It is easy to show that for all (b, i) satisfying Scott's inequality, a lattice polygon having b boundary points and i interior points can be constructed. Each pair (b, i) corresponds to a certain Ehrhart polynomial, and that Ehrhart polynomial is said to be realized by a lattice polygon having b boundary points and i interior points.

But what if we limit ourselves to lattice triangles? Can all Ehrhart polynomials of lattice polygons, i.e. those who can be realized by a lattice polygon, be realized by a lattice triangle as well? If not, under what conditions can they be? Are there any patterns of when an Ehrhart polynomial of a lattice polygon can be realized by a lattice triangle? This thesis will try to answer these questions.

In doing this, it will at the same time try to answer the question: given two non-negative integers b and i, under what conditions is it possible to construct a lattice triangle having b boundary points and i interior points?

To the thesis supervisor's knowledge, most of these questions have not been extensively investigated before. However, Higashitani [5, Theorem 0.1] has independently proved a result corresponding to Theorem 5.3 below in the more general case of lattice simplices in $\mathbb{R}^n$ with prime normalized volume. Theorem 5.3 is a special case of Higashitani's theorem, but it is proven independently in this thesis.

2 Preliminaries

2.1 Polytopes

Definition 2.1. We say that a polytope in $\mathbb{R}^n$ is a **lattice polytope** if all the vertices of the polytope has integer coordinates.

Definition 2.2. We will denote the set of vertices of a polytope P by $\mathbf{V_P}$.

Definition 2.3. Let P be a lattice polygon in $\mathbb{R}^2$. We will use the following notation:

- $\mathbf{a(P)}$ is the area of P
- $\mathbf{i(P)}$ is the number of lattice points that lies strictly in the interior of P
- $\mathbf{b(P)}$ is the number of lattice points on the boundary of P

2.2 Lattice equivalences

Definition 2.4. Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine, bijective transformation. We say that ϕ is a **lattice equivalence** if it maps $\mathbb{Z}^2$ onto $\mathbb{Z}^2$.

Proposition 2.1. *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a lattice equivalence. Then*

$$\phi(\mathbf{x}) = M\mathbf{x} + \mathbf{c} \tag{1}$$

where M is a 2×2-matrix with integer elements with

$$\det M = \pm 1 \tag{2}$$

and $\mathbf{c} \in \mathbb{Z}^2$.

Proof. A lattice equivalence is by definition an affine transformation, and affine transformations have the form (1).

Let

$$M = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Since ϕ is assumed to map $\mathbb{Z}^2$ to $\mathbb{Z}^2$, we require that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} k_{11} \\ k_{12} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} k_{21} \\ k_{22} \end{pmatrix} \in \mathbb{Z}^2$$

for all $k_{11}, k_{12} \in \mathbb{Z}$. In particular, this means that

$$(k_{11}, k_{12}) = (0,0) \implies c_1 = k_{21} \in \mathbb{Z} \wedge c_2 = k_{22} \in \mathbb{Z}$$

Hence, $\mathbf{c} \in \mathbb{Z}^2$. Furthermore,

$$(k_{11}, k_{12}) = (1,0) \implies a_{11} = k_{21} - c_1 \in \mathbb{Z} \;\wedge\; a_{21} = k_{22} - c_2 \in \mathbb{Z}$$

$$(k_{11}, k_{12}) = (0,1) \implies a_{12} = k_{21} - c_1 \in \mathbb{Z} \;\wedge\; a_{22} = k_{22} - c_2 \in \mathbb{Z}$$

Hence, M has only integer elements.

By definition, a lattice equivalence is a bijection. Hence, ϕ^{-1} exists, which implies that M^{-1} exists. M^{-1} has only integer elements, by the same argument as for M. Since M and M^{-1} have only integer elements, both $\det M$ and $\det M^{-1}$ are integers. But $\det M^{-1} = \frac{1}{\det M}$, which is a contradiction unless $\det M = \pm 1$. Thus we have proved (2). □

Proposition 2.2. *Lattice equivalences maps lattice triangles to lattice triangles.*

Proof. A lattice equivalence is an affine transformation. Affine transformations preserves lines. Therefore, it preserves triangles. In particular, it preserves lattice triangles, since lattice equivalences maps $\mathbb{Z}^2$ to $\mathbb{Z}^2$. □

Corollary 2.3. *The area, the number of interior points and the number of boundary points of a lattice polygon in $\mathbb{R}^2$ are preserved under lattice equivalences.*

Proof. Let $\phi = M\mathbf{x} + \mathbf{c}$ be a lattice equivalence.

Affine transformations maps lines to lines. Let E be one of the edges of a given lattice polygon and let $\mathbf{x_1}$ and $\mathbf{x_2}$ be two lattice points on E such that there are no lattice points between them on E. Then $\phi(\mathbf{x_1})$ and $\phi(\mathbf{x_2})$ are lattice points lying on the same edge of the image of the polygon under the lattice equivalence. Call this edge of the image $\phi(E)$. Assume that there is a point $\phi(\mathbf{x_0}) = \mathbf{y}$ between $\phi(\mathbf{x_1})$ and $\phi(\mathbf{x_2})$ on $\phi(E)$. Then $\phi^{-1}(\mathbf{y}) = \mathbf{x_0}$ is a lattice point between $\mathbf{x_1}$ and $\mathbf{x_2}$ on E. But this is a contradiction, since we assumed that there is no lattice point between $\mathbf{x_1}$ and $\mathbf{x_2}$ on E. Hence, the same number of lattice points lies on $\phi(E)$ and E for an arbitrary edge E of P, which means that ϕ preserves the number of boundary points of the whole polygon.

The area of the polygon is preserved since $\det M = \pm 1$, by Proposition 2.1.

From this it follows that the number of interior points are preserved, by Theorem 2.4, given below. $\square$

2.3 Ehrhart polynomials

The notions in this subsection (i.e. section 2.3) is from [2].

Definition 2.5. Let P be a lattice polytope in $\mathbb{R}^n$ with vertex set $V_P = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}\}$. The **t-dilation** of P, denoted tP, is the polytope defined by the vertex set

$$V_{tP} = \{(t\mathbf{v_1}, t\mathbf{v_2}, \ldots, t\mathbf{v_m}) \in \mathbb{R}^n \mid t \in \mathbb{Z}\}$$

Definition 2.6. The **Ehrhart polynomial of a lattice polytope P** (in $\mathbb{R}^n$), denoted $L_P(t)$, is defined by

$$L_P(t) := i(tP) + b(tP)$$

where tP is the t-dilation of P.

Definition 2.7. We say that an Ehrhart polynomial $L_P(t)$ is **realized by the lattice polytope P** (in $\mathbb{R}^n$) if $L_P(t) = i(tP) + b(tP)$ for all integers $t \geq 1$.

If a polynomial $p(t)$ is such that $p(t) = i(tP) + b(tP)$ for some lattice polygon P, then we say that $p(t)$ is **the Ehrhart polynomial of a lattice polygon**. Correspondingly, we say that $p(t)$ is **the Ehrhart polynomial of a lattice triangle** if $p(t) = i(tT) + b(tT)$ for some lattice triangle T.

2.4 Known results

The following known results will be used.

Theorem 2.4 (Pick's theorem [6])**.** *Let P be a lattice polygon. Then*

$$a(P) = i(P) + \frac{b(P)}{2} - 1$$

Theorem 2.5 (Scott's inequality [7])**.** *Let P be a lattice polygon such that $i(P) > 0$. Then*

$$b(P) \leq 2 \cdot i(P) + 7$$

and

$$b(P) = 2 \cdot i(P) + 7 \implies V_P = \{(0,0),(3,0),(0,3)\}$$

In other words, if P is not the lattice triangle with $V_P = \{(0,0),(3,0),(0,3)\}$, then $b(P) \leq 2 \cdot i(P) + 6$.

Theorem 2.6 ([4, pp. 4-5])**.** *Let b and i be non-negative integers. Then there is a lattice polygon P such that $b(P) = b$ and $i(P) = i$ if and only if one of the following conditions is true:*

(i) $i = 0$ *and* $b \geq 3$

(ii) $i = 1$ *and* $b = 9$

(iii) $i \geq 1$ *and* $3 \leq b \leq 2i + 6$

3 The normal form of a triangle with respect to a certain vertex

In this section, we will show that any lattice triangle in $\mathbb{R}^2$ can be transformed to what we will call a normal form, while preserving the number of boundary points, interior points and the area of the triangle. These are precisely the quantities we are interested in, so it will suffice to study lattice triangles which can be the normal form of some lattice triangle.

Definition 3.1. Let T be a lattice triangle. We say that $\mathbf{x} = (x_1, x_2) \in V_T$ is **the lowest left-most vertex of T** if and only if

$$\mathbf{x} \in \{(v_{11}, v_{12}) \in V_T \mid \forall (w_{11}, w_{12}) \in V_T : v_{11} \leq w_{11}\} =: S$$

and

$$x_2 = \min \{v_{12} \mid (v_{11}, v_{12}) \in S\}$$

See Figure 1.

The following lemma is a known fact. For a proof (of a sharper version of the lemma), see for example [1, p. 8].

Lemma 3.1. *Let $m, n \in \mathbb{Z}$. Then*

$$\gcd(m,n) = k \implies \exists a, b \in \mathbb{Z} : am + bn = k$$

and

$$\exists a, b \in \mathbb{Z} : am + bn = k \implies \gcd(m,n) \mid k$$

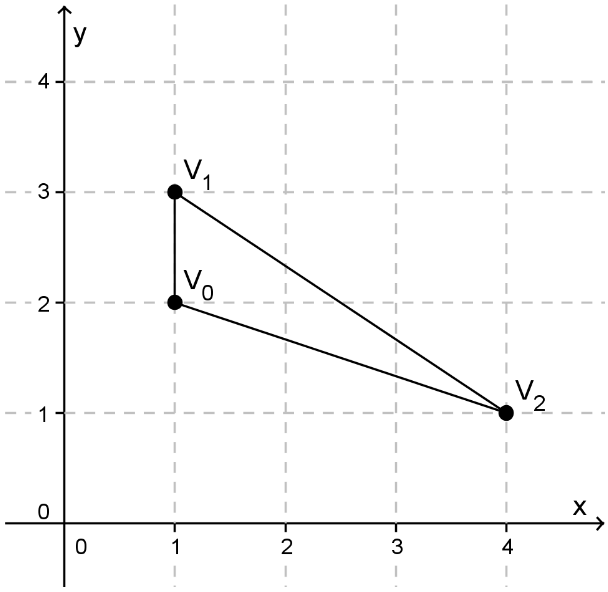

Figure 1: $\mathbf{v_0}$ and $\mathbf{v_1}$ has the same x-coordinate, but the y-coordinate of $\mathbf{v_0}$ is smaller. Hence, $\mathbf{v_0}$ is the lowest left-most vertex of this lattice triangle.

Theorem 3.2. *Let T be a lattice triangle in $\mathbb{R}^2$ with vertex set V_T. Let $\mathbf{x_0}$ be the lowest left-most vertex of T. Consider the two edges between $\mathbf{x_0}$ and one of the other vertices respectively. Let E_1 and E_2 be named in the following manner:*

- *If both edges lies on lines with finite slope: Let E_1 be the bottom edge and let E_2 be the top edge.*
- *If one edge is parallel to the y-axis: Let the edge parallel to the y-axis be called E_2, and let the other edge be called E_1.*

Let the vertex on the other side of E_i be called v_i, for $i = 1, 2$. See Figure 2. Let $\mathbf{x_i}$ be the lattice point closest to $\mathbf{x_0}$ on E_i, for $i = 1, 2$. See Figure 3. Furthermore, let $\phi(\mathbf{x}) = M(\mathbf{x} - \mathbf{x_0})$, where

$$M = \begin{pmatrix} a + k(x_{12} - x_{02}) & b - k(x_{11} - x_{01}) \\ -(x_{12} - x_{02}) & x_{11} - x_{01} \end{pmatrix} \tag{3}$$

where $a, b \in \mathbb{Z}$ is such that

$$a(x_{11} - x_{01}) + b(x_{12} - x_{01}) = 1 \tag{4}$$

and $k \in \mathbb{Z}$ is such that

$$a(x_{21} - x_{01}) + b(x_{22} - x_{02}) = (-(x_{12} - x_{02})(x_{21} - x_{01}) + (x_{11} - x_{01})(x_{22} - x_{02}))\, k + p \tag{5}$$

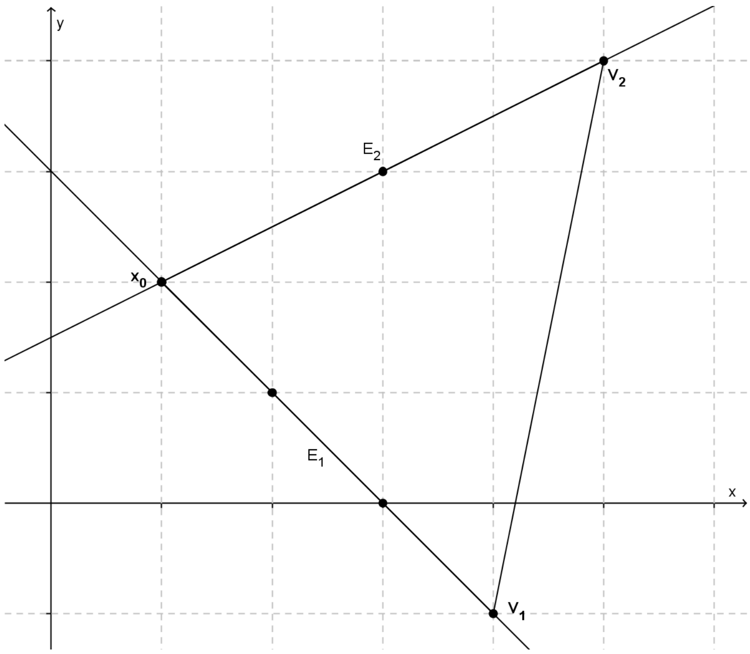

Figure 2: Naming the edges and vertices of a lattice triangle.

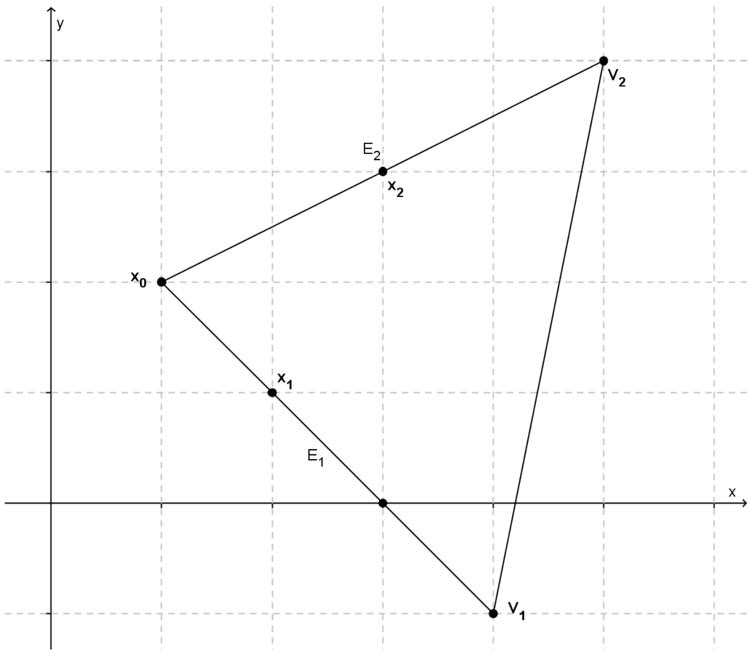

Figure 3: Naming the lattice points of a lattice triangle.

for a unique $p \in \mathbb{Z} : 0 \le p < -(x_{12} - x_{02})(x_{21} - x_{01}) + (x_{11} - x_{01})(x_{22} - x_{02})$.

Then ϕ *is the unique orientation preserving lattice equivalence* $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfying*

$$\phi(\mathbf{x_0}) = \mathbf{0} \tag{6}$$

$$\phi(\mathbf{x_1}) = (1, 0)^t \tag{7}$$

$$\phi(\mathbf{x_2}) = (p, q)^t \tag{8}$$

where $\gcd(p, q) = 1$ *and* $0 \le p < q$*. Moreover,* a*,* b *and* k *satisfying* (4) *and* (5) *can always be found.*

Proof. Let $\phi(\mathbf{x}) = M\mathbf{x} + \mathbf{c}$, where $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$, be a lattice equivalence satisfying (6), (7) and (8).

Then

$$m_{11}(x_{11} - x_{01}) + m_{12}(x_{12} - x_{02}) = 1 \tag{9}$$

$$m_{21}(x_{11} - x_{01}) + m_{22}(x_{12} - x_{02}) = 0 \tag{10}$$

$$m_{11}(x_{21} - x_{01}) + m_{12}(x_{22} - x_{02}) = p \tag{11}$$

$$m_{21}(x_{21} - x_{01}) + m_{22}(x_{22} - x_{02}) = q \tag{12}$$

$$\det M = m_{11}m_{22} - m_{12}m_{21} = 1 \tag{13}$$

where $0 \le p < q$ and $\gcd(p, q) = 1$.

We will show that (9)-(13) determines a unique M.

First of all, note that $x_{11} - x_{01}$ and $x_{12} - x_{02}$ can not both be zero, since $\mathbf{x_1} \ne \mathbf{x_0}$. This means that $\gcd(x_{11} - x_{01}, x_{12} - x_{02})$ always exists. We can therefore assume that $\gcd(x_{11} - x_{01}, x_{12} - x_{02}) = d > 1$. Then $x_{11} - x_{01} = dk_1$ and $x_{12} - x_{02} = dk_2$, for some $k_1, k_2 \in \mathbb{Z}$. This means that (k_1, k_2) is a lattice point on E_1 between $\mathbf{x_0}$ and $\mathbf{x_1}$. But $\mathbf{x_1}$ was chosen to be the lattice point closest to $\mathbf{x_0}$ on E_1. Hence

$$\gcd(x_{11} - x_{01}, x_{12} - x_{02}) = 1 \tag{14}$$

by contradiction. This implies that

$$\exists a, b \in \mathbb{Z} : a(x_{11} - x_{01}) + b(x_{12} - x_{02}) = 1 \tag{15}$$

by Lemma 3.1. In other words, (a, b) is a particular solution to the diophantine equation (9). Therefore, (9) has the general solution

$$(m_{11}, m_{12}) = (a + k(x_{12} - x_{02}), b - k(x_{11} - x_{01}))$$

for $k \in \mathbb{Z}$.

At least one of $x_{11} - x_{01}$ and $x_{12} - x_{02}$ is not equal to zero, since $\mathbf{x_1} \ne \mathbf{x_0}$. Let us assume that $x_{11} - x_{01} \ne 0$. Then (10) gives

$$m_{21} = -m_{22}\frac{x_{12} - x_{02}}{x_{11} - x_{01}}$$

which implies that

$$\exists k_3 \in \mathbb{Z} : m_{22} = k_3(x_{11} - x_{01})$$

by (14) and $m_{21} \in \mathbb{Z}$. This gives

$$m_{21} = -k_3(x_{12} - x_{02})$$

So far, we have

$$M = \begin{pmatrix} a + k(x_{12} - x_{02}) & b - k(x_{11} - x_{01}) \\ -k_3(x_{12} - x_{02}) & k_3(x_{11} - x_{01}) \end{pmatrix}$$

where a and b are known, while k and k_3 are so far unknown, integers. But it is easy to find k_3, since

$$\det M = k_3(a(x_{11} - x_{01}) + b(x_{12} - x_{02})) = k_3 = 1$$

by (13) and (15). We get

$$M = \begin{pmatrix} a + k(x_{12} - x_{02}) & b - k(x_{11} - x_{01}) \\ -(x_{12} - x_{02}) & x_{11} - x_{01} \end{pmatrix}$$

It remains to find k. Now,

$$a(x_{21} - x_{01}) + b(x_{22} - x_{02}) + k((x_{12} - x_{02})(x_{21} - x_{01}) - (x_{11} - x_{01})(x_{22} - x_{02})) = p$$

by (11). Set

$$q = (x_{11} - x_{01})(x_{22} - x_{02}) - (x_{12} - x_{02})(x_{21} - x_{01}) \tag{16}$$

We require that

$$q = (x_{11} - x_{01})(x_{22} - x_{02}) - (x_{12} - x_{02})(x_{21} - x_{01}) > 0$$

which is equivalent to

$$x_{22} - x_{02} > \frac{x_{12} - x_{02}}{x_{11} - x_{01}}(x_{21} - x_{01}) \tag{17}$$

where we have used that $x_{11} - x_{01} > 0$, by the way $\mathbf{x_1}$ and $\mathbf{x_0}$ are chosen. Furthermore, $x_{21} - x_{01} \geq 0$ by the way $\mathbf{x_2}$ and $\mathbf{x_0}$ are chosen. We have two cases.

- Assume that $x_{21} - x_{01} = 0$. Then $x_{22} - x_{02} = 1$, since we immediately hit a lattice point when moving along the y-axis. Then we require that $1 > 0$, by (17), which certainly is true.
- Assume that $x_{21} - x_{01} > 0$. Then we require that

  $$\frac{x_{22} - x_{02}}{x_{21} - x_{01}} > \frac{x_{12} - x_{02}}{x_{11} - x_{01}}$$

  In other words, we require that the slope of the line on which E_1 lies is smaller than the slope of the line on which E_2 lies. This is always true, by the way E_1 and E_2 are chosen.

Hence $q > 0$. Furthermore, we must show that $0 \leq p < q$ and $\gcd(p, q) = 1$. We have that

$$a(x_{21} - x_{01}) + b(x_{22} - x_{02}) = qk + p$$

By the division algorithm, k can be chosen such that $0 \leq p < q$, and this choice is unique. Also,

$$\begin{gathered} \mu_1 p + \mu_2 q \\ = [\mu_1 a + \mu_2(x_{12} - x_{02})]\,(x_{21} - x_{01}) + [\mu_1 b - \mu_2(x_{11} - x_{01})]\,(x_{22} - x_{02}) \\ = \lambda_1(x_{21} - x_{01}) + \lambda_2(x_{22} - x_{02}) = 1 \end{gathered}$$

for some $\lambda_1, \lambda_2 \in \mathbb{Z}$, since $\gcd(x_{21} - x_{01}, x_{22} - x_{02}) = 1$ by the same argument as for (14). But then

$$\mu_1 = \lambda_1(x_{11} - x_{01}) + \lambda_2(x_{12} - x_{02}) \in \mathbb{Z}$$

$$\mu_2 = \lambda_1 a - \lambda_2 b \in \mathbb{Z}$$

which implies that

$$\gcd(p, q) = 1$$

by Lemma 3.1.

We have shown that if $\phi(\mathbf{x}) = M(\mathbf{x} - \mathbf{x_0})$ is a lattice equivalence satisfying (6), (7) and (8), then

$$M = \begin{pmatrix} a + k(x_{12} - x_{02}) & b - k(x_{11} - x_{01}) \\ -(x_{12} - x_{02}) & x_{11} - x_{01} \end{pmatrix} \tag{18}$$

where $a, b, k \in \mathbb{Z}$ are chosen according to (4) and (5), and that such a choice is always possible.

It remains to show that M is unique. Assume that $\phi_1(\mathbf{x}) = M_1(\mathbf{x} - \mathbf{x_0})$ and $\phi_2(\mathbf{x}) = M_2(\mathbf{x} - \mathbf{x_0})$ both satisfy (6), (7) and (8). Then, by (18),

$$M_i = \begin{pmatrix} a_i + k_i(x_{22} - x_{02}) & b_i - k_i(x_{11} - x_{01}) \\ -(x_{22} - x_{02}) & x_{11} - x_{01} \end{pmatrix}$$

where $a_i, b_i, k_i \in \mathbb{Z}$ satisfies (4) and (5). Then

$$a_i(x_{11} - x_{01}) + b_i(x_{12} - x_{02}) = 1$$

for $i = 1, 2$. But then $(a_2, b_2) = (a_1 + k_0(x_{12} - x_{02}), b_1 - k_0(x_{11} - x_{01}))$ for some $k_0 \in \mathbb{Z}$, since if (a_1, b_1) is a particular solution to (9), then $(a_1 + k(x_{12} - x_{02}), b_1 - k(x_{11} - x_{01}))$, where $k \in \mathbb{Z}$, is the general solution. Hence

$$M_2 = \begin{pmatrix} a_1 + (k_0 + k_2)(x_{12} - x_{02}) & b_1 - (k_0 + k_2)(x_{11} - x_{01}) \\ -(x_{12} - x_{02}) & x_{11} - x_{01} \end{pmatrix}$$

By (11), k_0, k_1 and k_2 are such that both $m = k_1$ and $m = k_0 + k_2$ satisfies

$$a_1(x_{21} - x_{01}) + b_1(x_{22} - x_{02}) = qm + p$$

where $0 \leq p < q$ and q is defined by (16). But then m is unique, by the division algorithm. Hence, $k_1 = k_0 + k_2$.

This means that $M_1 = M_2$. In other words, $\phi_1 = \phi_2$. Thus we have shown that M is unique. $\square$

Definition 3.2. Let T be a lattice triangle. Pick a vertex $\mathbf{x}$ of T and let ϕ be the unique orientation preserving lattice equivalence corresponding to this $\mathbf{x}$. We say that (A, B, C), such that $\phi(T)$ has the vertices $(0,0),(A,0),(B,C)$, is the **normal form of T with respect to x**. See Figure 4.

We will use **the normal form of T** as shorthand for "the normal form of T with respect to the lowest left-most vertex of T". We will also allow ourselves use the notation $T = (A, B, C)$ for the normal form of T.

Regarding Definition 3.2, note the following:

(i) By this definition, the normal form of a given lattice triangle depends on the choice of vertex. This means that a lattice triangle does not have a unique normal form; rather, it has three possible normal forms. Since a lattice triangle certainly is isomorphic to itself, this means that isomorphic lattice triangles do not in general have the same normal form.

One could define normal form so that it has this property, by defining one of the three images of a lattice triangle under the different lattice equivalences to be *the* normal form of the triangle in question, according to some criterion.

This property is not necessary for our current purposes, however. We are only interested in the number of boundary points, interior points and the area of a lattice triangle. Since these properties are preserved by lattice equivalences, all possible normal forms of a triangle will have the same number of boundary points, interior points, and the same area.

(ii) An alternative way to define a unique normal form for lattice triangles T is to define it in terms of the Hermite normal form [3, Section 2.4.2] of the matrix $W = \begin{pmatrix} v_{01} & v_{11} & v_{21} \\ v_{02} & v_{12} & v_{22} \end{pmatrix}$, where $V_T = \{(v_{01}, v_{02}), (v_{11}, v_{12}), (v_{21}, v_{22})\}$.

Example 3.1. Let T be a lattice triangle with $V_T = \{(6,6),(2,2),(2,4)\}$. The lowest left-most vertex of T is $(2,2)$, so we set $\mathbf{x_0} = (2,2)$. Naming the lattice points as in Theorem 3.2, we set $\mathbf{x_1} = (3,3)$ and $\mathbf{x_2} = (2,3)$. See Figure 5. This gives $\mathbf{x_1} - \mathbf{x_0} = (1,1)$ and $\mathbf{x_2} - \mathbf{x_0} = (0,1)$.

By (4), we must find a particular solution of the diophantine equation $m_{11} \cdot 1 + m_{12} \cdot 1 = 1$. One such solution is $(m_{11}, m_{12}) = (1,0)$. Therefore, the general solution is $(m_{11}, m_{12}) = (1+n, -n)$, where $n \in \mathbb{Z}$.

Now, by (5), we must find $k \in \mathbb{Z}$ such that

$$1 \cdot 0 + 0 \cdot 1 = (-1 \cdot 0 + 1 \cdot 1)k + p$$

$$\Leftrightarrow 0 = k + p$$

where we require that $0 \le p < 1$. The unique solution is $k = 0$, which gives $p = 0$.

Let

$$M = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

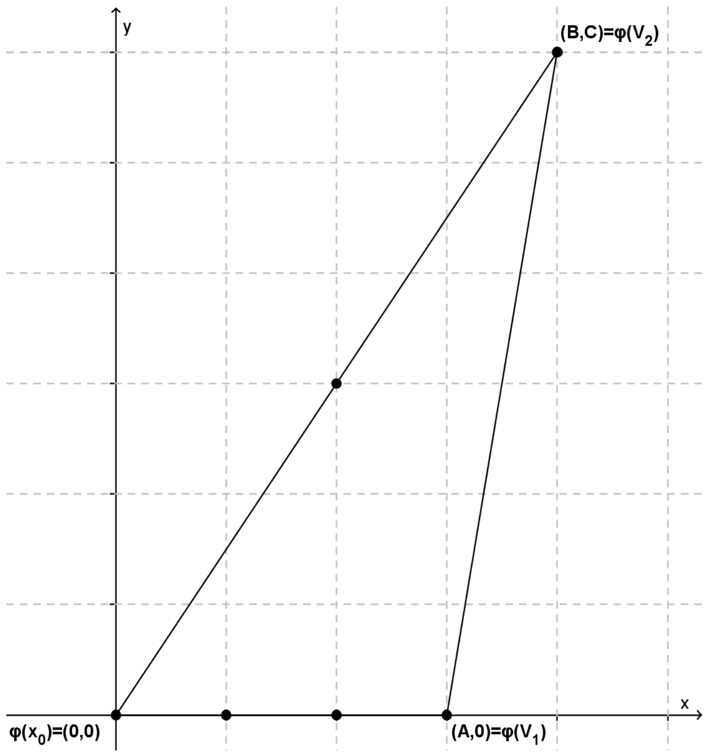

Figure 4: The normal form of a lattice triangle T with respect to the lowest left-most vertex of T. $\mathbf{x_0}, \mathbf{v_1}$ and $\mathbf{v_2}$ named according to Theorem 3.2.

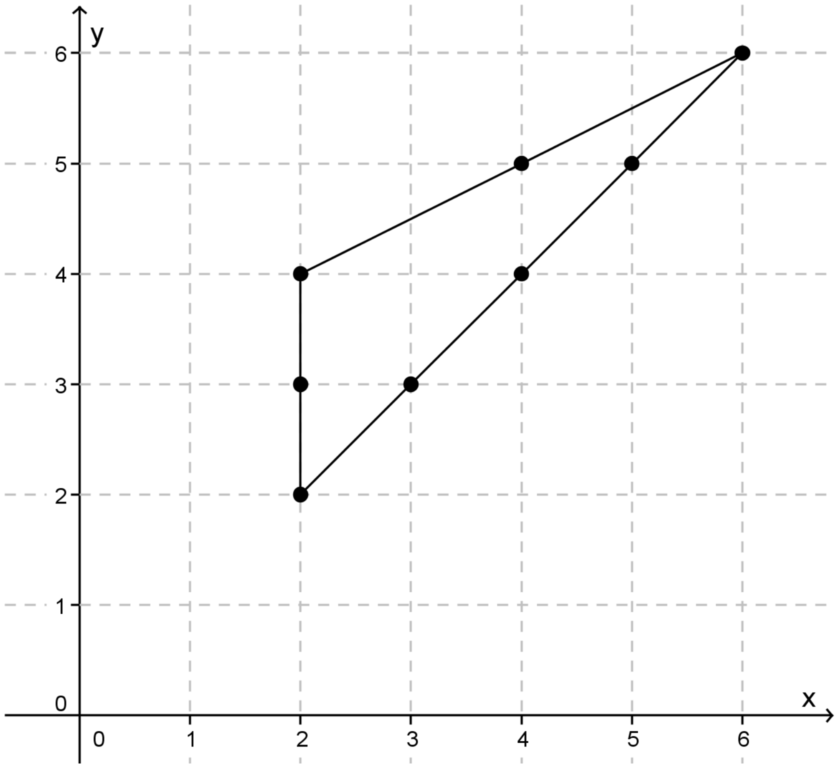

Figure 5: The lattice triangle defined in Example 3.1

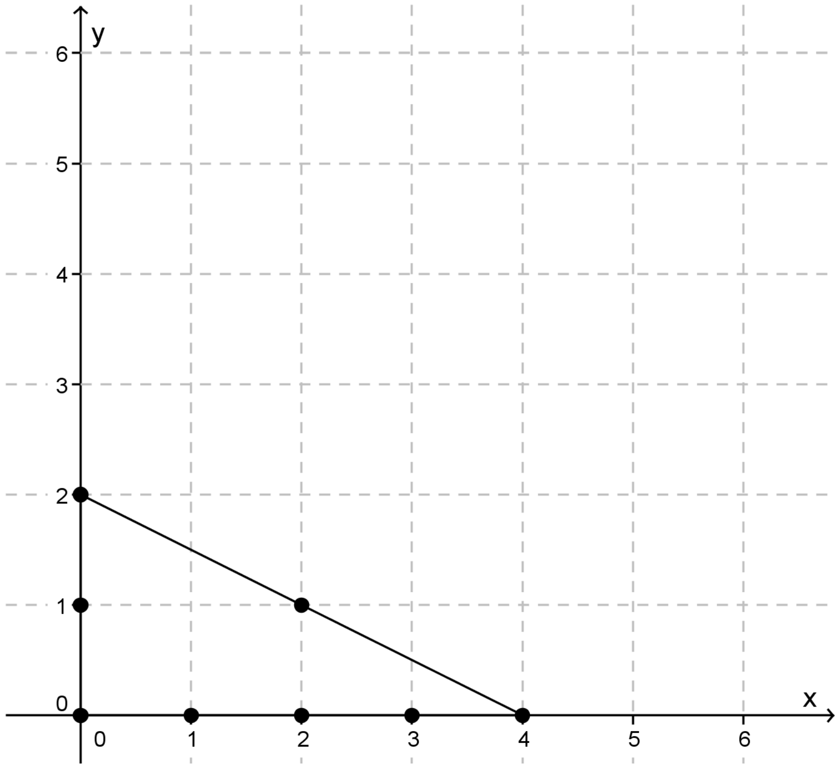

Figure 6: The normal form of the lattice triangle T defined in Example 3.1

Then $\phi(\mathbf{x}) = M(\mathbf{x} - \mathbf{x_0})$ is the unique lattice equivalence satisfying (6), (7) and (8) in Theorem 3.2, given the vertex $\mathbf{x_0}$.

T has the normal form has the normal form $(4, 0, 2)$. See Figure 6.

4 Formulas for the number of boundary points, interior points and area of a lattice triangle

4.1 The formulas

Theorem 4.1. *Let T be a lattice triangle with the normal form (A, B, C). Then*

$$a(T) = \frac{AC}{2} \tag{19}$$

$$b(T) = A + \gcd(B, C) + \gcd(B - A, C) \tag{20}$$

$$i(T) = \frac{AC}{2} - \frac{A + \gcd(B, C) + \gcd(B - A, C)}{2} + 1 \tag{21}$$

Conversely, if $\exists A, B, C$*, with* $A > 0$ *and* $0 \leq B < C$*, then for*

$$a = \frac{AC}{2} \tag{22}$$

$$b = A + \gcd(B, C) + \gcd(B - A, C) \tag{23}$$

$$i = \frac{AC}{2} - \frac{A + \gcd(B, C) + \gcd(B - A, C)}{2} + 1 \tag{24}$$

T *with* $V_T = \{(0, 0), (A, 0), (B, C)\}$ *is a lattice triangle with*

$$a(T) = a$$

$$b(T) = b$$

$$i(T) = i$$

Proof. Let T be a lattice triangle with normal form (A, B, C). By Corollary 2.3, we can assume that T is on its normal form.

A is the length of the base and C is the height of T. Hence, $a(T) = \frac{AC}{2}$. This proves (19).

We will now count the boundary points of T. There is a boundary point of T at the origin; let us count this one separately. Let E_1 be the edge between $(0, 0)$ and $(A, 0)$. Let E_2 be the edge between $(0, 0)$ and (B, C). Let E_3 be the edge between $(A, 0)$ and (B, C). Let b_i be the number of lattice points, excluding the origin, on E_i for $i = 1, 2$. Let b_3 be the number of lattice points, excluding $(A, 0)$ and (B, C) on E_3. Then $b(T) = 1 + b_1 + b_2 + b_3$. Let us compute the b_i.

- Obviously,

$$b_1 = A \tag{25}$$

- Consider the point (B, C). It lies on the line with equation $y = \frac{p}{q}x$ for some $p, q \in \mathbb{Z}$ with $\gcd(p, q) = 1$, by Theorem 3.2. Hence, $C = \frac{p}{q}B$. Since $C \in \mathbb{Z}$, this implies that $q = 1 \lor q|B$. Now,

$$q = 1 \implies C = Bp \implies B|C \implies \gcd(B, C) = B$$

$$\implies (B, C) = \gcd(B, C)(1, p)$$

and

$$q|B \implies B = kq \implies C = pk \implies (B, C) = k(p, q)$$

for some $k \in \mathbb{Z}$. But $\gcd(p, q) = 1$, so $\gcd(B, C) = k$. Hence

$$q|B \implies (B, C) = \gcd(B, C)(p, q)$$

Since the set of lattice points, excluding the origin, on this edge is the set $\{k(p, q) \mid k \in \{1, 2, \dots, \gcd(B, C)\}\}$, the number of lattice points on this edge is $\gcd(B, C)$. Hence,

$$b_2 = \gcd(B, C) \tag{26}$$

- By the same argument as above, the number of lattice points, excluding $(A, 0)$, on the edge from $(A, 0)$ to (B, C) is $\gcd(B - A, C)$. However, we also wish to exclude the point (B, C) on this edge. Hence

$$b_3 = \gcd(B - A, C) - 1 \tag{27}$$

By (25), (26) and (27)

$$\begin{aligned} b(T) &= 1 + A + \gcd(B, C) + \gcd(B - A, C) - 1 \\ &= A + \gcd(B, C) + \gcd(B - A, C) \end{aligned}$$

This proves (20).

Finally, (21) follows from (19), (20) and Theorem 2.4.

Thus we have proved the first part of the theorem.

On the other hand, assume we have a, b and i, and $A, B, C \in \mathbb{Z}$ with $A > 0$, $0 \le B < C$, such that (22), (23) and (24) are satisfied. Then T, with $V_T = \{(0, 0), (A, 0), (B, C)\}$ is a lattice triangle such that $a = a(T), b = b(T)$ and $i = i(T)$. This proves the second part of the theorem. □

4.2 Existence of lattice triangles for a given pair (b,i)

We can use Theorem 4.1 to make a brute force computation of for which pairs (b, i) there exists at least one lattice triangle T such that $b = b(T)$ and $i = i(T)$. Recall that Scott's inequality (Theorem 2.5) puts a bound on b in terms of i. Therefore, for every i, we only have to check the integers b satisfying Scott's inequality. For every i, we run through all integers b satisfying Scott's inequality. Given a pair (b, i), we want to find A, B, C such that $A > 0$, $0 \le B < C$ and (23) and (24) are satisfied. This restricts our choices of A and C, since

$$b + 2(i - 1) = AC$$

i.e. A and C must be integers whose product is $b + 2(i - 1)$. Also, B is bounded by C. Hence, given (b, i), there is a finite number of possible choices of A, B, C. Given (b, i), we can run through all possible choices of A, B, C, looking for a triple satisfying (23) and (24). If we let i run through all integers from 0 to some positive integer n and then make a scatter plot of the (b, i) for which a lattice triangle can be found, we get Figure 7 for $n = 100$ and Figure 8 for $n = 1000$.

4.3 Lines in the (b,i) plane

We note that, by the division algorithm, we can write $A = Ck + r$ for some integers k and r, where $0 \le r < C$. Note that $k \ne 0$ if $r = 0$, since $A > 0$.

The following is a corollary to Theorem 4.1.

Corollary 4.2. *Let T be a triangle with normal form $(Ck + r, B, C)$, where k and r are integers such that $0 \le r < C$ and $k \ge 0$, with $k > 0$ if $r = 0$. Then*

$$i(T) = \frac{C - 1}{2} \cdot b(T) - C \cdot \frac{\gcd(B, C) + \gcd(B - r, C)}{2} + 1 \tag{28}$$

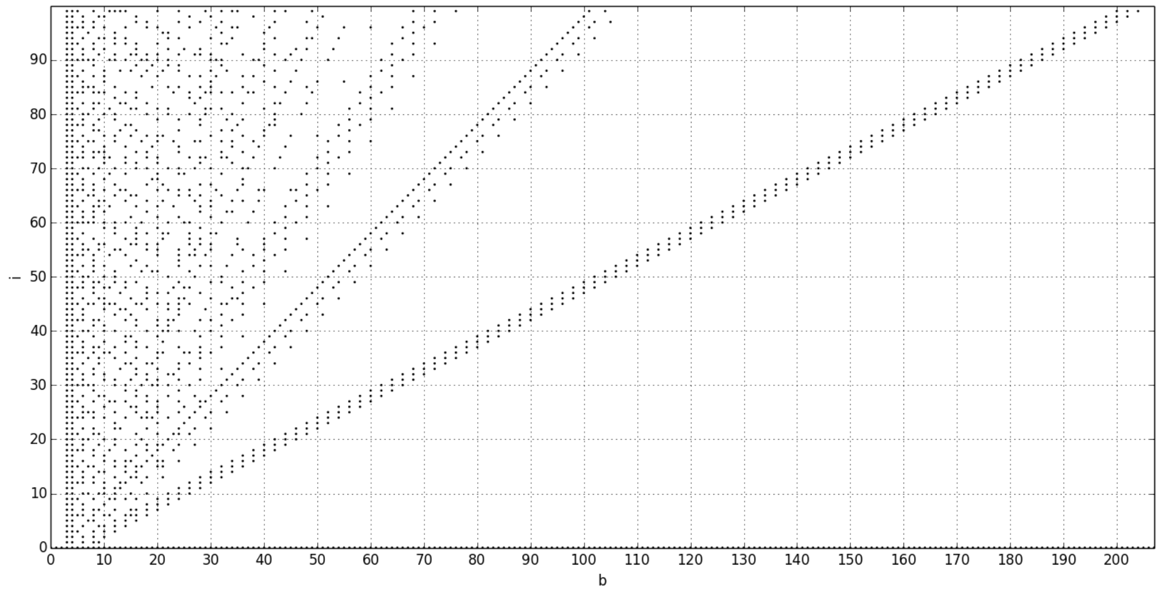

Figure 7: All (b, i) with $0 \leq i \leq 100$ such that there is a lattice triangle T with $b = b(T)$ and $i = i(T)$.

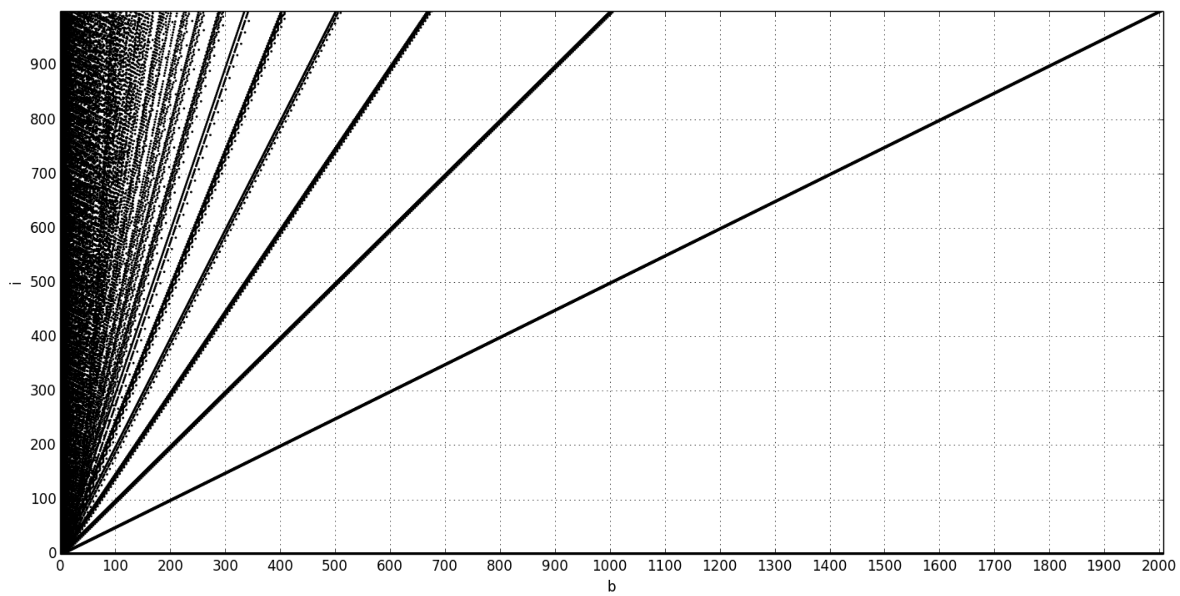

Figure 8: All (b, i) with $0 \leq i \leq 1000$ such that there is a lattice triangle T with $b = b(T)$ and $i = i(T)$.

where

$$b(T) = Ck + r + \gcd(B, C) + \gcd(B - r, C) \tag{29}$$

Proof. We note that

$$\gcd(B - (Ck + r), C) = \gcd(C \cdot (-k) + B - r, C) = \gcd(B - r, C)$$

(20) gives

$$\begin{aligned} b(T) &= Ck + r + \gcd(B, C) + \gcd(B - r, C) \\ \Leftrightarrow k &= \frac{b(T) - r - \gcd(B, C) - \gcd(B - r, C)}{C} \end{aligned} \tag{30}$$

(21) and (30) gives

$$\begin{aligned} i(T) &= \frac{C(Ck + r)}{2} - \frac{Ck + r + \gcd(B, C) + \gcd(B - r, C)}{2} + 1 \\ &= \frac{C - 1}{2} \cdot b(T) - C \cdot \frac{\gcd(B, C) + \gcd(B - r, C)}{2} + 1 \end{aligned}$$

□

Corollary 4.2 means that $C = c$, for some positive integer c, determines a family of parallel lines with the slope $\frac{c-1}{2}$ in the (b, i) plane, and all lattice triangles with normal form (A, B, c), for some $A = ck + r$, with $0 \leq r < c$, and B, lies on one of these lines. The intercept of a particular line is determined by B and r. But $0 \leq B < C$ and $0 \leq r < C$, so there are only finitely many lines for a given C. The set of intercepts for a family of lines can be determined by checking all possible choices of B and r given C. Note that different pairs (B, r) can determine the same intercept.

Note that not all points (b, i) on a certain line are such that there exists a lattice triangle T such that $b = b(T)$ and $i = i(T)$. Given a certain line, B and r are given. On this line, the points for which there is a lattice triangle are exactly those satisfying

$$b = Ck + r + \gcd(B, C) + \gcd(B - r, C)$$

for some $k \in \mathbb{Z}$ such that $k \geq 0$, with $k > 0$ if $r = 0$, by Theorem 4.1.

The preceding remarks motivates the following definitions.

Definition 4.1. Let $C > 0$ be an integer. We say that $i = sb + m$ is **a line (in the (b,i) plane) generated by C** if

$$s = \frac{C - 1}{2} \tag{31}$$

and $\exists B, r \in \{0, 1, \ldots, C - 1\}$ such that

$$m = 1 - C \cdot \frac{\gcd(B, C) + \gcd(B - r, C)}{2} \tag{32}$$

For a given C and a given pair (B, r) such that (31) and (32) are satisfied, we say that $i = sb + m$ is **the line (in the (b,i) plane) generated by C with intercept determined by (B,r)**, denoted $\mathbf{L_{C,B,r}}$.

Definition 4.2. Let C, B and r be given integers, such that $C > 0$ and $B, r \in \{0, 1, \dots, C-1\}$. We say that

$$L_{C,B,r} \;\cap\; \big\{(b,i) \in \mathbb{Z}^2 \mid b \geq 3 \;\wedge\; b = Ck + r + \gcd(B,C) + \gcd(B-r,C)$$

$$\text{for some integer } k \geq 0, \text{ with } k > 0 \text{ if } r = 0\big\}$$

is **the lattice triangle subset of $\mathbf{L_{C,B,r}}$**, denoted $\mathbf{S_{C,B,r}}$.

Note that $S_{C,B,r}$ is non-empty for all triples (C, B, r) satisfying the conditions in Definition 4.2. Given C, B and r, we get a line $L_{C,B,r}$ in the (b, i) plane. We can choose an integer k such that $b = Ck+r+\gcd(B,C)+\gcd(B-r,C) \geq 3$. For such a k, $T = (Ck + r, B, C)$ is a lattice triangle such that $(b(T), i(T)) = (b, i)$, where i is given by the equation of $L_{C,B,r}$. This means that (b, i) belongs to $S_{C,B,r}$, i.e. $S_{C,B,r}$ is non-empty.

Example 4.1. Let us investigate which lines are generated by $C = 4$. Recall Corollary 4.2. The lines generated by $C = 4$ all have the slope $\frac{3}{2}$. The intercepts are determined by B and r. Let $d_1 = \gcd(B, C)$ and $d_2 = \gcd(B - r, C)$. $\gcd(B, C)$ and $\gcd(B-r, C)$ are divisors of C, so d_1 and d_2 are divisors of C. If we first check all possible sums $d_1 + d_2$, and then for each given sum check if we can find $B, r \in \{0, 1, \dots, C-1\}$ such that $\gcd(B, C) = d_1 \wedge \gcd(B-r, C) = d_2$, we can find all intercepts.

For example, let $d_1 = 1$ and $d_2 = 2$. $\gcd(1, 4) = 1$ and $\gcd(-2, 4) = 2$, so this is satisfied by, for example, $B = 1$ and $r = 3$. This means that $L_{4,1,3}$ is the line $i = \frac{3}{2}b - 5$ is a a line generated by $C = 4$.

Checking all cases gives that

$$L_{4,1,0} : i = \frac{3}{2}b - 3$$

$$L_{4,1,3} : i = \frac{3}{2}b - 5$$

$$L_{4,2,0} : i = \frac{3}{2}b - 7$$

$$L_{4,1,1} : i = \frac{3}{2}b - 9$$

$$L_{4,0,2} : i = \frac{3}{2}b - 11$$

$$L_{4,0,0} : i = \frac{3}{2}b - 15$$

are the lines generated by $C = 4$. See Figure 9.

Furthermore

$$S_{4,1,0} = L_{4,1,0} \;\cap\; \big\{(b,i) \in \mathbb{Z}^2 \mid b = 4k + 2 \text{ for some integer } k > 0\big\}$$

$$S_{4,1,3} = L_{4,1,3} \;\cap\; \big\{(b,i) \in \mathbb{Z}^2 \mid b = 4k + 2 \text{ for some integer } k > 0\big\}$$

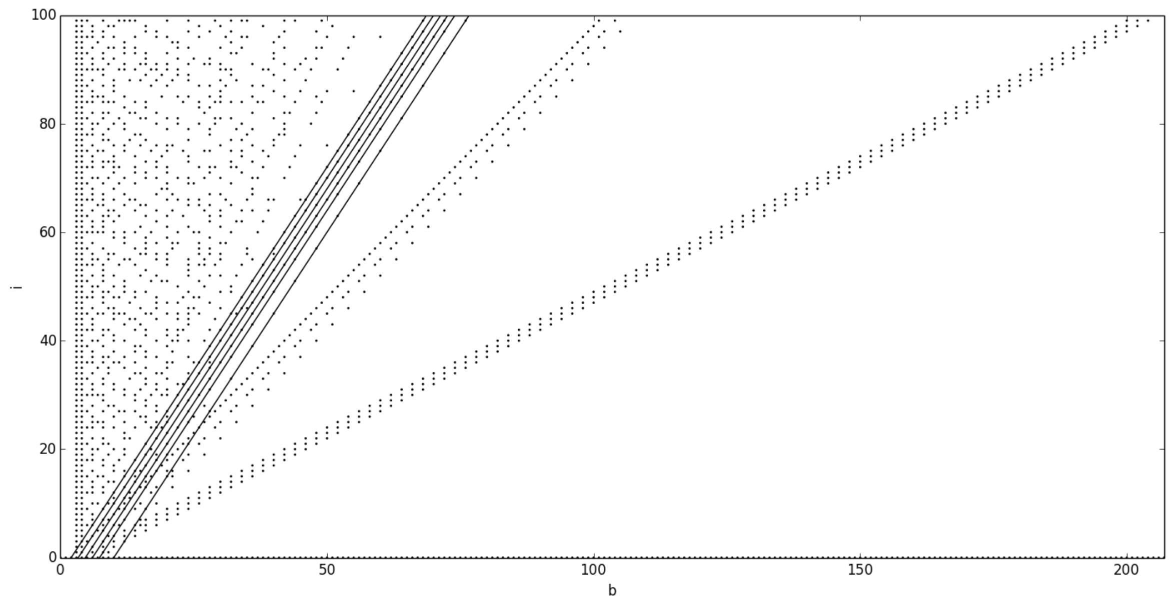

Figure 9: The solid lines are the lines generated by $C = 4$. The marked points on each line are precisely the points that constitute the lattice triangle subset of that particular line, for $i \leq 100$.

$$S_{4,2,0} = L_{4,2,0} \ \cap \ \left\{(b,i) \in \mathbb{Z}^2 \mid b = 4k \text{ for some integer } k > 0\right\}$$
$$S_{4,1,1} = L_{4,1,1} \ \cap \ \left\{(b,i) \in \mathbb{Z}^2 \mid b = 4k+2 \text{ for some integer } k > 0\right\}$$
$$S_{4,0,2} = L_{4,0,2} \ \cap \ \left\{(b,i) \in \mathbb{Z}^2 \mid b = 4k \text{ for some integer } k > 0\right\}$$
$$S_{4,0,0} = L_{4,0,0} \ \cap \ \left\{(b,i) \in \mathbb{Z}^2 \mid b = 4k \text{ for some integer } k > 0\right\}$$

where $k > 0$ in the cases of $S_{4,1,3}, S_{4,1,1}$ and $S_{4,0,2}$ follows from the fact that $b \geq 3$.

We now give a corollary to Corollary 4.2.

Corollary 4.3. *Let $C > 0$ be a given integer and let $B, r \in \{0, 1, \ldots, C-1\}$. Then the intercept m of $L_{C,B,r}$ is such that*

$$1 - C^2 \leq m \leq 1 - C$$

Moreover,

$$L_{C,B,r} \text{ has the intercept } 1 - C^2 \Leftrightarrow B = r = 0$$

and

$$\gcd(B,C) = \gcd(B-r,C) = 1 \Leftrightarrow L_{C,B,r} \text{ has the intercept 1-C.}$$

Proof. We have

$$i(T) = \frac{C-1}{2} \cdot b(T) + 1 - C \cdot \frac{\gcd(B,C) + \gcd(B-r,C)}{2}$$

by Corollary 4.2.

The intercept is as small as possible if $\gcd(B,C) = \gcd(B-r,C) = C$ which is the case if and only if $B = r = 0$. This gives the intercept $1 - C^2$.

The intercept is as large as possible if and only if $\gcd(B,C) = \gcd(B-r,C) = 1$. This gives the intercept $1 - C$. □

Corollary 4.3 motivates the following definition.

Definition 4.3. Let C be a given positive integer.

We say that $L_{C,0,0}$ is **the minimum line generated by C**, denoted $\mathbf{min_{L,C}}$.

We say that $L_{C,B,r}$ is **the maximum line generated by C**, denoted $\mathbf{max_{L,C}}$, if $\gcd(B,C) = \gcd(B-r,C) = 1$.

We say that

$$\left\{(b,i) \in \mathbb{Z}^2 \mid \frac{C-1}{2} b + 1 - C^2 \leq i \leq \frac{C-1}{2} b + 1 - C\right\}$$

- in other words, the (b,i) lying between the minimum and the maximum line generated by C - is **the region (in the (b,i) plane) generated by C**.

We give another corollary to Corollary 4.2.

Corollary 4.4. *Let p be a prime number. Then there are exactly three lines in the (b, i) plane generated by p. These are*

$$L_{p,B,r} = \left\{ (b,i) \in \mathbb{Z}^2 \mid i = \frac{p-1}{2} \cdot b + m \right\}$$

where

$$m = \begin{cases} 1-p, & \text{if } r \neq B \neq 0 \\ 1 - \frac{p(p+1)}{2}, & \text{if } r \neq B = 0 \ \vee \ r = B \neq 0 \\ 1-p^2, & \text{if } B = r = 0 \end{cases}$$

Moreover

$$S_{p,B,r} = L_{p,B,r} \cap \left\{ (b,i) \in \mathbb{Z}^2 \mid b = pk + r + m' \text{ for some } k \in \mathbb{Z} \right\}$$

where

$$m' = \begin{cases} 2, & r \neq B \neq 0 \\ p+1, & r \neq B = 0 \ \vee \ r = B \neq 0 \\ 2p, & B = r = 0 \end{cases}$$

Proof. $\gcd(B,p), \gcd(B-r,p) \in \{1,p\}$, since p is prime. This gives precisely the possibilities above. □

Theorem 4.5. *Let T be a triangle with normal form $(Ck+r, B, C)$. Let n be the number of lines in the (b, i)-plane generated by C. Then*

$$\pi(C) + 1 \leq n \leq \binom{\pi(C)}{2} + \pi(C) \tag{33}$$

where $\pi(C)$ is the number of divisors of C.

Proof. The lines generated by a certain C all have the same slope, but different intercepts. We see in Corollary 4.2 that the intercepts are determined by the sums $\gcd(B,C) + \gcd(B-r,C)$ for different B and r.

$\gcd(B,C)$ and $\gcd(B-r,C)$ are both divisors of C, so they can each take on $\pi(C)$ possible values. Either $\gcd(B,C) = \gcd(B-r,C)$, which can happen in $\pi(C)$ different ways, or $\gcd(B,C) \neq \gcd(B-r,C)$, which can happen in $\binom{\pi(C)}{2}$ different ways. Therefore, the maximum number of distinct sums $\gcd(B,C) + \gcd(B-r,C)$ is $\pi(C) + \binom{\pi(C)}{2}$. Note that we do not always have equality, since different choices of $\gcd(B,C)$ and $\gcd(B-r,C)$ can give the same sum $\gcd(B,C) + \gcd(B-r,C)$.

However, if $\gcd(B,C) = \gcd(B-r,C) = d_i | C$, then $\gcd(B,C) + \gcd(B-r,C) = 2d_i$, and $2d_i \neq 2d_j$ if $d_i \neq d_j$. This gives us at least $\pi(C)$ distinct sums. Assume that $D = \{d_1, d_2, \ldots, d_k\}$ is the divisors of C, and assume that d_2 is the smallest divisor of C not equal to 1. Then $1 + 1 < 1 + d_2 < 2d_2$, since $d_2 > 1$. In other words, there is at least one sum distinct from those where $\gcd(B,C) = \gcd(B,C)$, so the number of lines generated by C is at least $\pi(C) + 1$.

Thus we have proved that the number of distinct sums $\gcd(B,C) + \gcd(B-r,C)$ is at least $\pi(C) + 1$ and at most $\pi(C) + \binom{\pi(C)}{2}$. □

Example 4.2. Recall Example 4.1.

The divisors of 4 are $\{1, 2, 4\}$. Hence, $\pi(4) = 3$. According to Theorem 4.5, the number of lines generated by $C = 4$ is greater than or equal to 3 but less than or equal to 6. We saw in Example 4.1 that $C = 4$ generates six lines.

Furthermore,

$$\min_{L,4} = L_{4,0,0} : i = \frac{3}{2}b - 15$$

$$\max_{L,4} = L_{4,1,0} : i = \frac{3}{2}b - 3$$

5 Conditions on (b,i)

We note the following relation, which is just a reformulation of Pick's theorem (Theorem 2.4) using Theorem 4.1. It will be used repeatedly in this section.

Proposition 5.1. *Let T be a triangle with integer vertices and with normal form (A, B, C). Then*

$$b(T) + 2(i(T) - 1) = AC = 2 \cdot a(T)$$

5.1 Feasible composition of b with respect to i

In this section, we will look for conditions on (b, i) for there to be a triangle T such that $b = b(T)$ and $i = i(T)$.

Definition 5.1. Let $b \geq 3$ and $i \geq 0$ be integers. We say that $b = n_1 + n_2 + n_3$, where the n_j are positive integers, is a **feasible composition of b with respect to i** if the following conditions are satisfied:

$$n_j \leq b - 2, \text{ for } j = 1, 2, 3 \tag{34}$$

$$n_j | b + 2(i - 1), \text{for } j = 1, 2, 3 \tag{35}$$

$$\gcd(n_2, n_3) \mid n_1 \tag{36}$$

$$n_1 \text{ even } \implies [n_2 \text{ even } \Leftrightarrow n_3 \text{ even }] \tag{37}$$

$$n_1 \text{ odd } \implies \neg\,(n_2 \text{ even } \wedge n_3 \text{ even }) \tag{38}$$

$$\begin{gathered} n_{j_1} = n_{j_2} \text{ for some } j_1 \neq j_2 \implies n_{j_1} | n_{j_3}, \text{ where } j_3 \notin \{j_1, j_2\} \\ \text{and } j_1, j_2, j_3 \in \{1, 2, 3\} \end{gathered} \tag{39}$$

$$\frac{b + 2(i - 1)}{n_1} = p, \text{ where p prime } \implies n_2 \in \{1, p\} \wedge n_3 \in \{1, p\} \tag{40}$$

5.2 Necessary and sufficient conditions

Theorem 5.2. *Let $b \geq 3$ and $i \geq 0$ be integers.*

(i) Let T be a triangle with normal form (A, B, C) such that $b = b(T)$ and $i = i(T)$. Then there exists a feasible composition $b = n_1+n_2+n_3$ of b with respect to i, such that $n_1 = A$, $n_2 = \gcd(B, C)$ and $n_3 = \gcd(B - A, C)$.

(ii) Conversely, let $b = n_1 + n_2 + n_3$ be a feasible composition of b with respect to i. Assume that there exists integers A, B, C such that $A > 0, 0 \leq B < C$ and

$$\begin{cases} A = n_1 \\ \gcd(B, C) = n_2 \\ \gcd(B - A, C) = n_3 \end{cases} \tag{41}$$

Then T with $V_T = \{(0, 0), (A, 0), (B, C)\}$ is a lattice triangle such that $b(T) = b$ and $i(T) = i$.

Proof. To prove (i), we will show that all the conditions in the definition of a feasible composition of b with respect to i are necessary for there to be a triangle T such that $b = b(T)$ and $i = i(T)$.

Let T be a triangle with normal form (A, B, C). By Theorem 4.1, we have that $b(T) = A + \gcd(B, C) + \gcd(B - A, C)$. This shows that we must be able to find a composition of b into exactly three positive integers for there to be a triangle with $b = b(T)$.

For the rest of the proof, let $n_1 = A$, $n_2 = \gcd(B, C)$ and $n_3 = \gcd(B-A, C)$.

It is obvious that (34) is a necessary condition, since $n_i \geq 1$ for $i = 1, 2, 3$.

Now, $b(T)+2(i(T)-1) = 2a(T) = AC$, by Proposition 5.1. This shows that $n_1|b + 2(i - 1)$. Also, $n_2|C$ and $n_3|C$. Since $C|b + 2(i - 1)$, we conclude that $n_i|b + 2(i - 1)$ for $i = 1, 2, 3$. This proves that (35) is necessary.

$n_2 \mid B$ and $n_3 \mid B - n_1$, which means that $B = k_1 n_2$ and $B - n_1 = k_2 n_3$ for some $k_1, k_2 \in \mathbb{N}$, so $n_1 = B - (B - n_1) = k_1 n_2 - k_2 n_3$. Hence, $\gcd(n_2, n_3) \mid n_1$, by Lemma 3.1. We have proved that (36) is necessary.

Assume that n_1 is even. Then

$$n_3 \text{ even } \Leftrightarrow B - n_1 \text{ even } \wedge C \text{ even } \Leftrightarrow B \text{ even } \wedge C \text{ even } \Leftrightarrow n_2 \text{ even}$$

This proves that (37) is necessary.

Assume that n_1 is odd. Then

$$n_2 \text{ even } \implies B \text{ even} \wedge C \text{ even } \implies B - n_1 \text{ odd} \wedge C \text{ even } \implies n_3 \text{ odd}$$

Of course, this also means that n_2 is odd if n_3 is even. Thus, we have proved that (38) is necessary.

Assume that $n_{j_1} = n_{j_2}$, for some $j_1 \neq j_2$, where $j_1, j_2 \in \{1, 2, 3\}$. We have the following three chains of implications:

$$n_1 = n_2 \implies n_1|B \ \wedge \ n_1|C \implies n_1|B - n_1 \ \wedge \ n_1|C \implies n_1|n_3$$

$$n_1 = n_3 \implies n_1 | B - n_1 \ \wedge \ n_1 | C \implies n_1 | B \ \wedge \ n_1 | C \implies n_1 | n_2$$

$$n_2 = n_3 \implies n_2 | B \ \wedge \ n_2 | B - n_1 \implies n_2 | n_1$$

This proves that $n_{j_1} | n_{j_3}$, where $j_3 \in \{1, 2, 3\}$ and $j_3 \notin \{j_1, j_2\}$, for all $j_1 \neq j_2$. Thus, we have proved that (39) is necessary.

Assume that $\frac{b+2(i-1)}{n_1} = p$ where p is prime. Then $C = p$, by Proposition 5.1. This implies that

$$n_2 = \begin{cases} p, & B = 0 \\ 1, & 0 < B < p \end{cases} \ \wedge \ n_3 = \begin{cases} p, & A \equiv B \mod p \\ 1, & A \not\equiv B \mod p \end{cases}$$

Thus, we have proved that (40) is necessary.

This concludes the proof of (i).

Let A, B, C be integers such that $A > 0, 0 \leq B < C$ and satisfying (41). Let T be a lattice triangle with $V_T = \{(0,0), (A, 0), (B, C)\}$. Then

$$b(T) = A + \gcd(B, C) + \gcd(B - A, C) = n_1 + n_2 + n_3 = b$$

by (41) and

$$i(T) = a(T) - \frac{b(T)}{2} + 1 = \frac{b}{2} + i - 1 - \frac{b}{2} + 1 = i$$

where we have used that

$$a(T) = \frac{AC}{2} = \frac{b + 2(i-1)}{2} = \frac{b}{2} + i - 1$$

by Proposition 5.1. Thus we have proved statement (ii). □

5.3 Proving non-existence of lattice triangles for a given pair (b,i)

Theorem 5.2 can be used to prove that for a certain pair (b, i) there is no lattice triangle T such that $b(T) = b$ and $i(T) = i$, by showing that there is no feasible composition of b with respect to i.

Example 5.1. There is no triangle T such that $b(T) = 6$ and $i(T) = 3$.

Let $b = 6$ and $i = 3$. Then $b + 2(i-1) = 10$.

First, we use (34) and (35). The divisors of 10 less than or equal to $6 - 2 = 4$ are $D = \{1, 2\}$. We want to find a composition of 6 using only the numbers in D. The only such composition is $6 = 2 + 2 + 2$.

But $\frac{b+2(i-1)}{n_1} = \frac{10}{2} = 5$. Since 5 is a prime number, we require that $n_2, n_3 \in \{1, 5\}$ by (40). But $n_2 = n_3 = 2 \notin \{1, 5\}$. Therefore, $6 = 2 + 2 + 2$ is not a feasible composition of 6 with respect to 3.

Hence, there is no feasible compositions of 6 with respect to 3. This implies, by Theorem 5.2, that there is no triangle T such that $b(T) = 6$ and $i(T) = 3$.

Example 5.2. There is no triangle T such that $b(T) = 15$ and $i(T) = 11$.

Let $b = 15$ and $i = 11$. Then $b + 2(i-1) = 35$.

Again, we begin by using (34) and (35). The divisors of 35 which are less than or equal to $15 - 2 = 13$ are $D = \{1, 5, 7\}$. Then

$$\{(n_1, n_2, n_3) \mid 15 = n_1 + n_2 + n_3 \ \wedge \ n_i \in D, \text{ for } i = 1, 2, 3\}$$

$$= \{(5,5,5), (7,7,1), (7,1,7), (1,7,7)\}$$

We will show that none of these compositions is a feasible composition of 15 with respect to 11.

Let $(n_1, n_2, n_3) = (5,5,5)$. Then $\frac{b+2(i-1)}{n_1} = \frac{35}{5} = 7$. But 7 is a prime number. Then $n_2, n_3 \in \{1, 7\}$ by (40). But $n_2 = n_3 = 5 \notin \{1, 7\}$. Hence, $(n_1, n_2, n_3) = (5,5,5)$ is not a feasible composition of 15 with respect to 11.

The remaining compositions all have the property $n_i = 7 = n_j$ for some $i \neq j$. But $7 \nmid 1$. Hence, by (39), none of these compositions is a feasible composition.

Hence, there is no feasible composition of 15 with respect to 11. This implies, by Theorem 5.2, that there is no triangle T such that $b(T) = 15$ and $i(T) = 11$.

5.4 Lattice triangles with prime normalized area

Definition 5.2. The T be a triangle. We say that $2 \cdot a(T)$ is the **normalized area of T**.

Theorem 5.3. *Let $b \geq 3$ and $i \geq 0$ be integers such that $b + 2(i-1) = p$, where p is prime and $p \geq 3$. Then there exists a triangle T such that $b(T) = b$ and $i(T) = i$ if and only if $(b, i) \in \{(p+2, 0), (3, \frac{p-1}{2}\}$.*

Moreover, the triangles satisfying this condition are precisely those with the normal form

$$(A, B, C) \in \{(1, B, p) \mid B \in \{0, 1, \ldots, p-1\}\} \cup \{(p, 0, 1)\}$$

Specifically

$$(b(T), i(T)) = \begin{cases} (3, \frac{p-1}{2}), & (A, B, C) \in \{(1, B, p) \mid B \in \{2, 3, \ldots, p-1\}\} \\ (p+2, 0), & (A, B, C) \in \{(1, 0, p), (1, 1, p), (p, 0, 1)\}) \end{cases}$$

Proof. Let $b \geq 3$ and $i \geq 0$ be integers such that $b + 2(i-1) = p$ and let T be a triangle with normal form (A, B, C). Then $b + 2(i-1) = AC$ for positive integers A and C, by Proposition 5.1, which implies that

$$(A, C) \in \{(1, p), (p, 1)\}$$

Assume that $(A, C) = (p, 1)$. Then $B = 0$, since $0 \leq B < C$. Thus, the only possible choice is $T = (p, 0, 1)$. By using Theorem 4.1, we get

$$b((p, 0, 1)) = p + 2$$

and

$$i((p,0,1)) = 0$$

On the other hand, assume that $(A,C) = (1,p)$. Then $B \in \{0,1,\ldots,C-1\}$. We get

$$b((1,B,p)) = \begin{cases} p+2, & \text{if } B \in \{0,1\} \\ 3, & \text{if } B \in \{2,3,\ldots,p-1\} \end{cases}$$

and

$$i((1,B,p)) = \begin{cases} 0, & \text{if } B \in \{0,1\} \\ \frac{p-1}{2} & B \in \{2,3,\ldots,p-1\} \end{cases}$$

again by Theorem 4.1. □

Note that Theorem 5.3 is also a special case of a theorem proved by Higashitani [5, Theorem 0.1].

Given a prime number p, Theorem 5.3 gives us an enumeration of all lattice triangles having normalized area p.

Furthermore, Theorem 5.3 means that on the line $i = -\frac{b}{2} + (1 + \frac{p}{2})$ in the (b,i) plane, the only points (b,i) for which there exists a lattice triangle is $(b,i) \in \{(p+2,0), (3, \frac{p-1}{2})\}$. This gives us classes of (b,i) for which we know there are no lattice triangles.

Example 5.3. A lattice triangle has normalized area 5 if and only if its normal form belongs to the set

$$\{(1,0,5), (1,1,5), (1,2,5), (1,3,5), (1,4,5), (5,0,1)\}$$

There is no lattice triangle T such that $b(T) = 5$ and $i(T) = 1$, since $b(T) + 2(i(T)-1) = 5$ but $b(T) = 5 \notin \{3,7\}$.

6 Patterns of non-existence of lattice triangles

As noted at the end of the last section, Theorem 5.3 gives us classes of (b,i) for which there are no lattice triangles. In this section, we will conjecture the non-existence of lattice triangles for other large classes of (b,i).

Recall Corollary 4.2. Consider Figures 7 and 8. At least for small C, one can clearly see the lines generated by different C. Consider the lines generated by $C = c$ and $C = c+1$ for some integer $c \geq 2$. There seems to be some positive integer k such that for $b \geq k$ there are no points (b,i) in the area in between the maximum line generated by c and the minimum line generated by $c+1$. We know the equations for these lines, thanks to Corollary 4.2 and Corollary 4.3.

The following proposition is used in Conjecture 6.2.

Proposition 6.1. *Let C be a positive integer. Let (b,i) be the point where* $\max_{L,C}$ *and* $\min_{L,C+1}$ *intersect. Then*

(i) $b = 2(C^2 + C + 1)$ *and* $i = C(C+1)(C-1)$

(ii) T with $V_T = \{(0,0),(2C(C+1),0),(1,C)\}$ is a lattice triangle such that $b = b(T)$ and $i = i(T)$

Proof. Recall Definition 4.3. $\max_{L,C}$ has the equation

$$i = \frac{C-1}{2}b + 1 - C$$

and $\min_{L,C+1}$ has the equation

$$i = \frac{(C+1)-1}{2}b + 1 - (C+1)^2 = \frac{C}{2}b - (C^2+2C)$$

by Corollary 4.2 and Corollary 4.3. They intersect when

$$\frac{C-1}{2}b + 1 - C = \frac{C}{2}b - (C^2+2C)$$
$$\Leftrightarrow b = 2(C^2+C+1)$$

which gives

$$i = \frac{C}{2}2(C^2+C+1) - (C^2+2C) = C(C+1)(C-1)$$

This proves (i).

To prove (ii), we must show that $(b,i) = (2(C^2+C+1), C(C+1)(C-1))$ belongs to the lattice triangle subset of at least one of $\max_{L,C}$ or $\min_{L,C+1}$.

Consider $\max_{L,C}$. By Definition 4.2, the lattice triangle subset is

$$\left\{(b,i) \in \mathbb{Z}^2 \mid b = Ck + r + 2 \text{ for some non-negative } k \in \mathbb{Z}\right\}$$

where r is such that $\gcd(B-r, C) = 1$, where in turn B is such that $\gcd(B,C) = 1$. This is satisfied by, for example, the choice $B = 1$ and $r = 0$. Hence, there is a lattice triangle at $(b,i) = (2(C^2+C+1), C(C+1)(C-1))$ if we can find a positive integer k such that

$$2(C^2+C+1) = Ck + 2$$
$$\Leftrightarrow k = 2(C+1)$$

which obviously can always be done. This proves (ii).

□

We formulate the following conjecture.

Conjecture 6.2. *Let $C \geq 2$ be an integer.*

Then there are no lattice triangles T such that

$$b(T) > 2(C^2+C+1) \ \wedge \ \frac{C-1}{2}b(T) + 1 - C < i(T) < \frac{C}{2}b(T) - C(C+2)$$

Furthermore, by Proposition 6.1, there is lattice triangle T such that

$$b(T) = 2(C^2+C+1) \ \wedge \ i(T) = C(C+1)(C-1)$$

By computation, the conjecture has been verified to hold for triples (C,b,i) such that $C \leq 11, i \leq 1500$ and $b \leq 2 \cdot 1500 + 7 = 3007$.

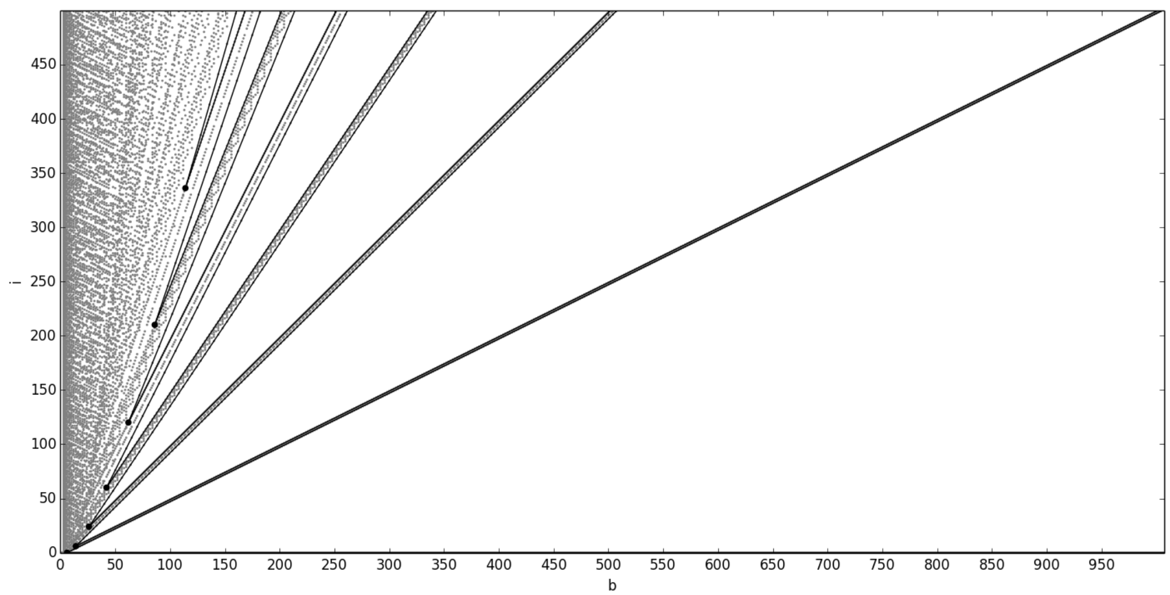

Figure 10: Empty regions, according to Conjecture 6.2. The indicated points are the points mentioned in the conjecture, for $C = 2, 3, 4, 5, 6, 7$.

See figure 10.

Note that for $C = 1$, the conjecture would claim there are no lattice triangles such that

$$b(T) > 6 \;\wedge\; 0 < i(T) < \frac{b(T)}{2} - 3$$

We see that

$$i(T) < \frac{b(T)}{2} - 3 \Leftrightarrow b(T) > 2i(T) - 6$$

which is almost true, by Scott's inequality (Theorem 2.5); however, by the same theorem we know that there is exactly one lattice triangle violating this, namely T with $V_T = \{(0,0),(3,0),(0,3)\}$. T is such that $b(T) = 9$ and $i(T) = 1$. With the exception of this triangle, the conjecture holds for $C = 1$ as well.

Let $c \geq 2$ be a given integer. According to Conjecture 6.2 there are no lattice triangles for (b,i) in the region between the maximum line generated by c and the minimum line generated $c+1$ for $b > 2(c^2+c+1)$. But consider Figure 11. According to the conjecture, there are no lattice triangles for (b,i) between the maximum line generated by 2 and the minimum line generated by 3 for $b > 14$ (set $C = 2$ in the conjecture). However, we see that regions generated by $c' > c$, here $c' = 30$, intersect the aforementioned region. In other words, there are lines generated by larger C's intersecting that region. This means it is not obvious that there are no lattice triangles in the region in question.

To prove the conjecture, one must show that there are no lattice triangles on the lines intersecting the regions claimed to have no lattice triangles, i.e. that for each line intersecting such a region, the intersection of the region and the lattice triangle subset of the line is empty.

7 Conclusions regarding Ehrhart polynomials

Recall Definition 2.6. The following theorem can be shown. We will prove it only for lattice triangles.

Theorem 7.1 ([2], pp. 38-40)**.** *The Ehrhart polynomial of a lattice polygon P in $\mathbb{R}^2$ is*

$$L_P(t) = a(P)t^2 + \frac{b(P)}{2}t + 1$$

Proof. Let T be a lattice triangle with normal form (A,B,C). Then tT has the normal form (tA,tB,tC). This gives

$$a(tT) = \frac{tA \cdot tC}{2} = \frac{AC}{2}t^2 = a(T)t^2$$

and

$$\begin{aligned} b(tT) &= tA + \gcd(tB,tC) + \gcd(tB - tA, tC) \\ &= t(A + \gcd(B,C) + \gcd(B - A, C) = b(T)t \end{aligned}$$

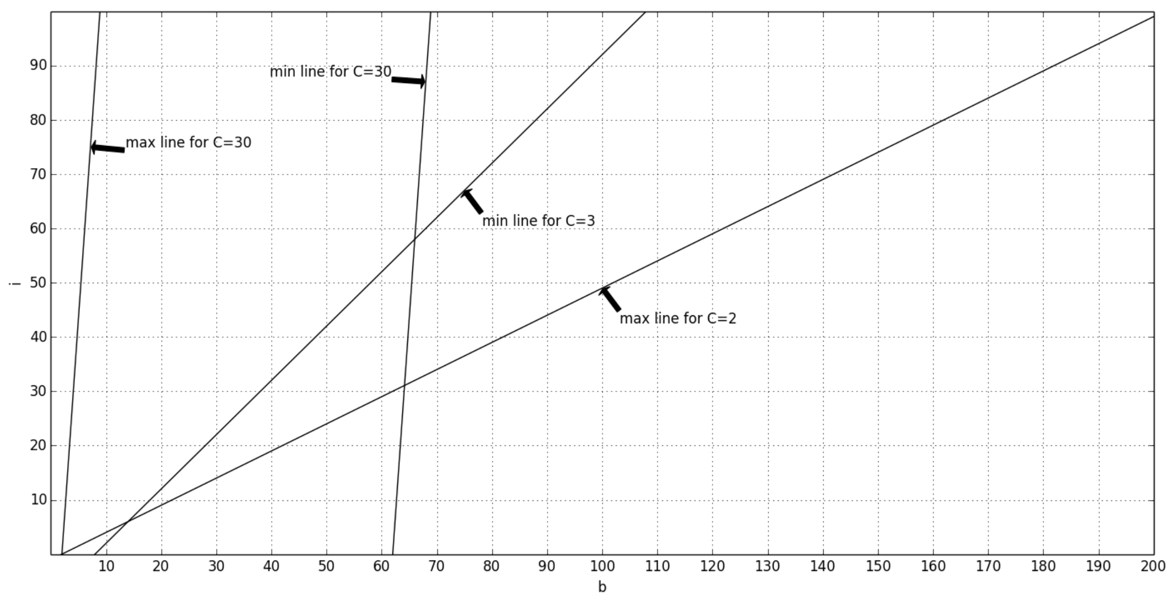

Figure 11: Example of how regions claimed to have no lattice triangles are intersected by regions generated by larger C's.

by Theorem 4.1. Finally,

$$L_P(t) = i(tT) + b(tT) = a(tT) - \frac{b(tT)}{2} + 1 + b(tT) = a(tT) + \frac{b(tT)}{2} + 1$$

$$= a(T)t^2 + \frac{b(T)}{2}t + 1$$

where we have used Pick's theorem (Theorem 2.4) for the second equality. We have thus proved the theorem for lattice triangles. □

Lemma 7.2. *Assume that P is a lattice polygon in $\mathbb{R}^2$ with Ehrhart polynomial $L_P(t) = \frac{p}{2}t^2 + \frac{b}{2}t + 1$, where $p \geq 3$ is a prime number. Then*

$$L_P(t) \text{ can be realized by a lattice triangle} \Leftrightarrow b \in \{p+2, 3\}$$

Proof. From the Ehrhart polynomial, we see that $a(P) = \frac{p}{2}$. Hence, the normalized area of P is prime. By Theorem 5.3, P can be a lattice triangle if and only if $b(P) \in \{3, p+2\}$. Hence, P can be a lattice triangle if and only if $b \in \{3, p+2\}$, by Theorem 7.1. □

Example 7.1. $L_P(t) = \frac{5}{2}t^2 + \frac{7}{2}t + 1$ can be realized by a triangle, since the coefficient of t is $p+2$, with $p = 5$.

$L_P(t) = \frac{5}{2}t^2 + \frac{3}{2}t + 1$ can be realized by a triangle, since the coefficient of t is 3.

However, $L_P(t) = \frac{5}{2}t^2 + \frac{9}{2}t + 1$ can not be realized by a triangle, since the coefficient of t is 9, which is neither $p+2$, with $p = 5$, nor 3.

Theorem 7.3. *There are an infinite number of Ehrhart polynomials of lattice polygons in $\mathbb{R}^2$ which can not realized by a lattice triangle.*

Proof. Let $q(t) = \frac{p}{2}t^2 + 5t + 1$, where p is prime. If this polynomial is realized by a lattice polygon P, then $a(P) = \frac{p}{2}$ and $b(P) = 5$. Such a polygon exists if

$$5 \leq 2i + 6 = 2\left(\frac{p}{2} - \frac{5}{2} + 1\right) + 6 \Leftrightarrow p \geq 2$$

by Theorem 2.6 and Theorem 2.4. This inequality is certainly satisfied by all primes p.

$q(t)$ can be realized by a lattice triangle if and only if $5 \in \{3, p+2\}$ by Lemma 7.2. This is satisfied if and only if $p = 3$.

But there are infinitely many primes larger than 3. Hence, there are infinitely many Ehrhart polynomials of lattice polygons which are not Ehrhart polynomials of lattice triangles. □

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