SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Equivariant Sheaves on Topological Categories

av

Johan Lindberg

2015 - No 7

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Equivariant Sheaves on Topological Categories

Johan Lindberg

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Henrik Forssell

2015

Abstract

The category $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves on an arbitrary topological category $\mathcal{C}$ can be constructed as a colimit in the 2-category of Grothendieck toposes and geometric morphisms, and is therefore a Grothendieck topos. In this thesis we investigate elementary properties of $\mathcal{C}$-spaces and equivariant sheaves, regarded as spaces respectively local homeomorphisms over the space of objects of $\mathcal{C}$ equipped with a continuous action, and how these properties depend on the openness of $\mathcal{C}$. We give direct proof, using Giraud's theorem, that $\mathrm{Sh}_{C_1}(C_0)$ is a Grothendieck topos for the case of a topological category $\mathcal{C}$ where the codomain function is assumed to be open, thus extending Moerdijk's brief sketch of a proof of this proposition. We also show that the category of equivariant sheaves with an open action is (equivalent to) an open subtopos of $\mathrm{Sh}_{C_1}(C_0)$, for an arbitrary topological category $\mathcal{C}$.

Moerdijk's site description for the equivariant sheaf topos of an open localic groupoid depends on defining an equivalence relation in terms of "open subgroupoids" of the underlying localic groupoid. We apply a similar equivalence relation to arbitrary topological groupoids over a fixed open topological groupoid $\mathcal{G}$. For the category of morphisms of topological groupoids $h : \mathcal{H} \to \mathcal{G}$ such that this equivalence relation is open (*i.e.* has an open quotient map), this is shown to define a functor Λ to the category of $\mathcal{G}$-spaces.

Every $\mathcal{G}$-space also determines a topological groupoid over $\mathcal{G}$ in a functorial way. Brown, Danesh-Naruie and Hardy have shown that this functor, which we denote by S, yields an equivalence between the category of $\mathcal{G}$-spaces and the category of topological covering morphisms to $\mathcal{G}$. We generalize this result to topological categories, which yields an alternative description of the category of equivariant sheaves on a topological category $\mathcal{C}$ as the category of local homeomorphic covering morphisms to $\mathcal{C}$. In the case of an open topological groupoid $\mathcal{G}$ we show that Λ is left adjoint to S. In this case, the equivalence by Brown, Danesh-Naruie and Hardy turns out to be a special case of the adjunction $\Lambda \dashv S$.

Acknowledgments

I would like to thank my supervisor Henrik Forssell for introducing me to this interesting subject, for many inspiring and thought provoking discussions, as well as for the help he has supplied.

I would also like to thank my father for his support this period.

Contents

Chapter 1

Introduction

An elementary topos can be described as a "generalized universe of sets". A Grothendieck topos is an elementary topos with some additional properties (the existence of a set of generators and the existence of all small coproducts), and is sometimes described as a "generalized space". The standard definition, however, is that a Grothendieck topos is a category equivalent to the category of sheaves of sets on a (small) site. Equivalently, a category is a Grothendieck topos iff it satisfies the conditions of Giraud's theorem.

Given a Grothendieck topos, a site, for which the category is equivalent to the category of sheaves on the site, is in general not unique. Giraud's theorem characterizes a Grothendieck topos in terms of a set of generators rather than referring to a particular site. The theorem can be used for proving that a certain category is a Grothedieck topos in cases when no explicit site description is available. However, given a category that satisfies the conditions of Giraud's theorem one can construct a canonical site.

By a topological category we mean a category where the set of objects and set of arrows are equipped with topologies that makes the structure maps continuous.[1] In other words, a topological category is a category object (or an internal category) in the category of topological spaces and continuous functions. A topological groupoid is a topological category where every arrow is invertible, and the operation of inverting an arrow is continuous.

An equivariant sheaf on a topological category (or groupoid) $\mathcal{C}$ is a sheaf (in the sense of a local homeomorphism) over the space of objects of $\mathcal{C}$ equipped with a continuous action. Such equivariant sheaves, together with the local homeomorphisms between them that respects the action, form a category. This category can be constructed as a colimit in the 2-category of Grothendieck toposes and geometric morphisms, and is therefore a Grothendieck topos (see [Moe88] and [Moe95]).

Similar to equivariant sheaves on $\mathcal{C}$, a $\mathcal{C}$-space is topological space over the space of objects of $\mathcal{C}$ equipped with a continuous action. The $\mathcal{C}$-spaces, together with the continuous functions between them that respects the action, also form a category.

[1] The term topological category has other, inequivalent, definitions compared to the one we shall adopt (*e.g.* [AHS90]).

1.1 Some related results

Representing Grothendieck toposes

A geometric morphism $p : \mathcal{E}_1 \to \mathcal{E}_2$ between toposes $\mathcal{E}_1, \mathcal{E}_2$ is a pair of adjoint functors $p^* \dashv p_*$, where the left adjoint p^*, called the inverse image, preserves finite limits. A point of a topos $\mathcal{E}$ is a geometric morphism from the topos of sets and functions, **Set**, to $\mathcal{E}$. In a sense, this is a generalization of the notion of a point in point-set topology.

A topos $\mathcal{E}$ is said to have enough points if the class of all inverse image functors p^* of points p of $\mathcal{E}$ is jointly conservative. In [BM98] it is shown that any Grothendieck topos with enough points is equivalent to the category of equivariant sheaves on some topological groupoid where the domain and codomain functions are open.

One may also consider "pointless spaces" called locales, were the primitive notion is that of a lattice of open sets, and localic groupoids. Any Grothendieck topos is known to be equivalent to the category of equivariant sheaves on some (open) localic groupoid [JT84] (the more recent publication [Tow14] offers a shorter proof of this proposition).

An application to mathematical logic

One application of equivariant sheaf toposes arise in connection to models of certain first-order theories via the notion of "classifying topos", which is briefly described below. Since any such classifying topos is a Grothedieck topos, it can be represented by the category of equivariant sheaves on a localic groupoid.

An interpretation of a first-order language L in a topos $\mathcal{E}$ is an extension of the notion of a set-theoretic L-structure expressed in diagrammatic form in **Set**. Given a theory $\mathbb{T}$ in L, one can in this way speak of models of $\mathbb{T}$ in a topos. Loosely speaking, a topos $\mathbf{Set}[\mathbb{T}]$ is said to be a classifying topos (over **Set**) for $\mathbb{T}$ models if there is an equivalence, natural in $\mathcal{E}$, between the category of geometric morphisms $\mathcal{E} \to \mathbf{Set}[\mathbb{T}]$ and the category of models of $\mathbb{T}$ in $\mathcal{E}$, for cocomplete toposes $\mathcal{E}$.

A coherent formula is a first-order formula built using connectives $\top, \bot, \wedge, \exists$ and $\vee$. By allowing infinitary disjunction with only finitely many free variables one obtains a geometric theory. A coherent (geometric) theory $\mathbb{T}$ is a set of sequents of coherent (geometric) formulas.

For any geometric theory there exists a classifying topos. Conversely, any topos is (equivalent to) the classifying topos of some geometric theory (see [Joh02b, D3.1]). Grothendieck toposes which occur as the classifying topos of a coherent theory are called coherent toposes. Deligne's theorem states that any coherent topos has enough points. Thus the classifying topos of any coherent theory can be represented as the category of equivarant sheaves on a topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$, denoted $\mathrm{Sh}_{G_1}(G_0)$.

If $\mathrm{Sh}_{G_1}(G_0)$ represents the classifying topos of a geometric theory $\mathbb{T}$ then the underlying topological groupoid can be taken to consist of $\mathbb{T}$ models and isomorphisms (of $\mathbb{T}$ models), see [BM98], [AF13]. The toposes with enough points are the classifying toposes of geometric theories with enough models, in the sense of that a sequent of geometric formulas is valid in $\mathbb{T}$ if it is valid in all models of $\mathbb{T}$ in **Set**.

A quotient theory of $\mathbb{T}$ can be described as a theory extension of $\mathbb{T}$ in the same language. In [For13], the known correspondence of quotient theories of a theory $\mathbb{T}$ and subtoposes of $\mathbf{Set}[\mathbb{T}]$ is extended to subgroupoids of $\mathcal{G}$ and subtoposes of $\mathrm{Sh}_{G_1}(G_0)$, where $\mathrm{Sh}_{G_1}(G_0)$ is the classifying topos of the theory $\mathbb{T}$. An intrinsic characterization of the subgroupoids $\mathcal{H} \hookrightarrow \mathcal{G}$ that are definable by quotient theories (of $\mathbb{T}$) in this way is also given in [For13].

1.2 This thesis

In this thesis we will treat category theory as performed within a classical universe of sets, with choice, and make extensive use of point-set arguments and results in point-set topology. This thesis presents details and contains proofs of some basic properties of equivariant sheaves that does not appear to have a similar summarized and detailed presentation accessible in the literature.

In Chapter 2 the basic notions of the subject are introduced. We list useful properties of open maps, local homeomorphisms, quotient maps and relevant forgetful functors collected from various sources.

Chapter 3 contains a proof that the category of equivariant sheaves on a topological category $\mathcal{C}$, denoted $\mathrm{Sh}_{C_1}(C_0)$, has all finite limits and all small colimits. In this chapter we also investigate how certain properties of equivariant sheaves (such as openness of the action) are related to the openness of the underlying topological category. We prove a canonical isomorphism of $\mathcal{C}$-spaces, which shows that each $\mathcal{C}$-space is essentially a quotient space with an action induced by composition of arrows in $\mathcal{C}$.

Chapter 4 contains a proof, using Giraud's theorem, that for a topological category $\mathcal{C}$ where the codomain function is open, $\mathrm{Sh}_{C_1}(C_0)$ is a Grothendieck topos. Published lecture notes by Moerdijk contains a brief sketch of a proof of this statement, also using Giraud's theorem [Moe95]. Our proof fleshes out Moerdijk's sketch and emphasizes how the generators can be seen to arise via the canonical isomorphism of $\mathcal{C}$-spaces proved in Chapter 3. We verify the other conditions in detail. Further, we also show that the category of equivariant sheaves with an open action is equivalent to an open subtopos of $\mathrm{Sh}_{C_1}(C_0)$, for an arbitrary topological category $\mathcal{C}$.

Chapter 5 first summarizes material from [BDNH76] concerning topological covering morphisms. These results are then extended to topological categories. In particular, the category of equivariant sheaves on a topological category $\mathcal{C}$ is shown to be equivalent to the category of local homeomorpic covering morphism to $\mathcal{C}$. Moerdijk's site description for the equivariant sheaf topos of an open localic groupoid in [Moe88] depends on defining an equivalence relation in terms of "open subgroupoids" of the underlying localic groupoid. We apply a similar equivalence relation to arbitrary topological groupoids over a fixed open topological groupoid $\mathcal{G}$. For the category of morphisms of topological groupoids $h : \mathcal{H} \to \mathcal{G}$ such that this equivalence relation is open (*i.e.* has an open quotient map), this is shown to define a functor Λ to the category of $\mathcal{G}$-spaces. We prove that Λ has a right adjoint and the adjunction restricts to the category of equivariant sheaves on $\mathcal{G}$ and the category of semi-local homeomorphic morphisms to $\mathcal{G}$. The equivalence of the category of topological covering morphisms to $\mathcal{G}$ and the category of $\mathcal{G}$-spaces, proved in [BDNH76], turns out to be a special case of this adjunction, when the topological groupoid $\mathcal{G}$ is open.

1.3 To the reader

This thesis is aimed at readers of the level equivalent to a master student in mathematics, assuming familiarity with the basics of category theory and the theory of (elementary and Grothendieck) toposes (as may be obtained via [Mac97] and [MM92]). Especially, the reader is assumed to be familiar with computing basic limits and colimits (products, equalizers, pullbacks and their duals) in the category of topological spaces and continuous functions.

When a non-trivial statement appearing in Chapters 3–5 of this thesis is known to the author to have been published somewhere else effort has been make this clear and supply an explicit reference to the publication in question.

The reader may wish to consult the index of notation, which is included at the end.

Chapter 2

Preliminaries

In this chapter we review and list some properties of the basic concepts of our subject matter.

2.1 Open maps, local homeomorphisms and quotient maps

We shall call a continuous function $f : X \to Y$ between topological spaces a map, and will in this case also say that X is a space over Y. The category of topological spaces and maps will be denoted **Sp**. It is well-known that this category is both complete and cocomplete and that the forgetful functor (of forgetting the topology) from **Sp** to the category of sets and functions, denoted **Set**, preserves both limits and colimits (*e.g.* [Mac97, V.9]).

For convenience we will often use the same symbol(s) for the restriction of a function to a subspace of its domain and to the original function. This will in some cases lead to the same symbol(s) being used to denote functions with different domains.

We will deal extensively with open maps and local homeomorphisms of topological spaces. A local homeomorphism $p : X \to Y$ is a map such that for each $x \in X$ there is an open set $U \subseteq X$ such that $x \in U$, $p(U)$ is open and the restriction of p to U, $p|_U$, is a homeomorphism onto its image. When we, in a diagram, wish to emphasize that a map is open or a local homeomorphism, we put a circle respectively a dot on the shaft of the arrow, as in diagram (2.1). However, in diagrams in categories where all arrows are local homeomorphisms, we may suppress this notation for readability.

Local homeomorphism are also called étale maps. We shall follow [Joh02b] and not use the term (*cf.* C1.3). We shall, however, use the abbreviation *LH* for local homeomorphism. The collection of topological spaces and LH's between them form a category **LH**.

Notice that every homeomorphism is an LH, and the inclusion $E \hookrightarrow X$ of an open subset $E \subseteq X$ (with the subspace topology) is an LH. Furthermore, the restriction of an LH $f : X \to Y$ to an open subset $E \subseteq X$ is an LH $E \to Y$.

As the properties of LH's and open maps of topological spaces, that we shall need, are not conveniently summarized in the standard literature we list these and supply proofs, or references to where proofs can be found:

Lemma 2.1. *Let X, Y and Z be topological spaces and the following diagram be a pullback square in* **Sp**

$$\begin{tikzcd} Y \times_Z X \arrow[r] \arrow[d, "k"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

(i) If f is open, then k is open.

(ii) If f is an LH, then k is an LH.

In other words, open maps and LH's are stable under pullback.

Proof: See [MM92, Lemma IX.6.1] and [MM92, Lemma II.9.1], respectively. □

The following lemma describes an equivalent charaterization of LH's, where Δ takes $x \mapsto (x, x)$.

Lemma 2.2. *Let X, Y be topological spaces. $f : X \to Y$ is an LH iff both f and the diagonal map $\Delta : X \to X \times_Y X$ are open.*

Proof: See [MM92, Ex. II.10]. □

Lemma 2.3. *Let X, Y and Z be topological spaces and the following diagram be commutative (i.e. $k = g \circ f$) in* **Sp**

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "k"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

(i) If k and g are LH's, then f is an LH.

(ii) If g is an LH and k is open, then f is open.

(iii) If f is surjective and k is open, then g is open.

(iv) If f is surjective and open and k is an LH, then f and g are LH's.

Proof: (i) : omitted, see [MM92, Ex. II.10].

(ii) : This property is mentioned in [Moe95, II.3]. Let $U \subseteq X$ be open and let $x \in U$. Then there exist an open subset $V_x \subseteq Y$ such that $f(x) \in V_x$, $g(V_x)$ is open and g restricted to V_x is a homeomorphism onto its image. Observe that $f\left(f^{-1}(V_x) \cap U\right) = f(U) \cap V_x$, so the set

$$\begin{aligned} W &= g\left(f(U) \cap V_x\right) \\ &= g \circ f\left(f^{-1}(V_x) \cap U\right) \end{aligned}$$

is open. Since $g|_W$ is injective on V_x we have that

$$\begin{aligned}(g|_{V_x})^{-1} \circ g|_{V_x}(f(U) \cap V_x) &= f(U) \cap V_x \\ &= g^{-1}(W) \cap V_x.\end{aligned}$$

So $V_x \cap f(U)$ is a subset of $f(U)$ which is an open neighborhood of $f(x)$. It follows that $f(U)$ is open. Hence f is open.

Using (i), Lemma 2.1 and Lemma 2.2 we can give an alternative proof. Consider the following diagram in **Sp**

$$\begin{tikzcd}
X \arrow[dr, dotted, "1_X \times_Z f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "1_X"'] & & \\
& X \times_Z Y \arrow[r, "\pi_Y"] \arrow[d, "\pi_X"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "g"] \\
& X \arrow[r, "k"'] & Z
\end{tikzcd} \tag{2.1}$$

Since g is an LH and k is open, π_X is an LH and π_Y is open. Then $1_X \times_Z f$ is an LH by (i), which is open by Lemma 2.2. Hence $f = \pi_Y \circ (1_X \times_Z f)$ is open as well.

(iii) : Let $U \subseteq Y$ be open, then as f is surjective $f\left[f^{-1}(U)\right] = U$. So $g(U) = g \circ f\left[f^{-1}(U)\right]$ is open.

(iv) : If follows from (iii) and Lemma 2.2 that g is open. Let $y \in Y$, then as f is surjective there is a $x \in X$ such that $f(x) = y$. Choose $V_x \subseteq X$ open such that $x \in V_x$ and $k|_{V_x}$ is a homeomorphism onto $k(V_x)$. Then $f(V_x)$ is an open neighborhood of y, and g is injective on this set, since k is injective on V_x. Thus g restricted to the open set $f(V_x)$ is open and injective and hence homeomorphism onto its image. So g is an LH. If follows from (i) that f is also an LH. □

Regarding (iii) in the preceeding lemma, we remark that a corresponding proposition holds for injective maps. That is, $g \circ f$ open and g injective implies f open (see *e.g.* [Bou89, Proposition I.5.1]).

The following lemma will also be useful.

Lemma 2.4. *Let X, X', Y and Z be topological spaces and $f : X \to Z$, $f' : X' \to Z$ and $k : X \to X'$ be maps such that $f = f' \circ k$. Let the following be pullback diagrams in* **Sp***:*

$$\begin{tikzcd}
Y \times_Z X \arrow[r, "\pi_X"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi'_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f'"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

If k is an LH, then the function $1_Y \times_Z k : Y \times_Z X \to Y \times_Z X'$ is an LH. If k is open, then $1_Y \times_Z k$ is open.

Proof: $1_Y \times_Z k$ denotes the unique map making the following diagram commute (in **Sp**)

$$
\begin{tikzcd}
Y \times_Z X \arrow[dr, dotted, "1_Y \times_Z k"] \arrow[drr, bend left, "k \circ \pi_X"] \arrow[ddr, bend right, "\pi_Y"'] & & \\
& Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi'_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f'"] \\
& Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

It follows from the so-called "pullback lemma" (or "two pullback lemma" or "pullback pasting lemma"), see *e.g.* [Gol84, 3.13] or [Mac97, Ex. III.4.8], that the top square in the following diagram (in **Sp**) is a pullback, since the outer rectangle and bottom square are pullbacks

$$
\begin{tikzcd}
Y \times_Z X \arrow[r, "\pi_X"] \arrow[d, "1_Y \times_Z k"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "k"] \\
Y \times_Z X' \arrow[r, "\pi_{X'}"] \arrow[d, "\pi_Y"'] \arrow[dr, phantom, "\lrcorner", very near start] & X' \arrow[d, "f'"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}
$$

From Lemma 2.1, $1_Y \times_Z k$ is open, respectively an LH, if k is. □

2.1.1 Quotient maps

If R is an equivalence relation on a space X, we denote the quotient space by X/R and the quotient map by $q : X \to X/R$. The equivalence class of an element $x \in X$ will be denoted $[x]_R$. For reference, we list a couple of basic facts about quotient maps.

Lemma 2.5. *Let X, Y be topological spaces and R be an equivalence relation on X. Then:*

(i) a function $f : X/R \to Y$ is continuous iff $f \circ q$ is continuous;

(ii) if g is a continuous function $X \to Y$ which is constant on the equivalence classes of R, then there exist a continuous function $f : X/R \to Y$ such that $g = f \circ q$.

Proof: See *e.g.* [GG99, Theorem 2.13.2–2.13.3]. □

Following [Bou89] we shall say that an equivalence relation is open if the corresponding quotient map is open.

Lemma 2.6. *For a topological space X, let R be an equivalence relation on X and $q : X \to X/R$ be the corresponding quotient map. Then:*

(i) q is open iff the restrictions of the projection maps $X \times X \to X$ to R are open;

(ii) R is open as a subset of $X \times X$ iff X/R is discrete;

(iii) q is open iff there exist an open map $k : X \to Y$ constant on the equivalence classes of R and such that R is an open subset of $X \times_Y X$.

Proof: (i): "$\Rightarrow$" If q is open, then as R is the pullback of q along itself, it follows from Lemma 2.1 that the projection maps π_1, π_2 are open:

$$\begin{tikzcd} R \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "q"] \\ X \arrow[r, "q"'] & X/R \end{tikzcd} \tag{2.2}$$

"$\Leftarrow$" Suppose the projection maps $\pi_1, \pi_2 : R \to X$ are open. For $U \subseteq X$ open we have that $q[U] \subseteq X/R$ is open if $q^{-1}(q[U])$ is open. But $q^{-1}(q[U])$ equals the set $\pi_2\left[\pi_1^{-1}(U)\right]$:

$$\begin{aligned} \pi_2\left[\pi_1^{-1}(U)\right] &= \pi_2\left[\{(x, y) \in R \mid x \in U\}\right] \\ &= \{y \in X \mid \exists x \in U\, [x \sim_R y]\}, \end{aligned}$$

which is open. Hence q is open.

(ii) : "$\Rightarrow$" If $R \subseteq X \times X$ is an open subset then the restrictions of the projection maps $X \times X \to X$ to R are open. By (i), q is open. By commutativity of

$$\begin{tikzcd} R \arrow[r, hook] \arrow[d, two heads, "q \circ \pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \times X \arrow[d, two heads, "q \times q"] \\ X/R \arrow[r, "\Delta"'] & X/R \times X/R \end{tikzcd}$$

where $\pi_1 : R \to X$ is the projection onto the first component, we get from Lemma 2.3 (iii) that the diagonal map Δ is open. Let $!_{X/R}$ be the unique map from X/R to the one point space. We have that $!_{X/R}$ is open and that $X/R \times X/R$ is the pullback of $!_{X/R}$ along itself. By Lemma 2.2, $!_{X/R}$ is an LH. This implies that X/R is discrete.

"$\Leftarrow$" If X/R is discrete, then the diagonal map $\Delta : X/R \to X/R \times X/R$ is open. From the following pullback and Lemma 2.1 we obtain that $R \subseteq X \times X$

is open:

$$\begin{tikzcd}
R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times X \arrow[d, "q \times q"] \\
X/R \arrow[r, "\Delta"'] & X/R \times X/R
\end{tikzcd}$$

(iii): "$\Rightarrow$" If q is open then q is such a map, for $R = X \times_{X/R} X$ is the pullback of $q : X \to X/R$ along itself, as in (2.2).

"$\Leftarrow$" Since k is open, the projection maps $\pi_1, \pi_2 : X \times_Y X \to X$ are open by Lemma 2.1. Since $R \subseteq X \times_Y X$ is open, the restrictions of π_1 and π_2 to R are open. By (i), q is open. $\square$

The following result is implicit in [For13] and [Moe88]:

Lemma 2.7. *Let X and Y be topological spaces and $k : X \to Y$ be an open map. If R is an equivalence relation on X such that k is constant on the equivalence classes of R and R is an open subset of $X \times_Y X$, then the induced map g, such that the diagram below commutes, is an LH.*

$$\begin{tikzcd}
X \arrow[rr, two heads, "q"] \arrow[dr, "k"'] & & X/R \arrow[dl, "g"] \\
& Y &
\end{tikzcd}$$

Proof: As q is surjective, it follows from Lemma 2.3 (iii) that g is open. From Lemma 2.6 (iii) it follows that q is open. By Lemma 2.2 it suffices to shows that the diagonal map $\Delta : X/R \to X/R \times_Y X/R$ is open to conclude that g is an LH. We show that $q \times_Y q : X \times_Y X \to X/R \times_Y X/R$ is open, and then it follows that Δ is open from the following commutative diagram and Lemma 2.3 (iii):

$$\begin{tikzcd}
R \arrow[r, hook] \arrow[d, two heads, "q \circ \pi_1"'] & X \times_Y X \arrow[d, "q \times_Y q"] \\
X/R \arrow[r, "\Delta"'] & X/R \times_Y X/R
\end{tikzcd}$$

where $\pi_1 : R \to X$ is the projection onto the first component.

Since $g \circ q = k$ and q is open, have that $1_X \times_Y q : X \times_Y X \to X \times_Y X/R$ is open by Lemma 2.4. A similar argument shows that $q \times_Y 1_{X/R} : X \times_Y X/R \to X/R \times_Y X/R$ is open. Hence $(q \times_Y 1_{X/R}) \circ (1_X \times_Y q) = q \times_Y q : X \times_Y X \to X/R \times_Y X/R$ is open. $\square$

2.2 Topological categories

A category where the set of objects and set of arrows are equipped with topologies that makes the structure maps continuous is called a topological category.

Alternatively, a topological category is a category object (or an internal category) in **Sp** (*cf.* [Mac97, XII.1] or [Joh02a, B2.3]). A topological groupoid is a topological category where every arrow is invertible, and the operation of inverting an arrow is continuous.

We shall denote a topological category by $\mathcal{C}$, or $\mathcal{C} : C_1 \rightrightarrows C_0$ when we wish to indicate that the space C_1 is the collection of arrows and the space C_0 is the collection of objects. When the category is a groupiod we instead use the symbols $\mathcal{G}, G_1$ and G_0 in the corresponding way. A topological category $\mathcal{C}$ thus corresponds to a diagram in **Sp** of the form

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m_C"] & C_1 \arrow[r, bend left, "t_C"] \arrow[r, bend right, "s_C"'] & C_0 \arrow[l, "u_C"'] \end{tikzcd}$$

where m_C is the composition, u_C is the insertion of identities, t_C is the codomain function and s_C is the domain function. For convenience we will, however, write $g \circ f$, 1_x and $f : x \to y$, for f, g in C_1 and x, y in C_0, in the usual way. For a groupoid we use $i_G : G_1 \to G_1$ for the inverse function $f \mapsto f^{-1}$.

A functor or morphism of topological categories $\phi : \mathcal{D} \to \mathcal{C}$ is a pair of maps $\phi_0 : D_0 \to C_0$ and $\phi_1 : D_1 \to C_1$ such that the expected diagrams commute (see [Mac97, XII.1]). Such morphisms are also called internal functors.

We shall denote the category of topological categories by **TCat** and the category of topological groupoids by **TGpd**.

2.3 Equivariant sheaves and $\mathcal{C}$-spaces

A $\mathcal{C}$-space on a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$ is a triple (e, E, α_e) where $e : E \to C_0$ is continuous and the action α_e is a continuous function $C_1 \times_{C_0} E \to E$ such that

$$\begin{aligned} e \circ \alpha_e(g, x) &= t_C(g), \\ \alpha_e(1_{e(x)}, x) &= x, \\ \alpha_e(f, \alpha_e(g, x)) &= \alpha_e(f \circ g, x), \end{aligned} \tag{2.3}$$

where the pullback $C_1 \times_{C_0} E$ is as in the diagram

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "\pi_E"] \arrow[d, "\pi_{C_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\ C_1 \arrow[r, "s_C"'] & C_0 \end{tikzcd}$$

A morphism of $\mathcal{C}$-spaces is a continuous function between spaces over C_0 that respect the action. That is, a morphism $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ is a map

$f : E \to A$ such that $e = a \circ f$ and the following diagram commutes

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "\alpha_e"] \arrow[d, "1_{C_1} \times_{C_0} f"'] & E \arrow[d, "f"] \\ C_1 \times_{C_0} A \arrow[r, "\alpha_a"'] & A \end{tikzcd} \tag{2.4}$$

We will also use the point-set equation expressed by the commutativity of the above diagram:

$$f \circ \alpha_e(k, x) = \alpha_a(k, f(x)) \tag{2.5}$$

where $(k, x) \in C_1 \times_{C_0} E$. We shall also call a morphism $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ an equivariant morphism or an equivarant map, and say that (2.4) and (2.5) expresses equivariance. The $\mathcal{C}$-spaces form a category that we denote $\mathbf{Sp}^{\mathcal{C}}$ for reasons that will be come clear in the next section.

An equivariant sheaf on $\mathcal{C}$, or a $\mathcal{C}$-sheaf, is a $\mathcal{C}$-space (e, E, α_e) where $e : E \to C_0$ is an LH. The equivariant sheaves on $\mathcal{C}$, and the equivariant maps between them, form a category denoted by $\mathrm{Sh}_{C_1}(C_0)$, and in the case of a groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$ by $\mathrm{Sh}_{G_1}(G_0)$.

By Lemma 2.3 (i) the morphisms in $\mathrm{Sh}_{C_1}(C_0)$ are also LH's, and we will also call such a map an equivariant LH.

2.3.1 Left $\mathcal{C}$-objects

Recall from [MM92] (or [Mac97]) that the internal functors in **Set** does not include functors $H : \mathbf{C} \to \mathbf{Set}$ (such as the hom-functors) for an internal category $\mathbf{C}$ in **Set**. The concept of such functors can be reformulated by replacing the object function $H_0 : C_0 \to \mathbf{Set}$ by a coproduct of sets and a projection, as in

$$F = \coprod_{c \in C_0} H_0(c) \to C_0, \quad (c, x) \mapsto c \text{ for } x \in H_0(c).$$

The arrow function H_1 can be described by a single function specifying the action of each arrow $f : c \to d$ in C_1 on elements $x \in H(c)$. This is an "action" $C_1 \times_{C_0} F \to F$ satisfying the conditions of (2.3).

This construction can be generalized to any category $\mathbf{E}$ with pullbacks and any internal category $\mathbf{C}$ in $\mathbf{E}$. A *left* $\mathbf{C}$*-object* in $\mathbf{E}$ is a triple (a, A, α_a) where $a : A \to C_0$ is a morphism in $\mathbf{E}$ and $\alpha_a : C_1 \times_{C_0} A \to A$ is a morphism in $\mathbf{E}$ that satisfies the conditions of (2.3) expressed in diagrammatic form.

A morphism $\phi : (e, E, \alpha_e) \to (a, A, \alpha_a)$ of left $\mathbf{C}$-objects is an arrow $\phi : E \to A$ in $\mathbf{E}$ that respects the action, in the sense of making the diagram (2.4) commute, and such that $e = a \circ \phi$. The left $\mathbf{C}$-objects in $\mathbf{E}$ form a category denoted $\mathbf{E}^{\mathbf{C}}$. If $\mathbf{E}$ is an elementary topos, this category $\mathbf{E}^{\mathbf{C}}$ of "internal presheaves" is again an elementary topos ([MM92, Theorem V.7.1]).

Thus, accordingly, we have the category of left $\mathcal{C}$-objects in **Sp** for any topological category $\mathcal{C}$. This category is clearly the category of $\mathcal{C}$-spaces.

We can without loss of generality restrict ourself to considering only left actions. For given an internal category $\mathbf{C}$ in a category $\mathbf{E}$ with pullbacks, there is an equivalence between the category of right $\mathbf{C}$-objects in $\mathbf{E}$ and left $\mathbf{C}^{\mathrm{op}}$-objects in $\mathbf{E}$ (*cf.* [MM92, V.7]).

2.3.2 Locales and sober topological spaces

We recall some properties of locales and sober topological spaces from [MM92]. A closed subset Y of topological space S is called irreducible if whenever F_1 and F_2 are closed sets such that $Y = F_1 \cup F_2$ then $Y = F_1$ or $Y = F_2$. A topological space S is called sober if every nonempty irreducible closed set is the closure of a unique point.

A frame is a lattice with all finite meets and all joins and that satisfies the infinite distributive law $U \wedge \bigvee_i V_i = \bigvee_i U \wedge V_i$. The category of locales **Loc** is the opposite of the category of frames and morphisms of frames. The functor $\mathrm{Loc} : \mathbf{Sp} \to \mathbf{Loc}$ that associates to a topological space S its locale $\mathrm{Loc}(S)$ of open sets has a right adjoint $\mathrm{pt} : \mathbf{Loc} \to \mathbf{Sp}$ that to each locale X associate the "space" of points of X. A point of a locale X is by definition a morphism $1 \to X$, where 1 is the terminal object in the category of locales.

The unit of the adjunction $\mathrm{Loc} \dashv \mathrm{pt}$ is a homeomorphism iff the space S is sober. For sober (topological) spaces the points of the space S is in a bijective correspondence with the points of the locale $\mathrm{Loc}(S)$.

But locales may have no points at all. A locale X is said to be spatial (or have enough points) when the counit of the adjunction $\mathrm{Loc} \dashv \mathrm{pt}$ is an isomorphism of locales. This is equivalent to X being isomorphic to $\mathrm{Loc}(S)$ for some topological space S. The full subcategory of **Sp** of sober topological spaces is equivalent to the full subcategory of **Loc** of spatial locales.

It is sometimes assumed that all considered topological spaces are sober (*e.g.* [Moe95, I.2], [Joh02b, C1.2]). We shall, however, make no such assumption.

2.4 Topos

In this thesis, topos will henceforth mean Grothendieck topos. The category $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves on a topological category $\mathcal{C}$ is known to be a topos. Existing proofs of the general case (for an arbitrary topological category $\mathcal{C}$) depend on the construction of $\mathrm{Sh}_{C_1}(C_0)$ as a colimit in the 2-category of (Grothendieck) toposes and geometric morphisms (and the existence of such colimits), *cf.* [Moe95, II.3] and [Moe88].

In this thesis we study properties of $\mathrm{Sh}_{C_1}(C_0)$ and, in the case of a topological category where the codomain map is assumed to be open, show, in a more direct and "elementary" way, that it indeed is a topos using Giraud's theorem, instead of as a colimit of toposes.

In [Moe95] Moerdijk gives a brief sketch of proof that $\mathrm{Sh}_{C_1}(C_0)$ is topos, also for a topological category $\mathcal{C}$ where the codomain map is assumed to be open, using Giraud's theorem.

2.5 Some useful functors

We list some relevant functors and some of their properties. As already mentioned, the forgetful functor $\mathbf{Sp} \to \mathbf{Set}$, which forgets the topology, preserves both limits and colimits ([Mac97, V.9]). There is a well-known equivalence of categories $\mathbf{LH}/X \cong \mathrm{Sh}(X)$ (*e.g.* [MM92, Corollary II.6.3]). In particular, $\mathbf{LH}/X$ has all small limits and colimits ([MM92, II.8, Proposition II.2.2]). The inclusion functor $i : \mathbf{LH}/X \hookrightarrow \mathbf{Sp}/X$ has a right adjoint and preserves finite limits ([MM92, II.9,Corollary II.6.3]).

If a category $\mathbf{E}$ has finite limits then the forgetful functor $\mathbf{E}/B \to \mathbf{E}$, taking an object A over B to A, has a right adjoint ([MM92, I.9]) and preserves pullbacks ([Joh02a, A1.2]). Furthermore, a slice category $\mathbf{E}/B$ has finite limits iff $\mathbf{E}$ has pullbacks ([Joh02a, A1.2.6]). Hence the forgetful functor $\mathbf{Sp}/X \to \mathbf{Sp}$ preserves colimits and pullbacks.

For an internal category $\mathbf{C}$ in a category $\mathbf{E}$ with pullbacks, the forgetful functor $U_E : \mathbf{E}^{\mathbf{C}} \to \mathbf{E}/C_0$ of forgetting the action has a left adjoint ([MM92, Theorem V.7.2]).

In the next chapter we prove that the forgetful functor $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$, of forgetting the action, preserves finite limits and small colimits. We denote the functor $\mathbf{Sp}^{\mathcal{C}} \to \mathbf{Sp}/C_0$ which forgets the action by U'.

We remark that there is also the forgetful functor $V : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{Set}^{\mathcal{C}}$ which forgets the topology. As noticed in *e.g.* [For13], for the case of topological groupoids, V is conservative and the inverse image part of geometric morphism. We will will not prove these results for V as we will not use them. Given the explicit construction of finite limits and small colimits in $\mathrm{Sh}_{C_1}(C_0)$ in the proof of Theorem 3.8 it is straightforward to prove that V preserve these limits. One may then proceed as in Corollary 3.10 to show that V has a right adjoint.

Lemma 2.8. *For a morphism f in $\mathbf{LH}/C_0$ or $\mathbf{Sp}/C_0$:*

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "e"'] & & A \arrow[dl, "a"] \\ & C_0 & \end{tikzcd}$$

(i) f is monic iff $f : E \to A$ is an injective function,

(ii) f is epic iff $f : E \to A$ is a surjective function,

(iii) f is an isomorphism iff $f : E \to A$ is a homeomorphism.

Proof: Monics, epics and isomorphisms in **Set** are the injective, surjective and bijective functions, respectively.

"$\Rightarrow$": Recall that in an arbitrary category a morphism $f : E \to A$ is monic iff the following diagram is a pullback

$$\begin{tikzcd} E \arrow[r, "1_E"] \arrow[d, "1_E"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "f"] \\ E \arrow[r, "f"'] & A \end{tikzcd} \tag{2.6}$$

and $f : E \to A$ is an epic iff the following diagram is a pushout:

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "f"'] & A \arrow[d, "1_A"] \\ A \arrow[r, "1_A"'] \arrow[ur, phantom, "\lrcorner"] & A \end{tikzcd} \tag{2.7}$$

By the text preceeding the lemma, the following inclusion functor and forgetful functors all preserve pullbacks and colimits

$$\mathbf{LH}/C_0 \hookrightarrow \mathbf{Sp}/C_0 \to \mathbf{Sp} \to \mathbf{Set}. \tag{2.8}$$

Hence, if f is monic the diagram in (2.6) is pullback in **Set**, so f is an injective function. If f is an epic, the diagram in (2.7) is a pushout in **Set**, so f is surjective function. An isomorphism in $\mathbf{LH}/C_0$ or $\mathbf{Sp}/C_0$ is a continuous function with a continuous inverse, which is a homeomorphism.

"$\Leftarrow$": Faithful functors reflect monics and epics. The inclusion functor $\mathbf{LH}/C_0 \hookrightarrow \mathbf{Sp}/C_0$ and the forgetful functors $\mathbf{Sp}/C_0 \to \mathbf{Sp}$ and $\mathbf{Sp} \to \mathbf{Set}$ are obviously all faithful. Composites of faithful functors are also faithful. So if f in $\mathbf{LH}/C_0$ or $\mathbf{Sp}/C_0$ is an injective (surjective) function, then f is monic (epic). If f is homeomorphism then f is clearly an isomorphism in $\mathbf{LH}/C_0$ or $\mathbf{Sp}/C_0$. $\square$

Regarding U, the following is also noticed in *e.g.* [For13]:

Proposition 2.9. *The forgetful functors* $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$ *and* $U' : \mathbf{Sp}^{\mathcal{C}} \to \mathbf{Sp}/C_0$*, of forgetting the action, are conservative.*

Proof: We must show that the functors are faithful and reflect isomorphisms. It is clear that U and U' are faithful. To show that they reflect isomorphisms, let $\phi : (e, E, \alpha_e) \to (a, A, \alpha_a)$ be a morphism in $\mathbf{Sp}^{\mathcal{C}}$ (or in $\mathrm{Sh}_{C_1}(C_0)$) such that $\phi : E \to A$ is an isomorphism in $\mathbf{Sp}/C_0$ (or $\mathbf{LH}/C_0$). Then ϕ is a homeomorphism. If $(f, y) \in C_1 \times_{C_0} A$, $x \in E$ and $\phi(x) = y$ then since

$$\phi \circ \alpha_e(f, x) = \alpha_a(f, \phi(x))$$

we have

$$\alpha_e(f, \phi^{-1}(y)) = \phi^{-1} \circ \alpha_a(f, y)$$

so $\phi^{-1} : A \to E$ is also equivariant and hence is an isomorphism in $\mathbf{Sp}^{\mathcal{C}}$ (or in $\mathrm{Sh}_{C_1}(C_0)$). $\square$

Chapter 3

Properties of equivariant sheaves on topological categories

In this chapter we prove a canonical isomorphism of $\mathcal{C}$-spaces and investigate how certain properties of equivariant sheaves depend on the openness of the underlying topological category. We prove the existence of finite limits and all (small) colimits in $\mathrm{Sh}_{C_1}(C_0)$ and that the forgetful functor $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$ (of forgetting the action) preserve these (co)limits.

3.1 A canonical isomorphism of $\mathcal{C}$-spaces

By the following theorem, every $\mathcal{C}$-space can be regarded as a quotient space with an action induced by the operation of composition of arrows in $\mathcal{C}$.

Theorem 3.1. *For a $\mathcal{C}$-space (e, E, α_e), let $D = C_1 \times_{C_0} E$ be the following pullback*

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "\pi_E"] \arrow[d, "\pi_{C_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\ C_1 \arrow[r, "s_C"'] & C_0 \end{tikzcd} \tag{3.1}$$

Let R be the equivalence relation on D given by

$$(f, x) \sim_R (g, y) \quad \textit{iff} \quad \alpha_e(f, x) = \alpha_e(g, y), \tag{3.2}$$

and D/R be the quotient space. Then D/R is a $\mathcal{C}$-space when equipped with

$$\begin{aligned} [t_C] &: D/R \to C_0, & [(f, x)]_R &\mapsto t_c(f), \\ \alpha_d &: C_1 \times_{C_0} D/R \to D/R, & (g, [(f, x)]_R) &\mapsto [(g \circ f, x)]_R, \end{aligned}$$

and the function $[\alpha_e] : D/R \to E$, induced by the action α_e on E and taking $[(f, x)]_R \mapsto \alpha_e(f, x)$, is an isomorphism of $\mathcal{C}$-spaces $([t_C], D/R, \alpha_d) \cong (e, E, \alpha_e)$.

Proof: The function $[\alpha_e]$ is clearly well defined, it is also continuous since $[\alpha_e] \circ q = \alpha_e : D \to E$ is continuous (Lemma 2.5), where $q : D \to D/R$ is the quotient map.

Furthermore, $[\alpha_e]$ has a continuous inverse given by $x \mapsto [(1_{e(x)}, x)]_R$, which is the composition of the following maps

$$E \xrightarrow[\sim]{e \times_{C_0} 1_E} C_0 \times_{C_0} E \xrightarrow{u_C \times_{C_0} 1_E} C_1 \times_{C_0} E \xrightarrow{q} D/R\ .$$

Indeed, we have that $(f, x) \sim_R (1_{t_C(f)}, \alpha_e(f, x))$.

This shows that $[\alpha_e]$ is a homeomorphism. The function $[t_C]$ is also well-defined, and since the following diagram commutes

$$\begin{tikzcd} D/R \arrow[rr, "{[\alpha_e]}", "\sim"'] \arrow[dr, "{[t_C]}"'] & & E \arrow[dl, "e"] \\ & C_0 & \end{tikzcd}$$

we have that $[t_C]$, which sends $[(f, x)]_R$ to $t_C(f)$, is continuous. Furthermore, $[t_C]$ an LH if e is an LH.

The quotient D/R carries a natural action α_d, which is induced by α_e:

$$\begin{tikzcd} C_1 \times_{C_0} D/R \arrow[r, dotted, "\alpha_d"] \arrow[d, "1_{C_1} \times_{C_0} {[\alpha_e]}"'] & D/R & (g, [(f, x)]_R) \arrow[r, dotted, mapsto] \arrow[d, mapsto] & {[(g \circ f, x)]_R} \\ C_1 \times_{C_0} E \arrow[r, "\alpha_e"'] & E \arrow[u, "{[\alpha_e]^{-1}}"'] & (g, \alpha_e(f, x)) \arrow[r, mapsto] & \alpha_e(g \circ f, x) \arrow[u, mapsto] \end{tikzcd}$$

This shows continuity of α_d which clearly also satisfies the conditions of being an action, given in (2.3).

Now, $[\alpha_e] : D/R \to E$ respects the action by construction and since the forgetful functor $U' : \mathbf{Sp}^{\mathcal{C}} \to \mathbf{Sp}/C_0$ is conservative (Proposition 2.9), $[\alpha_e]$ is an isomorphism of $\mathcal{C}$-spaces. $\square$

In particular, the conclusion of the theorem applies to equivariant sheaves on $\mathcal{C}$. For (e, E, α_e) in $\mathrm{Sh}_{C_1}(C_0)$ we thus have $(e, E, \alpha_e) \cong ([t_C], D/R, \alpha_d)$.

Proposition 3.2. *For $(e, E, \alpha_e) \in \mathbf{Sp}^{\mathcal{C}}$, let $D = C_1 \times_{C_0} E$ be the pullback in* (3.1) *and R be the equivalence relation on D in* (3.2)*. Then the quotient map $q : D \to D/R$ is open iff the action α_e is open.*

Proof: The statement follows from the following commutative diagram

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[d, two heads, "q"'] \arrow[dr, "\alpha_e"] & \\ D/R \arrow[r, "\sim", "{[\alpha_e]}"'] & E \end{tikzcd}$$

$\square$

3.2 Openess of $\mathcal{C}$ and equivariant sheaves

Note that when $e : E \to C_0$ is an LH, the projection $\pi_{C_1} : C_1 \times_{C_0} E \to C_1$ is an LH:

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "\pi_E"] \arrow[d, "\pi_{C_1}"' , "\bullet" description] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e", "\bullet" description] \\ C_1 \arrow[r, "s_C"'] & C_0 \end{tikzcd} \tag{3.3}$$

Proposition 3.3. *For* $(e, E, \alpha_e) \in \mathrm{Sh}_{C_1}(C_0)$,

(i) if t_C *is open then* α_e *is open,*

(ii) if e *is surjective, then* α_e *is open iff* t_C *is open.*

Proof: (i): Consider the following diagram:

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "\pi_{C_1}", "\bullet" description] \arrow[d, "\alpha_e"', two heads] & C_1 \arrow[d, "t_C", two heads] \\ E \arrow[r, "e"', "\bullet" description] & C_0 \end{tikzcd} \tag{3.4}$$

If t_C is open, so is $t_C \circ \pi_{C_1}$. As e is an LH it follows from Lemma 2.3 (ii) that α_e is open.
(ii): "$\Leftarrow$": This is (i). "$\Rightarrow$": In the diagram (3.4), if α_e is open then $t_C \circ \pi_{C_1} = e \circ \alpha_e$ is open. If e is surjective we have that π_{C_1} is also surjective. It follows from Lemma 2.3 (iii) that t_C is open.

(ii) also follows from Proposition 3.6 (ii) with the collection consisting of only (e, E, α_e). $\square$

If the topological category is a groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$ more can be shown. We say that a topological groupoid is *open* when the domain and codomain functions are open maps. For an open localic groupoid $\mathcal{G}$, the statements of the following proposition are mentioned in [Moe88]. We supply direct proof for these statements, for the case of topological groupoids.

Proposition 3.4. *For a topological groupoid* $\mathcal{G} : G_1 \rightrightarrows G_0$*:*

(i) s_G *is open iff* t_G *is open;*

(ii) if $\mathcal{G}$ *is open then* m_G *is open;*

(iii) if $\mathcal{G}$ *is open and* (e, E, α_e) *is a* $\mathcal{G}$*-space, then* α_e *is open.*

Proof: (i): The inverse map i_G is a homemorphism $G_1 \to G_1$ such that $i_G^{-1} = i_G$, and $s_G = t_G \circ i_G$.

(iii): Consider the following commutative diagram in **Sp**

$$\begin{tikzcd}
G_1 \times_{G_0} E \arrow[dr, dotted, "\theta", "\sim"'] \arrow[drr, bend left, "\alpha_e"] \arrow[ddr, bend right, "i_G \circ \pi_{G_1}"'] & & \\
& G_1 \times_{G_0} E \arrow[r, "\pi_2"] \arrow[d, "\pi_{G_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\
& G_1 \arrow[r, "s_G"'] & G_0
\end{tikzcd} \tag{3.5}$$

where θ is the unique map such that $\alpha_e = \pi_2 \circ \theta$ and $i_G \circ \pi_{G_1} = \pi_{G_1} \circ \theta$. The map θ is thus given by $\theta(f, x) = (f^{-1}, \alpha_e(f, x))$. We have that $\theta \circ \theta(f, x) = (f, x)$, so θ is a homeomorphism. It follows that α_e is open.

(ii): Follows from (iii) with the $\mathcal{G}$-space (t_G, G_1, m_G). $\square$

Proposition 3.5. *For an equivariant morphism* $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ *in* $\mathrm{Sh}_{C_1}(C_0)$*,*

(i) if α_a *is open then* α_e *is open;*

(ii) if α_e *is open and* f *surjective, then* α_a *is open.*

Proof: The observations (i) and (ii) can be deduced from the basic diagram expressing equivariance of f together with Lemma 2.3 and Lemma 2.4:

$$\begin{tikzcd}
C_1 \times_{C_0} E \arrow[r, "\alpha_e"] \arrow[d, "1_{C_1} \times_{C_0} f"'] & E \arrow[d, "f"] \\
C_1 \times_{C_0} A \arrow[r, "\alpha_a"'] & A
\end{tikzcd} \tag{3.6}$$

(i): $1_{C_1} \times_{C_0} f$ is an LH by Lemma 2.4, so if α_a is open, we have that $f \circ \alpha_e$ is open. Since f is an LH, α_e is open by Lemma 2.3 (ii).

(ii): If f is surjective, so is $1_{C_1} \times_{C_0} f$. Since $\alpha_a \circ (1_{C_1} \times_{C_0} f)$ is open, it follows from Lemma 2.3 (iii) that α_a is open. $\square$

Proposition 3.6. *Let* $\mathcal{C} : C_1 \rightrightarrows C_0$ *be a topological category.*

(i) For (e, E, α_e) *in* $\mathrm{Sh}_{C_1}(C_0)$, α_e *is open iff* t_C *restricted to the (open) set* $s_C^{-1}(e(E))$ *is open.*

(ii) If C_0 *can be covered by the union* $\bigcup e_i(E_i)$ *of the images* $e_i(E_i)$ *of a (small) collection of equivariant sheaves* (e_i, E_i, α_i) *where each action* α_i *is open, then* t_C *is open.*

Proof: (i): Let $e(E) = U$. Since (e, E, α_e) is an equivariant sheaf, all arrows in $\mathcal{C}$ starting in U also end in U. In other words, we have that $t_C \circ s_C^{-1}(U) = U$. Furthermore, the set $s_C^{-1}(U)$ is closed under composition of arrows and contains

$u_C(U)$ as a subset. We thus obtain a subcategory $\mathcal{C}_U$ of $\mathcal{C}$ with the space of objects U and space of arrows $s_C^{-1}(U)$ with the structure maps of $\mathcal{C}$ restricted to these sets (and to $s_C^{-1}(U) \times_U s_C^{-1}(U)$ in case of composition).

Observe that $e : E \to C_0$ is an LH also when regarded as function with codomain U. Indeed, let $e_U : E \to U$ be e with codmain U, so that $e_U(x) = e(x)$. As $U \subseteq C_0$ is open, the inclusion $i_U : U \to C_0$ is an LH. It follows that e_U is an LH, since $e = i_U \circ e_U$.

Furthermore, $s_C^{-1}(U) \times_U E$ is identical to $C_1 \times_{C_0} E$ as a subset of $C_1 \times E$, so α_e is defines an action $\alpha_e : s_C^{-1}(U) \times_U E \to E$. This means that (e_u, E, α_e) is an equivariant sheaf on $\mathcal{C}_U$. The statement now follows from Proposition 3.3 (*ii*).

(*ii*): If (e_i, E_i, α_i), for $i \in I$ for some (small) set I, is a collection of equivariant sheaves on $\mathcal{C}$ such that each α_i is open and the images $e_i(E_i) = U_i$ covers C_0, then the open sets $s_C^{-1}(U_i)$ covers C_1 and t_C restricted any of the sets $s_C^{-1}(U_i)$ is open by (*i*). Then, for $V \subseteq C_1$ open we have

$$\begin{aligned} t_C(V) &= t_C(V \cap C_1) \\ &= t_C\left(\bigcup_{i\in I}\left(V \cap s_C^{-1}(U_i)\right)\right) \\ &= \bigcup_{i \in I} t_C\left(V \cap s_C^{-1}(U_i)\right), \end{aligned}$$

which is open. Hence t_C is open.

Using that coproducts exists in $\mathrm{Sh}_{C_1}(C_0)$, which is proved in Theorem 3.8 below, we can give another proof. The action on the coproduct $(e, \coprod_j E_j, \alpha_e) = \coprod_{j\in J}(e_j, E_j, \alpha_j)$ satisfies

$$\alpha_e(U) = \bigcup_{j \in J} i_j \circ \alpha_j\left((1_{C_1} \times_{C_0} i_j)^{-1}(U)\right)$$

for $U \subseteq C_1 \times_{C_0} \coprod_{j\in J} E_j$, where $i_j : E_j \to \coprod_{j\in J} E_j$ are the coproduct inclusions. Since i_j is an LH, we have that α_e is open if each α_j is open. The statement now follows from Proposition 3.3 (*ii*). □

A topological category where the domain map is an LH is called s-étale in [Moe95]. That a topological groupoid $\mathcal{G}$ is s-étale iff all the structure maps are LH's is mentioned in [Moe95, II.4]. We give a direct proof of this statement in the next proposition, which also shows that in this case $\mathrm{Sh}_{G_1}(G_0) = \mathbf{LH}^{\mathcal{G}}$.

From the following proposition we can conclude that in the case of a s-étale topological groupoid $\mathcal{G}$, the forgetful functor $U : \mathrm{Sh}_{G_1}(G_0) \to \mathbf{LH}/G_0$ has a left adjoint (see Section 2.5). In fact, U also has a right adjoint, for we shall see in Corollary 3.10 that $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$ has a right adjoint for any topological category $\mathcal{C} : C_1 \rightrightarrows C_0$.

Proposition 3.7. *Let $\mathcal{G} : G_1 \rightrightarrows G_0$ be a topological groupoid where t_G or s_G is an LH. Then $\mathcal{G}$ is a groupoid object in* **LH** *and $\mathrm{Sh}_{G_1}(G_0)$ is the category $\mathbf{LH}^{\mathcal{G}}$.*

Proof: **LH** has pullbacks, which are the pullback in **Sp** (see Lemma 2.1).

We have that s_G is an LH iff t_G is an LH, as i_G is a homeomorphism and $s_G = t_G \circ i_G$. In the diagram (3.5), let $(e, E, \alpha_e) = (t_G, G_1, m_G)$. If s_G and t_G are LH's, then π_{G_1} and π_2 are LH's. So $m_G = \pi_2 \circ \theta$ is also an LH. Since $1_{G_0} = t_G \circ u_G$, u_G is an LH. Hence all the structure maps of $\mathcal{G}$ are LH's. It is straightforward to verify that the arrows in the diagrams expressing that $\mathcal{G}$ is an internal groupoid are all LH's (see [Mac97, XII.1]).

For an equivariant sheaf (e, E, α_e) on $\mathcal{G}$, the projection $\pi_{G_1} : G_1 \times_{G_0} E \to G_1$ is an LH. Since $t_G \circ \pi_{G_1} = e \circ \alpha_e$, the action α_e is also an LH. Furthermore, if ϕ is an equivariant morphism, then $1_{G_1} \times_{G_0} \phi$ is an LH by Lemma 2.4.

The notion of "left **C**-objects" can be applied to any category with pullbacks. It is now clear from Section 2.3.1 that $\mathrm{Sh}_{G_1}(G_0)$ is the category of left $\mathcal{G}$-objects in **LH**. $\square$

3.3 Finite limits and colimits in $\mathrm{Sh}_{C_1}(C_0)$

In [Moe95, Proposition II.3.2] Moerdijk gives a brief description of how to construct finite limits and colimits in $\mathrm{Sh}_{C_1}(C_0)$. For the corresponding diagram in $\mathbf{LH}/C_0$ there is a unique action making the limit in $\mathbf{LH}/C_0$ a limit in $\mathrm{Sh}_{C_1}(C_0)$. We explicitly construct these limits in the proof of the following theorem.

Theorem 3.8. $\mathrm{Sh}_{C_1}(C_0)$ *has all finite limits and all small colimits and the forgetful functor U preserves these (co)limits.*

Proof: It will be clear from construction that U preserves the (co)limits in question.

Finite limits

It is sufficient to show that $\mathrm{Sh}_{C_1}(C_0)$ has pullbacks and a terminal object, since this implies that $\mathrm{Sh}_{C_1}(C_0)$ has all finite limits.

The terminal object in $\mathbf{LH}/C_0$ is $1_{C_0} : C_0 \to C_0$. Given an LH $e : E \to C_0$, the unique morphism in $\mathbf{LH}/C_0$ to the terminal object is e. If $1_{C_0} : C_0 \to C_0$ has an action α_1, requiring equivarance of the map $e : E \to C_0$, for an equivarant sheaf (e, E, α_e), we would have that $\alpha_1(f, s_C(f)) = t_C(f)$.

This is indeed an action on C_0. With the pullback $C_1 \times_{C_0} C_0$ of 1_{C_0} along s_C, we have that $\alpha_1 = t_C \circ \pi_{C_1} : C_1 \times_{C_0} C_0 \to C_0$, where $\pi_{C_1} : C_1 \times_{C_0} C_0 \to C_1$ is the projection. The map α_1 obviously satisfies the conditions of being an action, given in (2.3), and if (e, E, α_e) is an object in $\mathrm{Sh}_{C_1}(C_0)$ then $e : E \to C_0$ is also the unique equivariant morphism in $\mathrm{Sh}_{C_1}(C_0)$ to $(1_{C_0}, C_0, \alpha_1)$. So $\mathrm{Sh}_{C_1}(C_0)$ has a terminal object.

Concerning pullbacks, recall from Section 2.5 that the following inclusion functor and forgetful functors all preserve pullbacks

$$\mathbf{LH}/C_0 \hookrightarrow \mathbf{Sp}/C_0 \to \mathbf{Sp} \to \mathbf{Set}. \tag{3.7}$$

Given a pair of equivariant morphisms

$$(a, A, \alpha_a) \xrightarrow{\ f\ } (e, E, \alpha_e) \xleftarrow{\ g\ } (b, B, \alpha_b)$$

we obtain the corresponding pullback in $\mathbf{LH}/C_0$, which is the sheaf $P = A \times_E B$ with map $p : P \to C_0$ the indicated arrow making following diagram commutative:

$$\begin{tikzcd}
P \arrow[rr, "\pi_B"] \arrow[dr, dotted, "p"] \arrow[dd, "\pi_A"'] & & B \arrow[dl, "b"'] \arrow[dd, "g"] \\
& C_0 & \\
A \arrow[ur, "a"] \arrow[rr, "f"'] & & E \arrow[ul, "e"']
\end{tikzcd}$$

The equivariance of f and g imply that for an element $(k, (x, y)) \in C_1 \times_{C_0} P$ we have

$$\begin{aligned}
g \circ \alpha_b(k, y) &= \alpha_e(k, g(y)) \\
&= \alpha_e(k, f(x)) \\
&= f \circ \alpha_a(k, x).
\end{aligned}$$

We obtain the action on P as the unique map making the following diagram commute in **Sp**:

$$\begin{tikzcd}
C_1 \times_{C_0} P \arrow[dr, dotted, "\alpha_p"] \arrow[drr, bend left, "\alpha_b \circ (1_{C_1} \times_{C_0} \pi_B)"] \arrow[ddr, bend right, "\alpha_a \circ (1_{C_1} \times_{C_0} \pi_A)"'] & & \\
& P \arrow[r, "\pi_B"] \arrow[d, "\pi_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & E
\end{tikzcd}$$

On elements α_p is given by

$$\alpha_p(k, (x, y)) = (\alpha_a(k, x), \alpha_b(k, y)).$$

By construction, this action makes the projections π_A and π_B equivariant. Since α_a and α_b are actions, α_p satisfies the conditions of being an action in (2.3).

To show that this is a limit in $\mathrm{Sh}_{C_1}(C_0)$ suppose $r : (p', P', \alpha'_p) \to (a, A, \alpha_a)$ and $s : (p', P', \alpha'_p) \to (b, B, \alpha_b)$ are equivariant morphisms such that $f \circ r = g \circ s$. Since P is a pullback in $\mathbf{LH}/C_0$ we obtain a unique LH $\theta : P' \to P$ such that $\pi_A \circ \theta = r$ and $\pi_B \circ \theta = s$. So $\theta(z) = (r(z), s(z))$ for $z \in P'$. It remains to show that θ is equivariant.

But r and s are by assumption equivariant so, for $(k, z) \in C_1 \times_{C_0} P'$,

$$\begin{aligned}
\theta \circ \alpha'_p(k, z) &= \left(r \circ \alpha'_p(k, z), s \circ \alpha'_p(k, z)\right) \\
&= (\alpha_a(k, r(z)), \alpha_b(k, s(z))) \\
&= \alpha_p(k, \theta(z)).
\end{aligned}$$

This completes the proof of the existence of finite limits.

Colimits

It is sufficent to show the existence of all small coproducts and coequalizers for all parallel pair of arrows. Recall that the inclusion functor and the forgetful functors in (3.7) all preserve colimits.

The map $s_C : C_1 \to C_0$ induces a functor $s_C^* : \mathbf{LH}/C_0 \to \mathbf{LH}/C_1$ by taking pullbacks along s_C. This functor is the inverse image of a geometric morphism between the toposes $\mathbf{LH}/C_0$ and $\mathbf{LH}/C_1$ ([MM92, Theorem II.9.2]). In particular, s_C^* has a right adjoint and hence preserves colimits.

We can, however, observe directly that the initial object in $\mathbf{LH}/C_0$, which is the empty space $\emptyset$ over C_0, together with the empty action yields the initial object in $\mathrm{Sh}_{C_1}(C_0)$. The unique map to any $\mathcal{C}$-sheaf is also the empty map, which is clearly equivariant.

We now consider arbitrary coproducts. Let $\{(e_j, E_j, \alpha_j)\}_{j\in J}$ be a collection of equivariant sheaves, where J is a set. Let $e : E \to C_0$ be the coproduct of $e_j : E_j \to C_0$ in $\mathbf{LH}/C_0$, with i_j the corresponding coproduct inclusions. By the above remark, this is the coproduct in $\mathbf{Sp}$ (and $\mathbf{Set}$), so $E = \coprod_{j\in J} E_j$. We shall write the disjoint union as consisting of elements (j, y) where $j \in J$ and $y \in E_j$. Then $e : \coprod E_j \to C_0$ takes $(j, y) \mapsto e_j(y)$ and the coproduct inclusions satisfy $i_j(y) = (j, y)$, for $y \in E_j$.

Notice that if α_e is an action on E such that all the coproduct inclusions i_j are equivariant, then for each $j \in J$ we have

$$\alpha_e \circ (1_{C_1} \times_{C_0} i_j) = i_j \circ \alpha_j. \tag{3.8}$$

This determines α_e on the objects of $C_1 \times_{C_0} E$.

As s_C^* preserves colimits we have that the induced map θ such that the following diagram commutes, for each $j \in J$, is a homeomorphism

$$\begin{tikzcd}
C_1 \times_{C_0} E_j \arrow[r, "i'_j"] \arrow[dr, "1_{C_1} \times_{C_0} i_j"'] & \coprod_j (C_1 \times_{C_0} E_j) \arrow[d, "\theta", "\sim"'] \\
& C_1 \times_{C_0} \coprod_j E_j
\end{tikzcd}$$

where i'_j are the coproduct inclusions $C_1 \times_{C_0} E_j \to \coprod_j (C_1 \times_{C_0} E_j)$.

The collection of maps α_j induces a unique map α_e in $\mathbf{Sp}$ such that the following diagram commutes, for each $j \in J$,

$$\begin{tikzcd}
C_1 \times_{C_0} E_j \arrow[r, "1_{C_1} \times_{C_0} i_j"] \arrow[dr, "i_j \circ \alpha_j"'] & C_1 \times_{C_0} \coprod_j E_j \arrow[d, "\alpha_e"] \\
& \coprod_j E_j
\end{tikzcd}$$

This action satisfies equation (3.8), and thus makes each i_j equivariant. We verify that α_e satisfies the conditions of being an action, where $z \in E_j$ and

$e_j(z) = w$,

$$\begin{aligned}
e \circ \alpha_e(g, (j, z)) &= e \circ i_j \circ \alpha_j(g, z) \\
&= e_j \circ \alpha_j(g, z) \\
&= t_C(g), \\
\alpha_e(1_w, (j, z)) &= i_j \circ \alpha_j(1_w, z) \\
&= (j, z), \\
\alpha_e(h, \alpha_e(g, (j, z))) &= \alpha_e(h, (j, \alpha_j(g, z))) \\
&= i_j \circ \alpha_j(h, \alpha_j(g, z)) \\
&= i_j \circ \alpha_j(h \circ g, z) \\
&= \alpha_e(h \circ g, (j, z)).
\end{aligned}$$

To prove that (e, E, α_e) is a colimit in $\mathrm{Sh}_{C_1}(C_0)$, let $i''_j : (e_j, E_j, \alpha_j) \to (y, Y, \alpha_y)$ be another cocone. Then this yields a unique morphism $\phi : E \to Y$ in $\mathbf{LH}/C_0$ such that $\phi \circ i_j = i''_j$ for each $j \in J$. And since

$$i''_j \circ \alpha_j = \alpha_y \circ (1_{C_1} \times_{C_0} i''_j)$$

holds by assumption for all $j \in J$, we get that ϕ is equivariant, that is:

$$\phi \circ \alpha_e = \alpha_y \circ (1_{C_1} \times_{C_0} \phi).$$

Indeed, for $z \in E_j$

$$\begin{aligned}
\phi \circ \alpha_e\,(g, (j, z)) &= \phi \circ i_j \circ \alpha_j(g, z) \\
&= i''_j \circ \alpha_j(g, z) \\
&= \alpha_y(g, i''_j(z)) \\
&= \alpha_y(g, \phi \circ i_j(z)) \\
&= \alpha_y(g, \phi(j, z)).
\end{aligned}$$

Hence ϕ is a morphism of equivariant sheaves. This completes the proof of the existence of coproducts.

We thus turn to coequalizers. Since the sequence of functors in (3.7) preserves colimits, a coequalizer diagram in $\mathbf{LH}/X$ yields a coequalizer diagram in $\mathbf{Sp}$ and $\mathbf{Set}$. Let $f, g : (e, E, \alpha_e) \to (a, A, \alpha_a)$ be a pair of parallel arrows in $\mathrm{Sh}_{C_1}(C_0)$. Then the coequalizer $q : A \to A/R$ of f and g exist in $\mathbf{LH}/C_0$ and makes the following diagram commute

$$\begin{tikzcd}
E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr, "e"'] & A \arrow[r, two heads, "q"] \arrow[d, "a"] & A/R \arrow[dl, "r"] \\
& C_0 &
\end{tikzcd}$$

where R is the least equivalence relation generated by the relation R' on A where $f(z) \sim_{R'} g(z)$ for $z \in E$.

Since s_C^* preserves colimits, we have that

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, shift left, "1_{C_1} \times_{C_0} f"] \arrow[r, shift right, "1_{C_1} \times_{C_0} g"'] & C_1 \times_{C_0} A \arrow[r, two heads, "1_{C_1} \times_{C_0} q"] & C_1 \times_{C_0} A/R \end{tikzcd}$$

is a co-equalizer diagram (over C_1) in $\mathbf{LH}/C_1$, and in $\mathbf{Sp}$.

The equivariance of f and g implies that for $(k, z) \in C_1 \times_{C_0} E$

$$\begin{aligned} \alpha_a(k, f(z)) &= f \circ \alpha_e(k, z), \\ \alpha_a(k, g(z)) &= g \circ \alpha_e(k, z). \end{aligned}$$

So $\alpha_a(k, f(z)) \sim_{R'} \alpha_a(k, g(z))$, and hence $q \circ \alpha_a(k, f(z)) = q \circ \alpha_a(k, g(z))$. We now obtain the action α_q on $C_1 \times_{C_0} A/R$ as the unique map making the following diagram commute in $\mathbf{Sp}$

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, shift left, "1_{C_1} \times_{C_0} f"] \arrow[r, shift right, "1_{C_1} \times_{C_0} g"'] & C_1 \times_{C_0} A \arrow[r, two heads, "1_{C_1} \times_{C_0} q"] \arrow[dr, "q \circ \alpha_a"'] & C_1 \times_{C_0} A/R \arrow[d, dotted, "\alpha_q"] \\ & & A/R \end{tikzcd}$$

Then, by construction, q is equivariant. The map α_q, furthermore, satisfies the conditions of an being action:

$$\begin{aligned} r \circ \alpha_q(k, [x]_R) &= r \circ \alpha_q \circ (1_{C_1} \times_{C_0} q)(k, x) \\ &= r \circ q \circ \alpha_a(k, x) \\ &= a \circ \alpha_a(k, x) \\ &= t_C(k), \\ \alpha_q(1_{r([x]_R)}, [x]_R) &= q \circ \alpha_a(1_{a(x)}, x) \\ &= [x]_R, \\ \alpha_q(h, \alpha_q(k, [x]_R)) &= \alpha_q(h, q \circ \alpha_a(k, x)) \\ &= q \circ \alpha_a(h \circ k, x) \\ &= \alpha_q(h \circ k, [x]_R). \end{aligned}$$

To show that $(r, A/R, \alpha_q)$ is a colimit in $\mathrm{Sh}_{C_1}(C_0)$, suppose $s : (a, A, \alpha_a) \to (b, B, \alpha_b)$ is an equivariant morphism that satisfy $s \circ f = s \circ g$. Then there exist a unique map ϕ in $\mathbf{LH}/C_0$ such that the following diagram commutes

$$\begin{tikzcd} E \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \arrow[r, two heads, "q"] \arrow[dr, "s"'] & A/R \arrow[d, dotted, "\phi"] \\ & & B \end{tikzcd}$$

It remains to show that ϕ respects the action. That is, that

$$\phi \circ \alpha_q(h, [x]_R) = \alpha_b(h, \phi([x]_R)).$$

But

$$\begin{aligned}\alpha_b(h, \phi([x]_R)) &= \alpha_b(h, \phi \circ q(x)) \\ &= \alpha_b(h, s(x)) \\ &= s \circ \alpha_a(h, x) \\ &= \phi \circ q \circ \alpha_a(h, x) \\ &= \phi \circ \alpha_q(h, [x]_R),\end{aligned}$$

so ϕ is an equivariant morphism. This completes the proof of Theorem 3.8. □

Corollary 3.9. *For a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$, let $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ be a morphism in $\mathrm{Sh}_{C_1}(C_0)$. Then,*

(i) f is monic iff $f : E \to A$ is an injective function,

(ii) f is epic iff $f : E \to A$ is a surjective function,

(iii) f is an isomorphism iff $f : E \to A$ is a bijective function.

Proof: "$\Rightarrow$" If f is monic (epic) in $\mathrm{Sh}_{C_1}(C_0)$ then the diagram corresponding to (2.6) (respectively (2.7)) is a pullback (pushout). Since U preserves finite limits (and colimits), f is monic (epic) in $\mathbf{LH}/C_0$. By Lemma 2.8, f is an injective (surjective) function. Any functor preserves isomorphisms, so if f is an isomorphism in $\mathrm{Sh}_{C_1}(C_0)$, then f is an isomorphism in $\mathbf{LH}/C_0$ and hence a homeomorphism.

"$\Leftarrow$" U is conservative by Proposition 2.9 and hence reflect isomorphisms in $\mathbf{LH}/C_0$. In particular, U is faithful and faithful functors reflect epics and monics. The implication now follows from Lemma 2.8. □

The following is also noticed in *e.g.* [For13]:

Corollary 3.10. *For a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$, the forgetful functor $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$ is the inverse image part of a geometric morphism $\mathbf{LH}/C_0 \to \mathrm{Sh}_{C_1}(C_0)$. In particular, U has a right adjoint.*

Proof: $\mathrm{Sh}_{C_1}(C_0)$ is a topos by Section 2.4 and since $U : \mathrm{Sh}_{C_1}(C_0) \to \mathbf{LH}/C_0$ preserves colimits, it follows from the Special Adjoint Functor Theorem that U has a right adjoint (*cf.* [Joh02b, C2.2.10]). Since U also preserve finite limits it is the inverse image part of geometric morphism $\mathbf{LH}/C_0 \to \mathrm{Sh}_{C_1}(C_0)$. □

The above corollary uses that $\mathrm{Sh}_{C_1}(C_0)$, for an arbitrary topological category $\mathcal{C} : C_1 \rightrightarrows C_0$, is a topos. This corollary will not be used in Chapters 4–5.

Chapter 4

Giraud's theorem

In this chapter we show that the category $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves on arbitrary topological category $\mathcal{C}$ satisfies the conditions of Giraud's theorem, except for the existence of set of generators. Restricted to the case when the codomain function of $\mathcal{C}$ is assumed to be open we prove the existence of a set of generators, and hence that $\mathrm{Sh}_{C_1}(C_0)$ is a topos.

An explicit site description or a construction of a set of generators for the general case appear not to have been published.

We also prove that the category of equivariant sheaves with an open action is equivalent to an open substopos of $\mathrm{Sh}_{C_1}(C_0)$, for an arbitrary topological category $\mathcal{C}$.

4.1 Giraud's theorem

Giraud's theorem for Grothendieck toposes, as stated in [MM92, Appendix], is as follows.

Theorem 4.1. *(Giraud) A category $\mathcal{E}$ with small hom-sets and all finite limits is a Grothendieck topos iff it has the following properties:*

(i) $\mathcal{E}$ has all small coproducts, and they are disjoint and stable under pullback,

(ii) every epimorphism in $\mathcal{E}$ is a coequalizer,

(iii) every equivalence relation $R \rightrightarrows E$ in $\mathcal{E}$ is a kernel pair and has a quotient,

(iv) every exact fork $R \rightrightarrows E \to Q$ is stably exact,

(v) there is a small set of objects of $\mathcal{E}$ which generate $\mathcal{E}$.

To clarify the meaning of the conditions (i) and (iv), we recall some definitions from [MM92, Appendix]. A "fork" is a commutative diagram of the form

$$R \underset{\partial_2}{\overset{\partial_1}{\rightrightarrows}} E \xrightarrow{q} Q\,. \tag{4.1}$$

A fork is said to be exact if q is the coequalizer of ∂_1 and ∂_2, while these form the kernel pair of q. The diagram (4.1) is stably exact if it remains exact after taking pullbacks along any map $Q' \to Q$ in $\mathcal{E}$, that is, when the diagram

$$R \times_Q Q' \rightrightarrows E \times_Q Q' \xrightarrow{q \times_Q 1_{Q'}} Q \times_Q Q' \cong Q', \tag{4.2}$$

obtained from (4.1) by pullback is again exact.

A coproduct $E = \coprod_{j \in J} E_j$ of a family of objects E_j in $\mathcal{E}$ is disjoint when every coproduct inclusion $i_j : E_j \to E$ is monic and for every $j \neq k$ in J the pullback $E_j \times_E E_k$ is the initial object in $\mathcal{E}$.

A coproduct $E = \coprod_{j \in J} E_j$ in $\mathcal{E}$ is stable under pullback if for any morphisms $E_j \to Y$, for $j \in J$, and $E' \to Y$, there is an isomorphism $E' \times_Y \coprod E_j \cong \coprod (E' \times_Y E_j)$. This is equivalent to that for any morphism $E' \to E$ in $\mathcal{E}$, the pullbacks $E' \times_E E_j$ along the coproduct inclusions $i_j : E_j \to E$ yield an isomorphism $E' \cong \coprod_{j \in J}(E' \times_E E_j)$.

4.2 The category of equivariant sheaves on a topological category

Notice that $\mathrm{Sh}_{C_1}(C_0)$ has small hom-sets, since **Sp** does. Finite limits exist in $\mathrm{Sh}_{C_1}(C_0)$ by Theorem 3.8. For convenience we prove the properties $(i) - (iv)$ of Theorem 4.1 for $\mathrm{Sh}_{C_1}(C_0)$ in separate propositions.

Proposition 4.2. *The category* $\mathrm{Sh}_{C_1}(C_0)$ *of equivariant sheaves on a topological category* $\mathcal{C} : C_1 \rightrightarrows C_0$ *has all small coproducts, and they are all disjoint and stable under pullback.*

Proof: By Theorem 3.8, $\mathrm{Sh}_{C_1}(C_0)$ has all finite limits and small colimits and these are preserved by the forgetful functor U.

For a small set J, let $f_j : (e_j, E_j, \alpha_j) \to (a, A, \alpha_a)$ and $f' : (e', E', \alpha'_e) \to (a, A, \alpha_a)$ be a collection of morphisms in $\mathrm{Sh}_{C_1}(C_0)$. Let (e, E, α_e) denote the coproduct $\coprod_{j \in J}(e_j, E_j, \alpha_j)$ and $i_j : (e_j, E_j, \alpha_j) \to (e, E, \alpha_e)$ be the coproduct inclusions and $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ be the unique map such that $f \circ i_j = f_j$ for each j.

Let $(b_j, E' \times_A E_j, \alpha'_j)$ be the pullback of f_j along f', and $(b, E' \times_A E, \alpha_b)$ be the pullback of f along f' in $\mathrm{Sh}_{C_1}(C_0)$. The collection of equivariant maps $1_{E'} \times_A i_j : (b_j, E' \times_A E_j, \alpha'_j) \to (b, E' \times_A E, \alpha_b)$ induces a unique equivariant map $\theta : \coprod_{j \in J}(b_j, E' \times_A E_j, \alpha'_j) \to (b, E' \times_A E, \alpha_b)$ such that $\theta \circ i'_j = 1_{E'} \times_A i_j$ for each $j \in J$, where i'_j are the coproduct inclusions $(b_j, E' \times_A E_j, \alpha'_j) \to \coprod_{j \in J}(b_j, E' \times_A E_j, \alpha'_j)$.

Since U preserves finite limits and all colimits, and $\mathbf{LH}/C_0$ is a topos, the map θ is an isomorphism in $\mathbf{LH}/C_0$. Since U is conservative (Proposition 2.9), we have that θ is an isomorphism in $\mathrm{Sh}_{C_1}(C_0)$. Hence coproducts in $\mathrm{Sh}_{C_1}(C_0)$ are stable under pullback.

We now show that coproducts in $\mathrm{Sh}_{C_1}(C_0)$ are disjoint. Coproducts in $\mathrm{Sh}_{C_1}(C_0)$ are preserved by the forgetful functor U and, since $\mathbf{LH}/C_0$ is a topos, the corresponding coproduct in $\mathbf{LH}/C_0$ is disjoint. In particular, this means that the coproduct inclusions functions i_j are monic in $\mathbf{LH}/C_0$. Since the forgetful functor U is faithful (it is conservative by Proposition 2.9), it reflects monics, so each i_j is monic in $\mathrm{Sh}_{C_1}(C_0)$ as well.

For $j \neq k$ in J, let (a, A, α_a) be the pullback of i_j along i_k in $\mathrm{Sh}_{C_1}(C_0)$. By applying U, we have that $A = \emptyset$ as coproducts in $\mathbf{LH}/C_0$ are disjoint. There is only one action on $\emptyset$, namely the empty action. Thus (a, A, α_a) is the initial object in $\mathrm{Sh}_{C_1}(C_0)$. □

Proposition 4.3. *Every epimorphism in* $\mathrm{Sh}_{C_1}(C_0)$ *is a coequalizer.*

Proof: Let $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ be an epimorphism in $\mathrm{Sh}_{C_1}(C_0)$. Since the forgetful functor U preserves colimits, f is an epimorphism in $\mathbf{LH}/C_0$. Hence f is a coequalizer in $\mathbf{LH}/C_0$. To proceed, we state and prove a relation between kernel pairs and coequalizers mentioned in [MM92, Appendix] in a lemma.

Lemma 4.4. *In a finitely complete category, if*

$$\begin{tikzcd} B \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & B' \arrow[r, "q"] & Q \end{tikzcd}$$

is a coequalizer diagram, then $q : B' \to Q$ *is also the coequalizer of its kernel pair.*

Proof: Consider the following diagram, where $d_1, d_2 : B' \times_Q B' \to B'$ is the kernel pair of q and θ is the unique map making the diagram commutative

$$\begin{tikzcd} B \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "b_1"] \arrow[ddr, bend right, "b_2"'] & & \\ & B' \times_Q B' \arrow[r, "d_1"] \arrow[d, "d_2"'] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d, "q"] \\ & B' \arrow[r, "q"'] & Q \end{tikzcd}$$

Then $b_1 = d_1 \circ \theta$ and $b_2 = d_2 \circ \theta$. If $q' : B' \to Q'$ is any arrow such that

$$\begin{tikzcd} B' \times_Q B' \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_2"'] & B' \arrow[r, "q'"] & Q' \end{tikzcd}$$

commutes, then we have that $q' \circ d_1 \circ \theta = q' \circ d_2 \circ \theta$ so also the following diagram is commutative

$$\begin{tikzcd} B \arrow[r, shift left, "b_1"] \arrow[r, shift right, "b_2"'] & B' \arrow[r, "q'"] & Q' . \end{tikzcd}$$

Hence there is a unique arrow $p : Q \to Q'$ such that $q' = p \circ q$, and so q is the coequalizer of its kernel pair. □

Let $(e', E \times_A E, \alpha')$ be the pullback of f along f in $\mathrm{Sh}_{C_1}(C_0)$, with equivariant projection maps $\partial_1, \partial_2 : E \times_A E \to E$. Let $k : (e, E, \alpha_e) \to (q, Q, \alpha_q)$ be the coequalizer of ∂_1 and ∂_2 in $\mathrm{Sh}_{C_1}(C_0)$. Then, since $f \circ \partial_1 = f \circ \partial_2$, there is a unique equivariant map $g : (q, Q, \alpha_q) \to (a, A, \alpha_a)$ such that $f = g \circ k$.

Since the forgetful functor U preserves finite limits and colimits (Theorem 3.8), ∂_1, ∂_2 form the kernel pair of f while, by the preceding lemma, f

is the coequalizer of ∂_1 and ∂_2 in $\mathbf{LH}/C_0$. So g is an isomorphism in $\mathbf{LH}/C_0$. Since U is conservative (Proposition 2.9), we have that $(q, Q, \alpha_q) \cong (a, A, \alpha_a)$ and f is a coequalizer in $\mathrm{Sh}_{C_1}(C_0)$. $\square$

Proposition 4.5. *Every equivalence relation* $(r, R, \alpha_r) \rightrightarrows (e, E, \alpha_e)$ *in* $\mathrm{Sh}_{C_1}(C_0)$ *is a kernel pair and has a quotient.*

Proof: Let $(\partial_1, \partial_2) : (r, R, \alpha_r) \rightarrowtail (e, E, \alpha_e) \times (e, E, \alpha_e)$ be the representative for an equivalence relation in $\mathrm{Sh}_{C_1}(C_0)$. It is straightforward to verify that $(\partial_1, \partial_2) : R \rightarrowtail E \times_{C_0} E$ is (the representative for) an equivalence relation in $\mathbf{LH}/C_0$.

The quotient map q of ∂_1 and ∂_2 exists in $\mathrm{Sh}_{C_1}(C_0)$ by Theorem 3.8 and is preserved by the forgetful functor U. Since $\mathbf{LH}/C_0$ is a topos, ∂_1 and ∂_2 form the kernel pair of some arrow $u : E \to D$, where D is a sheaf over C_0. To continue, we prove a property of the quotient map of a kernel pair mentioned in [MM92, Appendix].

Lemma 4.6. *The kernel pair of some arrow is also the kernel pair of its quotient map, when this morphism exist.*

Proof: We use the same notation as in the proof of Proposition 4.5. By assumption, $\partial_1, \partial_2 : R \rightrightarrows E$ is the kernel pair of $u : E \to D$. Let $q : E \to E/R$ be the corresponding quotient map.

Then as the following diagram commutes

$$\begin{tikzcd} R \arrow[r, shift left, "\partial_1"] \arrow[r, shift right, "\partial_2"'] & E \arrow[r, "u"] & D \end{tikzcd}, \tag{4.3}$$

there is a unique $f : E/R \to D$ such that $u = f \circ q$. Now, if $b_1, b_2 : B \to E$ are two arrows such that $q \circ b_1 = q \circ b_2$, then

$$\begin{aligned} u \circ b_1 &= f \circ q \circ b_1 \\ &= f \circ q \circ b_2 \\ &= u \circ b_2. \end{aligned}$$

So there is a unique arrow $\theta : B \to R$ such that the following diagram commute

$$\begin{tikzcd} B \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "b_1"] \arrow[ddr, bend right, "b_2"'] & & \\ & R \arrow[r, "\partial_1"] \arrow[d, "\partial_2"'] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "u"] \\ & E \arrow[r, "u"'] & D \end{tikzcd}$$

In particular, $b_1 = \partial_1 \circ \theta$ and $b_2 = \partial_2 \circ \theta$. Then θ is also the unique arrow making the diagram below commute

$$\begin{tikzcd}
B \arrow[rrd, bend left, "b_1"] \arrow[rd, dotted, "\theta"] \arrow[rdd, bend right, "b_2"'] & & \\
& R \arrow[r, "\partial_1"] \arrow[d, "\partial_2"'] & E \arrow[d, "q"] \\
& E \arrow[r, "q"'] & E/R
\end{tikzcd}$$

This shows that R is the pullback of the quotient map, and that ∂_1 and ∂_2 is the kernel pair of the quotient map q. $\square$

Let the pullback of q along q in $\mathrm{Sh}_{C_1}(C_0)$ be (a, A, α_a) with projections maps $\pi_1, \pi_2 : (a, A, \alpha_a) \to (e, E, \alpha_e)$. Since $q \circ \partial_1 = q \circ \partial_2$ there is a unique equivariant map $g : (r, R, \alpha_r) \to (a, A, \alpha_a)$ such that $\partial_1 = \pi_1 \circ g$ and $\partial_2 = \pi_2 \circ g$.

The forgetful functor U preserves these limits and, by the preceding lemma, ∂_1, ∂_2 is the kernel pair of q in $\mathbf{LH}/C_0$. Hence g is an isomorphism in $\mathbf{LH}/C_0$. Since U is conservative (Proposition 2.9), g is an isomorphism in $\mathrm{Sh}_{C_1}(C_0)$. Hence ∂_1, ∂_2 is a kernel pair in $\mathrm{Sh}_{C_1}(C_0)$. $\square$

Proposition 4.7. *Every exact fork in* $\mathrm{Sh}_{C_1}(C_0)$ *is stably exact.*

Proof: Let the following diagram be an exact fork of equivariant sheaves

$$\begin{tikzcd}
(r, R, \alpha_r) \arrow[r, shift left, "\partial_1"] \arrow[r, shift right, "\partial_2"'] & (e, E, \alpha_e) \arrow[r, "p"] & (q, Q, \alpha_q).
\end{tikzcd} \tag{4.4}$$

If $r : (q', Q', \alpha'_q) \to (q, Q, \alpha_q)$ is an equivariant morphism, we obtain the following diagram by pulling back (4.4) along r in $\mathrm{Sh}_{C_1}(C_0)$

$$\begin{tikzcd}
(r', R \times_Q Q', \alpha'_r) \arrow[r, shift left, "\partial_1 \times_Q 1_{Q'}"] \arrow[r, shift right, "\partial_2 \times_Q 1_{Q'}"'] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (e', E \times_Q Q', \alpha'_e) \arrow[r, "\pi_{Q'}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (q', Q', \alpha'_q) \arrow[d, "r"] \\
(r, R, \alpha_r) \arrow[r, shift left, "\partial_1"] \arrow[r, shift right, "\partial_2"'] & (e, E, \alpha_e) \arrow[r, "p"] & (q, Q, \alpha_q)
\end{tikzcd} \tag{4.5}$$

Indeed, by the "pullback lemma" (*cf.* the proof of Lemma 2.4) the left-hand rectangle in (4.5) is a pullback since the right-hand and outer rectangles are pullbacks.

Let $f : (e', E \times_Q Q', \alpha'_e) \to (a, A, \alpha_a)$ be the coequalizer of $\partial_1 \times_Q 1_{Q'}$ and $\partial_2 \times_Q 1_{Q'}$ in $\mathrm{Sh}_{C_1}(C_0)$. Since $\pi_{Q'} \circ (\partial_1 \times_Q 1_{Q'}) = \pi_{Q'} \circ (\partial_2 \times_Q 1_{Q'})$ there is a unique equivariant LH $g : (a, A, \alpha_a) \to (q', Q', \alpha'_q)$ such that $g \circ f = \pi_{Q'}$.

Let (a', A', α'_a) be the pullback of $\pi_{Q'}$ along itself in $\mathrm{Sh}_{C_1}(C_0)$, with projection maps $\pi_1, \pi_2 : (a', A', \alpha'_a) \to (e', E \times_Q Q', \alpha'_e)$. Since $\pi_{Q'} \circ (\partial_1 \times_Q 1_{Q'}) = \pi_{Q'} \circ (\partial_2 \times_Q 1_{Q'})$ there is a unique equivariant LH $k : (r', R \times_Q Q', \alpha'_r) \to (a', A', \alpha'_a)$ such that $\pi_1 \circ k = \partial_1 \times_Q 1_{Q'}$ and $\pi_2 \circ k = \partial_2 \times_Q 1_{Q'}$.

By Theorem 3.8 finite limits and all colimits are preserved by U. Since $\mathbf{LH}/C_0$ is a topos, the image of U on the upper "fork" in (4.5), *i.e.*

$$R \times_Q Q' \underset{\partial_2 \times_Q 1_{Q'}}{\overset{\partial_1 \times_Q 1_{Q'}}{\rightrightarrows}} E \times_Q Q' \xrightarrow{\pi_{Q'}} Q' \,, \tag{4.6}$$

is exact in $\mathbf{LH}/C_0$. So g and k are isomorphisms in $\mathbf{LH}/C_0$. Since U is conservative (Proposition 2.9), we have that g and k are isomorphisms in $\mathrm{Sh}_{C_1}(C_0)$. This means that $\partial_1 \times_Q 1_{Q'}, \partial_2 \times_Q 1_{Q'}$ is the kernel pair of $\pi_{Q'}$ and that $\pi_{Q'}$ is the coequalizer of $\partial_1 \times_Q 1_{Q'}$ and $\partial_2 \times_Q 1_{Q'}$. $\square$

The main difficulty of proving Giraud's theorem for the category of equivariant sheaves on an arbitrary topological category $\mathcal{C}$ lies in finding a set of generators, condition (v). Notice, incidentally, that any elements that generates the equivariant sheaves on $\mathcal{C}$ with open actions (if there are any) must themselves have open actions, by Proposition 3.5 (i).

Let $\underline{\mathrm{Sh}}_{C_1}(C_0)$ be the full subcategory of $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves with open actions.

Proposition 4.8. *For a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$, $\underline{\mathrm{Sh}}_{C_1}(C_0)$ has a set of generators.*

Proof: We will describe a collection $\mathfrak{G}$ of equivariant sheaves with open actions such that for any (e, E, α_e) in $\underline{\mathrm{Sh}}_{C_1}(C_0)$ and $z \in E$ there is a $(g, G, \alpha_g) \in \mathfrak{G}$ and an equivariant morphism $\phi : (g, G, \alpha_g) \to (e, E, \alpha_e)$ such that $z \in \phi(G)$. For (e, E, α_e) and (a, A, α_a) in $\underline{\mathrm{Sh}}_{C_1}(C_0)$, two parallel arrows $f, k : (e, E, \alpha_e) \to (a, A, \alpha_a)$ are equal if and only if $f(z) = k(z)$ for all $z \in E$. This means that $f = k$ iff $f \circ \phi = k \circ \phi$ for all arrows $\phi : (g, G, \alpha_g) \to (e, E, \alpha_e)$ for elements (g, G, α_e) in $\mathfrak{G}$. In other words, $\mathfrak{G}$ will be a collection of generators of $\underline{\mathrm{Sh}}_{C_1}(C_0)$. This collection will be shown to be a set.

Let $(e, E, \alpha_e) \in \underline{\mathrm{Sh}}_{C_1}(C_0)$ and $z \in E$. Since e is an LH, there is an open set $U \subseteq C_0$ and a continuous section $\sigma : U \to E$ such that $z \in \sigma(U)$ and $\sigma(U)$ is open (see *e.g.* [MM92, Proposition II.6.1]). Let $D = C_1 \times_{C_0} E$ be the pullback of e along s_C and R be the equivalence relation on D in (3.2) (*i.e.* where $(h, x) \sim_R (p, y)$ iff $\alpha_e(h, x) = \alpha_e(p, y)$).

Since the action α_e is open, the quotient map $q : D \to D/R$ is open by Proposition 3.2. And since $\sigma(U) \subseteq E$ is open we have that $D_\sigma = C_1 \times_{C_0} \sigma(U) \subseteq D$ is open, as follows from the following pullback diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d, "\circ" marking] \arrow[dr, phantom, "\lrcorner", very near start] & \sigma(U) \arrow[d, hook, "\circ" marking] \\ D \arrow[r, "\pi_E"'] & E \end{tikzcd}$$

where $\pi_E : D \to E$ is the projection. Hence $D_\sigma/R = q\left(C_1 \times_{C_0} \sigma(U)\right)$ is open in D/R, so the restriction of the LH $[t_C] : D/R \to C_0$ (see Theorem 3.1) to D_σ/R is an LH $[t_C]_\sigma : D_\sigma/R \to C_0$.

Since $D_\sigma/R \subseteq D/R$ is open, we also have that $C_1 \times_{C_0} D_\sigma/R \subseteq C_1 \times_{C_0} D/R$ is open:

$$\begin{tikzcd}
C_1 \times_{C_0} D_\sigma/R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D_\sigma/R \arrow[d, hook] \\
C_1 \times_{C_0} D/R \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D/R \arrow[d, "{[t_C]}"] \\
C_1 \arrow[r, "s_C"'] & C_0
\end{tikzcd}$$

Given the isomorphism $[\alpha_e] : ([t_C], D/R, \alpha_d) \to (e, E, \alpha_e)$ in $\mathrm{Sh}_{C_1}(C_0)$ of Theorem 3.1, the action α_d on D/R is open by Proposition 3.5 (i), since α_e is open. The restriction of α_d to $C_1 \times_{C_0} D_\sigma/R$ therefore yields a continuous open function $\alpha_\sigma : C_1 \times_{C_0} D_\sigma/R \to D/R$. Since the image of α_σ lies in D_σ/R, and $D_\sigma/R \subseteq D/R$ is open and has the subspace topology, $\alpha_\sigma : C_1 \times_{C_0} D_\sigma/R \to D_\sigma/R$ is open and continuous. As α_d is an action on D/R, it follows that α_σ satisfies the conditions of an being action (given in (2.3)) on D_σ/R.

The restriction of the equivariant LH $[\alpha_e] : D/R \to E$ to the open subset $D_\sigma/R \subseteq D/R$ yields an equivariant LH $[\alpha_e] : D_\sigma/R \to E$. Furthermore, we have that $z \in [\alpha_e]\,(D_\sigma/R)$ since $[(1_{e(z)}, z)]_R \in D_\sigma/R$ and $[\alpha_e]\big([(1_{e(z)}, z)]_R\big) = z$.

As the set of generators $\mathfrak{G}$, we choose a collection of equivariant sheaves isomorphic to sheaves of the form $([t_C]_\sigma, D_\sigma/R, \alpha_\sigma)$ as described above. $\mathfrak{G}$ is chosen as follows:

Observe that $C_1 \times_{C_0} U$ is homeomorphic to $C_1 \times_{C_0} \sigma(U)$ with mutually inverse continuous functions given by

$$\begin{tikzcd}
C_1 \times_{C_0} U \arrow[r, shift left, "1_{C_1} \times_{C_0} \sigma"] & C_1 \times_{C_0} \sigma(U) \arrow[l, shift left, "1_{C_1} \times_{C_0} e"]
\end{tikzcd} .$$

The equivalence relation R on D, restricted to D_σ, induces an equivalence relation R' on $C_1 \times_{C_0} U$. The equivalence relation R' is given by $(h, x) \sim_{R'} (p, y)$ iff $(h, \sigma(x)) \sim_R (p, \sigma(y))$, that is, iff $\alpha_e(h, \sigma(x)) = \alpha_e(p, \sigma(y))$. It follows that the map $\theta : (C_1 \times_{C_0} U)/R' \to D_\sigma/R$ which takes $[(h, x)]_{R'} \mapsto [(h, \sigma(x))]_R$ is a homeomorphism.

Via the homeomorphism θ, $(C_1 \times_{C_0} U)/R'$ is a sheaf over C_0:

$$\begin{tikzcd}
(C_1 \times_{C_0} U)/R' \arrow[r, "\theta", "\sim"'] \arrow[dr, dotted, "{[t_C]_U}"'] & D_\sigma/R \arrow[d, "{[t_C]_\sigma}"] \\
& C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
{[(h, x)]_{R'}} \arrow[r, mapsto] \arrow[dr, dotted, mapsto] & {[(h, \sigma(x))]_R} \arrow[d, mapsto] \\
& t_C(f)
\end{tikzcd}$$

The action α_σ on D_σ/R induces an open action α_U on $(C_1 \times_{C_0} U)/R'$:

$$\begin{tikzcd}
C_1 \times_{C_0} (C_1 \times_{C_0} U)/R' \arrow[r, dotted, "\alpha_U"] \arrow[d, "1_{C_1} \times_{C_0} \theta"'] & (C_1 \times_{C_0} U)/R' \\
C_1 \times_{C_0} D_\sigma/R \arrow[r, "\alpha_\sigma"'] & D_\sigma/R \arrow[u, "\theta^{-1}"']
\end{tikzcd}$$

which on elements is given by

$$\begin{tikzcd}
(h', [(h,x)]_{R'}) \arrow[r, dotted, mapsto] \arrow[d, mapsto] & {[(h' \circ h, x)]_{R'}} \\
(h', [(h,\sigma(x))]_R) \arrow[r, mapsto] & {[(h' \circ h, \sigma(x))]_R} \arrow[u, mapsto]
\end{tikzcd}$$

By construction, θ is equivariant. Since the forgetful functor U is conservative (Proposition 2.9), θ is an isomorphism in $\underline{\mathrm{Sh}}_{C_1}(C_0)$ between $([t_C]_U, (C_1 \times_{C_0} U)/R', \alpha_U)$ and $([t_C]_\sigma, D_\sigma/R, \alpha_\sigma)$.

Let $\mathfrak{G}$ be the collection of triples $([t_C]_U, (C_1 \times_{C_0} U)/R', \alpha_U)$ that are equivariant sheaves, where $U \subseteq C_0$ is open, R' is an equivalence relation on $C_1 \times_{C_0} U$ (such that $[t_C]_U$ and α_U are well-defined), $[t_C]_U$ is the function $[(h,x)]_{R'} \mapsto t_C(h)$ and α_U is the operation $(h', [(h,x)]_{R'}) \mapsto [(h' \circ h, x)]_{R'}$. The collection of spaces $C_1 \times_{C_0} U$, for $U \subseteq C_0$ open, form a set and the collection of equivalence relations to put on each $C_1 \times_{C_0} U$ is also a set. Hence $\mathfrak{G}$ is a set. This completes the proof. $\square$

Restricting to the case when the codomain map is open we can now prove the following.

Theorem 4.9. *For a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$ where the codomain map is open, the category $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves on $\mathcal{C}$ is a topos.*

Proof: When t_C is open the action on each equivariant sheaf on $\mathcal{C}$ is open by Proposition 3.3. So $\underline{\mathrm{Sh}}_{C_1}(C_0)$ is the category $\mathrm{Sh}_{C_1}(C_0)$.

Using Giraud's theorem, Theorem 4.1, the statement follows from Proposition 4.2, 4.3, 4.5, 4.7 and 4.8. $\square$

We remark that although our proof of the existence of a set of generators is somewhat different from Moerdijk's description in [Moe95], they yield essentially the same collection of generators.

If t_C is not open, it is also clear that $\underline{\mathrm{Sh}}_{C_1}(C_0)$ does not equal $\mathrm{Sh}_{C_1}(C_0)$, for the action on the terminal object (see Theorem 3.8) is open iff t_C is open (Proposition 3.3).

A monomorphism $U \rightarrowtail 1$ in an (elementary) topos $\mathcal{E}$ induces a geometric morphism $\mathcal{E}/U \to \mathcal{E}/1 \cong \mathcal{E}$ which is an inclusion (or embedding), see [Joh02a, A4.5] . The direct image of this geometric morphism determines a subtopos of $\mathcal{E}$. Such a slice category of the form $\mathcal{E}/U$ for a monomorphism $U \rightarrowtail 1$ is called an open subtopos of $\mathcal{E}$.

Using that $\mathrm{Sh}_{C_1}(C_0)$ is known to be a Grothendieck topos for an arbitrary topological category $\mathcal{C}$ (see Section 2.4), the following theorem says that $\underline{\mathrm{Sh}}_{C_1}(C_0)$ is equivalent to an open subtopos of $\mathrm{Sh}_{C_1}(C_0)$. However, the inclusion functor $\underline{\mathrm{Sh}}_{C_1}(C_0) \hookrightarrow \mathrm{Sh}_{C_1}(C_0)$ is not to be confused with the direct image of the induced geometric morphism.

Theorem 4.10. *For a topological category* $\mathcal{C} : C_1 \rightrightarrows C_0$*,* $\underline{\mathrm{Sh}}_{C_1}(C_0)$ *is (equivalent to) an open subtopos of* $\mathrm{Sh}_{C_1}(C_0)$*.*

Proof: Let $1 = (1_{C_0}, C_0, \alpha_1)$ be the terminal object in $\mathrm{Sh}_{C_1}(C_0)$. Using Proposition 3.6, the image of each equivariant sheaf with an open action is an open set $V \subseteq C_0$ such that

$$t_C\left(s_C^{-1}(V)\right) = V \text{ and } t_C \text{ restricted to } s_C^{-1}(V) \text{ is open.} \tag{4.7}$$

Let U be the union of all open subsets $V \subseteq C_0$ that satisfies (4.7). Then U also satisfies (4.7). Using Proposition 3.6, the equivariant sheaf (i_U, U, α_U) where $i_U : U \hookrightarrow C_0$ is the inclusion, $\alpha_U = t_C \circ \pi_{C_1} : C_1 \times_{C_0} U \to U$ and π_{C_1} is the projection onto C_1, has an open action.

The inclusion i_U is an injective equivariant LH $(i_U, U, \alpha_U) \to 1$, and hence a monic arrow in $\mathrm{Sh}_{C_1}(C_0)$ (by Corollary 3.9). So $\mathrm{Sh}_{C_1}(C_0)/(i_U, U, \alpha_U)$ is an open subtopos of $\mathrm{Sh}_{C_1}(C_0)$.

Since α_U is open, for any equivariant morphism $f : (e, E, \alpha_e) \to (i_U, U, \alpha_U)$ we have that α_e is open by Proposition 3.5 (i). Hence the forgetful functor $F : \mathrm{Sh}_{C_1}(C_0)/(i_U, U, \alpha_U) \to \mathrm{Sh}_{C_1}(C_0)$, taking $(e, E, \alpha_e) \to (i_U, U, \alpha_U)$ to (e, E, α_e), is a functor $F : \mathrm{Sh}_{C_1}(C_0)/(i_U, U, \alpha_U) \to \underline{\mathrm{Sh}}_{C_1}(C_0)$.

Obviously, F is faithful. Furthermore, any (e, E, α_e) in $\underline{\mathrm{Sh}}_{C_1}(C_0)$ is in the image of F. Indeed, it follows from Proposition 3.6 (i) that any such (e, E, α_e) is a sheaf over a subset of U. The map $e_U : E \to U$ such that $e_U(x) = e(x)$ for $x \in E$ is an equivariant LH such that $e = i_U \circ e_U$. F is also full, for if $f : (e, E, \alpha_e) \to (a, A, \alpha_a)$ is a morphism in $\underline{\mathrm{Sh}}_{C_1}(C_0)$, then as both e and a factor through U, f is a morphism in $\mathrm{Sh}_{C_1}(C_0)/(i_U, U, \alpha_U)$. This means that F is an equivalence $\mathrm{Sh}_{C_1}(C_0)/(i_U, U, \alpha_U) \cong \underline{\mathrm{Sh}}_{C_1}(C_0)$. $\square$

Chapter 5

Covering morphisms and adjoints

For a topological groupoid $\mathcal{G}$ there is a functor, which we will denote S, mapping $\mathcal{G}$-spaces to topological groupoids over $\mathcal{G}$. At the same time, a set of generators for $\mathrm{Sh}_{G_1}(G_0)$, where $\mathcal{G}$ is open, can be obtained (as certain quotient spaces derived) from the open subgroupoids of $\mathcal{G}$. The main purpose of this chapter is to investigate how these constructions are related and if the method of obtaining a set of generators for $\mathrm{Sh}_{G_1}(G_0)$ from the subgroupoids of $\mathcal{G}$ can be put in a more general framework.

In Section 5.1 we summarize material from [BDNH76] concerning topological covering morphisms and $\mathcal{G}$-spaces, which we then, in Section 5.2, extend from topological groupoids to topological categories, local homeomorphic covering morphisms and equivariant sheaves. The conclusion from these sections is the two equivalences, for a topological category $\mathcal{C}$, between the category of $\mathcal{C}$-spaces and the category of topological covering morphisms to $\mathcal{C}$ and between $\mathrm{Sh}_{C_1}(C_0)$ and the category of local homeomorphic covering morphisms to $\mathcal{C}$.

In Section 5.3 we consider an open topological groupoid $\mathcal{G}$ and generalize, from open subgroupoids of $\mathcal{G}$ to arbitrary groupoids over $\mathcal{G}$, the construction of a set of generators for $\mathrm{Sh}_{G_1}(G_0)$. We define subcategories of $\mathbf{TGpd}/\mathcal{G}$ for which this construction defines a functor Λ to $\mathbf{Sp}^{\mathcal{G}}$ respectively $\mathrm{Sh}_{G_1}(G_0)$.

Section 5.4 contains a proof that Λ is left adjoint to S. In Section 5.5 we show that the open subcategories of an arbitrary topological category $\mathcal{C}$ generate (in a slightly non-standard sense) the category of local homeomorphic covering morphisms to $\mathcal{C}$. This result combined with the adjunction $\Lambda \dashv S$ yields an alternative proof that $\mathrm{Sh}_{G_1}(G_0)$ has a set of generators, for the case of an open topological groupoid $\mathcal{G}$.

5.1 Topological covering morphisms and $\mathcal{G}$-spaces

Given a topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$ a functor from the category of $\mathcal{G}$-spaces to the slice category $\mathbf{TGpd}/\mathcal{G}$ is described in [BDNH76] (and [Moe88]). We outline this construction. The functor in question will be denoted S, and takes a $\mathcal{G}$-space (a, A, α_a) to the pair $(\hat{A}, \hat{a})$ where $\hat{a}$ is a groupoid morphism of $\hat{A} : A_1 \rightrightarrows A_0$ into $\mathcal{G}$. The groupoid $\hat{A}$ has object space $A_0 = A$ and arrow space

$A_1 = G_1 \times_{G_0} A$ with the pullback given by

$$\begin{tikzcd} G_1 \times_{G_0} A \arrow[r, "\pi_A"] \arrow[d, "\pi^A_{G_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "a"] \\ G_1 \arrow[r, "s_G"'] & G_0 \end{tikzcd} \tag{5.1}$$

The groupoid operations, illustrated in the diagram

$$\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "m_A"] & A_1 \arrow[loop above, "i_A"] \arrow[r, bend left, "t_A"] \arrow[r, bend right, "s_A"'] & A_0 \arrow[l, "u_A"'] \end{tikzcd} \tag{5.2}$$

are on elements given by

$$\begin{aligned} s_A(k,x) &= x, \\ t_A(k,x) &= \alpha_a(k,x), \\ u_A(x) &= (1_{a(x)}, x), \\ m_A((k,x),(p,y)) &= (k \circ p, y), \end{aligned} \tag{5.3}$$

and $i_A(k,x) = (k^{-1}, \alpha_a(k,x))$. The components of the groupoid morphism $\hat{a}$ are

$$\begin{aligned} \hat{a}_0 &= a, \\ \hat{a}_1 &= \pi^A_{G_1}. \end{aligned} \tag{5.4}$$

For a morphism $k : (e, E, \alpha_e) \to (a, A, \alpha_a)$ we have $S(k) : (\hat{E}, \hat{e}) \to (\hat{A}, \hat{a})$ with components $S(k)_0 = k$ and $S(k)_1 = 1_{G_1} \times_{G_0} k$.

In [BDNH76] it is shown that the category of $\mathcal{G}$-spaces is equivalent to the category of *topological covering morphisms* to $\mathcal{G}$. A morphism $\phi : \mathcal{H} \to \mathcal{G}$ in **TGpd** is called a topological covering morphism if the function $\psi = \phi_1 \times_{G_0} s_H : H_1 \to G_1 \times_{G_0} H_0$ is a homeomorphism, where ψ is the unique function such that following diagram commutes in **Sp**:

$$\begin{tikzcd} H_1 \arrow[dr, dotted, "\psi"] \arrow[drr, bend left, "s_H"] \arrow[ddr, bend right, "\phi_1"'] & & \\ & G_1 \times_{G_0} H_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "\phi_0"] \\ & G_1 \arrow[r, "s_G"'] & G_0 \end{tikzcd} \tag{5.5}$$

Following [BDNH76] we shall also say that $\mathcal{H}$ is a covering grupoid of $\mathcal{G}$ when there exist a topological covering morphism $\phi : \mathcal{H} \to \mathcal{G}$.

Denote the full subcategory of $\mathbf{TGpd}/\mathcal{G}$ with topological covering morphisms into $\mathcal{G}$ by $\mathbf{TCov}/\mathcal{G}$. Then all the arrows in $\mathbf{TCov}/\mathcal{G}$ are topological covering morphisms, for in a commutative triangle of groupoid morphisms of the form

$$\begin{tikzcd} \mathcal{H}' \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{H} \arrow[dl, "g"] \\ & \mathcal{G} & \end{tikzcd}$$

with g a topological covering morphism, then f is a topological covering morphism iff p is a topological covering morphism ([BDNH76, Proposition 1]).

The image of S on a $\mathcal{G}$-space is a covering groupoid of $\mathcal{G}$. The functor S is one half of the equivalence $\mathbf{TCov}/\mathcal{G} \cong \mathbf{Sp}^{\mathcal{G}}$. The other half of this equivalence is the functor $\Gamma : \mathbf{TCov}/\mathcal{G} \to \mathbf{Sp}^{\mathcal{G}}$, which takes an object $(\mathcal{H}, h)$ and a morphism $\phi : (\mathcal{H}', h') \to (\mathcal{H}, h)$ of $\mathbf{TCov}/\mathcal{G}$ into

$$\begin{aligned} \Gamma(\mathcal{H}, h) &= (h_0, H_0, t_H \circ \psi^{-1}), \\ \Gamma(\phi) &= \phi_0, \end{aligned} \tag{5.6}$$

where ψ is the homeomorphism $H_1 \xrightarrow{\sim} G_1 \times_{G_0} H_0$. That $\Gamma(\phi)$ is an equivariant morphism will be clear from the proof Theorem 5.5.

5.2 Topological covering morphisms of topological categories and local homeomorphic coverings

We now generalize the constructions in the previous section to topological categories. We begin by observing that leaving out the inverse arrow function the functor S may be defined as a functor from the category of $\mathcal{C}$-spaces to $\mathbf{TCat}/\mathcal{C}$ for a topological category $\mathcal{C}$ by the equations in (5.3) and (5.4). The definition of a topological covering morphism can be applied to morphisms of topological categories exactly as for topological groupoids.

For topological categories $\mathcal{C}, \mathcal{D}$, that $\phi : \mathcal{D} \to \mathcal{C}$ is a topological covering morphism amounts to the following.

Lemma 5.1. *A morphism of topological categories $\phi : \mathcal{D} \to \mathcal{C}$ is a topological covering morphism iff the diagram*

$$\begin{tikzcd} D_1 \arrow[r, "s_D"] \arrow[d, "\phi_1"'] & D_0 \arrow[d, "\phi_0"] \\ C_1 \arrow[r, "s_C"'] & C_0 \end{tikzcd} \tag{5.7}$$

is a pullback.

Proof: The statement follows from that pullbacks are unique up to isomorphism. □

We can also extend Proposition 1 in [BDNH76] to topological categories:

Proposition 5.2. *Given a commutative diagram of topological categories*

$$\begin{tikzcd} \mathcal{D}' \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{D} \arrow[dl, "g"] \\ & \mathcal{C} & \end{tikzcd}$$

where g is a topological covering morphism. Then p is a topological covering morphism iff f is a topological covering morphism.

Proof: We supply a different and shorter proof than the corresponding proof in [BDNH76]. With $\mathcal{D}' : D_1' \rightrightarrows D_0'$, $\mathcal{D} : D_1 \rightrightarrows D_0$ and $\mathcal{C} : C_1 \rightrightarrows C_0$, consider the commutative diagram

$$\begin{tikzcd} D_1' \arrow[r, "s_{D'}"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_0' \arrow[d, "f_0"] \\ D_1 \arrow[r, "s_D"] \arrow[d, "g_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & D_0 \arrow[d, "g_0"] \\ C_1 \arrow[r, "s_C"'] & C_0 \end{tikzcd}$$

Using Lemma 5.1, the statement now follows from the "pullback lemma" (see the proof of Lemma 2.4). □

Definition 5.3. We define a *local homeomorphic covering morphism*, or shortly an *LHc* morphism, to be a topological covering morphism with components that are LH's. That is, $\phi : \mathcal{D} \to \mathcal{C}$ is an LHc morphism if ϕ is a topological covering morphism and $\phi_0 : D_0 \to C_0$ and $\phi_1 : D_1 \to C_1$ are LH's.

Notice that if ϕ is topological covering morphism and ϕ_0 is an LH, then by Lemma 5.1 and Lemma 2.1 this implies that ϕ_1 also is an LH, so ϕ is an LHc morphism in this case.

Definition 5.4. Let $\mathbf{TCov}/\mathcal{C}$ and $\mathbf{LHCov}/\mathcal{C}$ be the full subcategories of $\mathbf{TCat}/\mathcal{C}$ of topological covering morphisms into $\mathcal{C}$ and LHc morphisms into $\mathcal{C}$, respectively.

By Proposition 5.2 the arrows in $\mathbf{TCov}/\mathcal{C}$ are also topological covering morphisms, and using Lemma 2.3 (i), the arrows in $\mathbf{LHCov}/\mathcal{C}$ are LHc morphisms.

The following theorem is a generalization (to topological categories) and an extension (regarding the restriction to $\mathrm{Sh}_{C_1}(C_0)$) of Theorem 2 in [BDNH76].

Theorem 5.5. *For a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$ there is an equivalence of categories $\mathbf{TCov}/\mathcal{C} \cong \mathbf{Sp}^{\mathcal{C}}$ and $\mathbf{LHCov}/\mathcal{C} \cong \mathrm{Sh}_{C_1}(C_0)$:*

$$\begin{tikzcd} \mathbf{TCov}/\mathcal{C} \arrow[r, shift left, "\Gamma"] & \mathbf{Sp}^{\mathcal{C}} \arrow[l, shift left, "S"] \arrow[l, phantom, "\sim"] \\ \mathbf{LHCov}/\mathcal{C} \arrow[u, hook] \arrow[r, shift left, "\Gamma"] & \mathrm{Sh}_{C_1}(C_0) \arrow[u, hook] \arrow[l, shift left, "S"] \arrow[l, phantom, "\sim"] \end{tikzcd}$$

Proof: The argument is essentially the same as in [BDNH76]. We have the functors $S : \mathbf{Sp}^{\mathcal{C}} \to \mathbf{TCat}/\mathcal{C}$ and $\Gamma : \mathbf{TCov}/\mathcal{C} \to \mathbf{Sp}^{\mathcal{C}}$ as in the case for groupoids. Each $\mathcal{C}$-space (a, A, α_a) yields a pair $(\hat{A}, \hat{a}) = S(a, A, \alpha_a)$ with $\hat{A}$ the category in (5.2) (without the inverse map) with structure maps as in (5.3) and the morphism $\hat{a}$ to $\mathcal{C}$ as in (5.4). The image of S lies in $\mathbf{TCov}/\mathcal{C}$ since the arrow $\hat{a}_1 \times_{C_0} s_A : A_1 \to C_1 \times_{C_0} A_0$ is the identity on $C_1 \times_{C_0} A$. So S is a functor $\mathbf{Sp}^{\mathcal{C}} \to \mathbf{TCov}/\mathcal{C}$.

Γ is given by (5.6). Thus each pair $(\mathcal{D}, d) \in \mathbf{TCov}/\mathcal{C}$ determines a $\mathcal{C}$-space $\Gamma(\mathcal{D}, d) = (d_0, D_0, t_D \circ \psi_d^{-1})$ where ψ_d is the homeomorphism $D_1 \xrightarrow{\sim} G_1 \times_{G_0} D_0$.

We have $\Gamma S(a, A, \alpha_a) = (a, A, \alpha_a)$. Indeed, the homeomorphism $A_1 \xrightarrow{\psi_{\hat{a}}} G_1 \times_{G_0} A_0$ is the identity and $t_A = \alpha_a$. So ΓS is the identity on $\mathbf{Sp}^{\mathcal{C}}$.

Conversely, let $S\Gamma(\mathcal{D}, d) = (\hat{\mathcal{D}}, \hat{d})$, where $\hat{\mathcal{D}} : G_1 \times_{G_0} D_0 \rightrightarrows D_0$. It is straightforward to verify that the morphism $\gamma_d : (\hat{\mathcal{D}}, \hat{d}) \to (\mathcal{D}, d)$ with components $(\gamma_d)_1 = \psi_d^{-1}$, where ψ_d is the homeomorphism $D_1 \to C_1 \times_{C_0} D_0$, and $(\gamma_d)_0 = 1_{D_0}$ is an morphism in $\mathbf{TCov}/\mathcal{C}$ with an inverse that has components ψ_d and 1_{D_0}.

It remains to show naturality of $\gamma : S\Gamma \dot{\to} 1$, where 1 is the identity functor on $\mathbf{TCov}/\mathcal{C}$. For $\phi : (\mathcal{D}, d) \to (\mathcal{D}', d')$ we need to show that $\gamma_{d'} \circ S\Gamma(\phi) = \phi \circ \gamma_d$. On objects this is trivial. Concerning the arrow components, we have that $(S\Gamma(\phi))_1 = 1_{C_1} \times_{C_0} \phi_0$. Consider the map $\psi_{d'} \circ \phi_1 \circ \psi_d^{-1}$ which is the dotted arrow in the diagram

$$\begin{tikzcd} C_1 \times_{C_0} D_0 \arrow[r, "\psi_d^{-1}", "\sim"'] \arrow[d, dotted] & D_1 \arrow[d, "\phi_1"] \\ C_1 \times_{C_0} D'_0 & D'_1 \arrow[l, "\sim"', "\psi_{d'}"] \end{tikzcd}$$

For $(f, x) \in C_1 \times_{C_0} D_0$ if $\psi_d^{-1}(f, x) = f'$ then $d_1(f') = f$ and $s_D(f') = x$. And since

$$\begin{aligned}\psi_{d'} \circ \phi_1(f') &= (d_1' \circ \phi_1(f'), s_{D'} \circ \phi_1(f')) \\ &= (d_1(f'), \phi_0 \circ s_D(f')) \\ &= (f, \phi_0(x))\end{aligned}$$

we have that $\psi_{d'} \circ \phi_1 \circ \psi_d^{-1} = 1_{C_1} \times_{C_0} \phi_0$. Then $\phi_1 \circ \psi_d^{-1} = \psi_{d'}^{-1} \circ (1_{C_1} \times_{C_0} \phi_0)$, so by the left rectangle of the commutative diagram

$$\begin{tikzcd} C_1 \times_{C_0} D_0 \arrow[r, "\psi_d^{-1}", "\sim"'] \arrow[d, "1_{C_1} \times_{C_0} \phi_0"'] & D_1 \arrow[r, "t_D"] \arrow[d, "\phi_1"] & D_0 \arrow[d, "\phi_0"] \\ C_1 \times_{C_0} D_0' \arrow[r, "\sim", "\psi_{d'}^{-1}"'] & D_1' \arrow[r, "t_{D'}"'] & D_0' \end{tikzcd}$$

we have that γ is a natural transformation. Commutativity of the outer rectangle expresses equivariance of $\Gamma(\phi) = \phi_0$.

If (e, E, α_e) is an equivariant sheaf and $S(e, E, \alpha_e) = (\hat{E}, \hat{e})$ then that both components of $\hat{e}$ are LH's, so $\hat{e}$ is an LHc morphism. The image of Γ on an object $(\mathcal{D}, d)$ in $\mathbf{LHCov}/\mathcal{C}$ is obviously a $\mathcal{C}$-sheaf. $\square$

Corollary 5.6. *$S : \mathbf{Sp}^{\mathcal{C}} \to \mathbf{TCat}/\mathcal{C}$ is full and faithful.*

Proof: This follows from that $\mathbf{TCov}/\mathcal{C}$ is a full subcategory of $\mathbf{TCat}/\mathcal{C}$ and S is a part of an equivalence of categories $\mathbf{TCov}/\mathcal{C} \cong \mathbf{Sp}^{\mathcal{C}}$.

The following direct proof shows why S is full. Let $\phi : S(e, E, \alpha_e) \to S(a, A, \alpha_a)$, $S(e, E, \alpha_e) = (\hat{E}, \hat{e})$ and $S(a, A, \alpha_a) = (\hat{A}, \hat{a})$. As ϕ is a morphism between the topological categories $\hat{E}$ and $\hat{A}$ we have commutativity of the following diagrams

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "s_E = \pi_E"] \arrow[d, "\phi_1"'] & E \arrow[d, "\phi_0"] \\ C_1 \times_{C_0} A \arrow[r, "s_A = \pi_A"'] & A \end{tikzcd} \qquad \begin{tikzcd} C_1 \times_{C_0} E \arrow[r, "t_E = \alpha_e"] \arrow[d, "\phi_1"'] & E \arrow[d, "\phi_0"] \\ C_1 \times_{C_0} A \arrow[r, "t_A = \alpha_a"'] & A \end{tikzcd} \tag{5.8}$$

where π_E and π_A are the obvious projections. Since ϕ is a morphism between $\hat{E}$ and $\hat{A}$ over $\mathcal{C}$ we have commutativity of

$$\begin{tikzcd} C_1 \times_{C_0} E \arrow[rr, "\phi_1"] \arrow[dr, "\pi^E_{C_1}"'] & & C_1 \times_{C_0} A \arrow[dl, "\pi^A_{C_1}"] \\ & C_1 & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "\phi_0"] \arrow[dr, "e"'] & & A \arrow[dl, "a"] \\ & C_0 & \end{tikzcd} \tag{5.9}$$

where $\pi^E_{C_1}$ and $\pi^A_{C_1}$ are the projections onto the first component. The diagrams to the left in (5.8) and (5.9) shows that $\phi_1 = 1_{C_1} \times_{C_0} \phi_0$. Using this, the diagrams to the right in (5.8) and (5.9) shows that ϕ_0 is a an equivariant map $(e, E, \alpha_e) \to (a, A, \alpha_a)$. So we have $\phi = S(\phi_0)$ and S is full. $\square$

5.3 Construction of a functor $\mathbf{qTGpd}/\mathcal{G} \to \mathbf{Sp}^{\mathcal{G}}$

Moerdijk's site description for the equivariant sheaf topos of an open localic groupoid in [Moe88] depends on defining an equivalence relation in terms of "open subgroupoids" of the underlying groupoid (also see [For13]). In this section we apply a similar equivalence relation to arbitrary objects in $\mathbf{TGpd}/\mathcal{G}$, for a topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$.

Given an object $(\mathcal{H}, h)$ in $\mathbf{TGpd}/\mathcal{G}$, with morphism $h : \mathcal{H} \to \mathcal{G}$, take the pullback $D_h = G_1 \times_{G_0} H_0$:

$$\begin{tikzcd} G_1 \times_{G_0} H_0 \arrow[r, "\pi_{H_0}"] \arrow[d, "\pi_{G_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "h_0"] \\ G_1 \arrow[r, "s_G"'] & G_0 \end{tikzcd} \tag{5.10}$$

Define a relation R_h on D_h, where

$$\begin{aligned} (f, x) \sim_{R_h} (g, y) \quad \text{iff} \quad & t_G(f) = t_G(g) \quad \text{and} \\ & \exists k \in H_1 \left[s_H(k) = x \wedge t_H(k) = y \wedge f = g \circ h_1(k) \right]. \end{aligned} \tag{5.11}$$

In other words, (f, x) is related to (g, y) when there exists an arrow $k : x \to y$ in H_1 such that the following diagram commutes

$$\begin{tikzcd} h_0(x) \arrow[d, "h_1(k)"'] \arrow[dr, "f"] & \\ h_0(y) \arrow[r, "g"'] & \bullet \end{tikzcd}$$

This relation R_h is reflexive, symmetric and transitive, so R_h is an equivalence relation on D_h. Let D_h/R_h denote the corresponding quotient space.

Definition 5.7. A *semi-local homeomorphic groupoid morphism*, or simply a *semi-LH morphism*, is a morphism h of topological groupoids with the arrow component h_1 open and the object component h_0 an LH. Let $\mathbf{sLHTGpd}(\mathcal{G})$ denote the full subcategory of $\mathbf{TGpd}/\mathcal{G}$ with objects pairs $(\mathcal{H}, h)$ where $h : \mathcal{H} \to \mathcal{G}$ is a semi-LH morphism.

We summarize how semi-LH's and LHc's are related in a lemma. From the diagram (5.12) below it is clear that the notion of a semi-LH morphism can be regarding as a weakning of that of an LHc morphism, where the map $\psi = h_1 \times_{G_0} s_H : H_1 \to G_1 \times_{G_0} H_0$ is required to be open instead of a homeomorphism.

Lemma 5.8. *Let $h : \mathcal{H} \to \mathcal{G}$ be a semi-LH morphism between topological groupoids $\mathcal{H}, \mathcal{G}$. Then:*

(i) if $\mathcal{G}$ is open, then $\mathcal{H}$ is open;

(ii) if h is also covering morphism, then h is an LHc morphism.

Furthermore, if $h' : \mathcal{H} \to \mathcal{G}$ is an LHc morphism, then h' is a semi-LH morphism.

Proof: (i): Since h_0 is an LH, the projection $\pi_{G_1} : G_1 \times_{G_0} H_0 \to G_1$ is an LH:

$$\begin{tikzcd}
H_1 \arrow[drr, dotted, "h_1 \times_{G_0} s_H"] \arrow[drrr, bend left, "s_H"] \arrow[ddrr, bend right, "h_1"'] & & & \\
& & G_1 \times_{G_0} H_0 \arrow[r, "\pi_{H_0}"] \arrow[d, "\pi_{G_1}"'] \arrow[dr, phantom, "\lrcorner"] & H_0 \arrow[d, "h_0"] \\
& & G_1 \arrow[r, "s_G"'] & G_0
\end{tikzcd} \tag{5.12}$$

Since h_1 is open we obtain from Lemma 2.3 (ii) that $h_1 \times_{G_0} s_H$ is open. Then $s_H = \pi_{H_0} \circ (h_1 \times_{G_0} s_H)$ is open. So $\mathcal{H}$ is open by Proposition 3.4.

(ii): Since h_0 is an LH, the statement follows from the comment following the definition of an LHc morphism.

The last statement follows from that any LH is open. $\square$

Proposition 5.9. *Let $\mathcal{G} : G_1 \rightrightarrows G_0$ be an open topological groupoid. For $(\mathcal{H}, h) \in \mathbf{sLHTGpd}(\mathcal{G})$, the equivalence relation R_h in* (5.11) *on $D_h = G_1 \times_{G_0} H_0$ is an open subset $R_h \subseteq D_h \times_{G_0} D_h$, where D_h and $D_h \times_{G_0} D_h$ are the following pullbacks (in* $\mathbf{Sp}$*)*

$$\begin{tikzcd}
G_1 \times_{G_0} H_0 \arrow[r, "\pi_{H_0}"] \arrow[d, "\pi_{G_1}"'] \arrow[dr, phantom, "\lrcorner"] & H_0 \arrow[d, "h_0"] \\
G_1 \arrow[r, "s_G"'] & G_0
\end{tikzcd}
\qquad
\begin{tikzcd}
D_h \times_{G_0} D_h \arrow[r, "\pi_2^D"] \arrow[d, "\pi_1^D"'] \arrow[dr, phantom, "\lrcorner"] & D_h \arrow[d, "t_G \circ \pi_{G_1}"] \\
D_h \arrow[r, "t_G \circ \pi_{G_1}"'] & G_0
\end{tikzcd} \tag{5.13}$$

Proof: We drop the subscript h on D and R in this proof. From (5.13), the projection maps $\pi_1^D, \pi_2^D : D \times_{G_0} D \to D$ and the map $t_G \circ \pi_{G_1} : D \to G_0$ are open.

Let θ be the composition of maps

$$D \times_{G_0} D \xrightarrow{(i_G \circ \pi_{G_1}) \times_{G_0} \pi_{G_1}} G_1 \times_{G_0} G_1 \xrightarrow{m_G} G_1$$

then $\theta((f, x), (g, y)) = f^{-1} \circ g$. In the proof of Lemma 5.8 we showed that the map $h_1 \times_{G_0} s_H : H_1 \to D$ is open for $(\mathcal{H}, h) \in \mathbf{sLHTGpd}(\mathcal{G})$. Using this we get that the map $\psi' = h_1 \times_{G_0} s_H \times_{G_0} t_H : H_1 \to D \times_{G_0} H_0$ is open from Lemma 2.3

(ii) and the commutative diagram

$$\begin{tikzcd}
H_1 \arrow[drr, bend left, "t_H"] \arrow[dr, dotted, "\psi'"] \arrow[ddr, bend right, "h_1 \times_{G_0} s_H"'] & & \\
& D \times_{G_0} H_0 \arrow[r] \arrow[d, "\pi_D^H"'] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "h_0"] \\
& D \arrow[r, "t_G \circ \pi_{G_1}"'] & G_0
\end{tikzcd}$$

where π_D^H is an LH since h_0 is an LH. The unique map $\theta' : D \times_{G_0} D \to D$ such that the following diagram commute in **Sp**:

$$\begin{tikzcd}
D \times_{G_0} D \arrow[drr, bend left, "\pi_{H_0} \circ \pi_2^D"] \arrow[dr, dotted, "\theta'"] \arrow[ddr, bend right, "\theta"'] & & \\
& D \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "h_0"] \\
& G_1 \arrow[r, "s_G"'] & G_0
\end{tikzcd}$$

is continuous and takes $((f,x),(g,y)) \mapsto (f^{-1} \circ g, y)$. Similarly we have that the function $\theta'' : D \times_{G_0} D \to D \times_{G_0} H_0$ which takes $((f,x),(g,y)) \mapsto ((f^{-1} \circ g, y), x)$ is continuous:

$$\begin{tikzcd}
D \times_{G_0} D \arrow[drr, bend left, "\pi_{H_0} \circ \pi_1^D"] \arrow[dr, dotted, "\theta''"] \arrow[ddr, bend right, "\theta'"'] & & \\
& D \times_{G_0} H_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "h_0"] \\
& D \arrow[r, "t_G \circ \pi_{G_1}"'] & G_0
\end{tikzcd}$$

Now, from the pullback

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & D \times_{G_0} D \arrow[d, "\theta''"] \\
H_1 \arrow[r, "\psi'"'] & D \times_{G_0} H_0
\end{tikzcd}$$

we see that $R \subseteq D \times_{G_0} D$ is open. $\Box$

Definition 5.10. Let $\mathbf{qTGpd}(\mathcal{G})$ denote the full subcategory of $\mathbf{TGpd}/\mathcal{G}$ with objects pairs $(\mathcal{H}, h)$ such that the equivalence relation R_h in (5.11) is open.

Recall from Section 2.1.1 that we call an equivalence relation open if the corresponding quotient map is open. For an open topological groupoid $\mathcal{G}$, Lemma 5.8 and Proposition 5.11 show that we have the following relationship of categories

$$\begin{tikzcd} \mathbf{TCov}/\mathcal{G} \arrow[r, hook] & \mathbf{qTGpd}(\mathcal{G}) \arrow[r, hook] & \mathbf{TGpd}/\mathcal{G} \\ \mathbf{LHCov}/\mathcal{G} \arrow[r, hook] \arrow[u, hook] & \mathbf{sLHTGpd}(\mathcal{G}) \arrow[u, hook] & \end{tikzcd}$$

Proposition 5.11. *For a morphism of topological groupoids $h : \mathcal{H} \to \mathcal{G}$ where $\mathcal{G}$ is open, let q be the quotient map $q : D_h \to D_h/R_h$ of the equivalence relation of* (5.11) *on $D_h = G_1 \times_{G_0} H_0$.*

(i) If h is a semi-LH morphism, then q is open;

(ii) if h is a topological covering morphism, then q is open.

Proof: (i): $t_G \circ \pi_{G_1} : D_h \to G_0$ is open, where π_{G_1} is the projection onto G_1, and constant on the equivalence classes of R_h. Furthermore, $R_h \subseteq D_h \times_{G_0} D_h$ is an open subset by Proposition 5.9. By Lemma 2.6 (iii), q is open.

(ii): The equivalence relation R_h in (5.11) on D_h coincides with the equivalence relation in (3.2) in Theorem 3.1 for the $\mathcal{G}$-space $\Gamma(\mathcal{H}, h) = (h_0, H_0, t_H \circ \psi_h^{-1})$, where $\psi_h : H_1 \to G_1 \times_{G_0} H_0$ is the homeomorphism associated with the topological covering morphism h. That is, we have that

$$(f, x) \sim_{R_h} (g, y) \quad \text{iff} \quad t_H \circ \psi_h^{-1}(f, x) = t_H \circ \psi_h^{-1}(g, y), \tag{5.14}$$

for $(f, x), (g, y)$ in $G_1 \times_{G_0} H_0$.

Indeed, if $(f, x) \sim_{R_h} (g, y)$ then there an arrow $k : x \to y$ in H_1 such that $f = g \circ h_1(k)$. Let $g_y = \psi_h^{-1}(g, y)$, then $h_1(g_y) = g$ and $s_H(g_y) = y$, so $g_y \circ k$ is defined. Since $h_1(g_y \circ k) = h_1(g_y) \circ h_1(k) = f$ and $s_H(g_y \circ k) = x$ we have that $\psi_h^{-1}(f, x) = g_y \circ k$. Hence $t_H \circ \psi_h^{-1}(g, y) = t_H \circ \psi_h^{-1}(f, x)$.

If $t_H \circ \psi_h^{-1}(g, y) = t_H \circ \psi_h^{-1}(f, x)$, $g_y = \psi_h^{-1}(g, y)$ and $f_x = \psi_h^{-1}(f, x)$ then $g_y^{-1} \circ f_x$ is defined and satisfies

$$h_1(g_y^{-1} \circ f_x) = h_1(g_y^{-1}) \circ h_1(f_x) = g^{-1} \circ f.$$

Hence, with $k = g_y^{-1} \circ f_x$ in H_1 we have $f = g \circ h_1(k)$ and $s_H(k) = s_H(f_x) = x$ and $t_H(k) = t_H(g_y^{-1}) = s_H(g_y) = y$. So $(f, x) \sim_{R_h} (g, y)$.

Since the groupoid $\mathcal{G}$ is open and h is a topological covering morphism, it follows from Lemma 5.1 and Proposition 3.4 that $\mathcal{H}$ is open. Hence the action of $(h_0, H_0, t_H \circ \psi_h^{-1})$ is open. It follows from Proposition 3.2 that q is open. $\Box$

For $(\mathcal{H}, h)$ in $\mathbf{TGpd}/\mathcal{G}$ and the equivalence relation R_h in (5.11) on $D_h = G_1 \times_{G_0} H_0$, let D_h/R_h denote the corresponding quotient space and $[t_G]_h$:

$D_h/R_h \to G_0$ be the function $[(f,x)]_{R_h} \mapsto t_G(f)$. Then $[t_G]_h$ is clearly well-defined.

For $G_1 \times_{G_0} D_h/R_h$, the pullback of $[t_G]_h$ along s_G, define $\alpha_h : G_1 \times_{G_0} D_h/R_h \to D_h/R_h$ to be the function $(k, [(f,x)]_{R_h}) \mapsto [(k \circ f, x)]_{R_h}$. It is clear from the definition of R_h that α_h is also well-defined.

Proposition 5.12. *Let $\mathcal{G} : G_1 \rightrightarrows G_0$ be an open topological groupoid. There is a functor $\Lambda : \mathbf{qTGpd}(\mathcal{G}) \to \mathbf{Sp}^{\mathcal{G}}$ defined on objects $(\mathcal{H}, h)$ in $\mathbf{qTGpd}(\mathcal{G})$ by*

$$\Lambda(\mathcal{H}, h) = ([t_G]_h, D_h/R_h, \alpha_h),$$

and on morphisms $\phi : (\mathcal{H}, h) \to (\mathcal{H}', h')$ in $\mathbf{qTGpd}(\mathcal{G})$ by

$$\Lambda(\phi)\left([(f,x)]_{R_h}\right) = \left[(f, \phi_0(x))\right]_{R_{h'}}.$$

Proof: We drop the subscript h while showing that the assignment defines a $\mathcal{G}$-space.

Make $D = G_1 \times_{G_0} H_0$ from (5.10) into a space over G_0 via $t_G \circ \pi_{G_1} : D \to G_0$. By Lemma 2.5 the function $[t_G] : D/R \to G_0$ is continuous since it makes the following diagram commute

$$\begin{tikzcd} D \arrow[rr, two heads, "q"] \arrow[dr, "t_G \circ \pi_{G_1}"'] & & D/R \arrow[dl, "{[t_G]}"] \\ & G_0 & \end{tikzcd} \tag{5.15}$$

where $q : D \to D/R$ is the quotient map.

To show that α is continuous, let m denote the unique map making the diagram below commute in $\mathbf{Sp}$:

$$\begin{tikzcd} G_1 \times_{G_0} D \arrow[dr, dotted, "m"] \arrow[drr, bend left, "\pi_{H_0} \circ \pi_2"] \arrow[ddr, bend right, "m_G \circ (1_{G_1} \times_{G_0} \pi_{G_1})"'] & & \\ & D = G_1 \times_{G_0} H_0 \arrow[r, "\pi_{H_0}"] \arrow[d, "\pi_{G_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & H_0 \arrow[d, "h_0"] \\ & G_1 \arrow[r, "s_G"'] & G_0 \end{tikzcd}$$

where $\pi_2 : G_1 \times_{G_0} D \to D$ is the projection. Then $m(g, (f,x)) = (g \circ f, x)$ for $(g, (f,x)) \in G_1 \times_{G_0} D$.

Since q is open and (5.15) commutes, $1_{G_1} \times_{G_0} q : G_1 \times_{G_0} D \to G_1 \times_{G_0} D/R$ is open by Lemma 2.4. From the following commutative diagram we show that

α is continuous

$$\begin{tikzcd} G_1 \times_{G_0} D \arrow[r, "m"] \arrow[d, two heads, "1_{G_1} \times_{G_0} q"'] & D \arrow[d, two heads, "q"] \\ G_1 \times_{G_0} D/R \arrow[r, "\alpha"'] & D/R \end{tikzcd}$$

If $U \subseteq D/R$ is open, we get that $(q \circ m)^{-1}(U) = (\alpha \circ (1_{G_1} \times_{G_0} q))^{-1}(U) = (1_{G_1} \times_{G_0} q)^{-1}\left(\alpha^{-1}(U)\right)$ is open. But $1_{G_1} \times_{G_0} q$ is surjective and open, and hence $(1_{G_1} \times_{G_0} q)[(1_{G_1} \times_{G_0} q)^{-1}\left(\alpha^{-1}(U)\right)] = \alpha^{-1}(U)$ is open. Thus α is continuous.

For completeness we write out the straightforward verification that α satisfies the conditions of being an action:

$$\begin{aligned} [t_G] \circ \alpha(k, [(f,x)]_R) &= t_G(k), \\ \alpha(1_{t_G(f)}, [(f,x)]_R) &= [(f,x)]_R, \\ \alpha(p, \alpha(k, [(f,x)]_R)) &= [(p \circ k \circ f, x)]_R \\ &= \alpha(p \circ k, [(f,x)]_R). \end{aligned}$$

Now consider an arrow $\phi : (\mathcal{H}, h) \to (\mathcal{H}', h')$. We write

$$\begin{aligned} \Lambda(\mathcal{H}, h) &= ([t_G]_h, D_h/R_h, \alpha_h), \\ \Lambda(\mathcal{H}', h') &= ([t_G]_{h'}, D_{h'}/R_{h'}, \alpha_{h'}), \end{aligned}$$

where $D_h = G_1 \times_{G_0} H_0$, $D_{h'} = G_1 \times_{G_0} H'_0$ and R_h and $R_{h'}$ are the equivalence relations on D_h and $D_{h'}$ corresponding to (5.11).

The function $\Lambda(\phi)$ is well defined, for if $(f,x) \sim_{R_h} (g,y)$ then there is an arrow $k : x \to y$ in H_1 such that the following diagram commutes

$$\begin{tikzcd} h_0(x) \arrow[dr, "f"] \arrow[dd, "h_1(k)"'] & \\ & \bullet \\ h_0(y) \arrow[ur, "g"'] & \end{tikzcd}$$

and since $h_0 = h'_0 \circ \phi_0$ and $h_1 = h'_1 \circ \phi_1$ we get $(f, \phi_0(x)) \sim_{R_{h'}} (g, \phi_0(y))$ via $\phi_1(k) : \phi_0(x) \to \phi_0(y)$. So the function $[(f,x)]_{R_h} \mapsto [(f, \phi_0(x))]_{R_{h'}}$ is well defined and makes the following diagram commute

$$\begin{tikzcd} D_h/R_h \arrow[rr, "\Lambda(\phi)"] \arrow[dr, "{[t_G]_h}"'] & & D_{h'}/R_{h'} \arrow[dl, "{[t_G]_{h'}}"] \\ & G_0 & \end{tikzcd}$$

Now, $q_{h'} \circ (1_{G_1} \times_{G_0} \phi_0) : D_h \to D_{h'}/R_{h'}$ is continuous and, as shown above, constant on the equivalence classes of $R_{h'}$, where $q_{h'} : D_{h'} \to D_{h'}/R_{h'}$ is the quotient map. So the function $\Lambda(\phi) : D_h/R_h \to D_{h'}/R_{h'}$ is continuous by Lemma 2.5.

It remains to show that $\Lambda(\phi)$ respects the action, which is a straightforward verification:

$$
\begin{tikzcd}
G_1 \times_{G_0} D_h/R_h \arrow[r, "\alpha_h"] \arrow[d, "1_{G_1} \times_{G_0} \Lambda(\phi)"'] & D_h/R_h \arrow[d, "\Lambda(\phi)"] & (k, [(g,x)]_{R_h}) \arrow[r, mapsto] \arrow[d, mapsto] & {[(k \circ g, x)]_{R_h}} \arrow[d, mapsto] \\
G_1 \times_{G_0} D_{h'}/R_{h'} \arrow[r, "\alpha_{h'}"'] & D_{h'}/R_{h'} & (k, [(g, \phi_0(x))]_{R_{h'}}) \arrow[r, mapsto] & {[(k \circ g, \phi_0(x))]_{R_{h'}}}
\end{tikzcd}
$$

Clearly $\Lambda(1_{(\mathcal{H},h)}) = 1_{\Lambda(\mathcal{H},h)}$ and $\Lambda(\phi \circ \varphi) = \Lambda(\phi) \circ \Lambda(\varphi)$ holds for ϕ, φ such that $\phi \circ \varphi$ is defined. This completes the proof. □

We remark that coequalizers in **Sp** are not stable under pullback (see [DK70]). Thus we can not in general conclude that $G_1 \times_{G_0} D_h/R_h$ is (homeomorphic to) the quotient of $G_1 \times_{G_0} D_h$ under the equivalence relation induced by the equivalence relation R_h on D_h.

Proposition 5.13. *Let $\mathcal{G} : G_1 \rightrightarrows G_0$ be an open topological groupoid. If $h : \mathcal{H} \to \mathcal{G}$ is a semi-LH morphism then $\Lambda(\mathcal{H}, h) \in \mathrm{Sh}_{G_1}(G_0)$.*

Proof: From Proposition 5.9, the equivalence relation R_h in (5.11) is an open subset $R_h \subseteq D_h \times_{G_0} D_h$. Since $t_G \circ \pi_{G_1}$ in (5.15) is open, $[t_G]_h : D_h/R_h \to G_0$ is an LH by Lemma 2.7. Hence $\Lambda(\mathcal{H}, h)$ is an equivariant sheaf. □

5.4 Adjunction $\Lambda \dashv S$

For a $\mathcal{G}$-space (e, E, α_e), the image $S(e, E, \alpha_e)$ is covering groupoid of $\mathcal{G}$. When the topological groupoid $\mathcal{G}$ is open, we have that $S(e, E, \alpha_e)$ is an object in $\mathbf{qTGpd}(\mathcal{G})$ by Proposition 5.11. With some abuse of notation, we use S, as defined in the previous sections, to denote this functor $\mathbf{Sp}^{\mathcal{G}} \to \mathbf{qTGpd}(\mathcal{G})$.

The next theorem shows that we have following pair of adjoints for an open topological groupoid $\mathcal{G}$:

$$
\begin{tikzcd}
\mathbf{qTGpd}(\mathcal{G}) \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathbf{Sp}^{\mathcal{G}} \arrow[l, shift left, "S"] \\
\mathbf{sLHTGpd}(\mathcal{G}) \arrow[u, hook] \arrow[r, shift left, "\Lambda"] \arrow[r, phantom, "\perp"] & \mathrm{Sh}_{G_1}(G_0) \arrow[l, shift left, "S"] \arrow[u, hook]
\end{tikzcd}
$$

Theorem 5.14. *For an open topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$, Λ is left adjoint to $S : \mathbf{Sp}^{\mathcal{G}} \to \mathbf{qTGpd}(\mathcal{G})$. Furthermore, Λ restricted to* **sLHTGpd**$(\mathcal{G})$ *is left adjoint to S restricted to* $\mathrm{Sh}_{G_1}(G_0)$.

Proof: In this proof we write $\Lambda(\mathcal{H}, h) = ([t_G]_h, D_h/R_h, \alpha_h)$, as in Proposition 5.12, and $S(e, E, \alpha_e) = (\hat{E}, \hat{e})$, as in Section 5.1. First we observe that for a morphism $g : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ the following diagrams commutes as $g : \mathcal{H} \to \hat{E}$ is a morphism of groupoids over $\mathcal{G}$

$$\begin{tikzcd}
H_1 \arrow[r, "s_H"] \arrow[d, "g_1"'] & H_0 \arrow[d, "g_0"] \\
G_1 \times_{G_0} E \arrow[r, "\pi_E = s_E"'] & E
\end{tikzcd}
\qquad
\begin{tikzcd}
H_1 \arrow[rr, "g_1"] \arrow[dr, "h_1"'] & & G_1 \times_{G_0} E \arrow[dl, "\pi^E_{G_1} = \hat{e}_1"] \\
& G_1 &
\end{tikzcd}$$

where π_E and $\pi^E_{G_1}$ are the projections onto the second and first component, respectively. Thus for $k \in H_1$ the arrow component of g is given by

$$g_1(k) = (h_1(k), g_0 \circ s_H(k)). \tag{5.16}$$

We now describe the bijection

$$\hom_{\mathrm{Sp}^{\mathcal{G}}}(\Lambda(\mathcal{H}, h), (e, E, \alpha_e)) \xrightarrow{\varphi} \hom_{\mathbf{qTGpd}(\mathcal{G})}((\mathcal{H}, h), S(e, E, \alpha_e))$$

The correspondence is as follows. Associate to $f : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$ the morphism $\varphi(f) = g : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ with components

$$\begin{aligned}
g_0 &: H_0 \to E, & g_0(x) &= f\left(\left[(1_{h_0(x)}, x)\right]_{R_h}\right), \\
g_1 &: H_1 \to G_1 \times_{G_0} E, & g_1(k) &= (h_1(k), g_0 \circ s_H(k)).
\end{aligned}$$

To show continuity of the components of g, we start with g_0. This is just the composition

$$H_0 \xrightarrow{\sim} G_0 \times_{G_0} H_0 \xrightarrow{u_G \times_{G_0} 1_{H_0}} G_1 \times_{G_0} H_0 \xrightarrow{q_h} (G_1 \times_{G_0} H_0)/R_h \xrightarrow{f} E$$

where $q_h : D_h \to D_h/R_h$ is the quotient map. Concerning g_1, notice that $s_G \circ h_1(k) = h_0 \circ s_H(k) = e \circ g_0 \circ s_H(k)$. We thus obtain $g_1 = h_1 \times (g_0 \circ s_H)$ as the unique map making the following diagram commute (in **Sp**):

$$\begin{tikzcd}
H_1 \arrow[dr, dotted, "h_1 \times_{G_0} (g_0 \circ s_H)"] \arrow[drr, bend left, "g_0 \circ s_H"] \arrow[ddr, bend right, "h_1"'] & & \\
& G_1 \times_{G_0} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d, "e"] \\
& G_1 \arrow[r, "s_G"'] & G_0
\end{tikzcd}$$

To show that $g : \mathcal{H} \to \hat{E}$ defines a groupoid morphism we must show commutativity of the left rectangle and the outer, middle and inner right rectangles of

$$\begin{tikzcd}
H_1 \times_{H_0} H_1 \arrow[r, "m_H"] \arrow[d, "g_1 \times_{E_0} g_1"'] & H_1 \arrow[r, shift left=2, "t_H"] \arrow[r, shift right=2, "s_H"'] \arrow[d, "g_1"] & H_0 \arrow[l, "u_H" description] \arrow[d, "g_0"] \\
E_1 \times_{E_0} E_1 \arrow[r, "m_E"] & E_1 \arrow[r, shift left=2, "s_E"] \arrow[r, shift right=2, "t_E"'] & E_0 \arrow[l, "u_E" description]
\end{tikzcd}$$

From this it then follows that $g_1 \circ i_H = i_E \circ g_1$. That the identity $g_0 \circ s_H = s_E \circ g_1$ holds follows directly from the definitions. To see that $g_0 \circ t_H = t_E \circ g_1$, use that f respect the action in the following way

$$\begin{aligned}
t_E \circ g_1(k) &= \alpha_e\left(h_1(k), g_0 \circ s_H(k)\right) \\
&= \alpha_e\left(h_1(k), f\left(\left[\left(1_{h_0 \circ s_H(k)}, s_H(k)\right)\right]_{R_h}\right)\right) \\
&= f \circ \alpha_h\left(h_1(k), \left[\left(1_{h_0 \circ s_H(k)}, s_H(k)\right)\right]_{R_h}\right) \\
&= f\left([(h_1(k), s_H(k))]_{R_h}\right) \\
&= f\left(\left[\left(1_{h_0 \circ t_H(k)}, t_H(k)\right)\right]_{R_h}\right) \\
&= g_0 \circ t_H(k).
\end{aligned}$$

where we've used that $(h_1(k), s_H(k)) \sim_{R_h} \left(1_{h_0 \circ t_H(k)}, t_H(k)\right)$ via $k : s_H(k) \to t_H(k)$:

$$\begin{tikzcd}
h_0 \circ s_H(k) \arrow[rr, "h_1(k)"] \arrow[d, "h_1(k)"'] & & h_0 \circ t_H(k) \\
h_0 \circ t_H(k) \arrow[urr, "1_{h_0 \circ t_H(k)}"'] & &
\end{tikzcd}$$

Concerning the identity maps, it holds that $g_1 \circ u_H(x) = u_E \circ g_0(x)$:

$$\begin{aligned}
g_1(1_x) &= (h_1(1_x), g_0(x)) \\
&= \left(1_{h_0(x)}, g_0(x)\right) \\
&= u_E \circ g_0(x).
\end{aligned}$$

It remains to show that $g_1 \circ m_H = m_E \circ (g_1 \times_{E_0} g_1)$, which is straightforward:

$$\begin{aligned}
g_1 \circ m_H(k, p) &= (h_1(k \circ p), g_0 \circ s_H(k \circ p)) \\
&= (h_1(k) \circ h_1(p), g_0 \circ s_H(p)) \\
&= m_E \circ (g_1 \times_{E_0} g_1)(k, p).
\end{aligned}$$

This shows that $g : \mathcal{H} \to \hat{E}$ is a morphism of groupoids. Observe that we also have $h = \hat{e} \circ g$ as the following diagrams commutes

$$\begin{tikzcd}
x \arrow[r, mapsto, "g_0"] \arrow[dr, mapsto, "h_0"'] & f([(1_{h_0(x)}, x)]) \arrow[d, mapsto, "e"] \\
& h_0(x)
\end{tikzcd}
\qquad
\begin{tikzcd}
k \arrow[r, mapsto, "g_1"] \arrow[dr, mapsto, "h_1"'] & (h_1(k), g_0 \circ s_H(k)) \arrow[d, mapsto, "\pi^E_{G_1}"] \\
& h_1(k)
\end{tikzcd}$$

This completes the proof that g is a morphism in $\mathbf{qTGpd}(\mathcal{G})$.

Conversely, to $g : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ associate $f : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$ where $f([(k, x)]_{R_h}) = \alpha_e(k, g_0(x))$, then the following diagram commutes

$$\begin{tikzcd} D_h/R_h \arrow[rr, "f"] \arrow[dr, "{[t_G]_h}"'] & & E \arrow[dl, "e"] \\ & G_0 & \end{tikzcd}$$

We now show that f is well-defined, then continuity of f follows from Lemma 2.5 as f makes the diagram below commute

$$\begin{tikzcd} D_h \arrow[d, two heads, "q_h"'] \arrow[dr, "\alpha_e \circ (1_{G_1} \times_{G_0} g_0)"] & \\ D_h/R_h \arrow[r, "f"'] & E \end{tikzcd}$$

In addition, notice that as $[t_G]_h = e \circ f$, if e and $[t_G]_h$ are LH's then continuity of f will imply that f is an LH.

If $(k, x) \sim_{R_h} (p, y)$ then there is an arrow $u : x \to y$ in H_1 such that $k = p \circ h_1(u)$, as in

$$\begin{tikzcd} h_0(x) \arrow[d, "h_1(u)"'] \arrow[dr, "k"] & \\ h_0(y) \arrow[r, "p"'] & \bullet \end{tikzcd}$$

Since $g : \mathcal{H} \to \hat{E}$ is morphism of groupoids we have that $t_E \circ g_1 = g_0 \circ t_H$. Using that g_1 satisfies (5.16) and that $t_E = \alpha_e$, we thus have that $g_0 \circ t_H(u) = \alpha_e(h_1(u), g_0 \circ s_H(u))$. From this it follows that f is well defined:

$$\begin{aligned} \alpha_e(k, g_0(x)) &= \alpha_e(p \circ h_1(u), g_0 \circ s_H(u))) \\ &= \alpha_e(p, \alpha_e(h_1(u), g_0 \circ s_H(u))) \\ &= \alpha_e(p, g_0 \circ t_H(u)) \\ &= \alpha_e(p, g_0(y)). \end{aligned}$$

It remains to show that f respects the action, which is straightforward:

$$\begin{tikzcd} G_1 \times_{G_0} D_h/R_h \arrow[r, "\alpha_h"] \arrow[d, "1_{G_1} \times_{G_0} f"'] & D_h/R_h \arrow[d, "f"] & (p, [(k, x)]_{R_h}) \arrow[r, mapsto] \arrow[d, mapsto] & {[(p \circ k, x)]_{R_h}} \arrow[d, mapsto] \\ G_1 \times_{G_0} E \arrow[r, "\alpha_e"'] & E & (p, \alpha_e(k, g_0(x))) \arrow[r, mapsto] & \alpha_e(p \circ k, g_0(x)) \end{tikzcd}$$

This shows that f is an arrow in $\mathbf{Sp}^{\mathcal{G}}$ (and in $\mathrm{Sh}_{C_1}(C_0)$ if $\Lambda(\mathcal{H}, h)$, (e, E, α_e) are in $\mathrm{Sh}_{C_1}(C_0)$).

This correspondence is bijective. From $g : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ we, accordingly, get $f : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$ and then $\varphi(f) = \bar{g} : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ has components

$$\begin{aligned}
\bar{g}_0(x) &= f\left(\left[\left(1_{h_0(x)}, x\right)\right]_{R_h}\right) \\
&= \alpha_e(1_{h_0(x)}, g_0(x)) \\
&= g_0(x), \\
\bar{g}_1(k) &= (h_1(k), \bar{g}_0 \circ s_H(k)) \\
&= (h_1(k), g_0 \circ s_H(k)) \\
&= g_1(k).
\end{aligned}$$

where we used that $\bar{g}_1$ and g_1 satisfies (5.16). So $\bar{g} = g$. Conversely, from $f : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$ we get $\varphi(f) = g : (\mathcal{H}, h) \to (\hat{E}, \hat{e})$ and then the correspondence yield $\bar{f} : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$ where

$$\begin{aligned}
\bar{f}\left([(k, x)]_{R_h}\right) &= \alpha_e(k, g_0(x)) \\
&= \alpha_e\left(k, f\left(\left[\left(1_{h_0(x)}, x\right)\right]_{R_h}\right)\right) \\
&= f \circ \alpha_h\left(k, \left[(1_{h_0(x)}, x)\right]_{R_h}\right) \\
&= f\left([(k, x)]_{R_h}\right).
\end{aligned}$$

Hence $\bar{f} = f$ and the correspondence is bijective. It remains to show naturality in the arguments $(\mathcal{H}, h)$ and (e, E, α_e).

To this end, let $\phi : (\mathcal{H}, h) \to (\mathcal{H}', h')$ be an arrow in $\mathbf{qTGpd}(\mathcal{G})$. Commutativity of the diagram

$$\begin{tikzcd}
\hom\left(\Lambda(\mathcal{H}', h'), (e, E, \alpha_e)\right) \arrow[r, "\varphi"] \arrow[d, "\hom(\Lambda(\phi), (e, E, \alpha_e))"'] & \hom\left((\mathcal{H}', h'), S(e, E, \alpha_e)\right) \arrow[d, "\hom(\phi, S(e, E, \alpha_e))"] \\
\hom\left(\Lambda(\mathcal{H}, h), (e, E, \alpha_e)\right) \arrow[r, "\varphi"'] & \hom\left((\mathcal{H}, h), S(e, E, \alpha_e)\right)
\end{tikzcd}$$

holds if for any $k : \Lambda(\mathcal{H}', h') \to (e, E, \alpha_e)$ we have $\varphi(k \circ \Lambda(\phi)) = \varphi(k) \circ \phi$. This identity indeed holds; equality on the object components follows from

$$\begin{aligned}
(\varphi(k) \circ \phi)_0(x) &= k\left(\left[\left(1_{h_0' \circ \phi_0(x)}, \phi_0(x)\right)\right]_{R_{h'}}\right), \\
\varphi(k \circ \Lambda(\phi))_0(x) &= k \circ \Lambda(\phi)\left(\left[\left(1_{h_0(x)}, x\right)\right]_{R_h}\right) \\
&= k\left(\left[\left(1_{h_0' \circ \phi_0(x)}, \phi_0(x)\right)\right]_{R_{h'}}\right),
\end{aligned}$$

and equality on the arrow components follows from

$$\begin{aligned}
(\varphi(k) \circ \phi)_1(p) &= (h_1' \circ \phi_1(p), \varphi(k)_0 \circ s_{H'} \circ \phi_1(p)) \\
&= (h_1' \circ \phi_1(p), \varphi(k)_0 \circ \phi_0 \circ s_H(p)), \\
\varphi(k \circ \Lambda(\phi))_1(p) &= (h_1(p), \varphi(k \circ \Lambda(\phi))_0 \circ s_H(p)) \\
&= (h_1(p), (\varphi(k) \circ \phi)_0 \circ s_H(p)).
\end{aligned}$$

Now, let $k : (e, E, \alpha_e) \to (a, A, \alpha_a)$, then the following diagram also commutes

$$\begin{tikzcd}
\hom\left(\Lambda(\mathcal{H}, h), (e, E, \alpha_e)\right) \arrow[r, "\varphi"] \arrow[d, "\hom(\Lambda(\mathcal{H},h),k)"'] & \hom\left((\mathcal{H}, h), S(e, E, \alpha_e)\right) \arrow[d, "\hom((\mathcal{H},h),S(k))"] \\
\hom\left(\Lambda(\mathcal{H}, h), (a, A, \alpha_a)\right) \arrow[r, "\varphi"'] & \hom\left((\mathcal{H}, h), S(a, A, \alpha_a)\right)
\end{tikzcd}$$

as $\varphi(k \circ f) = S(k) \circ \varphi(f)$ for $f : \Lambda(\mathcal{H}, h) \to (e, E, \alpha_e)$. Explicitly, on objects we have

$$\begin{aligned}
\left(S(k) \circ \varphi(f)\right)_0 (x) &= k \circ f\left(\left[(1_{h_0(x)}, x)\right]_{R_h}\right), \\
\varphi(k \circ f)_0 (x) &= k \circ f\left(\left[(1_{h_0(x)}, x)\right]_{R_h}\right),
\end{aligned}$$

and on arrows

$$\begin{aligned}
\left(S(k) \circ \varphi(f)\right)_1 (p) &= \left(h_1(p), k \circ \varphi(f)_0 \circ s_H(p)\right) \\
\varphi(k \circ f)_1(p) &= \left(h_1(p), \varphi(k \circ f)_0 \circ s_H(p)\right) \\
&= \left(h_1(p), k \circ \varphi(f)_0 \circ s_H(p)\right).
\end{aligned}$$

This shows that φ is natural in both arguments and proves the adjunction $\Lambda \dashv S$.

By Proposition 5.13, for $(\mathcal{H}, h)$ in $\mathbf{sLHTGpd}(\mathcal{G})$ we have $\Lambda(\mathcal{H}, h) \in \mathrm{Sh}_{G_1}(G_0)$. The components of $\hat{e}$ (in $S(e, E, \alpha_e) = (\hat{E}, \hat{e})$) are LH's for $(e, E, \alpha_e) \in \mathrm{Sh}_{G_1}(G_0)$, so we have $(\hat{E}, \hat{e}) \in \mathbf{sLHTGpd}(\mathcal{G})$. Both $\mathbf{sLHTGpd}(\mathcal{G})$ and $\mathrm{Sh}_{G_1}(G_0)$ are full subcategories of $\mathbf{qTGpd}(\mathcal{G})$ respectively $\mathbf{Sp}^{\mathcal{G}}$. It follows that Λ restricted to $\mathbf{sLHTGpd}(\mathcal{G})$ is left adjoint to S restricted to $\mathrm{Sh}_{G_1}(G_0)$. □

Corollary 5.15. *For an open topological grouopoid $\mathcal{G}$, Λ is naturally isomorphic to Γ on $\mathbf{TCov}/\mathcal{G}$.*

Proof: $S : \mathbf{Sp}^{\mathcal{G}} \to \mathbf{TCov}/\mathcal{G}$ is part of an equivalence of categories (via Γ). This implies that S is a part of an adjoint equivalence, [Mac97, Theorem IV.4.1]. The left adjoint of a functor is unique up to natural isomorphism, [Mac97, Corollary IV.1.1]. □

5.4.1 Unit and conunit of the adjunction $\Lambda \dashv S$

The unit of the adjunction $\Lambda \dashv S$ is $\eta : 1 \stackrel{.}{\to} S\Lambda$, where 1 is the identity on $\mathbf{qTGpd}(\mathcal{G})$, with components

$$\begin{aligned}
\left(\eta_{(\mathcal{H},h)}\right)_0 &: H_0 \to D_h/R_h, & x &\mapsto \left[(1_{h_0(x)}, x)\right]_{R_h}, \\
\left(\eta_{(\mathcal{H},h)}\right)_1 &: H_1 \to G_1 \times_{G_0} D_h/R_h, & k &\mapsto \left(h_1(k), \left[(1_{h_0 \circ s_H(k)}, s_H(k))\right]_{R_h}\right).
\end{aligned}$$

The counit $\varepsilon : \Lambda S \stackrel{.}{\to} 1_{\mathbf{Sp}^{\mathcal{G}}}$ has components

$$\varepsilon_{(e,E,\alpha_e)} : \Lambda S(e, E, \alpha_e) \to (e, E, \alpha_e), \qquad [(k, x)]_{R_{\hat{e}}} \mapsto \alpha_e(k, x).$$

Proposition 5.16. *Each $\mathcal{G}-$*space *is isomorphic to $\Lambda(\mathcal{H}, h)$ for some $(\mathcal{H}, h)$ in $\mathbf{qTGpd}(\mathcal{G})$. Each $\mathcal{G}-$*sheaf *is isomorphic to $\Lambda(\mathcal{H}, h)$ for some $(\mathcal{H}, h)$ in* $\mathbf{sLHTGpd}(\mathcal{G})$.

Proof: By Corollary 5.6, the right adjoint S is full and faithful. It follows that the components of the conunit are isomorphisms [Mac97, Theorem IV.3.1]. Hence (e, E, α_e) in $\mathbf{Sp}^{\mathcal{G}}$ is isomorphic to $\Lambda S(e, E, \alpha_e)$ and (a, A, α_a) in $\mathrm{Sh}_{C_1}(C_0)$ is isomorphic to $\Lambda S(a, A, \alpha_a)$. $\square$

Given the isomorphism of $\mathcal{G}$-spaces in Theorem 3.1 and the isomorphism obtained via the counit of the adjunction $\Lambda \dashv S$, we now describe how these two are related. In fact, they express the same isomorphism. For if the pair $(\hat{E}, \hat{e})$ is the image of S on the $\mathcal{G}$-space (e, E, α_e) the equivalence relation $R_{\hat{e}}$ in (5.11) reduces to the equivalence relation R in Theorem 3.1, given by (3.2). Although part of this was used in the proof of Proposition 5.11 for the image of Γ on an object in $\mathbf{TCov}/\mathcal{G}$, we write out the details for a $\mathcal{G}$-space (e, E, α_e).

If $(f, x), (g, y) \in G_1 \times_{G_0} E$ are such that $(f, x) \sim_{R_{\hat{e}}} (g, y)$, then $t_G(f) = t_G(g)$ and there is an arrow $k : x \to y$ in $E_1 = G_1 \times_{G_0} E$ such that

$$\begin{tikzcd} \hat{e}_0(x) \arrow[d, "\hat{e}_1(k)"'] \arrow[dr, "f"] & \\ \hat{e}_0(y) \arrow[r, "g"'] & \bullet \end{tikzcd}$$

commutes. So $\hat{e}_1(k) = g^{-1} \circ f$. Hence $k = (g^{-1} \circ f, x)$ and $y = t_E(k) = \alpha_e(g^{-1} \circ f, x)$. We now have that

$$\begin{aligned} \alpha_e(f, x) &= \alpha_e(g \circ g^{-1} \circ f, x) \\ &= \alpha_e(g, \alpha_e(g^{-1} \circ f, x)) \\ &= \alpha_e(g, y), \end{aligned}$$

and so $(f, x) \sim_R (g, y)$.

Conversely, if $\alpha_e(f, x) = \alpha_e(g, y)$ then $t_G(f) = t_G(g)$ and $k = (g^{-1} \circ f, x)$ is an arrow $x \to y$ in $\hat{E}$ such that $f = g \circ \hat{e}_1(k)$. Hence $(f, x) \sim_{R_{\hat{e}}} (g, y)$.

5.5 Generators for LHCov/$\mathcal{G}$ and $\mathrm{Sh}_{G_1}(G_0)$

In this section we show how a set of generators for $\mathbf{LHCov}/\mathcal{G}$ and $\mathrm{Sh}_{G_1}(G_0)$, for an open topological groupoid $\mathcal{G}$, can be dervied from the adjunction $\Lambda \dashv S$.

Definition 5.17. *Let $\mathcal{C} : C_1 \rightrightarrows C_0$ be a topological category. An open subcategory of $\mathcal{C}$ is a pair of open subsets $U_0 \subseteq C_0$, $U_1 \subseteq C_1$ such that*

(i) U_1 is closed under composition,

(ii) $u_C(U_0) \subseteq U_1$,

(iii) $s_C(U_1), t_C(U_1) \subseteq U_0$.

If $\mathcal{C}$ is a groupoid and U_1 is also closed under taking inverses, then the pair U_0, U_1 is called an open subgroupoid of $\mathcal{C}$.

For a topological category (groupoid) $\mathcal{C}$, let $\mathfrak{O}(\mathcal{C})$ denote the set of all open subcategories (subgroupoids) of $\mathcal{C}$. Although the open subcategories of $\mathcal{C}$ aren't necessarily in $\mathbf{LHCov}/\mathcal{C}$ the next proposition could be said to show that the open subcategories of $\mathcal{C}$ generate $\mathbf{LHCov}/\mathcal{C}$.

Proposition 5.18. *Let $\mathcal{C}$ be a topological category. For any parallel pair of morphisms $f, g : (\mathcal{D}, d) \to (\mathcal{D}', d')$ in $\mathbf{LHCov}/\mathcal{C}$, $f = g$ iff $f \circ \phi = g \circ \phi$ for all $\phi : (\mathcal{U}, u) \to (\mathcal{D}, d)$ (in $\mathbf{TCat}/\mathcal{C}$) with $(\mathcal{U}, u)$ any element in $\mathfrak{O}(\mathcal{C})$.*

Proof: We first make some observations regarding morphisms in $\mathbf{TCat}/\mathcal{C}$ with codomain objects in $\mathbf{LHCov}/\mathcal{C}$.

Let $(\hat{E}, \hat{e})$ be the image of S on an equivariant sheaf (e, E, α_e). From the argument preceding (5.16) we can conclude that the arrow component of a morphism $g : (\mathcal{D}, d) \to (\hat{E}, \hat{e})$ in $\mathbf{TCat}/\mathcal{C}$ is given by

$$g_1(k) = (d_1(k), g_0 \circ s_D(k))$$

for $k \in D_1$. That is, such a morphism is determined by its object component g_0. Now, let $d' : \mathcal{D}' \to \mathcal{C}$ be an object in $\mathbf{LHCov}/\mathcal{C}$ and $\psi_{d'}$ be the corresponding homeomorphism $D'_1 \to G_1 \times_{G_0} D'_0$. From the proof of Theorem 5.5 we have that $\gamma_{d'} : (\hat{\mathcal{D}}', \hat{d}') \to (\mathcal{D}', d')$, where $(\hat{\mathcal{D}}', \hat{d}') = S\Gamma(\mathcal{D}', d')$, with components $(\gamma_{d'})_0 = 1_{D'_0}$ and $(\gamma_{d'})_1 = \psi_{d'}^{-1}$ is an isomorphism in $\mathbf{TCat}/\mathcal{C}$ with an inverse $\gamma_{d'}^{-1}$ that has components $(\gamma_{d'}^{-1})_0 = 1_{D'_0}$ and $(\gamma_{d'}^{-1})_1 = \psi_{d'}$.

For any morphism $g : (\mathcal{D}, d) \to (\mathcal{D}', d')$ in $\mathbf{TCat}/\mathcal{C}$ we thus have that $\gamma_{d'}^{-1} \circ g : (\mathcal{D}, d) \to (\hat{\mathcal{D}}', \hat{d}')$ has arrow component

$$\begin{aligned}(\gamma_{d'}^{-1} \circ g)_1(k) &= (d_1(k), (\gamma_{d'}^{-1} \circ g)_0 \circ s_D(k)) \\ &= (d_1(k), g_0 \circ s_D(k)).\end{aligned}$$

So $g = \gamma_{d'} \circ \gamma_{d'}^{-1} \circ g$ has arrow component $g_1 = \psi_{d'}^{-1}(d_1(k), g_0 \circ s_D(k))$ and is in this way determined by its object component.

Now, let $f, g : (\mathcal{D}, d) \to (\mathcal{D}', d')$ be a pair of parallel morphisms in $\mathbf{LHCov}/\mathcal{C}$. Since the morphisms f and g are determined by their object components, it is sufficient to show that if $g_0 \circ \phi_0 = f_0 \circ \phi_0$ for all morphisms $\phi : (\mathcal{U}, u) \to (\mathcal{D}, d)$ in $\mathbf{TCat}/\mathcal{C}$, for any $(\mathcal{U}, u)$ in $\mathfrak{O}(\mathcal{C})$, then $f_0 = g_0$.

Let x be an element in D_0 and $\sigma : U_0 \to D_0$ be a section through x, where $U_0 \subseteq C_0$ is open. Let $N = s_C^{-1}(U_0) \cap t_C^{-1}(U_0)$; then U_0 and N defines an open subcategory $\mathcal{U}'$ of $\mathcal{C}$, the full subcategory of C on U_0. Let u' denote the inclusion of $\mathcal{U}'$ into $\mathcal{C}$. If $\phi : (\mathcal{U}', u') \to (\mathcal{D}, d)$ is a morphism in $\mathbf{TCat}/\mathcal{C}$ with object component $\phi_0 = \sigma$ then the arrow component is given by $\phi_1(k) = \psi_d^{-1}(k, \sigma \circ s_C(k))$ for $k \in N$. If ϕ is to be a morphism of topological categories we must also require that $\phi_0 \circ t_C = t_D \circ \phi_1$, that is

$$\sigma \circ t_C(k) = t_D \circ \psi_d^{-1}(k, \sigma \circ s_C(k)) \tag{5.17}$$

for $k \in N$. Accordingly, we restrict attention to the subset of N consisting of elements such that this condition is satisfied and define $U_1 = \{k \in N \mid k \text{ satisfies } (5.17)\}$. We now show that U_1 and U_0 define a subcategory of $\mathcal{C}$:

(i) U_1 is closed under composition: For arrows k, p in U_1 such that $k \circ p$ is defined (in $\mathcal{C}$) let

$$k' = \psi_d^{-1}(k, \sigma \circ s_C(k)), \tag{5.18}$$

$$p' = \psi_d^{-1}(p, \sigma \circ s_C(p)). \tag{5.19}$$

Then $k' \circ p'$ is defined in $\mathcal{D}$ since k and p satisfies (5.17). Since

$$\begin{aligned} \psi_d(k' \circ p') &= (d_1(k' \circ p'), s_D(p')) \\ &= (d_1(k') \circ d_1(p'), \sigma \circ s_C(p)) \\ &= (k \circ p, \sigma \circ s_C(k \circ p)) \end{aligned} \tag{5.20}$$

we have $t_D \circ \psi_d^{-1}(k \circ p, \sigma \circ s_C(k \circ p)) = t_D(k')$, which equals $\sigma \circ t_C(k)$ by (5.17). Hence $k \circ p$ is in U_1.

(ii) For $x \in D_0$ we have $\psi_d^{-1}(1_{d_0(x)}, x) = 1_x$. Hence $u_C(U_0) \subseteq U_1$.

(iii) That $s_C(U_1), t_C(U_1) \subseteq U_0$ is clear since $U_1 \subseteq s_C^{-1}(U_0) \cap t_C^{-1}(U_0)$.

We now show that U_1 is an open subset of C_1. Since $d_0 \circ \sigma$ is the identity on U_0 and, for k in N,

$$\begin{aligned} d_0 \circ t_D \circ \psi_d^{-1}(k, \sigma \circ s_C(k)) &= t_C \circ d_1 \circ \psi_d^{-1}(k, \sigma \circ s_C(k)) \\ &= t_C(k) \end{aligned}$$

we obtain a unique map θ such that the following diagram commute in **Sp**

$$\begin{tikzcd}
N \arrow[dr, dotted, "\theta"] \arrow[drr, bend left, "t_D \circ \psi_d^{-1} \circ (1_N \times_{C_0} (\sigma \circ s_C))"] \arrow[ddr, bend right, "\sigma \circ t_C"'] & & \\
& D_0 \times_{C_0} D_0 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner"] & D_0 \arrow[d, "d_0"] \\
& D_0 \arrow[r, "d_0"'] & C_0
\end{tikzcd}$$

Then, for $k \in N$,

$$\theta(k) = \left(\sigma \circ t_C(k), t_D \circ \psi_d^{-1}(k, \sigma \circ s_C(k))\right).$$

We can now see that $U_1 \subseteq N$ is open:

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D_0 \arrow[d, "\Delta"] \\
N \arrow[r, "\theta"'] & D_0 \times_{C_0} D_0
\end{tikzcd}$$

where Δ is the diagonal. We thus have that $\mathcal{U} : U_1 \rightrightarrows U_0$ is an open subcategory of $\mathcal{C}$. Let u denote the inclusion of $\mathcal{U}$ into $\mathcal{C}$.

With $\phi : (\mathcal{U}, u) \to (\mathcal{D}, d)$ given by $\phi_0 = \sigma$ and $\phi_1(k) = \psi_d^{-1}(k, \sigma \circ s_C(k))$, for $k \in U_1$, we now have that $s_D \circ \phi_1 = \phi_0 \circ s_C$ and $t_D \circ \phi_1 = \phi_0 \circ t_C$ by construction. It is also true that $u_D \circ \phi_0 = \phi_1 \circ u_C$ as $1_{\sigma(x)} = \psi_d^{-1}(1_x, \sigma(x))$. Considering composition, let $k, p \in U_1$ be such that $k \circ p$ is defined. Let $k' = \phi_1(k)$ and $p' = \phi_1(p)$, as in (5.18) and (5.19). Then from (5.20) we have that $k' \circ p' = \phi_1(k \circ p)$.

By construction, $d_0 \circ \phi_0$ and $d_1 \circ \phi_1$ are the inclusion maps $U_0 \hookrightarrow C_0$ and $U_1 \hookrightarrow C_1$. This shows that ϕ is a morphism of categories over $\mathcal{C}$ and completes the proof. $\square$

Proposition 5.19. *Let $\mathcal{G}$ be an open topological groupoid. The set*

$$\{S\Lambda(\mathcal{U},u) \,|\, (\mathcal{U},u) \in \mathfrak{O}(\mathcal{G})\}$$

generates $\mathbf{LHCov}/\mathcal{G}$.

Proof: For each $(\mathcal{U},u)$ in $\mathfrak{O}(\mathcal{G})$, the unit η of the adjunction $\Lambda \dashv S$ yields an universal arrow from $(\mathcal{U},u)$ to S. For an equivariant sheaf (e,E,α_e), each arrow $\phi : (\mathcal{U},u) \to S(e,E,\alpha_e)$ (in $\mathbf{sLHTGpd}(\mathcal{G})$ and also in $\mathbf{TGpd}/\mathcal{G}$, since $\mathbf{sLHTGpd}(\mathcal{G})$ is a full subcategory) can then be written as $\phi = \phi' \circ \eta_u$ for a unique arrow $\phi' : S\Lambda(\mathcal{U},u) \to S(e,E,\alpha_e)$.

By Corollary 5.15 there is a natural isomorphism $\tau : \Gamma \dot{\to} \Lambda$ on $\mathbf{LHCov}/\mathcal{G}$. Then $S\tau$ is a natural isomorphism $S\Gamma \dot{\to} S\Lambda$ on $\mathbf{LHCov}/\mathcal{G}$. For $(\mathcal{H}',h')$ in $\mathbf{LHCov}/\mathcal{G}$ there is also an isomorphism $\gamma_{h'}$ between $S\Gamma(\mathcal{H}',h')$ and $(\mathcal{H}',h')$ (see the proof of Theorem 5.5).

If $f, g : (\mathcal{H}',h') \to (\mathcal{H},h)$ is a pair of morphisms in $\mathbf{LHCov}/\mathcal{G}$ such that $f \neq g$ then by Proposition 5.18 there is a morphism $\phi : (\mathcal{U},u) \to (\mathcal{H}',h')$, for some $(\mathcal{U},u)$ in $\mathfrak{O}(\mathcal{G})$, such that $f \circ \phi \neq g \circ \phi$. Let $\phi'' = (S\tau_{h'}) \circ \gamma_{h'}^{-1} \circ \phi$, then $\phi'' = \phi' \circ \eta_u$ for a unique $\phi' : S\Lambda(\mathcal{U},u) \to S\Lambda(\mathcal{H}',h')$:

$$\begin{tikzcd}
(\mathcal{U},u) \arrow[d, "\eta_u"'] \arrow[dr, "\phi''"] \arrow[drrr, "\phi"] & & & \\
S\Lambda(\mathcal{U},u) \arrow[r, "\phi'"'] & S\Lambda(\mathcal{H}',h') \arrow[r, "\sim", "(S\tau_{h'})^{-1}"'] & S\Gamma(\mathcal{H}',h') \arrow[r, "\sim", "\gamma_{h'}"'] & (\mathcal{H}',h')
\end{tikzcd}$$

We now have that $f \circ \gamma_{h'} \circ (S\tau_{h'})^{-1} \circ \phi' \neq g \circ \gamma_{h'} \circ (S\tau_{h'})^{-1} \circ \phi'$. $\square$

Proposition 5.20. *Let $\mathcal{G}$ be an open topological groupoid. The set*

$$\{\Lambda(\mathcal{U},u) \,|\, (\mathcal{U},u) \in \mathfrak{O}(\mathcal{G})\}$$

generates $\mathrm{Sh}_{G_1}(G_0)$.

Proof: The functor Γ is a part of an equivalence $\mathbf{LHCov}/\mathcal{G} \cong \mathrm{Sh}_{G_1}(G_0)$. Such a functor maps a set of generators in the one category into a set of generators in the other. Applying Γ to the set of generators of $\mathbf{LHCov}/\mathcal{C}$ in Proposition 5.19 and using that ΓS is the identity on $\mathrm{Sh}_{G_1}(G_0)$ (Theorem 5.5), the statement follows. $\square$

5.6 Summary

For an open topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$ the following diagram summarizes the results of the present chapter, where the inner and outer triangles of both top and bottom are commutative up to natural isomorphism:

$$\begin{tikzcd}
\mathbf{TCov}/\mathcal{G} \arrow[rr, hook, shift right] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\Gamma"] \arrow[dr, phantom, "\sim"] & & \mathbf{qTGpd}(\mathcal{G}) \arrow[ll, shift right, "S\Lambda"'] \arrow[r, hook] \arrow[dl, shift left, "\Lambda"] \arrow[dl, phantom, "\top"] & \mathbf{TGpd}/\mathcal{G} \\
& \mathbf{Sp}^{\mathcal{G}} \arrow[ul, shift left, "S"] \arrow[ur, shift left, "S"] & & \\
\mathbf{LHCov}/\mathcal{G} \arrow[uu, hook] \arrow[rr, hook, dotted, shift right] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left, "\Gamma"] \arrow[dr, phantom, "\sim"] & & \mathbf{sLHTGpd}(\mathcal{G}) \arrow[uu, hook] \arrow[ll, dotted, shift right, "S\Lambda"'] \arrow[dl, shift left, "\Lambda"] \arrow[dl, phantom, "\top"] & \\
& \mathrm{Sh}_{G_1}(G_0) \arrow[uu, hook] \arrow[ul, shift left, "S"] \arrow[ur, shift left, "S"] & &
\end{tikzcd}$$

The above diagram also displays $\mathbf{TCov}/\mathcal{G}$ ($\mathbf{LHCov}/\mathcal{G}$) as a reflective subcategory of $\mathbf{qTGpd}(\mathcal{G})$ ($\mathbf{sLHTGpd}(\mathcal{G})$).

In summary we have, with some abuse of notation, used S to denote the functors

$$\mathbf{Sp}^{\mathcal{G}} \to \mathbf{TCov}/\mathcal{G}, \quad \text{with restriction} \quad \mathrm{Sh}_{G_1}(G_0) \to \mathbf{LHCov}/\mathcal{G}, \tag{5.21}$$

and

$$\mathbf{Sp}^{\mathcal{G}} \to \mathbf{qTGpd}(\mathcal{G}), \quad \text{with restriction} \quad \mathrm{Sh}_{G_1}(G_0) \to \mathbf{sLHTGpd}(\mathcal{G}), \tag{5.22}$$

which operates in the same way. As the functors given in (5.21) S is a part of an equivalence and is both left and right adjoint to

$$\Gamma : \mathbf{TCov}/\mathcal{G} \to \mathbf{Sp}^{\mathcal{G}} \quad \text{respectively} \quad \Gamma : \mathbf{LHCov}/\mathcal{G} \to \mathrm{Sh}_{G_1}(G_0),$$

and as the functors given in (5.22) S is right adjoint to

$$\Lambda : \mathbf{qTGpd}(\mathcal{G}) \to \mathbf{Sp}^{\mathcal{G}} \quad \text{respectively} \quad \Lambda : \mathbf{sLHTGpd}(\mathcal{G}) \to \mathrm{Sh}_{G_1}(G_0).$$

Chapter 6

Future work

The main direction we would like to extend this work is the open question of finding a manageable set of generators for $\mathrm{Sh}_{C_1}(C_0)$ for an arbitrary topological category $\mathcal{C}$, instead of those arising from the construction of $\mathrm{Sh}_{C_1}(C_0)$ as a colimit of toposes (see [Moe88]).

As the other conditions of Giraud's theorem have been proved for the general case in Chapter 4, to give an elementary proof that $\mathrm{Sh}_{C_1}(C_0)$ is a (Grothendieck) topos, all that it is left to provide is an explicit description of a set of generators. This task has partly been accomplished in Proposition 4.8, where a set of generators are described for the full subcategory of $\mathrm{Sh}_{C_1}(C_0)$ with objects the equivariant sheaves with open actions.

Perhaps the existence of a set of generators could be proved via the results of Chapter 5. One may note that the proof in Section 5.5 of the existence of a set of generators for $\mathrm{Sh}_{G_1}(G_0)$, for an open topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$, depends only on the existence of a functor from a full subcategory of $\mathbf{TGpd}/\mathcal{G}$ that contains the open subgroupoids of $\mathcal{G}$ and the local homeomorphic covering morphisms to $\mathcal{G}$ into $\mathrm{Sh}_{G_1}(G_0)$, which is left adjoint to S. Proving the existence of such a functor would therefore be an alternative way of proving the existence of a set of generators of $\mathrm{Sh}_{C_1}(C_0)$ for an arbitrary topological category.

A suggestion for future work is to try equivalence relations different from the one given in (5.11), which might yield a functor similar to Λ, possibly for topological categories (instead of groupoids).

Bibliography

[AF13] S. Awodey and H. Forssell. *First-order logical duality.* Annals of Pure and Applied Logic 164, 2013.

[AHS90] J. Adámek, H. Herrlich, and G. Strecker. *Abstract and concrete categories.* John Wiley & Sons, 1990.

[BDNH76] R. Brown, G. Danesh-Naruie, and J.P.L Hardy. *Topological groupoids: II. Covering morphisms and G-spaces.* Math. Nachr. 74, 1976.

[BM98] C. Butz and I. Moerdijk. *Representing topoi by topological groupoids.* Journal of Pure and Applied Algebra 130, 1998.

[Bou89] N. Bourbaki. *Elements of mathematics, General Topology, Chapters 1-4.* Springer-Verlag, 1989.

[DK70] B.J. Day and G.M. Kelly. *On topological quotient maps preserved by pullbacks or products.* Proc. Camb. Philos. Soc. 67, 1970.

[For13] H. Forssell. *Quotient theories and subgroupoids.* Theory and applications of categories 28, 2013.

[GG99] T.W. Gamelin and R.E. Greene. *Introduction to Topology, 2nd ed.* Dover Publications, Inc., 1999.

[Gol84] R. I. Goldblatt. *Topoi: the categorical analysis of logic, 2nd ed.* North-Holland, 1984.

[Joh02a] P.T. Johnstone. *Sketches of an elephant - A Topos Theory Compendium, volume 1.* Number 43 in Oxford Logic Guides. Clarendon Press, 2002.

[Joh02b] P.T Johnstone. *Sketches of an elephant - A Topos Theory Compendium, volume 2.* Number 44 in Oxford Logic Guides. Clarendon Press, 2002.

[JT84] A. Joyal and M. Tierney. *An extension of the Galois theory of Grothendieck.* Mem. Amer. Math. Soc. 309, 1984.

[Mac97] S. MacLane. *Categories for the working mathematician, 2nd ed.* Springer-Verlag, 1997.

[MM92] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A first introduction to Topos Theory.* Springer-Verlag, 1992.

[Moe88] I. Moerdijk. *The classifying topos of a continuous groupoid. I.* Transactions of the American mathematical society 310(2), 1988.

[Moe95] I. Moerdijk. *Classifying spaces and classifying topoi.* Lecture Notes in Math. 1616. Springer-Verlag, 1995.

[Tow14] C. Townsend. *A short proof of the localic groupoid representation of Grothendieck toposes.* Proc. Amer. Math. Soc. 142(3), 2014.

Index of notation

Categories

Set	the category of sets and functions
Sp	the category of topological spaces and (continuous) maps
LH	the category of topological spaces and local homeomorphisms
$\mathbf{Sp}^{\mathcal{C}}$	the category of $\mathcal{C}$-spaces
$\mathrm{Sh}(X)$	the category of sheaves (of sets) on a topological space X
$\mathrm{Sh}_{C_1}(C_0)$	the category of equivariant sheaves on $\mathcal{C} : C_1 \rightrightarrows C_0$
$\underline{\mathrm{Sh}}_{C_1}(C_0)$	the full subcategory of $\mathrm{Sh}_{C_1}(C_0)$ of equivariant sheaves with open actions
TGpd	the category of topological groupoids
TCat	the category of topological categories
$\mathbf{TCov}/\mathcal{C}$	the category of topological covering morphisms into $\mathcal{C}$
$\mathbf{LHCov}/\mathcal{C}$	the category of local homeomorphic covering morphism into $\mathcal{C}$
$\mathbf{sLHGpd}(\mathcal{G})$	the full subcategory of $\mathbf{TGpd}/\mathcal{G}$ with objects semi-local homeomorphic morphisms into $\mathcal{G}$
$\mathbf{qTGpd}(\mathcal{G})$	the full subcategory of $\mathbf{TGpd}/\mathcal{G}$ with objects pairs $(\mathcal{H}, h)$ such that the equivalence relation in (5.11) is open, see Definition 5.10
Loc	the category of locales
$\mathbf{C}/\mathrm{c}$	slice category
$\mathbf{C}^{\mathbf{D}}$	functor category
$\mathbf{C}^{op}$	opposite category of $\mathbf{C}$
$\mathcal{G}, \mathcal{H}$	topological groupoids
$\mathcal{C}, \mathcal{D}$	topological categories
$\mathcal{G} : G_1 \rightrightarrows G_0$	topological groupoid $\mathcal{G}$ with object space G_1 and arrow space G_0
$\mathcal{C} : C_1 \rightrightarrows C_0$	topological category $\mathcal{C}$ with object space C_1 and arrow space C_0

Operations

$\times, \prod$	product
$\coprod$	coproduct
$\cong, \sim$	isomorphism

Functors

$T \dashv S$	T is left adjoint to S
$T \dot{\to} S$	natural transformation from T to S

Arrows

$\twoheadrightarrow$	epic arrow / epimorphism
$\rightarrowtail$	monic arrow / monomorphism
$\hookrightarrow$	inclusion
$-\!\circ\!\rightarrow$	open map
$-\!\bullet\!\rightarrow$	local homeomorphism

Other

1	terminal object or identity morphism
0	initial object
(a, A, α)	object in $\mathrm{Sh}_{C_1}(C_0)$ or $\mathbf{Sp}^{\mathcal{C}}$
$x \sim_R y$	x is related to y in the relation R
$[x]_R$	equivalence class of x in the equivalence relation R (if x is an element of the set on which R is defined)
$[\phi]$ $([\phi]_h)$	a function ϕ defined on a quotient (with index h)
$\square$	end of proof

Structure maps

The structure maps of a topological category $\mathcal{C} : C_1 \rightrightarrows C_0$ are denotes as in the diagram

$$\begin{tikzcd} C_1 \times_{C_0} C_1 \arrow[r, "m_C"] & C_1 \arrow[r, bend left, "t_C"] \arrow[r, bend right, "s_C"'] & C_0 \arrow[l, "u_C" description] \end{tikzcd}$$

The inverse component of a topological groupoid $\mathcal{G} : G_1 \rightrightarrows G_0$ is denoted $i_G : G_1 \to G_1$.

Abbreviations

LH	local homeomorphism
LHc	local homeomorphic covering
map	continuous function between topological spaces
semi-LH	semi-local homeomorphic
topos	Grothendieck topos, from Section 2.4 and onwards