SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Foundations of Graph Pebbling

av

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2015 - No 8

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Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

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2015

Abstract

Graph pebbling modeling started as a method for solving a combinatorial number theory conjecture by Erdős and Lemke. Using this method, Chung proved the conjecture in 1989. Since then, the literature has grown considerably. Several variations and possible applications have been discussed, in graph theory, computer science and network optimization.

The main focus in graph pebbling is graphs, mathematical structures modeling binary relations between vertices. To every vertex in some graph we assign a number of pebbles. If two pebbles are moved across an edge joining two distinct vertices, one pebble arrives and one pebble is lost. This is called a pebbling step.

The basic question in graph pebbling asks if one may from a given distribution of pebbles on a set of vertices move to another distribution on the same set via a series of pebbling steps.

In this Master's thesis we approach the above question using two models: a deterministic, which includes the notion of a pebbling number, and a probabilistic, which includes the notion of a threshold.

For both these models we clarify earlier proofs, and provide new proofs, of foundational theorems in graph pebbling. These results constitute the backbone for our discussion on recent research, which concentrates on generalizing and extending central notions in graph pebbling, for example the generalized idea of a pebbling number: the pi-pebbling function. Simultaneously, a corollary to the so called cover pebbling theorem is derived. This corollary lets us prove established, and newly found, theorems.

Regarding applications in graph pebbling, we argue that one should generalize existing results, and incorporate directed graphs into a bigger part of the theory. We suggest how this can be done.

Keywords: Combinatorics; Discrete probability theory; Graph theory; Graph pebbling; Pebbling number; Threshold; Cover pebbling number; Pi-pebbling function

Sammanfattning

Modellering med grafpebbling började som en metod för att lösa en kombinatorisk och talteoretisk förmodan av Erdős och Lemke. Genom att utnyttja denna metod bevisade Chung förmodandet 1989. Sedan dess har litteraturen växt avsevärt. Flera variationer och möjliga applikationer har diskuterats, i grafteori, datavetenskap och nätverksoptimering.

Huvudfokus för grafpebbling är grafer, matematiska strukturer som modellerar binära relationer mellan noder. Varje nod i grafen tilldelas ett antal pebblar. Om två pebbles (på svenska kiselstenar) förflyttas över en kant som förenar två noder, så anländer en pebble medan en försvinner. Detta kallas för ett pebblingssteg.

Den grundläggande frågan i grafpebbling är huruvida man kan från en given fördelning av pebbles på en mängd noder förflytta sig till en annan fördelning på samma mängd via en serie av pebblingsteg.

I denna masteruppsats angriper vi ovanstående fråga utifrån två modeller: en deterministisk, som inbegriper begreppet pebblingtal, och en sannolikhetsteoretisk, som inbegriper begreppet tröskel.

För båda dessa modeller klargör vi tidigare bevis, och tillhandahåller nya bevis, av fundamentala resultat i grafpebbling. Dessa resultat utgör ryggraden för vår diskussion om ny forskning som koncentrerar sig på att generalisera och utvidga centrala idéer i grafpebbling, till exempel den generaliserade idén av ett pebblingtal: pi-pebbling-funktionen. Samtidigt härleder vi ett korollarium till satsen om det så kallade täckande pebblingtalet. Detta korollarium låter oss bevisa fastlagda, och nyfunna, satser.

Apropå tillämpningar i grafpebbling så argumenterar vi för att man bör generalisera befintliga resultat, och inkorporera riktade grafer i en större del av teorin. Vi föreslår hur detta kan göras.

Nyckelord: Kombinatorik; Diskret sannolikhetsteori; Grafteori; Grafpebbling; Pebblingtal; Tröskel; Det täckande pebblingtalet; Pi-pebbling-funktionen

Acknowledgments

I want to give a special thanks to my supervisor Cecilia Holmgren for introducing me to graph pebbling and for being constructive. Instead of evaluating my work, or destructively criticising it, she seemed interested in suggesting those directions which I could favourably take. Moreover, her helpful comments certainly improved the comprehensibility of this thesis.

I would also like to thank Paul Vaderlind for his insightful comments. Vaderlind helped me make this thesis more user friendly and consistent.

Eventual mistakes are mine alone.

This Master's thesis consists of 30 ECTS credits at the Department of Mathematics at Stockholm University.

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Stockholm,
May 2015.

Contents

1 Introduction

Consider a finite connected graph G, and a distribution of pebbles on the vertices of G. Such a distribution is expressed by a configuration C on G. The configuration C describes how many pebbles there are at each vertex of G. The number of pebbles placed on each vertex is called the *size* of C; in other words, the size of C is the number of pebbles distributed on the vertices of G.

To construct a game we must first define the notion of a *pebbling step*: Whenever it is possible, two pebbles are removed from a vertex u and one pebble is placed on an adjacent vertex v. We say that the price at any vertex (for example u) is two, and that the price of a pebbling step is one.[1] The game, which this thesis is preoccupied with, is about trying to find, for any configuration C on G of some nonnegative size t, a series of pebbling steps that fills a given vertex r of G with one pebble; such a vertex r is often called the *target vertex*. This game is finite in the sense that t decreases by one for each pebbling step. To emphasize: the size of C is per definition a nonnegative integer.

Using the above definitions, we may describe graph pebbling as a game between two players on a graph G: Carl the *Configurer* and Patricia the *Pebbler*. To begin this game, Patricia buys t pebbles and then gives them to Carl who distributes a configuration C of t pebbles onto the vertices of G. Afterwards, Carl chooses a target vertex r. If Patricia can via a series of pebbling steps place at least one pebble on r, then we will say that Patricia has won the game and solved C, otherwise we will say that Carl won. Given information about the graph G, Patricia's challenge is to buy as few pebbles as possible (since they are quite expensive), while Carl's challenge is to distribute a configuration of size t which is as hard as possible for Patricia to solve.[2] [3]

[1]Two pebbles are removed at u, one pebble is added at v, and $-2+1=-1$. Hence, the price of a pebbling step is one.

[2]This paragraph was inspired by Hurlbert (2015).

[3]This game may be played with the computer based tool *Algoraph* (see `http://algoraph.cs.hope.edu/`) which "assist[s] in solving graph-related problems" such as

Graph pebbling may also be seen as a model of some transportation of resources that are consumed in the transition from one station to another. Thinking about such modeling problems one may be interested in whether some amount of resources (i.e. pebbles) can be moved, given that resources are lost in the process (i.e. paying the cost of some number of pebbling steps), from one set of geographical locations (i.e. set of vertices) to some other set of geographical locations.

One may regard questions about such modeling problems as deterministic or probabilistic. Because of this, two models are introduced in this thesis: a deterministic model and a probabilistic model.

For the deterministic model we want to examine the so called *pebbling number* for finite graphs G. Given a configuration C on G: if, for any vertex r of G, one can via a series of pebbling steps move at least one pebble to r, then C is said to be *solvable*. The pebbling number $\pi(G)$ of G is then defined as the minimum number t such that any configuration on G of size t is solvable. The deterministic model is mainly analyzed in section 2.2.1 and 3.

For the probabilistic model we define the notion of a *threshold*. A *threshold function* for a graph sequence $\mathscr{G} = (G_1, G_2, \ldots, G_n, \ldots)$, where G_n is a graph on n vertices, is any function $g(n)$ such that almost all configurations on G_n of size $t(n)$ is solvable if $\frac{t(n)}{g(n)} \to \infty$ as $n \to \infty$, and such that almost no configuration on G_n of size $t(n)$ is solvable if $\frac{t(n)}{g(n)} \to 0$ as $n \to \infty$. The threshold of $\mathscr{G}$ is the set of all threshold functions for $\mathscr{G}$.

The study of pebbling numbers and threshold functions will occupy most of this paper. In addition to this we study two more concepts that are central to the theory of graph pebbling (especially in recent years): firstly, the so called *cover pebbling number*, which tells us how many pebbles we need to distribute on some set of graph vertices to fill not only one vertex, but all vertices of the graph in question, with one pebble each; secondly, the so called *pi-pebbling function*, a generalization of the pebbling number which allows the price of a pebbling step to be greater than one.

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graph pebbling problems.

1.1 Purpose

The first aim of this thesis is to give foundational results regarding pebbling numbers and thresholds, and also to discuss generalizations of the pebbling number and variations on graph pebbling.

The second aim is to give the reader more explicit proofs of earlier foundational theorems in graph pebbling, and to give new proofs for established theorems.

The third aim is that this thesis functions as an introduction to graph pebbling, and is for this reason especially targeted at interested students in their last year of pursuing a Bachelor of Mathematics.

1.2 Central question

One central question we ask is: Can graph pebbling modeling, as a pure mathematical subject, be extended in such a way that it has a significant impact on applications and the field of applied mathematics?

1.3 History

Graph pebbling modeling came out as a result of finding a method for solving a combinatorial number theory conjecture by Erdős and Lemke (Czygrinow et al., 2002). They conjectured that given a positive integer d and integers $a_1, a_2, \ldots, a_d$, there exists a non-empty set $Q \subseteq \{1, 2, \ldots, d\}$ such that:

$$d \mid \sum_{i \in Q} a_i,$$

$$\text{and} \qquad \sum_{i \in Q} \gcd(a_i, d) \leq d.$$

Originally the conjecture was solved by Kleitman and Lemke (1988). Since the proof was detailed and contained an analysis of several cases, a solution to the conjecture via graph pebbling modeling was suggested by Lagarias and Saks in order to present a shorter and more comprehensible proof of Erdős' and Lemke's conjecture (Godbole et al., 2005). Chung

(1989) followed this suggestion and (with minor mistakes) proved the conjecture using graph pebbling modeling (a correction to Chung's proof was later given by Clarke et al. (1997)). In the same paper Chung developed the graph pebbling model by establishing further results concerning graph pebbling. Since then, the literature has grown considerably with over 50 published papers on the subject (Hurlbert, 2005).

The theory of graph pebbling has been formalized and generalized since Chung's paper. There are many variations on graph pebbling today, and possible applications are discussed. Several open problems and difficult conjectures exist. For example the conjecture of Graham that the pebbling number of a cartesian product of graphs is less than or equal to the product of the pebbling numbers for each graph.

A recent overview of graph pebbling may be found in (Hurlbert, 2014)[16].

1.4 Disposition

In section 2 we introduce the reader to graph theory (2.1) and graph pebbling (2.2). The latter depends on the former. Using the definitions of section 2.1 and 2.2, we derive some initial results concerning graph pebbling in section 2.2.1. These results will be the backbone of the analysis in section 3 and 4.

The main objective in section 3.1 is to derive the pebbling numbers for some well known graphs. In 3.2 we prove theorems concerning so called diameter 2 graphs. That is, section 3 deals with the deterministic model of graph pebbling.

In section 4 we analyze the probabilistic model of graph pebbling. Definitions and notations are introduced in 4.1, which will be sufficient for our analysis in section 4.2 and 4.3. Section 4.2 constitutes the groundwork for discussing thresholds for graph sequences in 4.3.

Section 5 explores recent research on graph pebbling. In section 5.1 we look at a generalization of the notion of a pebbling number: the cover pebbling number in 5.1.1 and the pi-pebbling function in 5.1.2. Applications and variations on graph pebbling are discussed in 5.2. Some suggestions on which directions the research on graph pebbling should take are discussed in 5.3; moreover: comments on graph pebbling on directed graphs are made in this section.

Section 6 summarizes this thesis and its main conclusions, and gives our answer to the question posed in 1.1 with which we began this venture.

Appendix A summarizes main results from sections 2.2.1, 3, 4 and 5.1.1.

Remark 1.1. *Proofs not referring to some publicized paper are provided by the author of this thesis.*[4]

1.5 Terminology

An index and a nomenclature is appended at the end to help the reader navigate and keep track of all the names and terms.

[4]More specifically, the author of this paper has proved Lemma 2.1, 2.2, 2.4, and Theorem 2.8, where formulations of the propositions, and ideas for proving the propositions, were provided by Hurlbert (2015)[17]. Moreover, the author was aware of Theorem 2.3, but proved the theorem independently. Theorem 2.6 and 2.7 are results which the author understood as implicit theorems in the research literature on graph pebbling, and because of this he made these implicit theorems explicit and proved them independently thereafter. The proof of Theorem 3.1 is the author's, but he had beforehand noticed the suggestion by Hurlbert (2015) that the theorem could be proven using mathematical induction. Theorem 3.2, 3.4–3.9 and Corollary 3.9.1 are independently proven by the author. The proof of 3.10 is a follow-up on a proof idea by Hurlbert (1991). The proofs in section 3.2.1 is a presentation and clarification of two proofs in Pachter et al. (1995); and the proofs in section 3.2.2 may be found in Clarke et al. (1997), but the structure of the succession of theorems leading to the final result Theorem 3.18 (Theorem 1.7 in the paper by Clarke et al. (1997)) is different in this thesis. Corollary 3.18.1 is found using a result stated in (Hurlbert, 1999). Also, some days before the publication of this thesis, the author came across a senior thesis on graph pebbling by Anna Blasiak (2008)[2] at Middlebury College which included a pedagogical approach to the proof of Lemma 3.14 and which deeply influenced the author's own proof of the lemma originally presented and proven in (Clarke et al., 1997). Theorem 4.2–4.4 and 4.7 were formulated by Czygrinow et al. (2002), but were never proven. Corollary 5.1.1 were proven by the author using Theorem 5.1 which was proven by Sjöstrand (2004). Theorem 5.2–5.8 were formulated by Sjöstrand, but never proven. Theorem 5.9 and 5.10 are independently proven by the author of this thesis. The proofs of Theorem 5.11–5.13 are essentially a presentation of the results originally found by Taylor (2005). All other proofs in this thesis are clarifications of earlier proofs from the articles in the reference list.

2 Preliminaries

2.1 Graph theory

In *graph theory* we study mathematical structures modeling binary relations. These structures are called *graphs*. There are two major types of graphs: undirected and directed graphs. In section 2 to 4 we will study undirected graphs. Directed graphs are discussed in section 5.

Graph theoretic definitions and notations will now be introduced.[5]

Definition 2.1 (Undirected graph)**.** *A finite undirected graph G is a pair of disjoint sets* $(V(G), E(G))$*, such that* $E(G)$ *is a subset of the set of unordered pairs of the finite set* $V(G)$*.*[6] [7]

For finite graphs G: $V(G)$ is called the set of *graph vertices*, or the vertex set, and $E(G)$ is called the set of *graph edges*, or the edge set.

That x is a vertex (edge) of G means that x is in $V(G)$ ($E(G)$).

An edge $\{x, y\}$ is said to *join* the vertices x and y. We will say that x in $V(G)$ is *adjacent* to y in $V(G)$ iff there is an edge $\{x, y\}$ in $E(G)$.

The number of vertices $|V(G)|$ in G is denoted by $n(G)$ – this number is called the *size* of G.[8]

See Figure 1 for an example of a graph.

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[5] An introduction to graph theory may be found in (Grimaldi, 2014: 565–640)[11], and in (Bollobás, 1979: 1–6)[5]. The definitions in this section were partly inspired by these two sources.

[6] An ordered pair (a, b) is a pair such that (a, b) is equal to (b, a) only if $b = a$. An unordered pair (i.e. set of two elements) $\{a, b\}$ is such that it is always equal to $\{b, a\}$. Whenever we talk about ordered or unordered pairs, we (implicitly) talk about ordered or unordered pairs of elements (such as the elements a and b).

[7] Note that $E(G)$ is possibly empty (contains no elements).

[8] If A is a set, then the *size* $|A|$ of A denotes the number of elements in A.

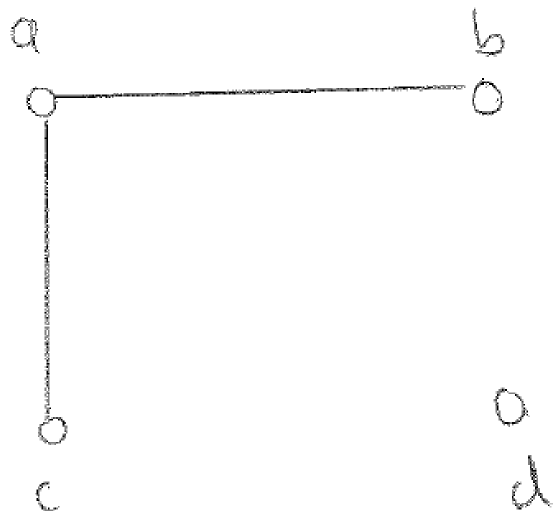

Figure 1: The above picture is a representation of the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ is the vertex set $V(G)$ of G, and $E = \{\{a, b\}, \{a, c\}\}$ is the edge set $E(G)$ of G. The vertices of G are a, b, c and d, and the edges of G are $\{a, b\}$ and $\{a, c\}$. The edge $\{a, b\}$ joins a and b, and $\{a, c\}$ joins a and c; hence, a is adjacent to b and c, while d is adjacent to no vertex. For this graph, the number of vertices is 4, so the size $n(G)$ of G is 4.

If not explicitly stated otherwise, a finite undirected graph G is called a graph.[9] For simplicity we call the graph vertices and graph edges of G the vertices and edges of G.

Furthermore we make the remark that we will only consider simple graphs in this thesis. *Simple graphs* are graphs which contain no *graph loops* (an edge which joins a vertex to itself) and no *multiple edges* (a set of two or more distinct edges that joins the same two distinct vertices).

Remark 2.1. *In this thesis we will only consider simple graphs.*

In Figure 2 a non-simple graph and a simple graph is represented.

[9]In section 5 we look at graphs that are not undirected, but *directed*.

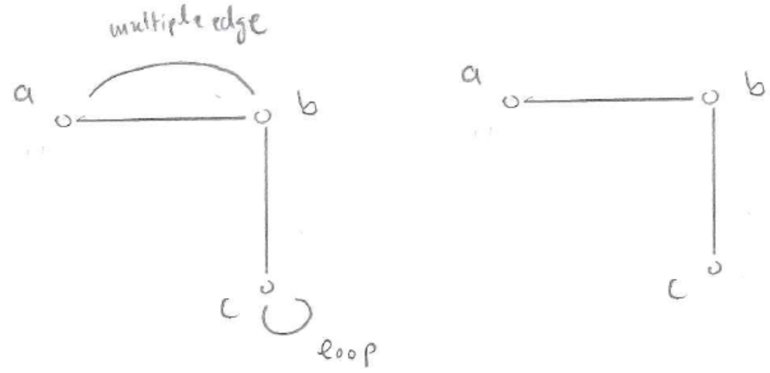

Figure 2: To the left: A non-simple graph with a multiple edge between a and b, and a graph loop at c. To the right: A simple graph with no multiple edge and no graph loop.

In Remark 2.2 at the end of section 2.2 we will state an additional specification, besides the specification stated in Remark 2.1 above, of the types of graphs studied in this thesis.

To continue:

Definition 2.2 (Subgraph). *A graph H is a subgraph of a graph G whenever $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.*[10]

Equivalently we may write $H \subseteq G$ if H is a subgraph of G.

Definition 2.3 (Spanning subgraph). *H is a spanning subgraph of G iff*[11] *H is a subgraph of G and $V(H) = V(G)$.*

A short definition is needed before we continue: The edge e is said to be *incident* with x, and *incident* with y, iff $e = \{x, y\}$.

Definition 2.4 (Vertex deletion). *If H is a subgraph of G, then $G - V(H)$ (equivalently $G - H$) denotes the subgraph of G obtained by deleting the vertices in $V(H)$ and all edges incident with them.*

[10]If A and B are two sets, then $A \subseteq B$ denotes the fact that A is a subset of A, i.e. that every element in A is an element in B.

[11]If and only if.

If x is a vertex in the vertex set of the graph G, then $G - x$ is short hand for $G - \{x\}$.

Definition 2.5 (Path). *A path P in G of length n on $n+1$ vertices is a series of distinct vertices $v_1, v_2, \ldots, v_{n+1}$, with $v_i \in V(G)$ for $i = 1, 2 \ldots, n$, $n \geq 1$, where $\{v_i, v_{i+1}\}$ is an edge for $i = 1, 2, \ldots, n$.*[12]

Definition 2.6 (Connectivity). *A graph G is connected iff for any two distinct vertices u and v in the vertex set of the graph G, there is a path from u to v. A graph G is disconnected (or unconnected) iff G is not connected.*

In this thesis we will only consider connected graphs. This will be motivated at the end of section 2.2.2.

See Figure 3 for an illustration of the concept of connectivity.

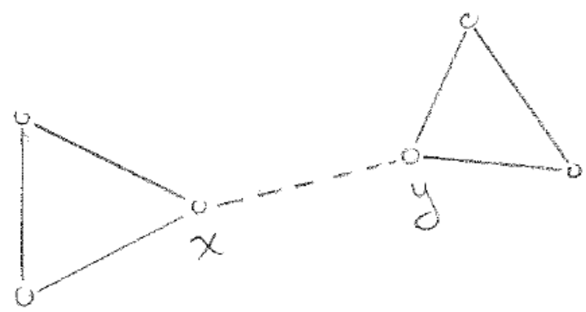

Figure 3: When the dashed edge $\{x, y\}$ is removed, this graph becomes disconnected, otherwise it is connected.

Definition 2.7 (Vertex-connectivity). *The vertex-connectivity $\kappa(G)$ of a graph G (other than a complete graph*[13]*) is the minimum number of vertices in $V(G)$ whose deletion disconnects G.*[14]

[12]Definition: $x \in X$ iff x is an element in the set X.

[13]A complete graph K_n on n vertices is a graph with diameter 1 in which there is a unique edge between any two distinct vertices of K_n.

[14]One may similarly define the *edge-connectivity* of a graph G as the minimum number of edges in $E(G)$ whose deletion disconnects G. Here one needs to define *edge deletion* first. In any way, the notion of edge-connectivity will not be used in this thesis.

Definition 2.8 (k-connectivity). *The graph G is k-connected iff $\kappa(G) \geq k$.*

Note that all connected graphs (except the complete graphs) are 1-connected.

Definition 2.9 (Distance and Diameter). *Let G be a connected graph and x, y two vertices in $V(G)$. The distance $dist(x, y)$ from x to y is the minimum length of a path in G from x to y.*

The diameter $diam(G)$ of G is the maximum distance between two vertices of $V(G)$, that is $\max_{x,y \in V(G)} dist(x, y)$. [15] [16]

G is said to be a *diameter k* graph iff $diam(G) = k$. We may also reformulate the notion of adjacency as: u is said to be *adjacent* to v iff $dist(u, v) = 1$.

Diameter 2 graphs are studied in section 3.2.

Definition 2.10 (Neighborhoods). *If G is a graph and v is a vertex of G, then the neighborhood $N(v)$ of v is the set of vertices in $V(G)$ adjacent to v. If $A \subseteq V(G)$, then $N(A)$ is the set of vertices not in A but adjacent to at least one $a \in A$.*[17]

We end this section with the notion of a tree and a cut vertex.

Definition 2.11 (Tree). *T is a tree iff for every pair of vertices in the graph T there is exactly one path between them.*[18]

Definition 2.12 (Spanning tree). *T is a spanning tree of G iff T is both a spanning subgraph of G and a tree.*

Definition 2.13 (Cut vertex). *If G is a connected graph and x is a vertex of G, then x is a cut-vertex for G iff $G - x$ is disconnected.*

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[15] $\max_{x \in D} f(x)$ denotes the maximum value of the function $f(x)$ taken over all elements x in D.

[16] One may extend this definition to include unconnected graphs by defining $dist(x, y)$ as ∞ when there is no path from x to y. But in this thesis, as mentioned before, we will only consider connected graphs. Hence, to extend Definition 2.9 in order to include unconnected graphs is superfluous.

[17] If A and B are two sets, then $A \cap B$ denotes the set of elements in A and B.

[18] Trees may also be defined through the notion of a *cycle*. A cycle of n vertices is a path of n vertices $v_1, v_2, \ldots, v_n$ were the additional edge $\{v_1, v_n\}$ joins the two vertices v_1 and v_n. Using this, we may say that a tree is an *acyclic* graph, a graph which contains no cycles (that is, there is no subset E of the edge set of T and subset V of the vertex set T such that the graph (V, E) constitutes a cycle).

2.2 Graph pebbling

In *graph pebbling* we distribute pebbles on vertices of a graph. The simplest operation on such a distribution is the pebbling step. This consists in a removal of two pebbles from one vertex u, and a placement of one pebble on some vertex v which is adjacent to u via an edge $\{u, v\}$.

A distribution of pebbles on some set of graph vertices, plus possible pebbling steps, constitutes basic rules for a kind of pebbling game. Several questions arise for this type of game. These questions are tackled in sections 2.2.1 and 3. Relevant definitions are here introduced so that we may answer such questions in a simpler language.

Definition 2.14 (Configurations). *A configuration on G is said to be a function* $C : V(G) \to \mathbb{Z}_{\geq 0}$.[19] *The expression* $C(v)$ *denotes the number of pebbles on the vertex* v *of* G. *If* $A \subseteq V(G)$, *then* $C(A) = \sum_{v \in A} C(v)$.

We call $C(V(G))$ *the size of* C *and denote it by* $|C|$.[20] *Let* $\mathcal{C}(G)$ *denote the set of possible configurations on* G. *If* H *is a graph, then* $C(H)$ *denotes* $C(V(H))$.

In other words: A configuration C on G is a function which describes the number of pebbles at each vertex of G, and the total number of pebbles $|C|$ (the size of C) which is accordingly distributed at the vertices of G.

Definition 2.15 (Pebbling step). *Given a configuration* C *on* G, *a pebbling step* $S_{u,v} : C \to \mathcal{C}(G)$ *from* u *to* v *satisfies*[21]

$$S_{u,v}(C)(x) = \begin{cases} C(x) - 2 & \text{if } x = u, \\ C(x) + 1 & \text{if } x = v, \\ C(x) & \text{otherwise.} \end{cases}$$

Hence, a pebbling step $S_{u,v}$ applied to a configuration C results in a new configuration $C' = S_{u,v}(C)$. The new configuration C' is said to be *derived* from C in this case.

[19] The set $\mathbb{Z}_{\geq 0}$ denotes the set of nonnegative integers.

[20] Note that the concept of size is defined for three types of abstract entities: sets, graphs and configurations. When applying the concept of size we will make it clear whether we consider the size of a graph, or the size of a configuration on a graph.

[21] This definition of a pebbling step is similar to one found in (Vuong & Wyckoff, 2004)[26].

Notice that this implies that $C(u) \geq 2$ must hold, otherwise $S_{u,v}(C)$ is not in $\mathcal{C}(G)$, because for all configurations C on G it holds that $C(x) \geq 0$ for all x in $V(G)$.

Informally one may define a pebbling step as follows: Given an edge $\{u, v\}$ in the edge set $E(G)$ of a graph G: Whenever u contains at least two pebbles, a pebbling step $u \to v$ from u to v is a removal of two pebbles from u and a placement of one pebble at v.

Definition 2.16 (Pebbling move)**.** *Given a configuration C on G, a pebble can be moved to the vertex r in $V(G)$ iff there exists a series of pebbling steps $S_1, S_2, \ldots, S_n$, $n \geq 1$, such that for $C' = S_1(S_2 \cdots (S_n(C)))$ we have $C'(r) \geq 1$.*

If one moves a pebble from a_1 to a_n, $n \geq 3$, passing the vertices a_k, $2 \leq k \leq n-1$, one may write $a_1 \to a_2 \to \cdots \to a_n$. Furthermore, let $a \to^t b$ denote the fact that one moves a positive number of t pebbles from a to the adjacent vertex b at a cost of t pebbles.[22]

When we want to move a pebble to some specified vertex r, we may call r the *target vertex*, or just the *target*.[23]

If we can move a pebble to r, we say that we can *pebble* r; and when we move a pebble to r, we say that we are *pebbling* r.

Definition 2.17 (Solvability)**.** *A configuration C on G is r-solvable iff one can move a pebble to r in $V(G)$. The configuration C is solvable iff it is r-solvable for every r in $V(G)$.*

C is r-unsolvable iff it is not r-solvable, and unsolvable iff it is not solvable.

Definition 2.18 (Pebbling number)**.** *Let G be a graph and r a vertex of G. The pebbling number $\pi(G, r)$ of G for (or with respect to) r is the smallest integer m such that for every configuration C on G of size m, C is r-solvable.*

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[22]What we pay, the cost, for the pebbling move $a \to^t b$ is t since the number of pebbles at a decreases by $2t$ (i.e. the price at a is $2t$) and the number of pebbles at b then increases by t.

[23]In Czygrinow et al. (2002) such a pebble r is called the *root vertex*. We use the term *target vertex* instead (which was also used by Pachter et al. (1995)), since the term "root vertex" is more associated with *rooted trees* in which a vertex, called the *root vertex*, is selected.

The pebbling number $\pi(G)$ of G is defined as $\max_{r \in V(G)} \pi(G, r)$. This is the smallest integer m such that for every configuration C on G of size m, C is solvable.[24]

We may define $\pi(G, r)$ as 1 plus the largest number m_r such that a configuration C on G of size m_r is r-unsolvable. That is, the pebbling number $\pi(G)$ may be defined as 1 plus $\max_{r \in V(G)} m_r$.

See Figure 4 for an illustration of graph pebbling, and an example of an unsolvable configuration.

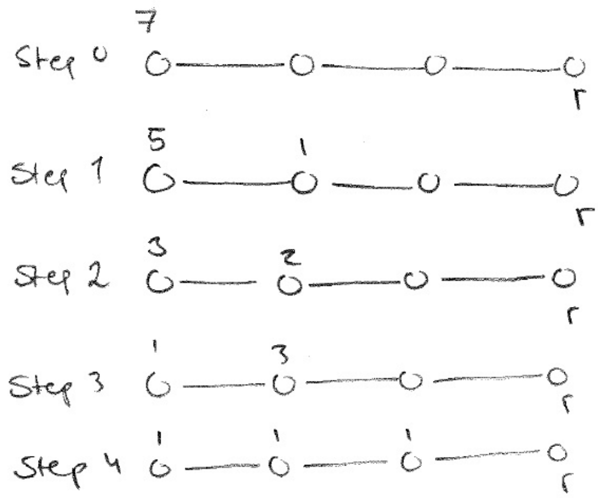

Figure 4: An illustration of four pebbling steps beginning with an initial configuration C of size $2^3 - 1$ on a path P_4 on four vertices. In this case, C is unsolvable, so the pebbling number $\pi(P_4)$ of P_4 is greater than $2^3 - 1$.

The set of graphs G for which $\pi(G) = n(G) + k$, $k \geq 0$, are called the *Class k* graphs. In section 3.2.1 we find that every graph of diameter 2 is of Class 0 or 1.

In section 3.2.2 we will use the notion of a *bad configuration*:

[24] It is proven in Theorem 2.5 in section 2.2.1 that every finite connected graph G has a pebbling number $\pi(G)$.

Definition 2.19. *C is bad configuration on G iff there is some vertex r in $V(G)$ such that the configuration C is r-unsolvable.*

Finally we add to Remark 2.1 an additional specification of the scope of this thesis:

Remark 2.2. *In this thesis we will only consider finite connected graphs.*

To motivate the above remark, consider the following argument: If G is disconnected (i.e. not connected), then there exist (at least) two graphs A and B for which there is no path from any vertex a in $V(A)$ to some vertex b in $V(B)$.[25] Thus one can not move a pebble to any b in $V(B)$ if all pebbles are distributed on the vertices of A. Hence, the pebbling number $\pi(G, b)$ does not exist for any vertex b in $V(B)$, implying that $\pi(G)$ does not exist.[26] (An example is presented in Figure 3 above: If the dashed edge $\{x, y\}$ is removed, then the pebbling number does not exist for x, nor any other vertex of the graph). Since the pebbling number of certain graphs will be the main subject of this thesis, we will not be interested in the graphs for which $\pi(G)$ does not exist. That is why we exclude unconnected graphs.

2.2.1 First results

Consider an arbitrary connected graph G. We begin by giving three bounds on the pebbling number $\pi(G)$ of G.

Lemma 2.1 (Vertex bound). $\pi(G) \geq n(G)$.[27]

Proof. Placing one pebble on each vertex of G except at the target gives an unsolvable configuration size $n(G) - 1$, since no vertex contains two pebbles, and so r can not be pebbled because a pebbling step costs at least two pebbles. □

[25] Here, A and B are called components of G.

[26] Maybe there are purposes in which one may allow this case. Then it may be reasonable to let $\pi(G) = \infty$.

[27] The vertex bound implies that every connected graph is of Class k for some nonnegative integer k.

Lemma 2.2 (Distance bound)**.** $\pi(G) \geq 2^{diam(G)}$.

Proof. Assume that $\pi(G) \leq 2^{diam(G)} - 1$, then one can place $2^{diam(G)} - 1$ pebbles on a vertex v with distance $diam(G)$ from the target r, and since, by Definition 2.9, the shortest path from v to r is of length $diam(G)$ we need at least $2^{diam(G)}$ pebbles placed at v to move one pebble to r, if no other vertex contains a pebble. Hence, there is an unsolvable configuration of size $2^{diam(G)} - 1$, and so by contradiction $\pi(G) \geq 2^{diam(G)}$. □

Recall *the principle of mathematical induction* which states that whenever $P_1, P_2, \ldots$ is a sequence of propositions such that if

(i) P_1 is true, and

(ii) P_{k+1} is true whenever P_k is true, for some positive integer k,

then P_n is true for all positive integers n (Beachy & Blair, 2006: 441)[3].

Theorem 2.3 (The generalized pigeonhole principle)**.** *Whenever $n > km$ items, $k \geq 1$, are distributed among m containers, then there is at least one container with at least $k+1$ items in it. The principle is called the pigeonhole principle when $k = 1$.*

Proof. Suppose $n > km$ items are distributed among m containers, but no container contains $k+1$ items. Then every container contains no more than k items. Then the total number of objects distributed is at most km and this is less than n, a contradiction since we distributed n items. Hence, at least one container contains $k+1$ items. □

Lemma 2.4 (Pigeonhole bound)**.** $\pi(G) \leq (2^{diam(G)} - 1)(n(G) - 1) + 1$.

Proof. When one has a configuration on G of size $(2^{diam(G)} - 1)(n(G) - 1) + 1$ one must, avoiding the target r in $V(G)$ with $C(r) = 0$, place at least $2^{diam(G)}$ pebbles at some vertex v in $V(G)$.[28] Then one may pebble r from v. □

[28]This follows from the generalized pigeonhole principle using $n = (2^{diam(G)} - 1)(N(G) - 1) + 1$, $k = 2^{diam(G)} - 1$ and $m = N(G) - 1$ (so that $n > km$ and one vertex contains $k + 1 = 2^{diam(G)}$ pebbles).

Corollary 2.4.1. $\pi(G) \leq 2^{diam(G)} n(G)$.

Proof.

$$\begin{aligned}\pi(G) &\leq (2^{diam(G)} - 1)(n(G) - 1) + 1 \\ &= 2^{diam(G)} n(G) - 2^{diam(G)} - n(G) + 1 + 1 \\ &\leq 2^{diam(G)} n(G),\end{aligned}$$

using Lemma 2.4 and noticing that $n(G)$ and $2^{diam(G)}$ are at least equal to 1. □

Summarizing Lemma 2.1, 2.2, 2.4 and Corollary 2.4.1, one has

Theorem 2.5.

$$\begin{aligned}\max(n(G), 2^{diam(G)}) \leq \pi(G) &\leq (2^{diam(G)} - 1)(n(G) - 1) + 1 \\ &\leq 2^{diam(G)} n(G).\end{aligned}$$

Hence, $\pi(G)$ exists for every connected graph G, and it is somewhere in the interval described by Theorem 2.5.

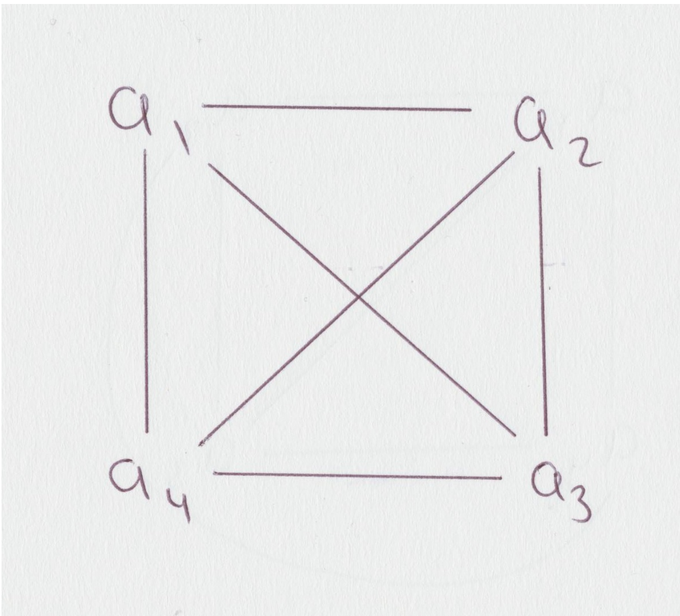

Figure 5: The complete graph K_4.

From Theorem 2.5 one can derive the pebbling number of K_n, where K_n is the complete graph on n vertices. That is a graph with diameter 1 in which there is a unique edge between any two distinct vertices (see Figure 5 for an example).

Corollary 2.5.1. $\pi(K_n) = n$.

Proof. Note that $diam(K_n) = 1$ and $n(K_n) = n$ by definition of K_n. Thus, the corollary follows from Theorem 2.5. □

Theorem 2.6. *Let t be a nonnegative integer. If $k + (2t + 1)$ pebbles are distributed on the neighborhood $N(r)$ of some vertex r, with $|N(r)| = k$, then at least $t + 1$ pebbles may be moved to r.*

Proof. Induction base: If $k + 1$ pebbles are distributed on k vertices, then by the pigeonhole principle there exist a vertex $a \in N(r)$ adjacent to r containing two pebbles. The pebbling step $a \to r$ then solves the problem. Hence, the theorem is true for $t = 0$.

Induction step: Assume that the theorem is true for some $0 \leq t < m$. Let $k + (2(t+1)+1)$ pebbles be placed at $N(r)$, then there is at least one vertex $a \in N(r)$ adjacent to r with two pebbles on it by the pigeonhole principle and the fact that $k + (2(t+1)+1) > k+1$ (since $t + 1 > 0$) pebbles are distributed on the k vertices of $N(r)$. Taking the pebbling step $a \to r$ at a price of two pebbles at a, we have at least one pebble at r and at least $k + (2(t+1)+1) - 2 = k + (2t+1)$ pebbles left distributed on $N(r)$.[29] Induction gives us that, additionally, at least t pebbles may be moved to r. Thus we may move pebbles to r so that it contains at least $t + 1$ pebbles.

Conclusion: By induction, the theorem is true for every nonnegative integer t. □

We end this section by stating two other important bounds.

Theorem 2.7 (Subgraph bound). *If H is a spanning subgraph of G, then $\pi(H) \geq \pi(G)$.*

[29] Even though the cost of the pebbling step $a \to r$ is one, the number of pebbles left distributed on $N(r)$ decreases by two, since the price at a is two, and the number of pebbles at r increases by one, but r is not in $N(r)$ while a is.

Proof. Every pebbling step possible in H is possible in G, since every edge in $E(H)$ is in $E(G)$ and $V(H) = V(G)$.[30] Thus, whenever a configuration on H is solvable, it is solvable on G. □

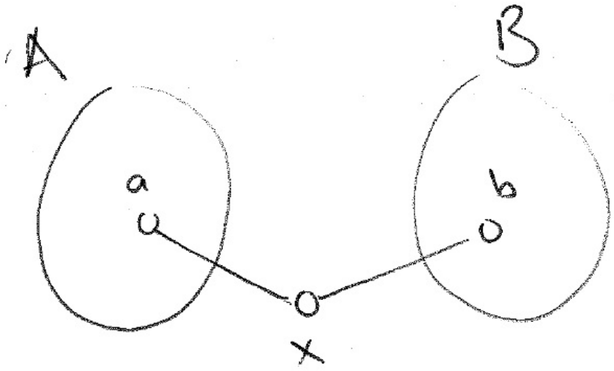

Figure 6: A graph with a cut vertex x.

Theorem 2.8 (Cut bound)**.** *If the graph G has a cut vertex x, then $\pi(G) > n(G)$.*

Proof. Consider the two components A and B of $G - x$ (see Figure 6). Pick two arbitrary vertices a in $V(A)$ and b in $V(B)$. Now place 3 pebbles at a and 1 pebble at each vertex of G except for x and b. This is a b-unsolvable configuration of size $n(G)$, since from the vertices of A one may at most move one pebble to x, resulting in a situation with no vertex with more than 1 pebble and b containing none. □

[30] The reverse statement does not hold, since G may contain edges not in $E(H)$.

3 Pebbling numbers

3.1 Pebbling numbers for some families of graphs

In this section we find the pebbling number for some families of graphs. Primarily the proofs take advantage of (1) the principle of mathematical induction (as in Theorem 3.1), (2) finding a subgraph for which the pebbling number is already known (as in Theorem 3.2), and (3) effective use of case analysis (as in Theorem 3.4).

3.1.1 Path graph

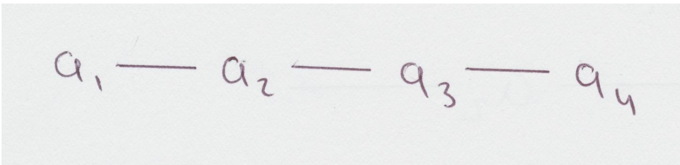

Figure 7: The path graph P_4.

Let P_n denote the path on n vertices (thus of length $n-1$). See Figure 7 for an example.

Theorem 3.1. $\pi(P_n) = 2^{n-1}$.

Proof. The proof is by mathematical induction.

Induction base: $\pi(P_1) = 1 = 2^0$ since $V(P_0) = 1$.

Induction step: Assume that $\pi(P_k) = 2^{k-1}$ for some $1 \leq k < n$. Notice that P_{k+1} is P_k with one vertex a_{k+1} adjacent to the last vertex a_k of P_k.

Case 1: If the target r is neither the first nor the last vertex of P_{k+1}, then the sequence of vertices, including r, to the left or right of r, can be seen as P_l with $1 \leq l \leq k$. Consider a configuration C on P_{k+1} of size 2^k. Since $2^k = 2 \cdot 2^{k-1}$ pebbles are put into exactly two sets of vertices, either the right path or the left path with respect to r contains $2^{k-1} \geq 2^{l-1}$ pebbles. The configuration is thus r-solvable, since either the left or right path P_l is r-solvable because $\pi(P_l) \leq \pi(P_k)$ by theorem 2.7 and since P_k is a spanning subgraph of P_l. So $\pi(P_l)$ is at most 2^{k-1} since $\pi(P_k) = 2^{k-1}$ by induction.

Case 2: If the target r is the last vertex, then in the worst configuration C there are no pebbles at r, and therefore the remaining 2^k pebbles are distributed at the vertices of $P_{k+1} - r = P_k$. One must now move two pebbles to the adjacent vertex v of r. In the worst case, one needs $\pi(P_k)$ pebbles to move one pebble to v, leaving no pebble behind.[31] Since $\pi(P_k) = 2^{k-1}$ by induction, and thus $2^k = 2\pi(P_k)$, we can move two pebbles to v and then one to r. Hence, C is r-solvable. By symmetry C is also r-solvable if r is the first vertex.

Conclusion: Case 1 and 2 completes the induction: Every configuration on P_n of at least size 2^{n-1} is solvable. Noting that $diam(P_n) = n - 1$ we have $\pi(P_n) \geq 2^{n-1}$ by the distance bound. Thus, $\pi(P_n) = 2^{n-1}$. □

[31] One such configuration on P_k is the one placing all 2^{k-1} pebbles at the opposite vertex of r (the other end).

3.1.2 Cycle graph

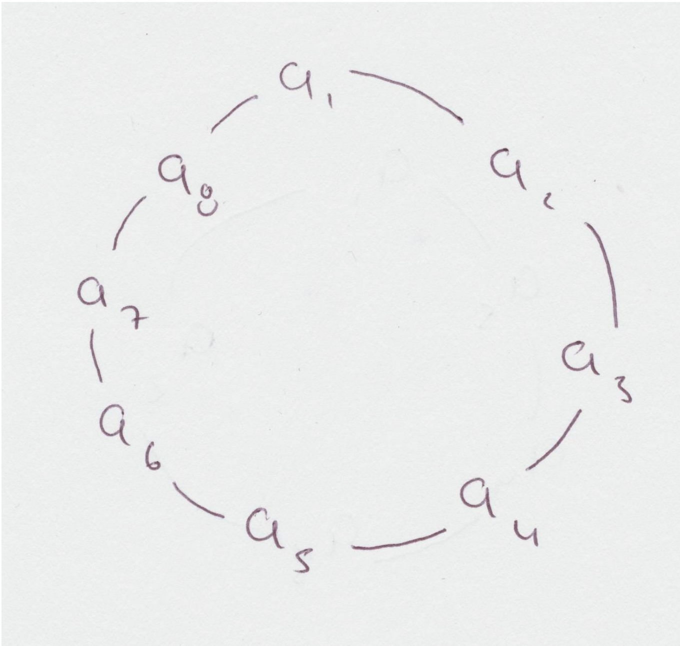

Figure 8: The cycle graph C_8.

Using Theorem 3.1 one can find the pebbling number of the cycle graph C_n for even numbers n. See Figure 8 respectively 9 for an example of an even cycle respectively an odd cycle.

Theorem 3.2. $\pi(C_{2k}) = 2^k, k \geq 1$.

Proof. Write C_{2k} as $a_1 \ldots a_{k-1}a_k a_{k+1} \ldots a_{2k}$, $k \geq 1$, and let a_1 be the target. A will denote the segment of vertices from a_2 to a_k and B the segment of vertices from a_{k+2} to a_{2k}. If one links the vertex a_1 to $a_2 \in A$ or $a_{2k} \in B$ one has the path P_k.

Consider a configuration C on C_{2k} of size 2^k, and recall Definition 2.14. If $C(A) \geq 2^{k-1}$ or $C(B) \geq 2^{k-1}$, then C is solvable since $\pi(P_k) = 2^{k-1}$ by Theorem 3.1 and since either A or B, together with a_1, may be seen as the path P_k. Thus, assume that $C(A), C(B) < 2^{k-1}$. Note that

$$|C| = C(A) + C(B) + C(a_{k+1})$$

since $C(a_1) = 0$.

One may always move $\lfloor C(a_{k+1})/2 \rfloor$ pebbles from a_{k+1} to either $a_k \in A$ or $a_{k+2} \in B$.[32] Hence, we may derive a configuration C' satisfying $C'(A) \geq 2^{k-1}$ or $C'(B) \geq 2^{k-1}$. This is because $C(A) + \lfloor C(a_{k+1})/2 \rfloor \geq 2^{k-1}$ or $C(B) + \lfloor C(a_{k+1})/2 \rfloor \geq 2^{k-1}$, since otherwise

$$C(A) + \lfloor C(a_{k+1})/2 \rfloor + C(B) + \lfloor C(a_{k+1})/2 \rfloor < 2^k - 2;$$

but this can not hold since

$$\begin{aligned} & C(A) + \lfloor C(a_{k+1})/2 \rfloor + C(B) + \lfloor C(a_{k+1})/2 \rfloor \\ & \geq C(A) + C(B) + C(a_{k+1})/2 - 1/2 + C(a_{k+1})/2 - 1/2 \\ & = |C| - 1 \\ & = 2^k - 1. \end{aligned}$$

So C is a_1-solvable since C' is. By symmetry C is r-solvable for every $r \in V(C_{2k})$.[33] Notice that $diam(C_{2k}) = k$, so $\pi(C_{2k}) \geq \max(2k, 2^k) = 2^k$ by Theorem 2.5. This proves that $\pi(C_{2k}) = 2^k$. $\square$

[32] The *floor function* $\lfloor \rfloor : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, maps a real number x to the largest integer not exceeding x. In other words, $\lfloor x \rfloor$ is the largest integer not greater than x.

[33] To see symmetry, just note that when we have found the $\pi(G, a_1)$, we may find $\pi(G, a_2)$ by rotating C_{2k} by $\frac{360}{2k}$ degrees to the left.

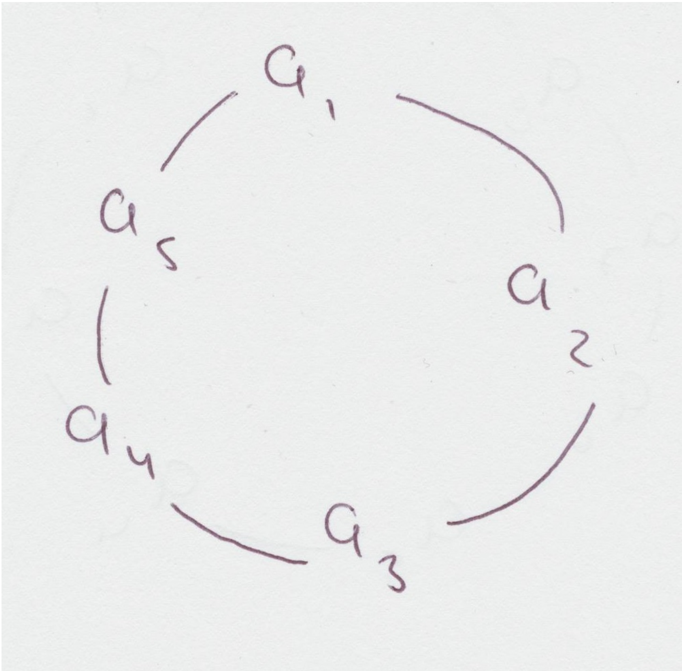

Figure 9: The cycle graph C_5.

The proof of Theorem 3.3 mirrors that of 3.2 in considering two paths in the cycle graph.

Theorem 3.3 (Pachter, 1995)**.** $\pi(C_{2k+1}) = 2\lfloor 2^{k+1}/3 \rfloor + 1, k \geq 1.$

Proof. Denote C_{2k+1} by $xa_{k-1}a_{k-2}\dots a_2a_1rb_1b_2\dots b_{k-1}y$, where r is the target. Let P_A denote the path $ra_1\dots a_{k-1}$ and P_B the path $rb_1\dots b_{k-1}$. First we show that $\pi(C_{2k+1}) > 2\lfloor 2^{k+1}/3 \rfloor$.

Suppose we are given $2\lfloor 2^{k+1}/3 \rfloor$ pebbles. Place $\lfloor 2^{k+1}/3 \rfloor$ pebbles at x and $\lfloor 2^{k+1}/3 \rfloor$ pebbles at y. This configuration C is r-unsolvable, since one can at most move $2^{k-1} - 1$ to either a_{k-1} or b_{k-1}, but one needs at least 2^{k-1} pebbles on one of them to move a pebble to r.

To see that one can move at most $2^{k-1} - 1$ pebbles to a_{k-1} or b_{k-1}, notice that $\lfloor 2^{k+1}/3 \rfloor \leq \frac{2^{k+1}}{3} - \frac{1}{3}$ and that one can move

$$\left\lfloor \frac{1}{2}(\frac{2^{k+1}}{3} - \frac{1}{3}) \right\rfloor \leq \frac{1}{2}(\frac{2^{k+1}}{3} - \frac{1}{3})$$

pebbles from x to a_{k-1}, and

$$\left\lfloor \frac{1}{4}(\frac{2^{k+1}}{3} - \frac{1}{3}) \right\rfloor \leq \frac{1}{4}(\frac{2^{k+1}}{3} - \frac{1}{3})$$

pebbles from y to a_{k-1}. Hence, one can move at most

$$\begin{aligned} \left\lfloor (1/2 + 1/4)(2^{k+1}/3 - 1/3) \right\rfloor &= \left\lfloor (3/4)(2^{k+1}/3 - 1/3) \right\rfloor \\ &= \left\lfloor 2^{k-1} - 1/4 \right\rfloor \\ &= 2^{k-1} - 1 \end{aligned}$$

pebbles to a_{k-1}. By symmetry, the same holds for b_{k-1}.
Hence, C is r-unsolvable. So $\pi(C_{2k+1}) > 2\lfloor 2^{k+1}/3 \rfloor$.
Let C be a configuration on C_{2k+1} of size $2\lfloor 2^{k+1}/3 \rfloor + 1$, then

$$C(P_A) + \left\lfloor \frac{C(x) + \lfloor C(y)/2 \rfloor}{2} \right\rfloor \leq 2^{k-1} - 1. \quad (3.1)$$

(3.1) holds since if the left hand side (LHS) of (3.1) is $\geq 2^{k-1}$, then one can move $\lfloor C(y)/2 \rfloor$ pebbles from y to x and then $\left\lfloor \frac{C(x)+\lfloor C(y)/2 \rfloor}{2} \right\rfloor$ pebbles from x to a_{k-1}, so that there are a total of at least 2^{k-1} pebbles on the vertices of P_A, implying that C on P_A is r-solvable. Hence, (3.1) may be assumed as we analyze all other cases.

Similarly

$$C(P_B) + \left\lfloor \frac{C(y) + \lfloor C(x)/2 \rfloor}{2} \right\rfloor \leq 2^{k-1} - 1.^{34} \quad (3.2)$$

(3.1) and (3.2) now gives

$$C(P_B) + C(P_B) + \left\lfloor \frac{C(x) + \lfloor C(y)/2 \rfloor}{2} \right\rfloor + \left\lfloor \frac{C(y) + \lfloor C(x)/2 \rfloor}{2} \right\rfloor \leq 2^k - 2. \quad (3.3)$$

[34] The second term on the LHS of (3.2) denotes the number of pebbles that may be taken to b_{k-1}.

Notice that $|C| = C(P_A) + C(P_B) + C(x) + C(y) = 2\lfloor 2^{k+1}/3 \rfloor + 1$. Since $C(P_A)$ or $C(P_B)$ is not affected by the floor function $\lfloor\rfloor : \mathbb{R} \to \mathbb{R}$, the LHS of (2.3) is minimized when $C(P_A) = C(P_B) = 0$.

$2\lfloor 2^{k+1}/3 \rfloor + 1$ is an odd number,[35] so only one of $C(x)$ or $C(y)$ is even. Without loss of generality assume that $C(x)$ is even.[36]

We want to move as many pebbles from x and y to a_{k-1}. Since $C(y)$ is odd, in the worst case we may have a pebble left on x and one on y (this happens whenever $C(y) \equiv 3 \pmod 4$).

If one wants to move as many pebbles as possible to b_{k-1}, y will have at most one pebble in the resulting configuration (whenever $C(x) \equiv 0 \pmod 4$), or no pebble (whenever $C(x) \equiv 0 \pmod 2$ and where $C(x)$ is not divisible by four).

Thus, when the LHS of (3.3) is to be minimized then

$$\left\lfloor \frac{C(x) + \lfloor C(y)/2 \rfloor}{2} \right\rfloor + \left\lfloor \frac{C(y) + \lfloor C(x)/2 \rfloor}{2} \right\rfloor = \frac{3}{4}C(x) + \frac{3}{4}C(y) - \frac{5}{4} \leq 2^k - 2, \tag{3.4}$$

The equality in (3.4) is motivated by (a) and (b) below.

(a): $\left\lfloor \frac{C(x)+\lfloor C(y)/2 \rfloor}{2} \right\rfloor$ denotes the case where $C(y) \equiv 3 \pmod 4$, hence $C(y) = 4m + 3$ for some m nonnegative integer in $\mathbb{Z}_{\geq 0}$. Using this we have

$$\begin{aligned} \left\lfloor \frac{C(x) + \lfloor C(y)/2 \rfloor}{2} \right\rfloor &= \left\lfloor \frac{C(x) + \lfloor 2m + 3/2 \rfloor}{2} \right\rfloor \\ &= \left\lfloor \frac{C(x) + 2m + 1}{2} \right\rfloor \\ &= C(x)/2 + m \\ &= C(x)/2 + C(y)/4 - 3/4. \end{aligned}$$

(b): $\left\lfloor \frac{C(y)+\lfloor C(x)/2 \rfloor}{2} \right\rfloor$ denotes the case where (i) $C(x) = 4m$ for some $m \in \mathbb{Z}_{\geq 0}$ or (ii) $C(x) = 2m$ for some odd $m \in \mathbb{Z}_{\geq 0}$.

[35] It is written on the form $2m + 1$ for some nonnegative integer m

[36] By symmetry the same argument works for $C(y)$ even.

For (i) we have

$$\left\lfloor \frac{C(y) + \lfloor C(x)/2 \rfloor}{2} \right\rfloor = \lfloor C(y)/2 + 2m/2 \rfloor$$
$$= C(y)/2 - 1/2 + m$$
$$= C(y)/2 + C(x)/4 - 1/2.$$

For (ii) we have

$$\left\lfloor \frac{C(y) + \lfloor C(x)/2 \rfloor}{2} \right\rfloor = \lfloor (C(y) + m)/2 \rfloor$$
$$= (C(y) + m)/2$$
$$= C(y)/2 + C(x)/4.$$

From (a) and (b.ii)[37] it follows that the minimum value of the LHS of (3.3) is

$$C(x)/2 + C(y)/4 - 3/4 + C(y)/2 + C(x)/4 - 1/2,$$

and this is equal to

$$\frac{3}{4}C(x) + \frac{3}{4}C(y) - 5/4.$$

This shows why the first equality in (3.4) holds.

Since

$$C(x) + C(y) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$$
$$\geq 2\left(\frac{2^{k+1}}{3} - \frac{2}{3}\right) + 1$$
$$= \frac{4}{3}(2^k - 1) + 1,$$

[37] The value calculated in (b.i) is less (1/2 less) than the value calculated in (b.ii).

we have

$$\begin{aligned}\frac{3}{4}(C(x)+C(y))-\frac{5}{4} &\geq (2^k-1)+\frac{3}{4}-\frac{5}{4}\\ &= 2^k-\frac{3}{2},\end{aligned}$$

and this contradicts the fact that the LHS of (3.4) is $\leq 2^k - 2$, because $2^k - \frac{3}{2} > 2^k - 2$.

Thus, by contradiction, either (3.1) or (3.2) is false, and in this case either C on P_A or P_B is r-solvable, and so is C_{2k+1} by implication. Hence, $\pi(C_{2k+1}, r) = 2\lfloor 2^{k+1}/3 \rfloor + 1$.

Since r was arbitrary, the theorem follows using symmetry. □

3.1.3 Wheel, friendship and fan graph

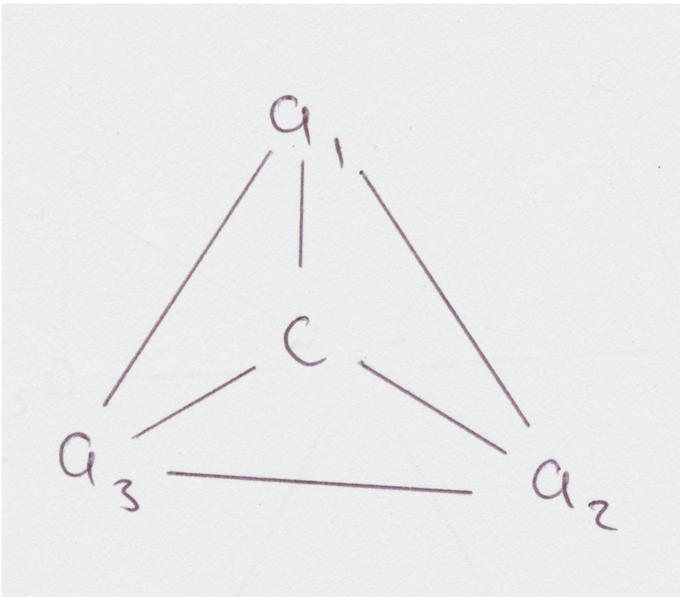

Figure 10: The wheel graph W_4.

We will now find the pebbling number for the wheel graph W_n with $n \geq 4$ vertices. W_n may be seen as containing a center vertex c adjacent to all vertices of a cycle graph C_{n-1}, which may be written as $a_1 \dots a_{n-1}$. See Figure 10 for an example.

Theorem 3.4. $\pi(W_n) = n$, $n \geq 4$.

Proof. Pick a configuration C on W_n, $n \geq 4$, of size n.

Case 1: The target is the center c of W_n. By the pigeonhole principle, and the fact that $|V(W_n - r)| = n - 1$, at least one vertex $v \neq r$ satisfies $C(v) \geq 2$. The vertex v is adjacent to c, so c may be pebbled. Thus, C is c-solvable.

Case 2: The target is not the center of W_n, say a_1.

Case 2a: If $C(c) \geq 2$, then $c \to a_1$ is a solution.

Case 2b: If $C(c) = 1$, then we have a distribution of $n - 1$ remaining pebbles (noting that $C(c) = 1$) on $n - 2$ vertices (all vertices but c and r). This gives us at least one vertex $v \neq c, a_1$ with two pebbles and distance 1 to c.[38] The pebbling move $v \to c \to a_1$ solves the problem.

Case 2c: If $C(c) = 0$, then consider the graph $G - c$. Since $|V(G - c)| = n - 1$ and $|C| = n$, there is at least one vertex $v \neq c$ with $C(v) \geq 2$.

If $C(v) \geq 4$, then $v \to^2 c \to r$ solves the problem.

If $C(v) = 3$, and no other $v' \neq c, v$ satisfies $C(v') \geq 2$ (else $v \to c, v' \to c, c \to r$ is a solution), then every vertex $v'' \neq c$ satisfies $C(v'') = 1$ except one, say $x \neq c, r$. Thus, one may take the path from v to r which does not include x or c.

If $C(v) = 2$, and no other vertex contains at least two pebbles, then every other vertex, except v and c, contains one pebble, and one may thus take the left or right path of the cycle C_{n-1} from v to r.

Using the vertex bound $\pi(W_n) \geq n$, case 1 and 2 gives us $\pi(W_n) = n$, $n \geq 4$. □

[38] That $x \neq y, z$ means that $x \neq y$ and $x \neq z$

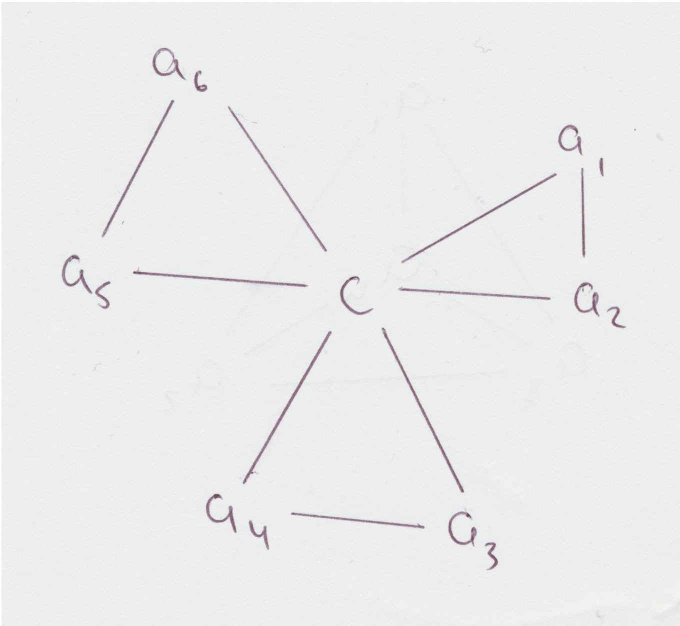

Figure 11: The friendship graph FR_3.

Lets look at another graph which is constructed using the structure of the cycle graph: FR_n. This is the friendship graph which consists in n copies of C_3, where each vertex is adjacent to a center vertex c and where c is a vertex in each copy of C_3. We will call A a copy of C_3 in FR_n iff A consists of two vertices adjacent to each other, each of them adjacent to the center c of FR_n.[39] Note that $n(FR_n) = (n(C_3) - 1)n + 1 = 2n + 1$.

In Figure 11 we have FR_3 with 7 vertices and three copies of C_3.

Theorem 3.5. $\pi(FR_n) = 2n + 2$, $n \geq 1$.

Proof. Placing no pebble on c and $r \neq c$, 3 on one vertex which is not adjacent to r and 1 pebble at all other vertices, one gets a r-unsolvable configuration of size $2n + 1$. Hence, $\pi(FR_n) \geq 2n + 2$.

Consider a configuration C on FR_n of size $2n + 2$.

Case 1: The target is the center c. If one places $2n + 1$ on the remaining $2n$ vertices, one gets one vertex v with at least two pebbles on it. Since the distance from v to c is 1, C is c-solvable.

[39] A is a copy of C_3 in FR_n in the sense that A is C_3 when all vertices in $V(A)$ is adjacent to the center c of FR_n.

Case 2: The target r is not c.

If $C(c) \geq 2$, then C is r-solvable ($c \to r$). If $C(c) = 1$, then C is solvable by the pigeonhole principle, since one has a distribution of $2n+1$ pebbles on $2n-1$ vertices, so there is at least one $v \neq r$ adjacent to c with $C(v) \geq 2$, giving the solution $v \to c \to r$. Assume that $C(c) = 0$ in case 2i and 2ii below.

If one places $2n+2$ pebbles on the vertices of $(FR_n - c) - r$, then one can not avoid having (2i) one copy A of C_3 in FR_n with $C(A) \geq 4$, or (2ii) two copies X and Y of C_3 in FR_n with $C(X), C(Y) \geq 3$. Since if neither (2i) or (2ii) holds, then $C(A) \leq 2$ for all copies A of C_3 in FR_n, except maybe one B with $C(B) = 3$, and so, since $C(c) = 0$, we have a distribution of no more than $3 + 2(n-1) = 2n+1$ pebbles, but this is a contradiction since C was of size $2n+2$.

Case 2i: If $C(A) = 4$ one may move one pebble to the center c from some vertex a in A. After the pebbling step there are $2n$ pebbles remaining on $2n-1$ vertices (excluding c and r), so that there is at least one vertex $x \neq c, r$ satisfying $C(x) \geq 2$. The pebbling move $a \to c$, $x \to c \to r$ now solves the problem. If $C(A) \geq 5$, where $A = \{a_1, a_2, c\}$ and $C(c) = 0$, then we may move two pebbles to c since $C(a_i) \geq 4$ for some $i = 1, 2$, or $C(a_1), C(a_2) \geq 2$.

Case 2ii: In this case one may move one pebble to the center from X and the same for Y, using $\pi(C_3) = 3$.

In both (2i) and (2ii), C is r-solvable.

By symmetry for target vertices not at the center, and the result of case 1, we have that every configuration C on FR_n of size $2n+2$ is solvable. $\square$

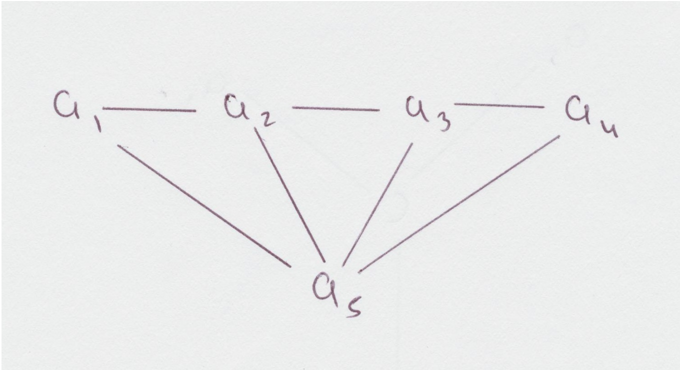

Figure 12: The fan graph F_5.

Let F_n denote the fan graph. This is the path P_{n-1} with an extra vertex a_n adjacent to all vertices of P_{n-1}. In Figure 12 we have F_5 with vertices a_1 to a_4 representing P_4, and where each vertex a_i, $1 \leq i \leq 4$, is adjacent to a_5.

Theorem 3.6. *$\pi(F_n) = n$, $n \geq 2$.*

Proof. Let C be a configuration on F_n of size n. Let the path P_{n-1} be denoted by $a_1, \ldots, a_{n-1}$ and let a_n be the vertex adjacent to all vertices in P_{n-1}.

Case 1: Let a_n be the target. A distribution of n pebbles on the vertices of P_{n-1} gives at least one vertex $a \in P_{n-1}$ with $C(a) \geq 2$. The pebbling step $a \to a_n$ is then a solution.

Suppose the target is a_1 in case 2 to 4.

Case 2: If $C(a_n) \geq 2$, then $a_n \to a_1$ is solution.

Case 3: If $C(a_n) = 1$, then $n-1$ pebbles are distributed on the remaining $n-2$ vertices of $P_{n-1} - a_1$, hence there is a $a \neq a_1, a_n$ with $C(a) \geq 2$ and it follows that the pebbling move $a \to a_n \to a_1$ is a solution.

Case 4: $C(a_n) = 0$ and $C(P_{n-1} - a_1) = n$. If there are two distinct vertices $a, a \neq a_1$ with $C(a), C(a') \geq 2$, then $a \to a_n, a' \to a_n \to a_1$ is a solution. Otherwise there is at most one $a \neq a_1, a_n$, with $2 \leq C(a) \leq 3$, and every other vertex $a' \neq a_1, a_n$ satisfies $C(a') \leq 1$. In this case $C(P_{n-1} - a_1) \leq 1(n-3) + 3$ (here $n-3$ is the number of vertices excluding a_1, a_n and a), and since the equality must hold we have $C(a') = 1$ for all $a' \neq a_1, a, a_n$. This gives us some path from a to a_1 using the fact that every pebbling step to an adjacent vertex closer to a_1 results in the possibility of another pebbling step. Moving pebbles in this way we see that C is a_1-solvable.

Case 2 to 4 can be applied to all other vertices a_i, $2 \leq i \leq n-1$. So $\pi(F_n, a_i) \leq n$ for $1 \leq i \leq n$. Summarizing this with case 1 we have $\pi(F_n) \leq n$.

Since $\pi(F_n) \leq n$ and $n(F_n) = n$, $\pi(F_n) = n$ follows by the vertex bound $\pi(F_n) \geq n$. □

3.1.4 Complete bipartite graph

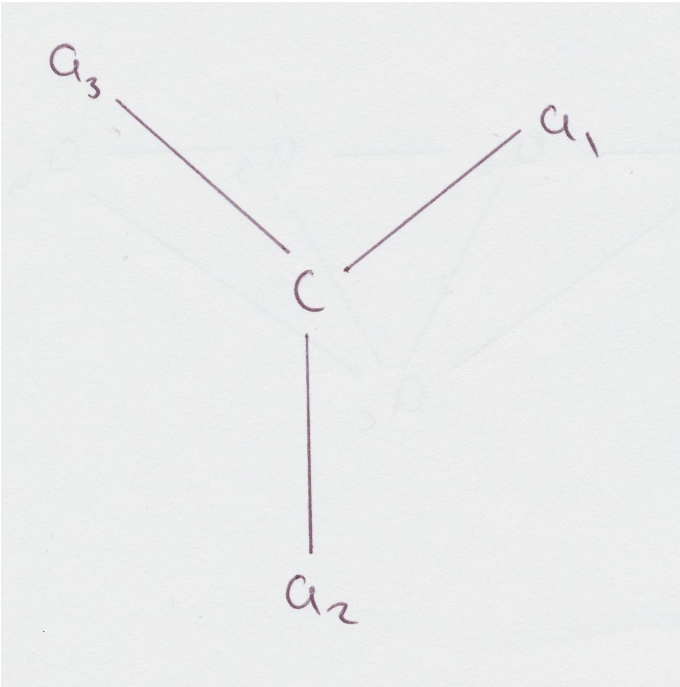

Figure 13: The star graph S_4.

A complete bipartite graph $K_{m,n}$, $m, n \geq 1$, is a graph which can be partitioned into two sets A and B such that for every two vertices $a \in A, b \in B$, $\{a, b\}$ is an edge in $E(K_{m,n})$, but where there are no edges between vertices in only A or only B. A special case is $K_{1,k-1}$: the star graph S_k on k vertices (an example is provided in Figure 13).

Theorem 3.7. $\pi(S_k) = k + 1, k \geq 3$.

Proof. The configuration placing 3 pebbles at some vertex $x \neq c$, and 1 pebble on all but one vertex $v \neq c$, is a v-unsolvable configuration of size k. Thus $\pi(S_k) \geq k + 1$.

Let C be a configuration on S_k of size $k + 1$.

Case 1: The target is the center c. When $k + 1$ pebbles are distributed on $k - 1$ vertices, at least one vertex $v \neq c$ adjacent to c satisfies $C(v) \geq 2$ by the pigeonhole principle. So $\pi(S_k, c) \leq k + 1$.

Case 2: The target r is not the center vertex. Consider a configuration C on S_k of size $k + 1$. If there is one pebble at the center, then there is at least one vertex v, among the remaining $k - 2$ vertices, adjacent to c and satisfying $C(v) \geq 2$; the pebbling move $a \to c \to r$ would then be a solution. Otherwise there is no pebble at the center. So if one places $k + 1$ pebbles at the remaining $k - 2$ vertices, by Theorem 2.6 and the equality $k + 1 = (k - 2) + (2 + 1)$, we may move two pebbles to the center, and then finally move one pebble to the target. Hence, $\pi(S_k, r) \leq k + 1$.

Case 1 and 2 gives $\pi(S_k) \leq k + 1$. This $\pi(S_k) = k + 1$ since $\pi(S_k) \geq k + 1$ as mentioned at the start of this proof. □

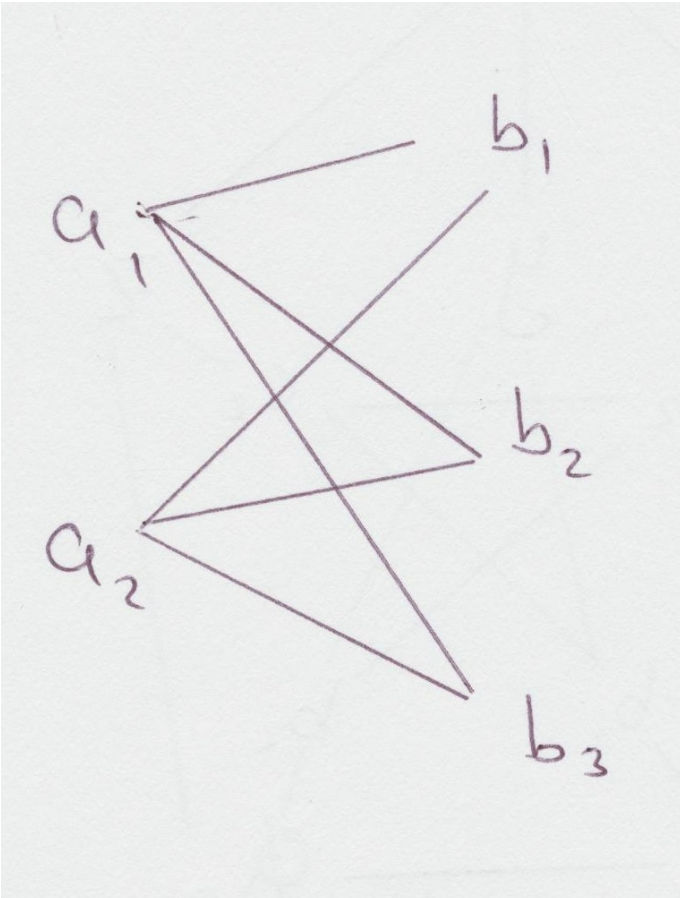

Figure 14: The complete bipartite graph $K_{2,3}$.

While Theorem 3.7 above shows that complete bipartite graphs on the form $K_{1,k}$, $k \geq 2$, are of Class 1, Theorem 3.8 and 3.9 below shows that $K_{n,m}$ is of Class 0 for all $n, m \geq 2$. Figure 14 shows an example of a bipartite graph $K_{2,3}$ of the latter form, and by Theorem 3.8 this graph is of Class 0 and its pebbling number is therefore $2 + 3 = 5$.

Theorem 3.8. $\pi(K_{2,k}) = k + 2$, $k \geq 2$.

Proof. $K_{2,k}$ may be seen as the graph in which every vertex in $A = \{a_1, a_2\}$ is linked to every vertex in $B = \{b_1, \ldots, b_k\}$, where no other edges exists except these. Let C be a configuration on $K_{2,k}$ of size $k + 2$.

Case 1: The target r is in A. Let $r = a_1$. Suppose $C(a_2) < 4$, for else $a_2 \to^2 b_1 \to a_1$ is a solution.

Case 1a: If $C(a_2) \leq 1$, then we have a distribution of at least $k+1$ pebbles on B with $|B| = k$. Using the pigeonhole principle there exists a vertex b in B which is adjacent to a_1 and satisfies $C(b) \geq 2$. So the pebbling step $b \to a_1$ solves our problem.

Case 1b: If $2 \leq C(a_2) \leq 3$, then we have a distribution of at least $k-1$ pebbles on B, for which there must exist a b in $V(B)$ with at least one pebble on it. $a_2 \to b \to a_1$ is then a solution.

1a, 1b and symmetry, gives us that C is a-solvable for every a in A.

Case 2: The target r is in B. Let $r = b_1$. If $C(a_i) \geq 2$ for some $i = 1, 2$, then $a_i \to b_1$ is a solution. So assume that $C(a_i) \leq 1$ for $i = 1, 2$. If $C(a_i) = 1$ for some $i = 1, 2$, then there is a distribution of at least k pebbles on $B - b_1$.[40] Thus there exists a vertex b in B with at least two pebbles on it, and $b \to a_i \to b_1$ is thus a solution. Otherwise $C(a_i) = 0$ for $i = 1, 2$, but then one places $k+2$ pebbles on the set $B - b_1$ of $k-1$ vertices, for which one can move two pebbles to a_1 by Theorem 2.6, and then one to b_1. Symmetry gives that every configuration C on $K_{2,k}$ of size $k+2$ is b-solvable for all vertices b in B.

The vertex bound gives $\pi(K_{2,k}) \geq k+2$. Hence, $\pi(K_{2,k}) = k+2$ by case 1 and 2. □

Theorem 3.9. *$\pi(K_{m,n}) = m + n$, $m, n \geq 3$.*

Proof. Consider a configuration C on $K_{m,n}$ of size $m+n$. Let $A = \{a_1, \ldots, a_m\}$ denote the left side of vertices of $K_{m,n}$ and $B = \{b_1, \ldots b_n\}$ the right hand side. Suppose the target vertex is a_1.

Case 1: There is a vertex b in B with at least two pebbles on it, then $b \to a_1$ is a solution.

Case 2: There is a vertex b in B with a pebble on it, and no other vertex b' in B with at least two pebbles on it. In this case $C(A - a_1) \geq m$ which implies, by the fact that $|A - a_1| = m - 1$, that there is a vertex $a \neq a_1$ in A with $C(a) \geq 2$, so that $a \to b \to a_1$ is a solution.

[40]If A and B are sets of elements, then $A - B$ is the set of elements in A but not in B. $A - x$ is shorthand for $A - \{x\}$.

Case 3: $C(b) = 0$ for all $b \in B$. Then $C(A - a_1) = m + n \geq m + 3$. Since we are distributing at least $m + 3$ pebbles on the set $A - a_1$ of $m - 1$ vertices, for which $(A - a_1) \subseteq N(b_1)$ holds, we can according to Theorem 2.6 move at least two pebbles to b_1 using $m + 3 > m + 2 = m - 1 + (2 + 1)$.

Thus, $\pi(K_{m,n}, a_1) \leq m + n$.

One may use the same argument if the target r is any other vertex in A or B, finding that $\pi(K_{m,n}) \leq m + n$. Thus $\pi(K_{m,n}) = m + n$ by the vertex bound $\pi(K_{m,n}) \geq m + n$. □

Using Theorem 3.1, with $P_2 = K_{1,1}$, Theorem 3.8 and 3.9 one finds that the following holds:

Corollary 3.9.1. $\pi(K_{n,n}) = 2n$, $n \geq 1$.

3.1.5 Petersen's graph and the m-dimensional cube

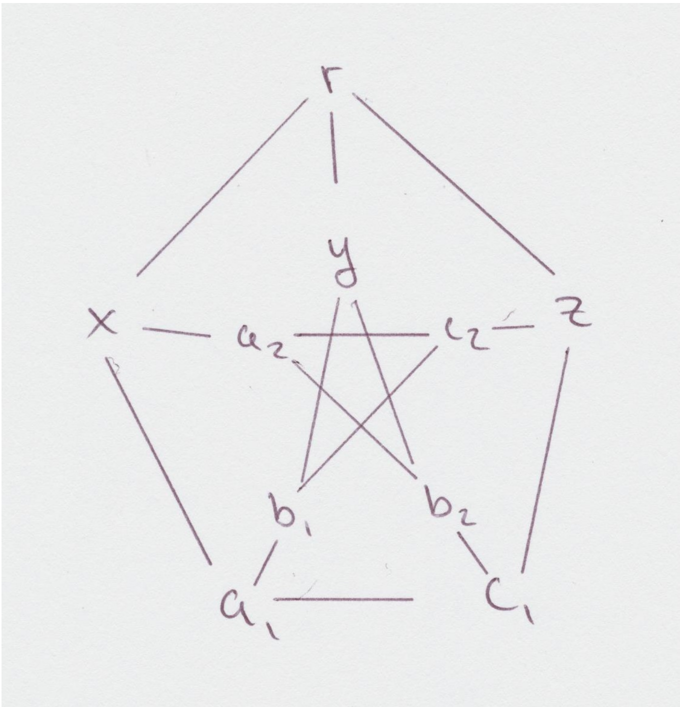

Figure 15: Petersen 's graph P.

By Theorem 3.3 we know that $\pi(C_5) = 5$. Using this we can prove that the pebbling number of Petersen's graph P is 10. See Figure 15 for an illustration of P.

The proof below was inspired by a comment in "A Survey of Graph Pebbling" (Hurlbert, 1991: 4)[14].

Theorem 3.10. *$\pi(P) = 10$.*

Proof. $n(P) = 10$ and the vertex bound gives us $\pi(P) \geq 10$, so if we can show that every configuration on P of size 10 is solvable, then $\pi(P) = 10$.

Pick a configuration C on P of size 10, with target r.

Case 1: If there is a neighbor of r, call it s, which contain one pebble, then one can draw two cycles C_5: A passing through r and B passing through s. Since $\pi(C_5) = 5$ we must have $C(A) \leq 4$ and $C(B) \geq 6$, for else we may move a pebble to r if $C(A) \geq 5$ because $\pi(A) = 5$. Since $C(B - s) \geq 5$ one can move a pebble to s, because $\pi(C_5) = 5$ and $B - s$ is part of a C_5-cycle, thus we will get $C(s) = 2$ and the we may move a pebble to r.

Case 2: Else there is no neighbor of r which contains a pebble. In this case we have 4 vertices (including r) which contain no pebble, since $\deg(r) = 3$. Now, r has three neighbors, each with degree 3, and ignoring r we see that their degree is 2. We notice that the remaining 6 vertices are the neighbors of the neighbors of r, and they come in pairs $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$.

A distribution of 10 pebbles on A, B and C gives that at least one of them, say A, contain 4 pebbles (note that A can not contain five pebbles or more, for then $C(a_1), C(a_2) \geq 2$, or $C(a_i) \geq 4$ for some $i = 1, 2$; in the first case we may move two pebbles from a_1 and a_2 to common neighbor of a_1, a_2 and r, in the other case we have a solution since $dist(a_i, r) = 2$ for $i = 1, 2$). One may immediately move two pebbles to a neighbor of r, and then to r, if both a_1 and a_2 contains 2 pebbles, or if one of them contains 4. Otherwise one of them contains 3 pebbles, say a_1, and the other 1 pebble. Ignoring the two pebbles on a_1, making it have 1 pebble, we have a distribution of 8 pebbles on the remaining 6 vertices which constitute a cycle graph C_6 in the following order M of vertices: $a_1c_1b_2a_2c_2b_1$ (consult Figure 15 above).

We will now show that we can move 1 pebble to a_1 so that it contains 4 pebbles, which shows that we can move a pebble to r through one of its neighbors.

For every configuration C on M of size 8 we assume that $C(a_1) = C(a_2) = 1$ by the above comments. $C(b_1), C(c_1) \leq 1$ must hold, for else $b_1 \to a_1$ or $c_1 \to a_1$ solves the problem. Since $C(a_2) = 1$ is assumed, $C(\{b_2, c_2\}) \geq 4$. If $C(b_2) = 3$, then $C(c_1) = 0$ for else $b_2 \to a_2 \to a_1$ is a solution. But then $C(c_1) \geq 2$ since $C(\{b_2, c_2\}) \geq 5$, implying that $c_2 \to a_2 \to b_2$ gives four pebbles at b_2, which solves the problem. A similar argument shows that one can move one pebble to a_1, or have four pebbles on c_2, if $C(c_2) = 3$. If $2 \leq C(b_2), C(c_2) \leq 3$ then $C(c_1) = C(b_1) = 0$, for else one can move a pebble to a_1. So $C(b_2) = C(c_2) = 3$. But then one can move a pebble to b_2 or c_2 through a_2, making b_2 or c_2 have four pebbles.

By case 2, C is either r-solvable or one pebble may be moved to a_1 so that it contains 2 pebbles. Remembering the 2 pebbles which we ignored, a_1 contains 4 pebbles and we can move a pebble to r via one of r's neighbors. By symmetry the same goes if a_2 contains 3 pebbles at the start. Using symmetry again, the same reasoning may be applied to B and C as it was applied to the case when A contained at four pebbles.

Using the results of case 1 and 2, and symmetry on r (Petersen's graph may be seen, with respect to every vertex, as an inner cycle graph C_5 where each vertex is adjacent to some vertex of an outer cycle graph C_5 which is disjoint from the inner cycle graph), we conclude that every configuration on P of size 10 is solvable. Hence, $\pi(P) = 10$. □

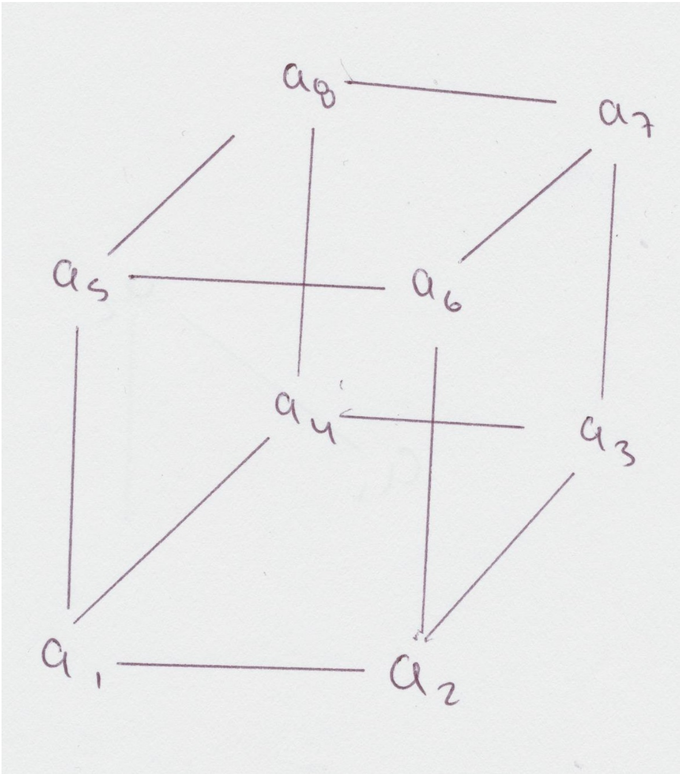

Figure 16: The 3-dimensional cube Q_3.

Before we end, I present the classic proof of the fact that $\pi(Q_m) = 2^m = |V(Q_m)|$ where Q_m is the m-dimensional cube (Chung, 1989: 468)[7]. In Figure 16 the 3-dimensional cube is represented.

Theorem 3.11 (Chung, 1989)**.** *$\pi(Q_m) = 2^m$ for $m \geq 1$.*

Proof. Define q_m as the number of vertices v of Q_m which contain an odd number of pebbles (so called odd vertices) for every configuration C on Q_m. The proof is by induction on m for the following two statements:

(i) If C is a configuration on Q_m of size 2^m, then C is r-solvable; and

(ii) if $|C| > 2^{m+1} - q_m$, then two pebbles can be moved to r.

Induction base: $\pi(Q_0) = 1 = 2^0$ since $V(Q_0) = 1$. Thus, (i) holds in the base case. Since $n(Q_0) = 1$, $q \leq 1$. So $2^{0+1} - q \geq 1$, and then $|C| > 1$

if $|C| > 2^{0+1} - q$, which implies that two pebbles are placed at the only vertex of Q_0. Hence, (ii) holds in the base case.

Induction step: Assume that $\pi(Q_k) = 2^k$ for all $k < m$. Partition Q_m into two $(m-1)$-cubes Q and Q'.[41] Pick a vertex r in $V(Q)$ and an adjacent vertex r' in $V(Q')$. Let q be the number of odd vertices in $V(Q)$ and q' the number of odd vertices in $V(Q')$. Set the size $C(V(Q))$ of C on Q equal to p, and $C(V(Q')) = p'$.

Assume that $C(V(Q_m)) = |C| \geq 2^m$. If $p \geq 2^{m-1}$, then induction on (i) gives us that one pebble may be moved to r, since $k = m - 1 < m$ and r is in $V(Q)$. Thus, assume that $p < 2^{m-1}$. Now, either (1a) $q' > p$ or (1b) $q' \leq p$.

Case 1a: If $q' > p$, then $p' = |C| - p = 2^m - p > 2^m - q'$. By induction on (ii) two pebbles can be moved to r'. From there one can move a pebble to r since r and r' are adjacent.

Case 1b: One can always move at least $(p' - q')/2$ pebbles to $V(Q)$, since taking away one pebble from each of the q' odd vertices, $(p' - q')$ pebbles are left distributed on the vertices of Q' for which all vertices contain an even number of pebbles. So we may move $(p' - q')/2$ pebbles in total from $V(Q')$ to $V(Q)$. Thus, in the resulting configuration one has

$$\begin{aligned} p + (p' - q')/2 &\geq p + (p' - p)/2 \\ &= (p + p')/2 \\ &\geq 2^{m-1}, \end{aligned}$$

using $q' \leq p$. From this it follows that Q is r-solvable by induction on (i).

Now (i) has been proven assuming (ii). We continue induction on (ii).

Suppose there are at least $|C| = p + p' > 2^{m+1} - q - q'$ pebbles distributed on $V(Q_m)$. We will show that given such configurations C on Q_m, two pebbles may be moved to r in $V(Q)$.

[41] Q and Q' are structurally identical, so Q_m is formed by taking equivalent vertices in Q and Q' and linking them together to form an edge of Q_m

Case 2a: If $p > 2^m - q$, then two pebbles can be moved to r using induction with $k = m - 1 < m$ for which $m = k + 1$.

Case 2b: $2^m - q \geq p \geq 2^{m-1}$. Since $p \geq 2^{m-1}$, Q is r-solvable, so at least one pebble may be moved to r. Notice that

$$\begin{aligned} p' &= |C| - p \\ &> 2^{m+1} - q - q' - p \\ &\geq 2^m + p + q - q - p - q' \\ &= 2^m - q', \end{aligned}$$

since $2^m \geq p + q$. By induction on (ii), two pebbles can be moved to the vertex r' in $V(Q')$ adjacent to r in $V(Q)$, so that one can move a pebble to r. In the resulting configuration, r contains at least two pebbles.

Case 2c: $p < 2^{m-1}$. Notice that for any positive integer t satisfying $p' \geq q' + 2t$, t pebbles may be moved to $V(Q)$, while $p' - 2t$ remain in $V(Q')$ (since t pebbling steps costs $2t$ pebbles).

From the relation $p + p' > 2^{m+1} - q - q'$ we have

$$\begin{aligned} p' > 2^{m+1} - q - q' - p &= (2^m - q') + (2^m - q - p) \\ &\geq q' + (2^m - q - p), \end{aligned} \tag{3.5}$$

in which (3.5) follows from using $q' \leq |V(Q')| = 2^{m-1}$, and thus $2q' \leq 2^m$, so $q' \leq 2^m - q'$.

Thinking about solving the equation $q' + 2t = q' + (2^m - q - p)$ we see that it is possible to move at least

$$\begin{aligned} t &= \lfloor (2^m - (q + p))/2 \rfloor \\ &= \left\lfloor 2^{m-1} - (q + p)/2 \right\rfloor \\ &= 2^{m-1} - \lceil (q + p)/2 \rceil \end{aligned}$$

pebbles to $V(Q)$. When we move t pebbles to the vertices of Q, we have a remaining of more than $2^m - q'$ pebbles at Q'. To see this we note that there are $p' - 2t$ remaining pebbles at the vertices of Q', and so we need to show that $p' - 2t > 2^m - q'$, but surely this is the case since $p' - (2^m - q - p) > (2^m - q')$ by (3.5) and

$$\begin{aligned} 2t &= 2^m - 2\lceil (q+p)/2 \rceil \\ &< (2^m - 2(q+p)/2) \\ &= 2^m - p - q. \end{aligned}$$

On the vertices $V(Q)$ we have a distribution of

$$\begin{aligned} p + t &= p + 2^{m-1} - \lceil (q+p)/2 \rceil \\ &= 2^{m-1} + \lfloor (p-q)/2 \rfloor \\ &\geq 2^{m-1} \end{aligned}$$

pebbles. By (i), inductively assuming (ii), it follows that one pebble can be moved to r, and since $p' > 2^m - q'$ we can move two pebbles to r' by induction on (ii), and then one to the adjacent vertex r so that in the final distribution there are two pebbles at r.

Conclusion: By the principle of induction it follows that (ii) is true for any m, and so is (i). This shows that $\pi(Q_m) \leq 2^m$; and since $\pi(Q_m) \geq \max(2^m, 2^m)$ (noting that $diam(Q_m) = m$), it follows that $\pi(Q_m) = 2^m$. □

3.2 Pebbling numbers for diameter 2 graphs

Two theorems for diameter 2 graphs G are proven. In section 3.2.1 it is shown that G must be of Class 0 or 1, and in section 3.2.2 it is shown that if G is 3-connected, then G is of Class 0, and from this an important corollary that almost all graphs are Class 0 follows.

The proofs in 3.2.1 are found in (Pachter et al., 1995: 70–1)[20]. Those in 3.2.2 are found in (Clarke et al., 1997: 121–24)[8].

Note that vertex-connectivity is defined for all graphs considered in this section, since no graph is equal to a complete graph (because we only consider diameter 2 graphs while a complete graph is a diameter 1 graph), and since the vertex-connectivity is defined for all graphs other than the complete graphs (confer Definition 2.7).

3.2.1 Every diameter 2 graph is of Class 0 or 1

To show that every diameter 2 graph is of Class 0 or 1 (Theorem 3.13), we first need to prove an important lemma:

Lemma 3.12 (Pachter et al., 1995)**.** *If G is a diameter 2 graph with $|V(G)| = n \geq 6$, then every configuration C of size at least n, such that at least three vertices receive at least two pebbles, is solvable.*

Proof. Assume that we have such a configuration C on G, that is: there are at least three vertices in $V(G)$ which contain at least two pebbles.

Case 1: G has a cut-vertex. Then there are two components A and B of G, and a vertex x of G neither among the vertices of A nor the vertices of B, such that there exists some edge $\{a, x\}$ with $a \in A$, and some edge $\{b, x\}$ with $b \in B$. Since $diam(G) = 2$, every vertex of A or B has an edge to x (for else there is an vertex a in $V(A)$ and a vertex b in $V(B)$ with $dist(a, b) \geq 3$ since one must pass x when moving pebbles between vertices in $V(A)$ and vertices in $V(B)$, but this would contradict the fact that $diam(G) = 2$).

If x contains two pebbles, the configuration is solvable. Else, there exists at least two vertices in A or at least two vertices in B containing at least two pebbles (since at least three vertices receive at least two pebbles). In either case we can pebble two pebbles to x and then one pebble to an arbitrary vertex in A or B.

Case 2: G is 2-connected. Let r be the target. We may assume that $C(x) \leq 1$ for all x in the neighborhood $N(r)$ of r, for else C is solvable.

Suppose there are at least three vertices x_i, $1 \leq i \leq k$ with $k \geq 3$, that satisfy $C(x_i) \geq 2$. No x_i is in $N(r)$ by our previous remark, and since $diam(G) = 2$ it follows that every x_i is adjacent to a neighbor y_i of r. If $y_i = y_j$ but $i \neq j$, then one may move a pebble to y_i from x_i and again from

x_j, giving the solution $y_i \to v$. Hence $y_i \neq y_j$ whenever $i \neq j$. Also all y_i are without pebbles, for else one may make the move $x_i \to y_i \to r$ for some i. Furthermore no x_i can contain 4 pebbles since $diam(G) = 2$.

Since every neighbor of r contains at most 1 pebble, and every neighbor y_i of x_i contains no pebbles, we must, since the size of the configuration is $n(G)$, have some x_j with 3 pebbles on it, in this way compensating for the fact that r contains no pebble. Relabelling gives us x_1 with 3 pebbles.

The vertex x_1 is not adjacent to any other x_j (else we would pebble x_1 so that it had 4 pebbles) so $dist(x_1, x_j) = 2$ for all $2 \leq j \leq k$ since $diam(G) = 2$. Hence there must exist $k - 1$ distinct vertices $z_{12}, z_{13}, \ldots, z_{1k}$ such that z_{1j} is adjacent to both x_1 and x_j for $2 \leq j \leq k$. Notice that no z_{1j} is a neighbor to r, for else $x_1 \to z_{1j}, x_j \to z_{1j}, z_{1j} \to r$ is a solution. Also, z_{1j} contains no pebble for all $2 \leq j \leq k$, for else one can pebble x_1 so that it contains 4 using the pebble move $x_j \to z_{1j} \to x_1$.

In other words: Every x_j for $2 \leq j \leq k$ is adjacent to two vertices y_j and z_{1j} that contains no pebbles. Noticing that the size of the configuration C is $n(G)$ we must compensate for this by saying that every x_i, $1 \leq i \leq k$, contains 3 pebbles.

As before: x_2 can not be adjacent to x_3 (for else $x_3 \to x_2$ gives x_2 four pebbles), so there is a vertex z_{23} which is not a neighbor of r and is adjacent to x_2 and x_3 and distinct from $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k, z_{12}, z_{13}, \ldots, z_{1k}$ and r, also z_{23} is without pebbles and this leaves us with one pebble to add to some x_i, $1 \leq i \leq k$, giving a vertex with 4 pebbles on it (we are pebbling $3k + 1$ pebbles on the set $\{x_1, x_2, \ldots, x_k\}$ of k pebbles). Thus C is r-solvable using $diam(G) = 2$ so that every vertex can be pebbled from x_i for some $i = 1, 2, \ldots, k$. Since r was arbitrary, C is solvable.

□

Given the above lemma we may give an upper bound on the pebbling number for an arbitrary diameter 2 graph:

Theorem 3.13 (Pachter et al., 1995)**.** $\pi(G) = n(G)$ *or* $\pi(G) = n(G) + 1$ *for all graphs G with diameter 2.*

Proof. The lower bound $n(G) \leq \pi(G)$ is given by the vertex bound Lemma 2.1. We will show that $\pi(G) \leq n(G) + 1$, from which the theorem follows.

Consider a configuration C on G of size $n(G) + 1$ where $diam(G) = 2$.

Case 1: G is 1-connected with $diam(G) = 2$. Then G has a cut vertex x. Let r be our target.

If $r = x$, then a placement of $n(G)$ pebbles on the two components A and B of $G - x$ gives one vertex a in $V(A)$ or one vertex b in $V(B)$ with at least two pebbles on it, in any case one may pebble x, because $dist(x, v) = 1$ for all $v \in V(A) \cup V(B)$ (otherwise $dist(a, b) \geq 3$ for some two vertices a in $V(A)$ and b in $V(B)$, for we must always pass x when finding a path from a to b).[42]

If the target vertex r is in $V(A)$ and C is a configuration on G of size $n(G) + 1$, then either (i) $C(x) \geq 2$, (ii) $C(x) = 1$, or (ii) $C(x) = 0$.

For (i) $x \to r$ is a solution.

For (ii) one has a distribution of $n + 1 - 1$ pebbles on $n - 1$ vertices, making at least one $v \in V(A) \cup V(B)$ satisfy $C(v) \geq 2$. $v \to x \to r$ now solves the problem.

For (iii) one has a distribution of $n + 1$ pebbles on $n - 2$ vertices, hence, by Theorem 2.6, two pebbles may be moved to x, and then one to r. In either case, C is solvable.

Case 2: G is 2-connected. Let r denote the target. The result is true for $|V(G)| \leq 5$ (tedious, but not hard calculations, verify this), so assume that $|V(G)| \geq 6$.

Distribute $n(G) + 1$ pebbles on $V(G)$. If there are three vertices with two pebbles or more, then G is solvable by Lemma 3.12. Hence, assume that no more than two vertices contains two or more pebbles. Notice that every neighbor of r contains no more than one pebble.

Case 2a: Two vertices x_1 and x_2 contains two or more pebbles. If there is a neighbor y to r, adjacent to x_1 and x_2, then $x_1 \to y$, $x_2 \to y$, $y \to r$. Hence, assume that x_i is adjacent to y_i for $i = 1, 2$ with $y_1 \neq y_2$. We must have y_1 and y_2 containing no pebbles (else $x_i \to y_i \to r$ is a solution for some $i = 1, 2$). Thus, either x_1 or x_2 must contain three pebbles (since

[42] If A and B are two sets, then $A \cup B$ is the set of elements in A or B or both.

we have a distribution of $n(G)$ pebbles on $N(G) - 3$ vertices, placing at most one pebbles on each vertex except x_1 and x_2, we must place at least 5 pebbles on x_1 and x_2). Suppose x_1 is this vertex. There must be a vertex $z \neq y_1, y_2$ adjacent to x_1 and x_2, for $diam(G) = 2$, but z cannot contain any pebble since then one can pebble x_1 so that it contains four pebbles, solving the problem. Hence, we have four vertices v, y_1, y_2 and z which contains no pebbles. Thus after distributing at most $n - 4$ pebbles on all vertices in $V(G)$, one needs to place 5 pebbles on x_1 and x_2 which contain one pebble each, so one of x_1 and x_2 must contain four pebbles, this solves the problem.

Case 2b: Only one vertex x contains two pebbles or more. Note that x is adjacent to some neighbor y of r, for else x is adjacent to r and $x \to r$ is a solution, or $dist(x, r) \geq 3$ which contradicts that $diam(G) = 2$. Furthermore, the vertex y contains no pebbles, for else $x \to y \to r$ solves the problem. Hence, one has a distribution of $n(G) + 1$ pebbles on $n(G) - 2$ vertices, where only x may contain more than one pebble, hence one must place at least 4 pebbles at x, and at most $n(G) - 3$ pebbles at all remaining vertices. Since the diameter of G is 2 we can move a pebble to r from x using the fact that x contains 4 pebbles.

Case 2c: No vertex x contains more than one pebble. This is not possible since we have a distribution of $n(G) + 1$ pebbles on a set of $n(G) + 1$ vertices, so at least one vertex must contain at least two pebbles by the pigeonhole principle.

In any case, C is solvable. The theorem now follows. □

h

3.2.2 Every 3-connected diameter 2 graph is of Class 0

This section aims at showing that every 3-connected diameter 2 graph is of Class 0. We begin by characterizing the bad configurations of size $n(G)$ on 2-connected diameter 2 graphs G (Lemma 3.14). Using this result we describe, in Theorem 3.15 and 3.16, a family of graphs $\mathcal{F}$ which includes every 2-connected diameter 2 graph G which is not of Class 0 (hence, G is of Class 1 by Theorem 3.13 and the fact that G has diameter 2). From this it will follow that a graph must be Class 0 if it is a 3-connected diameter 2 graph.

Begin by recalling Definition 2.8 and the notion of k-connectivity.

Given a configuration C on G, let $S_i = \{v \in G | C(v) = i\}$ and $s_i = |S_i|$. Also, let $A_1 A_2 \cdots A_l$, $2 \leq l \leq k$, denote the set of vertices adjacent to some $a_1 \in A_1, a_2 \in A_2, \ldots, a_{l-1} \in A_{l-1}$ and some $a_l \in A_l$, for any series $A_1, A_2, \ldots, A_k$, $k \geq 2$, of sets. So for example, if A and B are two sets, then AB denotes the set of vertices adjacent to some pair of vertices a in A and b in B.

Recall Definition 2.19 where we defined a bad configuration C on G as a configuration on G for which there is some target r in $V(G)$ such that C is r-unsolvable.

Lemma 3.14 (Clarke et al., 1997)**.** *For all 2-connected diameter 2 graphs G of Class 1 we have $s_2 = 0$ and $s_3 = 2$ for any bad configuration C on G of size $n(G)$.*[43]

Proof. Let C be a bad configuration of size $n(G)$ for the target r in $V(G)$ (that is, C is r-unsolvable).

First, observe the following:

1. Every configuration C_0 derived from C satisfies $C_0(v) < 4$ for all vertices v in $V(G)$.

   Otherwise, C is r-solvable since the diameter of G is 2, and so 4 pebbles at some vertex is enough to pebble any vertex of G.

2. $N(r) \cap (S_2 \cup S_3) = \varnothing$.[44]

   Otherwise there is a vertex v in $N(r) \cap (S_2 \cup S_3)$, and then one can pebble r from v since $C(v) \geq 2$ and v is in the neighborhood of r. This contradicts the fact that C is a bad configuration for r; hence, $N(r) \cap (S_2 \cup S_3) = \varnothing$.

[43] As stated in footnote 4: In structuring the proof of this lemma, the author was inspired by Blasiak (2008).

[44] The expression $\varnothing$ denotes the *empty set*, which is the set having no elements.

3. $S_2S_3\{r\} = S_3S_3\{r\} = S_2S_2\{r\} = \varnothing$.

   If $v \in S_2S_3\{r\}$, then one may move two pebbles to v using some vertex x in S_2 and some vertex y in S_3, and afterwards pebble r from v, contradicting the fact that C is r-unsolvable. Similarly there is no vertex v in $S_3S_3\{r\}$, for else one may move two pebbles to v from two distinct vertices of S_3 and then one pebble to r. Also, $S_2S_2\{r\} = \varnothing$, since otherwise there is a vertex v adjacent to two distinct vertices in S_2, and so one may move two pebbles to v, and then one to r, contradicting that C is r-unsolvable.

4. $S_2S_3S_3 = S_2S_2S_3 = \varnothing$.

   If v is in $S_2S_3S_3$, then we can move two pebbles to v from one vertex u in S_2, and one vertex v_1 in S_3, and then we can from v move one pebble to some other vertex $v_2 \neq v_1$ in S_3 so that v_2 contains 4 pebbles, contradicting (1). Hence, $S_2S_3S_3 = \varnothing$. Similarly, if v is in $S_2S_2S_3$ then one may move two pebbles to v from two distinct vertices in S_2, and then pebble some vertex u in S_3 from v, so that u contains 4 pebbles, contradicting (1) again.

5. $S_3 \cap N(S_2 \cup S_3) = \varnothing$.

   If there is a vertex v in $S_3 \cap N(S_2 \cup S_3)$, then one may move a pebble to v in S_3 from some vertex in $S_2 \cup S_3$ which v is adjacent to (such a vertex exists since v is in $N(S_2 \cup S_3)$ by assumption, otherwise $N(S_2 \cup S_3)$ must be equal to $\varnothing$, and then (5) holds anyway). This would give us a new configuration C_0 with $C_0(v) = 4$ from which one can pebble r; this is a contradiction since C was r-unsolvable by assumption. Hence, the distance between any pair of vertices $v \in S_3$ and $u \in S_2 \cup S_3$ is 2, and thus v can not be adjacent to any vertex in $S_2 \cup S_3$, and so v is not in $N(S_2 \cup S_3)$.

Using the above observations we may show that (a) the sets $\{r\}, \{r\}S_2, \{r\}S_3, S_2S_3$ and S_3S_3 are all disjoint, and that (b) all the sets listed in (a) are subsets of S_0. The arguments are as follows:

(a) The set $\{r\}$ is disjoint from $\{r\}S_2, \{r\}S_3, S_2S_3$ and S_3S_3, otherwise r is adjacent to itself (if $\{r\}$ is not disjoint from $\{r\}S_2$ or $\{r\}S_3$) which is not possible since it has distance 0 to itself, or r is a neighbor to some vertex in $S_2 \cup S_3$, contradicting (2).

The set $\{r\}S_2$ is disjoint from $\{r\}S_3$, since $\{r\}S_2 \cap \{r\}S_3 = \{r\}S_2S_3 = \varnothing$,[45] where the last equality follows by (3). Similarly $\{r\}S_3$ is disjoint from S_2S_3 and S_3S_3 since $\{r\}S_3 \cap S_2S_3 = \{r\}S_2S_3 = \varnothing$ and $\{r\}S_3 \cap S_3S_3 = \{r\}S_3S_3 = \varnothing$, where the last equalities follows from (3).

Also, $S_2S_3 \cap S_3S_3 = S_2S_3S_3 = \varnothing$ from (4), so S_2S_3 and S_3S_3 are disjoint sets.

(b) We can not pebble r, so $\{r\} \subseteq S_0$.

Furthermore, $\{r\}S_2$ and $\{r\}S_3$ are subsets of S_0, since if $v \in \{r\}S_i$, $i = 2, 3$, and $v \notin S_0$, then $C(v) \geq 1$ and so we may pebble v from S_i, $i = 2, 3$, so that v contains at least two pebbles, and then pebble r since v was adjacent to r by assumption. This contradicts the fact that C is r-unsolvable.

Similarly S_iS_3, $i = 2, 3$, are subsets of S_0, for else there is a vertex v with $C(v) \geq 1$ adjacent to some vertex u in S_i, $i = 2, 3$. From this one may pebble v from u so that v contains at least two pebbles, and then one may pebble that vertex x in S_3 which is distinct from u and adjacent to v, so that x contains 4 pebbles. This contradicts (4), and so it follows that S_iS_3, $i = 2, 3$, are subsets of S_0.

Counting the number of elements in each set $\{r\}, \{r\}S_2, \{r\}S_3, S_2S_3$ and S_3S_3, we have

(i) $|\{r\}| = 1$.

[45] If A, B and C are three nonempty sets, then $AB \cap BC = ABC$, since if $v \in AB$ and $v \in BC$ then v is adjacent to some vertex in A and some vertex in B, by being included in AB, and also adjacent to some vertex in B and some vertex in C, by being included in BC; it thus follows that v is adjacent to three vertices, one in A, one in B and one in C, and this means that v is in ABC.

(ii) $|\{r\}S_2| = |S_2|$ and $|\{r\}S_3| = |S_3|$, since r can not be adjacent to vertices in S_2 or S_3 by (2), but since the diameter is 2, each vertex in S_2 or S_3 must be adjacent to some vertex v that is also adjacent to r. Each such vertex v is distinct for each element of S_2 and S_3, otherwise there would be a vertex adjacent to r and two vertices in S_2 or S_3, which contradicts (3).

(iii) $|S_2S_3| = s_2s_3$. This follows by the following argument: By (5), S_2 and S_3 have no common edges, and since additionally the graph is of diameter two, there is a vertex v adjacent between any pair of vertices x in S_2 and y in S_3. Each such vertex v is unique for each x and y, because otherwise there is a vertex v adjacent to either two distinct vertices in S_2 and one in S_3, one vertex in S_2 and two distinct vertices in S_3, or two distinct vertices in S_2 and two distinct vertices in S_3; in any case, by (4), no alternative is possible, hence: each such vertex v is unique. Thus, for each x in S_2 there is a unique vertex v for all vertices in S_3, so $|S_2S_3| = s_2s_3$.

(iv) That $|S_3S_3| = \binom{s_3}{2}$ is shown by replacing S_2 by S_3 in the above argument (iii), and noting that we are here taking subsets of S_3 of size two when we determine $|S_3S_3|$.[46]

Thus, from (a) and (b), it follows that $\{r\}, \{r\}S_2, \{r\}S_3, S_2S_3$ and S_3S_3 are disjoint subsets of S_3, where the size of each set is determined by (i)–(iv). Hence,[47]

$$1 + s_2 + s_3 + s_2s_3 + \binom{s_3}{2} \leq s_0 = s_2 + 2s_3 \tag{3.6}$$

The equality in (3.6) follows from three facts: $|S_2 \cup S_3| = s_2 + s_3$, $C(S_2) = 2s_2$ and $C(S_3) = 3s_3$. So when distributing $n(G)$ pebbles on the vertices of G we may place 1 pebble at each vertex. Hence, when one has placed one

[46] $\binom{n}{k}$ is called the *binomial coefficient* of x^k in $(1+x)^n$, and is equal to $\frac{n!}{k!(n-k)!}$, where $n! = n(n-1)\cdots 1$, $n \geq 0$ is an integer, is the *factorial* of n.

[47] If A_i are disjoint subsets of B for $1 \leq i \leq n$, that is $A_i \cap A_j = \varnothing$ and $A_i \subseteq B$ for $1 \leq i, j \leq n$, $i \neq j$, then $|A_1 \cup A_2 \cdots \cup A_n| = \sum_{i=0}^{n} |A_i| \leq |B|$.

pebble at each vertex in S_2 and S_3 one has $2s_2 + 3s_3 - |S_2 \cup S_3| = s_2 + 2s_3$ pebbles left, and this number must be equal to s_0 since these are the remaining pebbles not yet placed on the vertices of S_0.

From (3.6) we have

$$s_3^2 - (3 - 2s_2)s_3 + 2 \leq 0 \Rightarrow {}^{48}3 \geq 2s_2 \Leftrightarrow s_2 \leq 1. \tag{3.7}$$

If $s_2 = 1$ then (3.7) translates into $s_3^2 - s_3 + 2 = (s_3 - \frac{1}{2})^2 + \frac{7}{4} \leq 0$, a contradiction since the left expression is greater than or equal to $\frac{7}{4}$, and hence greater than 0. So $s_2 = 0$.

$s_2 = 0$, with (3.7), gives us $s_3^2 - 3s_3 + 2 = (s_3 - 1)(s_3 - 2) \leq 0$, hence $s_3 = 1$ or $s_3 = 2$.[49] If $s_3 = 1$, then $s_0 = 2$, and $S_0 = \{r, v\}$. If there is a path from a vertex in S_3 to r, then it must pass v, for else there some x in $N(S_3)$ with $C(x) \geq 1$ and distance 1 to r, a contradiction since $s \to x \to r$ for some vertex s in $V(S_3)$ would then be a solution. Hence, all paths from the vertices in S_3 must go through v, making v a cut vertex. This implies that G is not 2-connected, a contradiction. So $s_3 = 2$. □

Now, define $\mathcal{F}$ as the family of 2-connected diameter 2 graphs of Class 1. The smallest graph in $\mathcal{F}$ is the cycle $C_6 = rapcqbr$ (in this order), in which the set $\{a, b, c\}$ induces at least two edges. Given $G \in \mathcal{F}$ and a graph H_p (respectively H_q) we can add the vertices of H_p (respectively H_q) to $V(G)$, including $E(H_p)$ (or $E(H_q)$), to obtain a new graph in $\mathcal{F}$, provided that every component of H_p (respectively H_q) has at least one vertex adjacent to p (respectively q) and that each vertex in H_p (respectively H_q) is adjacent to both a and c respectively b and c) an to not other vertex in $V(G)$. One may also obtain a new graph in $\mathcal{F}$ for any $G \in \mathcal{F}$ by adding $V(H_c)$ to $V(G)$ for graphs H_c for which each vertex of H_c is adjacent to c, to either a or b (or both), and to no other vertex in $V(G)$. We may do the same for graphs H_r (that is add $V(H_r)$ to $V(G)$ to obtain a new graph in $\mathcal{F}$) whenever each vertex of H_r is adjacent to both a and b, and to no other vertex of G, except possibly r.

[48]If $3 < 2s_2$ then $-(3 - 2s_2) > 0$ so $s_3^2 - (3 - 2s_2)s_3 + 2 > 0$, a contradiction.

[49]Both $s_3 = 0$ and $s_3 \geq 3$ gives $m \leq 0$, $m \geq 2$, which is false.

See Figure 17 for an illustration of graphs in $\mathcal{F}$: In Figure 17 a solid line between a graph H and a vertex x means that x is adjacent to all vertices in H. Moreover, at least two of the three dotted dotted edges between a, b and c exist, and the two arrows from H_c to a and b indicates that all vertices in H_c must be adjacent to a or b or both. Squiggly lines from p (respectively q) to H_p (respectively H_q) indicates that every component in H_p (respectively H_q) has at least one vertex adjacent to p (respectively q). Finally, the dotted arrow from H_r to r indicates that possibly, but not necessarily, there is a vertex in H_r which is adjacent to r.

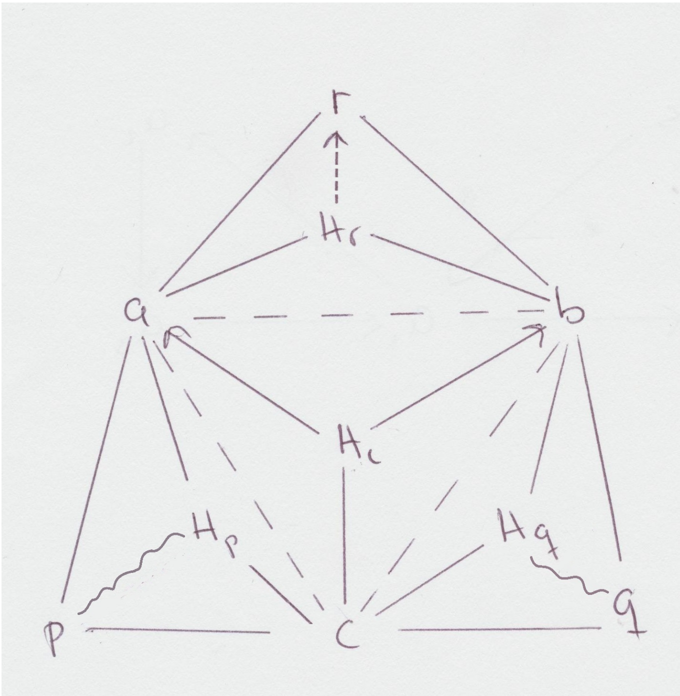

Figure 17: A graph in $\mathcal{F}$.

Before reading the proof of Theorem 3.15 below, recall Definition 2.7, the definition of the vertex-connectivity $\kappa(G)$ of a graph G.[50]

[50]The theorems that follow will be a presentation, and an explication, of the proofs found in (Clarke et al., 1997). However, at the same day as the deadline for this thesis, I, the author of this thesis, noticed, while reading "Graph Pebbling" (Blasiak, 2008), that the characterization of $\mathcal{F}$ as the family of 2-connected diameter 2 graphs of Class 1 is probably faulty. I think this since Blasiak (2008: p. 33) argued that there is a graph G^*

Theorem 3.15 (Clarke et al., 1997). *If $G \in \mathcal{F}$, then $diam(G) = 2$, $\kappa(G) = 2$, and G is of Class 1.*

Proof. One may consult Figure 17 to notice that $diam(G) = 2$ and $\kappa(G) = 2$. Thus, by Theorem 3.13, G is either of Class 0 or Class 1. The configuration C with $C(v) = 0$ for $v \in \{p, q, c, r\}$, $C(v) = 3$ for $v \in \{p, q\}$ and $C(v) = 1$ for all other vertices $v \in V(G)$, is r-unsolvable of size $0 \cdot 4 + 3 \cdot 2 + 1 \cdot (n(G) - 6) = n(G)$. Hence, G must be of Class 1. □

Theorem 3.16 (Clarke et al., 1997). *If $diam(G) = 2$, $\kappa(G) \geq 2$ and G is of Class 1, then $G \in \mathcal{F}$.*

Proof. Suppose $diam(G) = 2$, $\kappa(G) \geq 2$ and that C is a bad configuration on G of size $n(G)$. Using the results of Lemma 3.14 we may define $S_3 = \{p, q\}$, $\{r\}\{p\} = \{a\}$, $\{r\}\{q\} = \{b\}$ and $S_3S_3 = \{c\}$, so that $S_0 = \{a, b, c, r\}$. For all configurations C' derived from C we have $C'(r) = 0$ and $C'(v) < 4$ for all $v \in V(G)$ (else C is r-solvable because $diam(G) = 2$).

Suppose $\{a, b, c\}$ induce at most one edge, we will show that this supposition leads to a contradiction by showing that C is r-solvable in this case, it then follows that $\{a, b, c\}$ must induce at least two edges, and thus G is a graph in $\mathcal{F}$.

If c is neither adjacent to a nor b, then, because $dist(c, r) = 2$, there exists a $v \notin S_0 \cup S_3$ ($v \notin S_0$ since $v \neq a, b, r$ and $v \notin S_3$ since it is adjacent

which is not in $\mathcal{F}$ but which is a 2-connected diameter 2 graph of Class 1. After presenting this counterexample, Blasiak (2008) made a slight modification to the characterization of $\mathcal{F}$ which, she argues, correctly characterizes the family of 2-connected diameter 2 graphs of Class 1, and then she proved similar theorems for this family of graphs. I have not yet checked Blasiak's (2008) argument in detail, but I think it is correct. Theorem 3.16 is, then, probably false. However, as Blasiak (2008) herself notes, the proofs of the theorems with the family $\mathcal{F}$ of graphs which we here present are similar to the proofs of the theorems with Blasiak's (2008) slightly modified characterization of $\mathcal{F}$. Because of this, the reader is encouraged to first read the proofs of Theorem 3.15 to 3.18 in this thesis, and then read the paper by Blasiak (2008) and try to unravel what goes wrong in the proof of Theorem 3.16 presented in this thesis (something in the proof must be wrong if Blasiak's argument (2008), that there exists a 2-connected diameter 2 graph G^* of Class 1 which is not in $\mathcal{F}$, is true). If I had more time, I would try to clear up this problem. This footnote, then, is here to make the reader aware, and to indicate that I am aware, of this problem which I, because of time limitation, do not have time to remedy in this thesis.

to r), hence $C(v) \geq 1$, which is adjacent to c and r. But then one may move two pebbles to c from p and q, and then to r from c through v.

If b is neither adjacent to c or a, then there is a common neighbor $u \notin S_0 \cup S_3$ of p and b, since $dist(p, b) = 2$, that can be used to move a pebble from p to b through u, then moving a pebble from q to b gives a solution since b is adjacent to r.

The case where we assume that a is neither adjacent to b nor c, is symmetric with respect to the case where b is neither adjacent to neither a nor c.

Thus, all above cases leads to the contradiction that C is r-solvable, hence our assumption that $\{a, b, c\}$ induce at most one edge must be false. It must then hold that at least two edges are induced by $\{a, b, c\}$, and thus $G \in \mathcal{F}$. □

Theorem 3.17 (Clarke et al., 1997)**.** *For every diameter 2 graph G: G is of Class 0 iff* $\kappa(G) \geq 2$ *and* $G \notin \mathcal{F}$*.*

Proof. If $diam(G) = 2$ and G is of Class 0, then $G \notin \mathcal{F}$. Otherwise G would be of Class 1 by Theorem 3.16, a contradiction. The vertex-connectivity $\kappa(G)$ of G can not be equal to 1, if it was G would not be of Class 0 by the cut bound, Theorem 2.6.

If $diam(G) = 2$, $G \notin \mathcal{F}$ and $\kappa(G) \geq 2$, then G can not be of Class 1, since in this case $G \in \mathcal{F}$ by Theorem 3.16, a contradiction. □

Theorem 3.18 (Clarke et al., 1997)**.** *If* $diam(G) = 2$ *and* $\kappa(G) \geq 3$*, then G is of Class 0.*

Proof. $G \notin \mathcal{F}$ by Theorem 3.15 since $\kappa(G) \neq 2$, thus the right side of the equivalence in Theorem 3.17 is satisfied, so it follows that G is of Class 0. □

Theorem 3.18 gives us an alternative proof to Theorem 10 that the pebbling number of Petersen's graph is 10, since one may show that Petersen's graph in Figure 15 has diameter 2 and connectivity 3, thus by Theorem 3.18 it follows that $\pi(P) = n(P) = 10$.

A corollary to Theorem 3.18 is

Corollary 3.18.1. *Almost all graphs are Class 0.*

This follows because almost all graphs are Class 0, since almost all graphs have diameter 2, and at least connectivity 3, in the probabilistic sense (Hurlbert, 1999).

4 Thresholds

In this section a probabilistic model for graph pebbling is introduced. The presentation is mostly based on "On pebbling threshold functions for graph sequence" (Czygrinow et al., 2003)[9], especially the presentation of the proofs of the theorems in section 4.2.2 and 4.3.

In the probabilistic model of graph pebbling one randomly and uniformly selects an element from $\mathcal{G}_{n,t}$, the set of all configurations C on G_n of size $t = t(n)$. Since every pebble is the other alike,[51] a distribution of t pebbles on a set of n vertices is like a distribution of t *unlabeled* balls into n *labeled* urns, and the number of ways to put t unlabeled balls into n labeled urns is equal to $\binom{n+t-1}{t}$.[52] Thus, for fixed n and t, $|\mathcal{G}_{n,t}| = \binom{n+t-1}{t}$ is the number of configurations of size t on n vertices.

In section 4.1 we outline the relevant definitions and notations for working with the probabilistic model for graph pebbling. Afterwards we give several results relating to this subject.

Section 4.2 begins with a presentation of the existence theorem of a central mathematical object for the probabilistic graph pebbling model (4.2.1): a *threshold*. A threshold is a function which takes a sequence of graphs as input, and gives a set of functions $\mathbb{N} \to \mathbb{N}$ called *threshold functions* as its output. The existence theorem ensures us that every graph sequence has a threshold.

The main results in section 4.2.2 is the general characterization of the threshold functions for thresholds of graph sequences (Theorem 4.1), and the relation between the threshold of a graph sequence $\mathscr{H}$ and the threshold of a graph sequence $\mathscr{G}$, where the edge set of each graph in the se-

[51] The fact that the pebbles are indistinguishable follows from Definition 2.14, which tells us that if C is a configuration on G, and v a vertex of G, then $C(v)$ is a nonnegative integer. Hence, $C(v)$ just counts the number of pebbles on v without distinguishing them from each other; in other words, they are indistinguishable. If one could distinguish the pebbles, $C(v)$ would have to be an element in $\mathbb{Z}^n$ for $n \geq 2$, and then $C(v) = (1,2)$ in $\mathbb{Z}^2$, for example, would express the fact that v contains 1 pebble of sort 1, and 2 pebbles of sort 2. If the pebbles are distinct, the process of pebbling may be called *Maxwell Boltzmann* pebbling. See (Godbole et al., 2005)[10] for an extended discussion of *Maxwell Boltzmann* pebbling.

[52] See (Grimaldi, 2014: 24–5)[11] for an argument for this fact.

quence $\mathscr{H}$ of n vertices is a subset of the graph with n vertices in the graph sequence $\mathscr{G}$ (Theorem 4.5). The ambition is that section 4.2.1 simplifies some of the inferences made in section 4.2.2 and 4.3.

Section 4.2.3 gives elementary results regarding the probabilistic model. In particular we determine the threshold for the sequence of complete graphs (Theorem 4.6), and relate the pebbling number of the graphs in a given graph sequence $\mathscr{G}$ to the threshold of $\mathscr{G}$ (Theorem 4.7).

Section 4.2 is less technical than section 4.3.

Section 4.3 continues the characterization of thresholds for graph sequences: Section 4.3.1 gives a general bound for thresholds (Theorem 4.8), section 4.3 more narrowly specifies bounds for the sequence of paths and cycles (Theorem 4.11 and 4.12), and section 4.3.2 derives the threshold for the sequence of stars and wheels (Theorem 4.13 and Corollary 4.13.1).

The goal of section 4 is to show how probability theory can be applied in graph pebbling modeling, and to give an overview of relevant lemmas, theorems and their corollaries, for the probabilistic model.

Some of the most important results in this section are shown in Table 2 in Appendix B.

4.1 Definitions and notation

We begin our analysis of the probabilistic model of graph pebbling by presenting the definitions and notation which underpins it:

Definition 4.1 (Asymptotic notation)**.** *Let f and g be two functions $\mathbb{N} \to \mathbb{N}$, we say that $f \ll g$, or $g \gg f$, is equivalent to* $\lim_{n\to\infty} f(n)/g(n) = 0$.

The set $o(g)$ denotes $\{f|f \ll g\}$, and the set $\omega(f)$ denotes $\{g|f \ll g\}$.

Moreover, $f \in O(g)$ (or $g \in \Omega(f)$) holds whenever there exists constants $c, k > 0$ such that $f(n)/g(n) < c$ for all $n > k$.

Let $\Theta(g)$ be defined as $O(g) \cap \Omega(g)$. By the inequality $\Theta(f) \leq \Theta(g)$ we mean that $f_0 \in O(g_0)$ for every pair of functions $f_0 \in \Theta(f)$ and $g_0 \in \Theta(g)$.

Finally, the relation $f \lesssim g$ is equivalent to $\lim_{n\to\infty} \sup(f(n)/g(n)) \leq 1$.

Notice that by the above definition $f \in o(g)$ iff $g \in \omega(f)$.

Denote by G_n a graph on n vertices, let C be a configuration on G_n, and define $t = t(n) = |C|$ for every n. The pebbling number $\pi(G_n)$ of G_n is the least integer $t(n)$ such that G_n is solvable.

Let $P_{\mathscr{G}}(n, t(n))$ denote the probability that an element in $\mathcal{G}_{n,t}$, the set of all configurations C on G_n of size $t(n) = |C|$, chosen uniformly at random[53] is solvable.[54]

A *graph sequence* $\mathscr{G}$ is a sequence of graphs $(G_1, G_2, \ldots G_m, ...)$ where $V(G_m) = \{v_i | i = 1, 2, \ldots, n_m\}$. Often n_m is taken to be equal to m. In other cases it is taken to be equal to 2^m, as in the sequence of cubes $\mathscr{Q}$.

Definition 4.2 (Threshold function)**.** *For every graph sequence $\mathscr{G} = (G_1, G_2, \ldots, G_m, \ldots)$, g is called a threshold function for $\mathscr{G}$, equivalently g is in the threshold $th(\mathscr{G})$ of $\mathscr{G}$, iff as $m \to \infty$:*

(i) $P_{\mathscr{G}}(n_m, t(m)) \to 1$ whenever $t \gg g$, and

(ii) $P_{\mathscr{G}}(n_m, t(m)) \to 0$ whenever $t \ll g$.

If $n_m = m$, then Definition 4.2 reduces to the fact that $g \in th(\mathscr{G})$ iff as $m \to \infty$:

(i) $P_{\mathscr{G}}(m, t(m)) \to 1$ whenever $t \gg g$, and

(ii) $P_{\mathscr{G}}(m, t(m)) \to 0$ whenever $t \ll g$.

4.2 Groundwork

Before moving to the relatively technical section 4.3 we do the groundwork of an analysis of thresholds: Firstly, in section 4.2.1, we present an existence theorem for thresholds. Secondly, in section 4.2.2, we relate

[53]This concept, *uniformly randomly chosen*, may be informally described as follows: If Ω is a set of outcomes, then in a random experiment we say that we are uniformly randomly choosing an outcome in Ω iff every outcome in Ω has the same probability of being chosen. That is, every outcome is equally likely to happen. One may study this concept in (Alm & Britton, 2008: 14, 73)[1]. In our special case, each configuration in $\mathcal{G}_{n,t}$ is equally likely of being chosen.

[54]An introduction to probability theory, and random variables as discussed in Theorem 4.10 for example, may be found in (Ross, 2010: 1–96)[22].

the threshold to the big-Theta function Θ from Definition 4.1, and provide some preliminary results regarding generic graph sequences. Thirdly, in section 4.2.3, we calculate the threshold for the sequence of complete graphs and relate the threshold of some arbitrary graph sequence $\mathscr{G} = (G_1, G_2, \ldots)$ to the pebbling number of each graph G_n, $n \geq 1$, in $\mathscr{G}$.

4.2.1 Existence of thresholds

It is not obvious that every graph sequence has a nonempty threshold. For this reason we will assume the following:

Theorem 4.1 (Bekmetjev et al., 2003 [4]). *Every graph sequence $\mathscr{G}$ has a nonempty threshold $th(\mathscr{G})$.*

The proof of Theorem 4.1 is technical and several pages long in (Bekmetjev et al., 2003). A sketch of the proof is provided by Hurlbert (2014)[16]. Hurlbert (2014) mimics an earlier result by Bollobás and Thomason (1987)[6] which says that "monotone graph properties (...) have random graph thresholds". Without investigating the results by Bollobás and Thomason (1987), we will briefly describe Hurlbert's (2014) sketch of a proof of Theorem 4.1.

Essentially, Hurlbert (2014) utilizes the concept of *multisets*[55]. To further give the reader an idea of the proof we define $\mathscr{M}_n$ as the set of all multisets of the set $[n] = \{v_1, v_2, \ldots, v_n\}$ of n distinct vertices v_i, $1 \leq i \leq n$, and $\mathscr{M}_n(t)$ as the set of multisets in $\mathscr{M}_n$ of size t. Hurlbert (2014) directs our attention to $\mathscr{M}_n(t)$ since every "configuration of size t is simply a t-multiset [a multiset of size t (the sum of the multiplicities of each element in the multiset)] of n vertices"[56] that is: $\mathscr{M}_n$ is the set of all possible config-

[55] A multiset generalizes the concept of a set. It is a set-like object where order is ignored, and it allows multiple instances of its elements. For example $\{a, b\}$ is a multiset where a and b has multiplicity one; and $\{a, a, b\}$ is a multiset where a has multiplicity two, and b has multiplicity one. Here $\{a, a, b\}$ is equal to $\{b, a, a\}$ since order is ignored.

[56] To exemplify this statement by Hurlbert (2014) let G be a graph on 5 vertices $\{a, b, c, d, e, f\}$. Then the multiset $\{a, a, a, b, b, c, c, d, f, f, f, f\}$ describes a configuration on the vertices of G, where there are 3 pebbles on a, 2 pebbles on b, 2 pebbles on c, 1 pebble on e, and 4 pebbles on f. This is a configuration on G of size $3 + 2 + 2 + 1 + 4 = 12$.

urations on $[n]$. Furthermore, we say that $\mathscr{F}_n \subseteq \mathscr{M}_n$ is a *family* of $\mathscr{M}_n$. We may take $\mathscr{F}_n$ to be the family of all solvable configurations on $[n]$.

The proof sketch by Hurlbert (2014) may now be explained as the execution of the following two tasks: (1) finding the probability $P_t(\mathscr{F}_n)$ that a randomly chosen element of $\mathscr{M}_n(t)$ is in $\mathscr{F}_n$, and (2) showing that $t^*(n) = \min\{h|P_h(\mathscr{F}_n) \geq 1/2\}$ is in $th(\mathscr{F})$, where $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_n, \ldots)$ is a sequence of families. Completing task (1) we explicitly find the value of $P_t(\mathscr{F}_n)$ for arbitrary t and n,[57] and completing task (2) we find a threshold function for $\mathscr{F}$, and thus a threshold function for any series of sets $\mathcal{G}_{n,t}$, $n = 1, 2, \ldots$, of configurations C on G_n of size $t(n) = |C|$.

4.2.2 Preliminary results

In this section we relate the threshold to the big-Theta function Θ (Theorem 4.2), and present some preliminary results regarding generic graph sequences $\mathscr{G} = (G_1, G_2, \ldots, G_n, \ldots)$ and $\mathscr{H} = (H_1, H_2, \ldots, H_n, \ldots)$.

Theorem 4.2. *$th(\mathscr{G}) = \Theta(g)$ iff $g \in th(\mathscr{G})$.*

Proof. If $th(\mathscr{G}) = \Theta(g)$, then $g \in th(\mathscr{G})$ since $g \in \Theta(g)$ because $g(n)/g(n) = 1 < 2$ for all $n \geq 1$.

If $g \in th(\mathscr{G})$, then if $f \in th(\mathscr{G})$ we must have $f(n)/g(n) < c$ for all $n > k$ and some k (that is $f \in O(g)$, for else $f(n)/g(n) \to \infty$ as $n \to \infty$, so $f \gg g$. We may now pick t so that $f \gg t \gg g$, in this case $P_{\mathscr{G}}(n,t) \to 1$ since $t \gg g$ and $g \in th(\mathscr{G})$ and $P_{\mathscr{G}}(n,t) \to 0$ since $f \gg t$ and $f \in th(\mathscr{G})$ – a contradiction. Thus $f \in O(g)$. Similarly $f \in \Omega(g)$ (for else $g \gg f$ and a similar argument applies).

The theorem now follows. □

In the following, we assume that $V(H_n) = V(G_n)$ for any $n \geq 1$.

Theorem 4.3. *If $E(H_n) \subseteq E(G_n)$, then $P_{\mathscr{G}}(n,t) \geq P_{\mathscr{H}}(n,t)$.*

[57] In fact, Hurlbert (2014) calculates $P_t(\mathscr{F}_n)$ to $|\mathscr{F}_n(t)|/\binom{n+t-1}{n}$. This is understandable since $|\mathscr{F}_n(t)$ is the number of solvable configurations of size t on $[n]$, and $\binom{n+t-1}{n}$ is the number of possible configurations of size t on $[n]$; thus, dividing the former by the latter, we get the probability that a randomly chosen element in $\mathscr{M}_n(t)$ of all possible configurations of size t on $[n]$ is solvable, this follows by classic probability theory.

Proof. Notice that $V(H_n) = V(G_n) = n$. Since $E(H_n) \subseteq E(G_n)$ and the edge set of the two graphs coincide, every H_n-solvable distribution is G_n-solvable, because for every initial distribution of t pebbles on H_n every possible pebbling move in H_n is by implication possible in G_n. □

Lemma 4.4. *If $P_{\mathscr{G}}(n,t) \geq P_{\mathscr{H}}(n,t)$ for all n, then $th(\mathscr{G}) \leq th(\mathscr{H})$.*

Proof. Let $P_{\mathscr{G}}(n,t) \geq P_{\mathscr{H}}(n,t)$ for all n.

We need to show that $th(\mathscr{G}) \leq th(\mathscr{H})$. This is equivalent to $\Theta(g) \leq \Theta(h)$ for $g \in th(\mathscr{G})$ and $h \in th(\mathscr{H})$. By Definition 4.1 we need to show that whenever $g_0 \in \Theta(g) = th(\mathscr{G})$ and $h_0 \in \Theta(h) = th(\mathscr{H})$ (where the equalities follows by Theorem 4.2), it follows that $g_0 \in O(h_0)$.

Suppose $g_0 \notin O(h_0)$. Then $g_0/h_0 \to \infty$ as $n \to \infty$. Hence $g_0 \gg h_0$. Choose t so that $g_0 \gg t \gg h_0$. Since $g_0 \in th(\mathscr{G})$ and $h_0 \in th(\mathscr{H})$ we have $P_{\mathscr{G}}(n,t) \to 0$ ($g_0 \gg t$) and $P_{\mathscr{H}}(n,t) \to 1$ ($t \gg h_0$). So the inequality $P_{\mathscr{G}}(n,t) \geq P_{\mathscr{H}}(n,t)$ gives us $0 \geq 1$ in the limit. A contradiction. By reductio ad absurdum $g_0 \in O(h_0)$, and this ends the proof. □

Theorem 4.5. *If $E(H_n) \subseteq E(G_n)$ for all n, then $th(\mathscr{G}) \leq th(\mathscr{H})$.*

Proof. The theorem follows by Theorem 4.3 and Lemma 4.4. □

4.2.3 Elementary results

Now we will explicitly calculate the threshold for the sequence of complete graphs (Theorem 4.6), and relate the threshold of a graph sequence $\mathscr{G}$ to the pebbling number of each graph in $\mathscr{G}$ (Theorem 4.7).

Let $\mathscr{K} = (K_1, K_2, \ldots, K_n, \ldots)$ denote the sequence of complete graphs where K_n is a complete graph on n vertices. The first five graphs in this sequence may be seen in Figure 18.

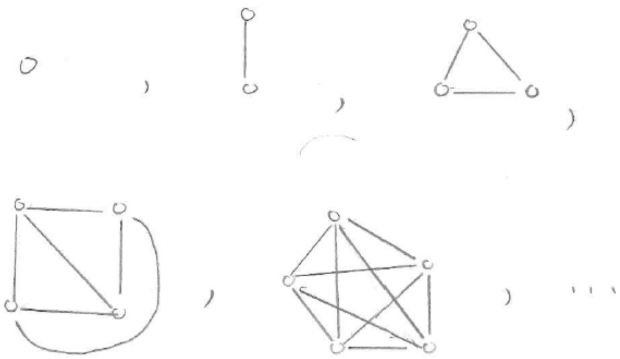

Figure 18: The first five graphs K_1, K_2, K_3, K_4, K_5 (in that order starting from the upper left corner) in the sequence $\mathscr{K} = (K_1, K_2, \ldots, K_n, \ldots)$ of complete graphs.

Theorem 4.6 (Czygrinow et al., 2002)**.** $th(\mathscr{K}) = \Theta(n^{1/2})$.

Proof. The number of configurations of size t on n vertices is equal to the number of ways to put t unlabeled balls into n labeled urns, and this number is equal to $\binom{n+t-1}{t}$.[58]

Moreover, if C is a configuration on K_n of size t, then for the set $B = \{\text{Configurations } C \text{ on } K_n |\ C \text{ is not solvable}\}$ we have $|B| = \binom{n}{t}$, since this is the number of distributions where there is at most one pebble at each vertex for $t(n) \leq n-1$, and each such distribution is unsolvable according to the proof of Lemma 2.1 (for $t(n) \geq n$, K_n is solvable by Corollary 2.5.1).

It follows that

$$P(B) = \frac{\binom{n}{t}}{\binom{n+t-1}{t}}.$$

Since

$$\frac{\binom{n}{t}}{\binom{n+t-1}{t}} = \frac{\frac{n!}{(n-t)!t!}}{\frac{(n+t-1)!}{(n-1)!t!}} = \frac{\frac{n!}{(n-t)!}}{\frac{(n+t-1)!}{(n-1)!}} = \frac{n(n-1)\cdots(n-t+1)}{(n+t-1)(n+t-2)\cdots(n-2)}$$

[58]See the second paragraph in section 4 for an extended reasoning of this fact.

we have

$$
\begin{aligned}
P(B) &= \frac{n(n-1)\cdots(n-t+1)}{(n+t-1)(n+t-2)\cdots(n-2)} = \left(\frac{(n+t-1)(n+t-2)\cdots(n-2)}{n(n-1)\cdots(n-t+1)}\right)^{-1} \\
&\leq \left(\frac{n+t-1}{n}\right)^{-t} = (1+t/n-1/n)^{-t} \leq (e^{t/n}-1/n)^{-t} \approx (e^{t/n})^{-t} \\
&= e^{-t^2/n},
\end{aligned}
$$

where we have used the series expansion $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, and where $1/n \approx 0$ for large n.[59]

Now,

$$
\lim_{n\to\infty} \frac{-t^2}{n} = \begin{cases} -\infty & \text{if } t \gg n^{1/2} \text{ (thus } t^2 \gg n), \\ 0 & \text{if } t \ll n^{1/2}. \end{cases}
$$

Hence,

$$
\lim_{n\to\infty} P(B) = \lim_{n\to\infty} e^{-t^2/n} = \begin{cases} 0 & \text{if } t \gg n^{1/2}, \\ 1 & \text{if } t \ll n^{1/2}, \end{cases}
$$

so that

$$
\lim_{n\to\infty} P_{\mathscr{K}}(n,t) = \lim_{n\to\infty} (1-P(B)) = \begin{cases} 1 & \text{if } t \gg n^{1/2}, \\ 0 & \text{if } t \ll n^{1/2}. \end{cases}
$$

The theorem follows by Definition 4.2.

□

Theorem 4.7. *For any graph sequence $\mathscr{G}$ and function $g(n) = \pi(G_n)$ we have $th(\mathscr{G}) \subseteq O(g)$.*

Proof. For every $g_0 \in th(\mathscr{G})$ we have to show that there exists constants $c, k > 0$ such that $g_0(n)/g(n) < c$ for all $n > k$.

If $g_0 \notin O(g)$, then $g_0(n)/g(n) \to \infty$ as $n \to \infty$. But then for an distribution on G_n of size $t(n)$ with $g_0(n) \gg t(n) \gg g(n)$ we have $P_{\mathscr{G}}(n,t) \to 1$,

[59]The series expansion of e^x is explained in *Principles of Mathematical Analysis* (Rudin, 1976: 178)[25].

since $t(n) \gg g(n) = \pi(G_n)$, and $P_{\mathscr{G}}(n,t) \to 0$ since $g_0(n) \gg t(n)$ and $g_0 \in th(\mathscr{G})$. □

Define $\mathscr{Q} = (Q_1, Q_2, \ldots, Q_m, \ldots)$ where Q_m the $m-$dimensional cube with 2^m vertices.

Corollary 4.7.1. *$th(\mathscr{Q}) \subseteq O(n(Q_m))$.*

Proof. $g(n_m) = \pi(Q_m) = 2^m = n(Q_m)$ by Theorem 2.11. □

Corollary 4.7.2. *If $diam(G_n) = 2$ for all n, then $th(\mathscr{G}) \subseteq O(n)$.*

Proof. $g(n) = \pi(G_n) \leq n + 1$ by Theorem 3.2. □

Corollary 4.7.3. *Define $d(n) = diam(G_n)$. $th(\mathscr{G}) \subseteq O(2^{d(n)}n)$.*

Proof. $g(n) = \pi(G_n) \leq 2^{d(n)}n$ by Corollary 2.4.1. □

For a special case of Corollary 4.7.3: If $d(n) \leq d$ for all n, then $th(\mathscr{G}) \subseteq O(n)$.

4.3 Thresholds for graph sequences

In this section we derive (i) a bound for the threshold of an arbitrary graph sequence (section 4.3.1), (ii) bounds for the thresholds of the sequence of paths and cycles (4.3.2), and (iii) the thresholds for the sequence of stars and wheels (4.3.3).

Note: To avoid applying the floor or ceiling function all too much, we let all large real constants, such as $1/\epsilon$ when $\epsilon > 0$ is small, be integers; that is, $1/\epsilon$ will denote the smallest integer greater than or equal to $1/\epsilon$.

4.3.1 General bound for thresholds

We begin by deriving a bound for the threshold of an arbitrary graph sequence.

Theorem 4.8 (Czygrinow et al., 2002)**.** *For all graph sequences $\mathscr{G}$: $th(\mathscr{G}) \subseteq o(n^{1+\epsilon})$ for all $\epsilon > 0$*

Proof. Pick an arbitrary constant $\epsilon > 0$. Chose uniformly at random a configuration C on G_n of size $t(n)$. Let $P(n)$ denote the probability that C is solvable.

The theorem follows if we can show that every configuration of size $t(n)$ with $t(n) \in \Omega(n^{1+\epsilon})$ implies that $P(n) \to 1$. Since in this case we have $t(n) \geq cn^{1+\epsilon}$ for some $c > 0$, and then we can not have $g(n) \gg t(n)$ for $g \in th(\mathscr{G})$ since this would give us $P(n) \to 0$ (since $g(n) \gg t(n)$ and $g \in th(\mathscr{G})$), but $P(n) \to 1$ (since $t(n) \in \Omega(n^{1+\epsilon})$). Thus $g \in o(n^{1+\epsilon})$ by contradiction.

Let H_l denote a graph on l vertices. Let T_l denote a spanning tree of H_l. As discussed in Theorem 4.2, every H_l-solvable distribution is T_l-solvable, hence $\pi(H_l) \leq \pi(T_l)$. In "Pebbling graphs" (Moews, 1992)[19] it is shown that $\pi(T_l) \leq \pi(P_l) = 2^{l-1}$, hence $\pi(H_l) < 2^l$. This result will be used below.

Define the following: let $\delta > 0$ be arbitrary, and $t \geq cn^{1+\epsilon}$ for some $c > 0$ and some fixed positive constant n. Let $l = (1+\delta)/\epsilon$ and $k = 2^l$. Consider a graph G_n with n vertices. Let $G_n(v)$ be a connected subgraph of G_n containing l vertices including $v \in V(G_n)$, furthermore let $|C_{G_n(v)}|$ denote the number of pebbles distributed on the vertices of $G_n(v)$.

$G_n(v)$ is called a l-neighborhood of G_n, and it is called k-bounded whenever $|C_{G_n(v)}| < k$. Below we show that there exists no k-bounded l-neighborhood. This implies that every l-neighborhood has at least k pebbles distributed on it. The above observation that $\pi(H_l) < 2^l$ implies that $\pi(G_n(v)) < 2^l$, so every configuration of size k is $G_n(v)$ is solvable.

$$\begin{aligned}
&P(\text{There exists a } k\text{-bounded } l\text{-neihbourhood}) \\
&\leq P(G_n(v) \text{ is } k\text{-bounded for every } v) \\
&= n\sum_{i=0}^{k-1} P(|G_n(v)| = i) = n\sum_{i=0}^{k-1} \frac{\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}}{\binom{n+t-1}{t}}. \qquad (4.1)
\end{aligned}$$

Here the first equality follows from the fact that if $G_n(v)$ is k-bounded, then there are less than k pebbles distributed on it.

$$P(|G_n(v)| = i) = \frac{\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}}{\binom{n+t-1}{t}}$$

holds, since (a) there are $\binom{n+t-1}{t}$ number of ways to distribute t unlabeled balls on n labeled vertices (i.e. the number of possible configurations on $G_n(v)$ of size t); and (b) multiplying $\binom{l+i-1}{i}$ – the number of ways to place i unlabeled pebbles on l labeled vertices (the vertices of $G_n(v)$) – with $\binom{n-l+t-i-1}{t-i}$ – the number of ways to place the remaining $t-i$ unlabeled pebbles on the $(n-l)$ remaining vertices of G_n – we have the number of favorable distributions for the event $\{|G_n(v)| = i\}$. Dividing the number of favorable distributions for the event $\{|G_n(v)| = i\}$ – which is $\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}$ by (b) – by the number of possible configurations on $G_n(v)$ of size t – which is $\binom{n+t-1}{t}$ by (a) – we get the above equation.

Continuing on (4.1) we have

$$\begin{aligned}
n\sum_{i=0}^{k-1} P(|G_n(v)| = i) &= n\sum_{i=0}^{k-1} \frac{\binom{l+i-1}{i}\binom{n-l+t-i-1}{t-i}}{\binom{n+t-1}{t}} \\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\prod_{j=0}^{i-1}\frac{t-j}{n-l+t-1} \\
&\leq \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\binom{n-l+t-1}{t}\left(\frac{t}{n-l+t-1}\right)^i \\
&= \frac{n}{\binom{n+t-1}{t}} \sum_{i=0}^{k-1} \binom{l+i-1}{i}\left(\frac{t}{n-l+t-1}\right)^i\binom{n+t-1}{t}\prod_{j=1}^{l}\frac{n-j}{n+t-j} \\
&\leq n\sum_{i=0}^{k-1}\binom{l+i-1}{i}\left(\frac{n}{t}\right)^l \\
&= C_1 n\left(\frac{n}{t}\right)^l \qquad (4.2)\\
&\leq C_1 c^{-l}(n^{1-\epsilon l}) \\
&= C_2 n^{-\delta}. \qquad (4.3)
\end{aligned}$$

In (4.2) and (4.3), $C_1 = \sum_{i=0}^{k-1}\binom{l+i-1}{i}$ and $C_2 = C_1 c^{-l}$ are constants with respect to n.

Since $\delta > 0$, $C_2 n^{-\delta} \to 0$ as $n \to \infty$. Thus with probability 0 there exists a k-bounded l-neighborhood. Hence, with probability 1 every l-neighborhood contains at least k pebbles. So $P(n) \to 1$ as $n \to \infty$, concluding what we wanted to show. □

4.3.2 Sequence of paths and cycles

After calculating a general bound for the threshold of graph sequences in Theorem 4.8, we continue by finding bounds for the thresholds of the sequence of paths.

Let $\mathscr{P}$ denote the sequence $(P_1, P_2, \ldots, P_n, \ldots)$ where P_n is the path on n vertices. To find $th(\mathscr{P})$ we need an important lemma, and Markov's inequality.

When Czygrinow et al. (2002) proved Theorem 4.11 below, they noticed that Lemma 4.9 below holds, but without proving this fact. Since Theorem 4.11 is dependent on Lemma 4.9, we prove the lemma in this thesis:

Lemma 4.9. *P_n (with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$) is solvable iff $\sum_{i=1}^{n} C(v_i)/2^{i-1} \geq 1$.*

Proof. Label the vertices of P_n as $v_1, v_2, \ldots, v_n$, where v_i is adjacent to v_{i+1} and v_{i-1} for $2 \leq i \leq n-1$, v_1 is adjacent to v_2, and v_{n-1} is adjacent to v_n. Define the weight W of C as $W = \sum_{i=1}^{n} C(v_i)/2^{i-1}$.[60]

If one moves a pebble from v_i to v_{i-1} the weight W, then all terms in W are constant except possibly $\frac{C(v_i)}{2^{i-1}}$ and $\frac{C(v_{i-1})}{2^{i-2}}$. As a result of the pebbling move $v_i \to v_{i-1}$, the sum $\frac{C(v_i)}{2^{i-1}} + \frac{C(v_{i-1})}{2^{i-2}}$ is changed to $\frac{C(v_i)-2}{2^{i-1}} + \frac{C(v_{i-1})+1}{2^{i-2}}$, where we have derived a new configuration C_0 with $C_0(v_i) = \frac{C(v_i)-2}{2^{i-1}}$, $C_0(v_{i-1}) = \frac{C(v_{i-1})+1}{2^{i-2}}$ and $C_0(v) = C(v)$ for all other vertices v.

The weight of C_0 is the same as the weight W of C since $\frac{C(v_i)-2}{2^{i-1}} + \frac{C(v_{i-1})+1}{2^{i-2}} = \frac{C(v_i)}{2^{i-1}} + \frac{C(v_{i-1})}{2^{i-2}}$. Hence, the value of the weight W of a con-

[60]Here $i-1$ denotes the distance form v_i to v_1, and 2^{i-1} is the number of pebbles on need to place at v_i to get to v_1, given that no other pebbles are placed at any other vertex in $V(P_n)$.

figuration on P_n does not change as one moves towards v_1. So if P_n is solvable we must in the end have a configuration C_1 derived from C satisfying $C_1(v_1) = 1$. Thus, $C_1(v_1)/2^{1-1} = 1$, and so the weight of W must be at least equal to 1. Thus, P_n can not be solvable if $W < 1$. In other words: P_n is solvable iff $W \geq 1$. □

To proceed we need to use Markov's inequality. The first part of the proof below is found in (Ross, 2010: 77-78)[22].

Theorem 4.10 (Markov's inequality)**.** *For any nonnegative random variable X and $a > 0$, $P(X \geq a) \leq \frac{E(X)}{a}$.*

Proof. If X is a nonnegative continuous random variable with density f, then for $a > 0$

$$\begin{aligned} E(X) &= \int_0^\infty x f(x) dx \\ &= \int_0^a x f(x) dx + \int_a^\infty x f(x) dx \\ &\geq 0 + \int_a^\infty x f(x) dx \\ &\geq \int_a^\infty a f(x) dx \\ &= a \int_a^\infty f(x) dx \\ &= a P(X \geq a), \end{aligned}$$

thus $P(X \geq a) \leq \frac{E(X)}{a}$.

If X is a nonnegative discrete random variable with probability mass function p, then

$$
\begin{aligned}
E(X) &= \sum_{x:p(x)>0} xp(x) \\
&= \sum_{x:p(x)>0,x<a} xp(x) + \sum_{x:p(x)>0,x\geq a} xp(x) \\
&\geq 0 + \sum_{x:p(x)>0,x\geq a} xp(x) \\
&\geq \sum_{x:p(x)>0,x\geq a} ap(x) \\
&= a \sum_{x:p(x)>0,x\geq a} p(x) \\
&= aP(X \geq a),
\end{aligned}
$$

thus $P(X \geq a) \leq \frac{E(X)}{a}$.[61] □

Using Lemma 4.9 and Markov's inequality, we may find the threshold for $\mathscr{P}$.

Theorem 4.11 (Czygrinow et al., 2002)**.** $th(\mathscr{P}) \subseteq \Omega(n)$.

Proof. Chose a configuration uniformly at random from all configurations on P_n of size $t(n)$. Let $P(n)$ denote the probability that the configuration is solvable.

If $t(n) \ll n$ implies $P(n) \to 0$, then the theorem follows. Since if $f \in th(\mathscr{P})$ but $f \notin \Omega(n)$ we would have $f(n) \ll n$, and we could pick $t(n)$ such that $f(n) \ll t(n) \ll n$ which would imply $P(n) \to 1$ (since $f(n) \ll t(n)$) and $P(n) \to 0$ since $t(n) \ll n$ by the yet to be proven result. So we show that $t(n) \ll n$ implies $P(n) \to 0$.

[61] Here we present another proof of Markov's inequality when X is a nonnegative discrete random variable: Let A be the event $\{X \geq a\}$. Let 1_A be equal to 1 whenever the relation described by A holds, and equal to 0 otherwise. $X \geq a1_A$ in this case. Since if $1_A = 1$ then $X \geq a$, which was true since $1_A = 1$ and thus the event A occurs, and if $1_A = 0$ then $X \geq 0$ which always holds since X is a nonnegative. Taking expectations on both sides of $X \geq a1_A$ we have $E(X) \geq E(a1_A) = aE(1_A) = a(0P(X < a) + 1P(X \geq a))$, so $E(X) \geq aP(X \geq a)$ and thus $P(X \geq a) \leq \frac{E(X)}{a}$.

We show that the probability that $W \geq 1$, and thus by Lemma 4.8 that P_n is solvable, tends to 0 as $n \to \infty$ whenever $t \ll n$.

Define $t(n) = n/\omega$ for some $\omega \to \infty$. Notice that $E(C(v_i)) = t/n$, since it is as likely to put a pebble on v_i as any other vertex v_j, $i \neq j$, and $n \cdot t/n = t =$ the number of pebbles distributed on P_n, thus $E(C(v_i)) = 1/\omega \to 0$ for every $i = 1, 2, \ldots, n$. Hence,[62]

$$
\begin{aligned}
E(W) &= \sum_{i=1}^{n} \frac{E(C(v_i)}{2^{i-1}} \\
&= 1/\omega \sum_{i=1}^{n} \frac{1}{2^{i-1}} \\
&< 2/\omega \to 0.
\end{aligned}
$$

By Markov's inequality $P(W \geq 1) \leq E(W)/1 \to 0$. □

Corollary 4.11.1. *For every $\epsilon > 0$, $th(\mathscr{P}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$.*

Proof. From Theorem 4.8 and 4.11. □

Corollary 4.11.1 says essentially that for every $\epsilon > 0$, every threshold function $t(n)$ of $th(\mathscr{P})$ can not be smaller than n or as large as $n^{1+\epsilon}$.[63]

Now, let $\mathscr{C} = (C_1, C_2, \ldots, C_n, \ldots)$ where C_n is the cycle graph on n vertices.

Theorem 4.12 (Czygrinow et al., 2002). *$th(\mathscr{C}) \subseteq \Omega(n)$.*

Proof. The proof is similar to that of Theorem 4.11. So pick a configuration uniformly at random from the set of all configurations on C_n of size $t(n)$ with $t(n) \ll n$, and set $t(n) = n/\omega$ for some $\omega \to \infty$. Label the vertices as $v_1, v_2, \ldots, v_n$, which denotes P_n with v_1 adjacent to v_n.

Define $W_1 = \sum_{i=1}^{k} C(v_i)/2^{i-1}$ and $W_= \sum_{i=k+1}^{n} C(v_i)/2^{n-i+1}$. By Lemma 4.8, the probability that C_n is solvable is equal to the probability that $W_1 \geq 1$ or $W_2 \geq 1$, since one can either solve C_n by making the path $v_1, v_2, \ldots, v_k$

[62] $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$, so $\sum_{i=0}^{R} \frac{1}{2^i} < 2$ for all integers $R > 0$.

[63] More specifically: there exists c, k such that $n/t(n) < c$ for all $n > k$, and $\lim_{n\to\infty} t(n)/n^{1+\epsilon} = 0$. Confer Definition 4.1.

solvable, which happens iff $W_1 \geq 1$; or one can make it so that 2 pebbles are placed at v_n for the path $v_{k+1}, v_{k+2}, \ldots, v_n$, which happens iff $W_2 \geq 1$.

Now $E(W_1) = \sum_{i=1}^{k} E(C(v_i))/2^{i-1} = 1/\omega \sum_{i=1}^{k} 1/2^{i-1} < 2/\omega \to 0$, and similarly $E(W_2) < \frac{2}{\omega} \to 0$.

Thus the probability that C_n is solvable is equivalent to $P(W \geq 1 \cup W_2 \geq 1) \leq P(W \geq 1) + P(W' \geq 1) \leq E(W)/1 + E(W')/1$, and this tends to 0. □

Corollary 4.12.1. *For every $\epsilon > 0$, $th(\mathscr{C}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$.*

Proof. From Theorem 4.8 and 4.12. □

4.3.3 Sequence of stars and wheels

In this section we derive the threshold for the sequence of stars. From this result the threshold for the sequence of wheels is easily found using Theorem 4.6 and Theorem 4.2.

Let $\mathscr{S}$ denote the sequence of stars $(S_1, S_2, \ldots, S_n, \ldots)$ where S_n is the star on n vertices.

Theorem 4.13 (Czygrinow et al., 2002)**.** $th(\mathscr{S}) = \Theta(n^{1/2})$.

Proof. For each n, choose a configuration uniformly at random from the set of all configurations on S_n of size $t(n)$. Let $P(n)$ be the probability that C is solvable.

Since $E(S_n) \subseteq E(K_n)$, Theorem 4.1 and 4.4 gives us that $t(n) \ll n^{1/2}$ implies $P(n) \to 0$. Now, we show that $t(n) \gg n^{1/2}$ implies that $P(n) \to 1$.

Define $t(n) = \omega n^{1/2}$ for some $\omega \to \infty$. By Theorem 2.5, $\pi(S_n) = n$. So assume that $t(n) < n$ for all n, else $P(n) \to 1$ and we are done.

Define $q = 1 - P(n)$. We will find an upper bound for q and show that this bound approaches 0 as $n \to \infty$.

If C is unsolvable, then one of (i)–(iii) holds:

(i) $C(c) \leq 1$ for the center c (else one may pebble to any vertex of S_n from c);

(ii) there exists at most one vertex $v \in V(S_n)$ with $C(v) \geq 2$ (if there exists two vertices v_1, v_2 for which $C(v_1), C(v_2) \geq 2$ we may pebble so that $C(c) \geq 2$, and then C is solvable, a contradiction), and

(iii) $C(v) < 4$ for all $v \in V(S_n)$ for $diam(S_n) = 2$.

Thus, the number of unsolvable configurations satisfying (i) to (iii) is no more than the number of configurations C having no v with $C(v) > 1$ plus the number of configurations having one, and only one, v with $2 \leq C(v) \leq 3$. This number is

$$\binom{n}{t} + (n-1)\binom{n-2}{t-2} + (n-1)\binom{n-2}{t-3}.$$ [64]

Since there are $\binom{n+t-1}{t}$ possible distributions of size t at the vertices of S_n, we have

$$\begin{aligned} q &= \frac{\binom{n}{t} + (n-1)\binom{n-2}{t-2} + (n-1)\binom{n-2}{t-3}}{\binom{n+t-1}{t}} \\ &= \left(1 + \frac{t(t-1)}{n} + \frac{t(t-1)(t-2)}{n(n-t+1)}\right) \frac{\binom{n}{t}}{\binom{n+t-1}{t}} \\ &< \left(1 + \frac{t^2}{n} + \frac{t^3}{n(n-t)}\right) \left(\frac{n}{n+t}\right)^{t-1} \\ &\overset{\text{(a)}}{\lesssim} \left(1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right) e^{-t(t-1)/(n+t)} \\ &\overset{\text{(b)}}{\lesssim} \left(1 + \omega^2 + \frac{\omega^3 n^{1/2}}{n - \omega n^{1/2}}\right) e^{-\omega^2/2}. \end{aligned} \tag{4.4}$$

[64] For $\binom{n}{t}$ one distributes so that each of the n labeled vertices contains no more than one pebbles. $\binom{n-2}{t-2}$ is the number of ways one can place two pebbles at one specified vertex, then one pebble at each vertex of the $n-2$ remaining vertices using the remaining $t-2$ pebbles; one can do this for any vertex except the center vertex, and so one can do it for $n-1$ vertices; summing gives us $(n-1)\binom{n-2}{t-2}$ number of ways to place two pebbles at one vertex, and one pebble on the remaining vertices. Regarding the expression $(n-1)\binom{n-2}{t-3}$ one does the same as for $(n-1)\binom{n-2}{t-2}$ but begins by placing three pebbles at some vertex $v \neq c$ so that only $t-3$ pebbles remain.

To see that (a) $(\frac{n}{n+t})^{t-1} \lesssim e^{-t(t-1)/(n+t)}$ we show that $\frac{n}{n+t} \lesssim e^{-t/(n+t)}$. Note that $\frac{n}{n+t}(e^{-t/(n+t)})^{-1} = \frac{n}{n+t}e^{t/(n+t)}$. Now, $n/(n+t) \to 0$ as $n \to \infty$ since the fact that $t > 0$ implies that $n+t > n$ and thus $n/(n+t) < 1$. Similarly $t/(n+t) < 1$ since $n > 0$ and $n+t > t$. Thus $\frac{n}{n+t}e^{t/(n+t)} \to 0e^0 = 0$ as $n \to \infty$. This shows that $(\frac{n}{n+t})^{t-1} \lesssim e^{-t(t-1)/(n+t)}$.

To see that (b) $e^{-t(t-1)/(n+t)} \lesssim e^{-\omega^2/2}$ we first remember that $\omega = t/n^{1/2}$, so $\omega^2 = t^2/n$. Hence,

$$\begin{aligned} e^{-t(t-1)/(n+t)}(e^{-\omega^2/2})^{-1} &= e^{-t(t-1)/(n+t)}e^{\omega^2/2} \\ &= e^{-t(t-1)/(n+t)+\omega^2/2} \\ &= e^{-t(t-1)/(n+t)+t^2/2n} \\ &\approx e^{-t^2/(n+t)+t^2/2n} \\ &\approx e^{-t^2/n+t^2/2n} \\ &= e^{-t^2/2n} \\ &= e^{-\omega^2/2} \end{aligned}$$

which goes to 0 as $\omega \to \infty$.

The expression in (4.4) tends to zero for $\omega \to \infty$. To show this, we consider two cases: Either (i) $\omega < n^{1/2} - 1$ or (ii) $\omega \geq n^{1/2} - 1$.

For (i) $n - \omega n^{1/2} > n^{1/2}$, so the right side of the last expression is at most $[1 + \omega^2 + \omega^3]e^{-\omega^2/2}$ which tends to 0 as $\omega \to \infty$.

For (ii) $n - \omega n^{1/2} = n - t \geq 1$ (since $t < n$), so the last expression is at most $[1 + \omega^2 + \omega^3 n^{1/2}]e^{-\omega^2/2} \leq [1 + \omega^2 + \omega^3(\omega + 1)]e^{-\omega^2/2}$ (using $\omega + 1 \geq n^{1/2}$) and this tends to 0 as $\omega \to \infty$.

Hence, $q \to 0$ whenever $t \gg n^{1/2}$. □

Let $\mathscr{W}$ be equal to $(W_1, W_2, \ldots, W_n, \ldots)$ where W_n is the wheel graph on n vertices.

Corollary 4.13.1 (Czygrinow et al., 2002). *$th(\mathscr{W}) = \Theta(n^{1/2})$.*

Proof. Notice that $E(S_n) \subseteq E(W_n) \subseteq E(K_n)$. Since the threshold for both

$\mathscr{S}$ and $\mathscr{K}$ is $\Omega(n^{1/2})$ (recall Theorem 4.6), the same must be true of $\mathscr{W}$ by Theorem 4.2. $\square$

5 About the research on graph pebbling

In section 1 to 4 we were concerned with the foundations of graph pebbling. The main topics have been the pebbling number for the deterministic model (3.1), and the threshold for the probabilistic model (4.3). Let us now discuss some recent research in graph pebbling (5.1), say something about variations and applications in the theory of graph pebbling (5.2), and finally lay forward normative statements on what research in graph pebbling should concentrate on (5.3).

5.1 Recent research

The pebbling number $\pi(G)$ is the minimum number of pebbles such that any configuration C on G is solvable. A configuration C on G is solvable iff we can move at least one pebble to any vertex. This analysis feels limited. Because if we define a *cover* of t vertices in $V(G)$ as a distribution in which there is at least one pebble on t vertices in $V(G)$, and if we, given a configuration C on the set A of vertices, define the fact that A can be *covered* as the fact we may from C derive a new configuration on A such there is a cover of all vertices in A, then we may extend the notion of solvability into the notion of *t-solvability*: The configuration C on G is t-solvable iff every subset A of $V(G)$ of size t may be covered. Let $\pi(G, t)$ denote the minimum number of pebbles such that any configuration C on G is t-solvable.

Another approach is based on pebbling price. Traditionally, when pebbling we had to "pay" one pebble for a pebbling step. But what if we let the price be arbitrary, and not necessarily equal to one? For example we may define the price at every vertex v_i in $V(G)$ as p_i, where $p_i \geq 2$ for $1 \leq i \leq n(G)$. In this way we can define a *price function* $P : V(G) \to \mathbb{Z}_{\geq 2}$ on G. A pebbling step from u to v may then be defined as the function

$S_{u,v} : C(G) \to \mathcal{C}(G)$ such that

$$S_{u,v}(C)(x) = \begin{cases} C(x) - P(x) & \text{if } x = u, \\ C(x) + 1 & \text{if } x = v, \\ C(x) & \text{else.} \end{cases}$$

Notice that $S(C)$ is in $\mathcal{C}(G)$, so $C(u) \geq P(u)$ must hold for a pebbling step to be possible.

Now we may generalize the notion of a pebbling number into the notion of a *pi-pebbling function* $\pi(G, P, t)$. This is the minimum number k such that any configuration C on G of size k with price function P is t-solvable. The expression $\pi(G, 2_P, n(G))$, where $2_P(v) = 2$ for any vertex v in $V(G)$, is called the *cover pebbling number* and is denoted by γ_G.

The pi-pebbling function was introduced by Taylor (2005)[24], and the cover pebbling number was studied by Sjöstrand (2004)[23]. In fact, Sjöstrand studied the cover pebbling number in a more general fashion by defining the notion of a *w-cover*: A w-cover is a distribution of pebbles such that each vertex v has at least $w(v)$ pebbles on it. What we defined as a cover is in Sjöstrand's notation a w-cover where $w(v) = 1$ for all vertices v. Using this remark, we see that we may extend the pi-pebbling function into a more generalized version, call it *the generalized pi-pebbling function* $\pi(G, P, t, w)$, which asks for the minimum number k such that for any configuration C on G of size k we may pebble t vertices v_i, $1 \leq i \leq t$, of $V(G)$ such that each vertex v_i contains at least $w(v_i)$ pebbles, or equivalently that each v_i satisfies $C(v_i) \geq w(v_i)$.[65] The expression $\pi(G, 2_P, n(G), w)$ will be denoted by $\gamma_G(w)$. Thus, the cover pebbling number γ_G is $\gamma_G(w)$ where $w(v) = 1$ for all v in $V(G)$.

Before proceeding to the next section, we must define the concept of a *directed graph*:

Definition 5.1 (Directed graph)**.** *A finite directed graph G is a pair of finite sets $(V(G), E(G))$ such that $E(G)$ is a subset of the set of ordered pairs of $V(G)$.*[66]

[65] Note that $\pi(G, P, t)$ is $\pi(G, P, t, w)$ where $w(v) = 1$ for all $v \in V(G)$.

[66] Consult footnote 6 for a definition of ordered (and unordered) pairs.

Informally, we may think of directed graphs as graphs where we allow one-way edges (a, b) which says that we can only go from vertex a to vertex b, but not from b to a. An example is presented in Figure 19.

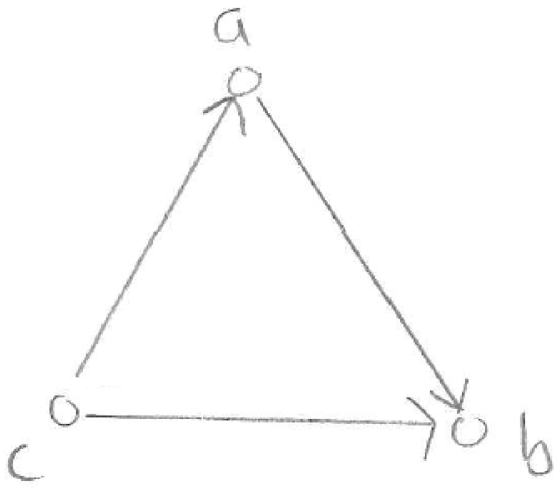

Figure 19: A representation of the directed graph $G = (V, E)$ with vertex set $V = \{a, b, c\}$ and edge set $E = \{(a, b), (c, a), (c, b)\}$.

5.1.1 Cover pebbling numbers

One of the most important theorems regarding cover pebbling numbers, *the cover pebbling theorem*, was proven by Sjöstrand (2004). For our purposes, we formulate a part of this theorem:

Theorem 5.1 (The cover pebbling theorem)**.** *Let G be a directed or undirected connected graph. Let w be a function $V(G) \to \mathbb{Z}_{\geq 1}$. To find $\gamma_G(w)$, it is sufficient to consider simple initial configurations where all pebbles are placed on one single vertex in $V(G)$.*

Moreover, we conjecture that the price function may be arbitrary in Theorem 5.1, since the proof of the cover pebbling theorem in (Sjöstrand, 2004) seems to be manipulable so that one may allow the price at every vertex to be arbitrary. That is, we conjecture that there is a similar theorem for $\pi(G, P, n(G), w)$ for any price function $P \neq 2_P$.

Conjecture 1. *Let G be a directed or undirected connected graph. Let w be a function $V(G) \to \mathbb{Z}_{\geq 1}$. To find $\pi(G, P, n(G), w)$ for any price function P, it is sufficient to consider simple initial configurations on G.*

Reflecting on Theorem 5.1 we may formulate it as follows: If $\mathcal{S}(G)$ is the set of simple initial configurations on G, then the w-cover pebbling number of G is found by finding the minimum number k such that we may from any configuration C in $\mathcal{S}(G)$ of size k obtain a w-cover.

Noting this, we will derive a corollary from Theorem 5.1. In (Sjöstrand, 2004) it is stated that, in the light of Theorem 5.1, it is easy to compute the cover pebbling number for P_n, C_n, Q_n, K_n, the complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$, where $n_1 \geq n_2 \geq \cdots \geq n_k$, and W_n. We will derive these cover pebbling numbers using Corollary 5.1.1, and also derive the cover pebbling number for the fan graph F_n and friendship graph FR_n.

Define the *cost* from u to v as $2^{dist(u,v)}$, and the cost $cost(u)$ of vertices u in $V(G)$ as the sum $\sum_{v \in V(G)} 2^{dist(u,v)}$.

Corollary 5.1.1. *The cover pebbling number γ_G of G is* $\max_{u \in V(G)} cost(u)$.

Proof. Consider a simple initial configuration on G of size k. By the definition of a simple initial configuration this configuration distributes all k pebbles on one vertex u. To move one pebble to $v \neq u$ we need $2^{dist(u,v)}$ pebbles on u. For this move $2^{dist(u,v)} - 1$ pebbles are consumed, and 1 pebble is placed on v. Ignoring the pebble on v, and calculating this as a cost, we see that $2^{dist(u,v)}$ is the cost for filling v with one pebble from u.

To move a pebble to $w \neq u, v$ we need to place $2^{dist(u,w)}$ pebbles at u, consuming all pebbles in this move together with a placement of one pebble at w. Continuing in this way for all vertices and summing we get $cost(u) - 1$. Placing one extra pebble at u gives $\sum_{v \in V(G)} 2^{dist(u,v)} = cost(u)$. Thus, any simple configuration of size $\max_{u \in V(G)} cost(u)$ placing all pebbles on any vertex u in $V(G)$ is $n(G)$-solvable. □

Theorem 5.2 to 5.8 proves the cover pebbling number for the graphs mentioned by Sjöstrand (2004) using Corollary 5.1.1.

Theorem 5.2. $\gamma_{P_n} = 2^n - 1$, $n \geq 1$.

Proof. Since the cost of the last (or first) vertex r is $2^0 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$,[67] and the cost is less than this number for any other vertex $v \neq r$ of P_n since they are generally closer to other vertices, the theorem follows from Corollary 5.1.1. □

Theorem 5.3. $\gamma_{C_{2n}} = 3(2^n - 1)$, $n \geq 1$.

Proof. The cost of any vertex u is $2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + \cdots + 2 \cdot 2^{n-1} + 2^n$, since u has distance 0 to itself, and distance k to two vertices for each $1 \leq k \leq n-1$, and distance n to one vertex. This is equal to

$$\begin{aligned} 1 + 4(1 + 2 + \cdots + 2^{n-2}) + 2^n &= 1 + 4(2^{n-1} - 1) + 2^n \\ &= 3(2^n - 1). \end{aligned}$$

□

Theorem 5.4. $\gamma_{C_{2n+1}} = 2^{n+2} - 3$, $n \geq 1$.

Proof. The cost of any vertex u of C_{2n+1} is $2^0 + 2 \cdot 2^1 + 2 \cdot 2^2 + \cdots + 2 \cdot 2^n$ since u has distance 0 to itself and distance k to two vertices for each $1 \leq k \leq n$. This cost is equal to

$$\begin{aligned} 1 + 4(1 + 2 + \cdots + 2^{n-1}) &= 1 + 4(2^n - 1) \\ &= 2^{n+2} - 3. \end{aligned}$$

□

Theorem 5.5. $\gamma_{Q_n} = 3^n$, $n \geq 0$.

Proof. The cost of any vertex is 3^n.

To see this, pick an arbitrary vertex u.

Induction base: For $n = 0$, clearly $cost(u) = 1$ for the only vertex u of Q_0.

Induction step: Suppose that $cost(u) = 3^n$ for any vertex u of Q_n for some $n \geq 1$. We show that $\gamma_{Q_{n+1}} = 3^{n+1}$. Partition Q_{n+1} into two cubes

[67] Follows from the geometric sum $G(n) = x^0 + x^1 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$, $n \geq 1$, with $x = 2$.

Q_n and Q'_n. Let u be a vertex of Q_n. The cost $cost(u, Q_{n+1})$ for u with respect to Q_{n+1} is the cost $cost(u, Q_n)$ times 3, since one needs $cost(u, Q_n)$ pebbles at u to cover every vertex of Q_n with one pebble, then to pebble the vertices of Q'_n one needs to pay the cost of moving two extra pebbles to every vertex in $V(Q_n)$ so that one may move a pebble from every vertex in $V(Q_n)$ to one, and only one, adjacent vertex in $V(Q'_n)$, and this costs us $2cost(u, Q_n)$ pebbles, since we need two pebbles at every vertex in $V(Q_n)$ for this to be possible.

Summing theses costs we get $cost(u, Q_{n+1}) = 3cost(u, Q_n)$. Using $cost(u, Q_n) = 3^n$ by induction, we conclude that $cost(u, Q_{n+1}) = 3^{n+1}$.

Conclusion: Thus, by the principle of mathematical induction,
$cost(u, Q_n) = 3^n$ is true for all $n \geq 0$. □

Theorem 5.6. $\gamma_{K_n} = 2n - 1, n \geq 1.$

Proof. It costs $2(n-1)$ pebbles to move one pebble to each vertex in $V(K_n)$ from any vertex v in $V(K_n)$, and then one finally places one vertex at v to get the cost $2(n-1) + 1 = 2n - 1$. □

Theorem 5.7. $\gamma_{K_{n_1,n_2,\ldots,n_k}} = 4n_1 + 2n_2 + \cdots + 2n_k - 3$, *where* $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1, k \geq 2.$

Proof. The cost of any vertex at the set with n_1 vertices is $4n_1 + 2n_2 + \cdots + 2n_k - 3$, and the cost for a vertex in the set of n_i, $2 \leq i \leq k$, vertices is

$$4n_i + 2 \sum_{j=1, j \neq i}^{k} n_j - 3 \leq 4n_1 + 2n_2 + \cdots + 2n_k - 3,$$

where the inequality follows from $n_1 \geq n_i$ for all $i \neq 1$.

To show that the cost is $4n_1 + 2n_2 + \cdots + 2n_k - 3$ for every vertex in the set of n_1 vertices (and thus by the above inequality that this is the maximum cost), consider a vertex u in the set of n_1 vertices. One needs $2n_2 + 2n_3 + \cdots + 2n_k$ pebbles to move one pebble to each of the vertex not in the set of vertices which u is included in, then one needs $4(n_1 - 1)$ pebbles at u to move one pebble to each vertex, except u, in the set of vertices which u is included in (since the distance is 2 to such vertices), then finally placing one pebble at u we have

$$\begin{aligned} cost(u) &= 4(n_1 - 1) + 2n_2 + 2n_3 + \cdots + 2n_k + 1 \\ &= 4n_1 + 2n_2 + \cdots + 2n_k - 3. \end{aligned}$$

□

Theorem 5.8. $\gamma_{W_n} = 4n - 9$, $n \geq 4$.

Proof. The cost of the center vertex is $2(n-1) + 1 = 2n - 1$. The cost of any other vertex u is

$$\begin{aligned} 1 + 2 \cdot 3 + 4(n-4) &= 1 + 6 + 4n - 16 \\ &= 4n - 9, \end{aligned}$$

since u has distance 0 to itself, three neighbors, and distance 2 to any other vertex thanks to the center vertex adjacent to all vertices of W_n.

So the maximum cost is $4n - 9$ since $4n - 9 \geq 2n - 1$ because $2n \geq 8$ for $n \geq 4$. □

Before continuing, I prove two theorems not mentioned by Sjöstrand (2004).

Theorem 5.9. $\gamma_{F_n} = 4n - 7$, $n \geq 2$.

Proof. Let F_n consists of the path $a_1 a_2 \cdots a_{n-1}$ with a_n adjacent to all vertices a_i, $1 \leq i \leq n-1$.

$cost(a_n) = 2(n-1) + 1$ since $dist(a_n, a_i) = 1$ for all $n-1$ vertices a_i, $1 \leq i \leq n-1$.

$cost(a_1) = cost(a_{n-1}) = 1 + 2 \cdot 2 + 4(n-3) = 4n - 7$ since both a_1 and a_{n-1} are adjacent to two vertices, and with distance two to any other vertex (except themselves).

$cost(a_i) = 1 + 2 \cdot 3 + 4(n-4) = 4n - 9$, $2 \leq i \leq n-2$, since a_i is adjacent to three vertices, and distance two to all other vertices (except itself).

The maximum of these numbers is $4n - 7$ for $n \geq 2$. $\square$

Theorem 5.10. $\gamma_{FR_n} = 8n - 3, n \geq 1$.

Proof. The cost of the center vertex is

$$\begin{aligned} 2(n(FR_n) - 1) + 1 &= 2n(FR_n) - 1 \\ &= 2(2n + 1) - 1 \\ &= 4n + 1, \end{aligned}$$

since it is adjacent to any other vertex, and distance 0 to itself.

The cost of any other vertex is

$$\begin{aligned} 2 \cdot 2 + 4(n(FR_n) - 3) + 1 &= 4 + 4(2n - 2) + 1 \\ &= 8n - 3, \end{aligned}$$

since these are adjacent to the center vertex and one vertex in some copy of C_3 in FR_n, and distance to all other vertices (except themselves).

Since $8n - 3 \geq 4n + 1$, because $4n \geq 4$ for $n \geq 1$, the maximum cost is $8n - 3$. $\square$

5.1.2 The pi-pebbling function

Going back to Taylor's (2005) pi-pebbling function we find an interesting theorem concerning $\pi(G, P, n(G))$. This theorem reduces the workload of finding $\pi(G, P, n(G) - 1)$.

Theorem 5.11. *For any graph G, $\pi(G, P, n(G) - 1) + 1 = \pi(G, P, n(G))$.*

Proof. Place $k + 1 = \pi(G, P, n(G) - 1) + 1$ on $V(G)$. Pick a vertex v_1 with at least 1 pebble. We may ignore one pebble on v_1 and consider the graph $G - v_1$. This graph may be covered since $|V(G - v_1)| = n(G) - 1$, $G - v_1 \subseteq G$ and we have a distribution of $k = \pi(G, P, n(G) - 1)$ pebbles on G.

One always cover $n - 1$ vertices when one covers n vertices, so $\pi(G, P, n(G)) > \pi(G, P, n(G) - 1)$.

The theorem now follows. $\square$

Taylor (2005) also found three bounds on $\pi(G, P, t)$ which reduced to Theorem 2.5 in the case $t = 1$ and price function 2_P. Furthermore, he computed $\pi(G, P, t)$ for complete graphs, path graphs and star graphs. Thus the pi-pebbling function was successfully used. We will state and prove two of these three bounds to illuminate how one may reason about the pi-pebbling function.

Theorem 5.12. *Number the vertices of the graph G such that the price p_i of a vertex v_i satisfies $p_i \leq p_{i+1}$, then for $n(G) = n \neq t$*

$$\sum_{i=t+1}^{n} (p_i - 1) + p_n(t-1) + 1 \leq \pi(G, P, t).$$

Proof. Place $(p_i - 1)$ pebbles on each v_i for $t + 1 \leq i \leq n - 1$. Place $p_n t - 1$ pebbles on v_n. Then there are t vertices v_i, $1 \leq i \leq t$, which contains no pebble, and we have a distribution of $\sum_{t+1}^{n-1}(p_i - 1) + p_n t - 1$ pebbles on $V(G)$. One can only move pebbles from v_n, and since $p_n t - 1$ pebbles are placed on v_n one may move one pebble from v_n no more than $t - 1$ times. So one can at most pebble $t - 1$ vertices from v_n (even if one passes v_i, $t + 1 \leq i \leq n - 1$, so one can not take advantage from them in this regard), but one needs to pebble t vertices. Hence,

$$\begin{aligned}\pi(G, P, t) &> \sum_{t+1}^{n-1}(p_i - 1) + p_n t - 1 \\ &= \sum_{i=t+1}^{n} (p_i - 1) + p_n(t-1).\end{aligned}$$

□

Theorem 5.13. *Let d be the diameter of the graph G and let $p_i \leq p_{i+1}$ for all vertices v_i in $V(G)$, then*

$$\pi(G, P, t) \leq t \left(\left(\prod_{i=n-(d-1)}^{n} (p_i) - 1 \right) (n-1) + 1 \right).$$

Proof. ($t = 1$) $\pi(G, P, 1) \leq (\prod_{i=n-(d-1)}^{n}(p_i) - 1)(n-1) + 1$, since if $(\prod_{i=n-(d-1)}^{n}(p_i) - 1)(n-1) + 1$ pebbles are placed on the vertices of G, either every vertex contains at least on pebble, or one vertex u contains $\prod_{i=n-(d-1)}^{n}(p_i)$ pebbles. From u one can pebble to any other vertex, since $diam(G) = d$ and using $\prod_{i=n-(d-1)}^{n}(p_i)$ pebbles we can traverse any path of length d since $p_{n-(d-1)}p_{n-(d-2)} \cdots p_n$ is the largest amount of pebbles one needs to traverse such a path.

($t \geq 2$) Since $\pi(G, P, 1)$ is the minimum number of pebbles we need to pay to pebble any vertex. Hence, we see that $\pi(G, P, 1) \leq t\pi(G, P, t)$ since we may use $\pi(G, P, 1)$ to pebble one arbitrary vertex, then ignore that this vertex has one pebble on it, and place $\pi(G, P, 1)$ pebbles again on $V(G)$ so as to pebble any other vertex. Continuing like this a number of t times gives the stated result. □

5.2 Variations and applications

I will now comment on some known variations and applications in graph pebbling modeling.

The *optimal pebbling number* $\pi_{opt}(G)$ of G is the minimum number of pebbles that may be distributed on the vertices of G such that at least one pebble may be moved to any vertex in $V(G)$. In (Wyels, 2003)[27] it was shown that $\pi_{opt}(C_{3t+r}) = 2t + r$ for nonnegative integers t and $0 \leq r \leq 2$. This was essentially proven by extending the proof that $\pi_{opt}(P_{3t+r}) = 2t + r$ for nonnegative integers t and $0 \leq r \leq 2$ (Pachter et al., 1995)[20].

Let G and H be two graphs. Then the Cartesian product $G \times H$ of G and H denotes the graph with vertex set $V(G) \times V(H)$, and edge set[68]

[68] Cartesian products for sets may be explained as follows: If A and B are two arbitrary sets, the Cartesian product $A \times B$ is the set of ordered pairs (a, b) such that $a \in A$ and $b \in B$. This may be generalized to include products of a number of n sets. Formally, the product of n sets $A_1, A_2, \ldots, A_n$ is $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | \forall i = 1, 2, \ldots, n, \quad a_i \in A_i\}$. The notation A^2 denotes $A \times A$, and A^n for positive integers n may is generalized in the obvious way.

$$E(G \times H) = \{((g,h),(g',h'))\,|g = g' \text{ and } (h,h') \in V(H)^2\}$$
$$\cup \{((g,h),(g',h'))\,|h = h' \text{ and } (g,g') \in V(G)^2\}.$$

A famous conjecture is the following:

Conjecture 2 (Graham's conjecture)**.** *For any two graphs G and H,*

$$\pi(G \times H) \leq \pi(G)\pi(H).$$

This conjecture was already formulated in 1989 by Chung. Interestingly, Wyels (2005) showed that

$$\pi_{opt}(G \times H) \leq \pi_{opt}(G)\pi_{opt}(H)$$

for any two graphs G and H.

Pebbling numbers for Cartesian products of graphs was studied in (Rongquan & Ju Young, 2000)[21]. They showed that

$$\pi(K_{m,n} \times G) \leq \pi(K_{m,n})\pi(G)$$

for any graph G.

Hurlbert (2005)[15] argues that since graph pebbling is very much alike games such as "Cops and Robbers" and "Chip-Firing", and since these games have been successfully applied in graph theory and theoretical computer science, one may expect a similar impact from the theory of graph pebbling. One may think of the cost of a pebbling step as a toll or as the loss of money, information, oil or electrical charge. Since the mentioned substances are often expressed in real numbers, we may prefer to consider configurations taking real values on graph vertices.

A variation of graph pebbling that allows one to move real valued "pebbles" is that of *efficient pebbling* (Pachter, 1995). Instead of moving pebbles, we may move sand, that is: a substance divisible into infinitesimal amounts. We place piles of sand on the vertices of some graph, then whenever we move sand from one vertex to another, we lose half of it. Thus, if we distribute $2^{diam(G)}$ units of sand on $V(G)$, we may move a unit of sand to any vertex of G.

We may manipulate the above game to fit other needs. For these reasons, let $\epsilon = \frac{1}{t} > 0$ where t is some positive integer. Place 2^{diamG} sand on the vertices $V(G)$ of G. A step consists in moving ϵ units of sand to a price of 2ϵ units of a sand at the vertex from which one moves sand. We ask for which graphs it is possible to move one unit of sand to any vertex of G. Let t be the least amount of pebbles such that for any distribution of $t2^{diam(G)}$ pebbles on $V(G)$ one can move one unit of sand to any vertex in $V(G)$. In this case we define the quanta $\epsilon_G = \frac{1}{t}$. If no such t exists, ϵ_G is set to be equal to 0.

An efficient graph G is a connected graph G for which $\epsilon_G > 0$. This tells us that if $2^{diam(G)}$ units of energy are distributed on $V(G)$, and if the energy comes in sufficiently small quanta ϵ_G, then one may move one unit of energy to any vertex of G.

Pachter et al. (1995) characterized the efficient graphs as those graphs which has a special property called the "antipodal property". He also showed that for every smallest integer t such that $2t \geq (n(G) - 2)$ for $G = K_n - e, e \in E(K_n)$,[69] we have $\epsilon_G = \frac{1}{t}$.

Pacther et al. (1995) also studied how the number of edges in a graph G relates to $\pi(G)$. The intuition is supposedly that the pebbling number of a graph G decreases as we add edges to $E(G)$ by joining pairs of vertices of G. What Pacther et al. (1999) showed was that if G is a connected graph with $n(G) \geq 4$, then: if $|E(G)| \geq \binom{n-1}{2}$, then $\pi(G) = n(G)$. We encourage the reader to study the relation between the number of edges in a graph G and the pebbling number of G.

One may finally be interested in studying *complexity in graph pebbling*. This subject tries to answer questions regarding how much time it takes

[69] The graph $K_n - e$, where $e \in E(K_n)$, is the graph obtained by deleting the vertex e from K_n, this is the subgraph of K_n containing the same vertices as K_n and all edges of K_n except e. In general, if F is a subset of E where $G = (V, E)$ is a graph, then $G - F$ is defined as $(V, E - F)$, where $E - F = \{x \in E | x \notin F\}$. "Deleting" F from G, i.e. performing the operation $G - F$ as previously defined, is called *edge deletion*.

to solve a graph pebbling problem, such as the following question: Given a diameter 2 graph G, how long does it take to compute that $\pi(G)$ is of Class 0 or 1?[70]

5.3 Future research and directed graphs

Considering Hurlbert's comment (2005) on real valued configurations, we may continue the generalization of the pi-pebbling function which was initiated by Taylor (2005), and let configurations C on graphs G to be functions from $V(G)$ to $\mathbb{R}_{\geq 0}$.[71]

Consider the problem of moving oil around a network. The nodes symbolizes destination, such as Stockholm and Paris. If there is an edge from Stockholm to Paris, one may transport oil from Stockholm to Paris (with a truck, boat, etcetera). The price of this edge is the amount of oil that is lost in the process of transporting oil from Stockholm to Paris. Maybe the oil company sets this cost to be the average loss of oil per ride. In any way, this edge should not necessarily be regarded as an undirected edge. What if the road from Stockholm to Paris is different from the road from Paris to Stockholm in significant ways regarding the loss of oil? If it is, then one may want to view the road from Paris to Stockholm as more or less pricey. Thus, if one wants to study applications such as these (transportations in networks), one should not neglect studying directed graphs. In this thesis we have mainly studied undirected graphs. This is true to the title of this thesis. The foundations of graph pebbling is almost exclusively busy with analyzing undirected graphs, and that should be remedied.

Investigations on directed graphs in graph pebbling have been done by Gundam and Higgins (2004)[12]. When pebbling on directed graphs, they argued, we should only consider *strongly connected directed graphs G*. A strongly connected directed graph $\vec{G}$ is defined as a graph $\vec{G}$ where any two vertices u and v of $\vec{G}$ are *mutually reachable*, where we define mutual reachability between two distinct vertices u and v as the existence of a

[70] The answer is that this can be done in $O(n(G)^4)$ time, see (Herscovici, 2011)[13].

[71] The set $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers.

path from u to v, and the existence of a path from v to u. Because of this we make the observation that strongly connected directed graphs contains no *sources*, an unreachable vertex, and no *sinks*, a vertex from which one cannot move pebbles.[72] For example, in Figure 19 above we see that b is a sink and c is a source.

Gundam and Higgins (2005) presented a theorem that may be used to classify strongly connected directed graphs. Using this theorem they characterized some graphs as strongly connected directed graphs, and derived the pebbling numbers for these graphs. Their paper is introductory. One should investigate graph pebbling on directed graphs in more detail, for the purpose of interest and for the purpose of future applications. Graph pebbling results concerning directed graphs should be an elementary part of the theory of graph pebbling.

See Figure 20 for an example of a directed graph with no source and no sink, hence, a strongly connected directed graph. Call the graph in Figure 20 the strongly connected directed friendship graph $\overrightarrow{FR}_2$ with vertices a_i, $1 \leq i \leq 5$. We will calculate the pebbling number of $\overrightarrow{FR}_2$ as an example of how one may solve graph pebbling problems on directed graphs.

The longest (directed) path to a_1 is of length 2. It can begin at either the left or right copy of C_3 in $\overrightarrow{FR}_2$. If $4 \cdot 2 - 1$ pebbles are distributed on $\overrightarrow{FR}_2$, one of the two copies contain 4 pebbles, from which one can pebble a_1 (since a_1 will be the end vertex of a directed path P_3: either in the path $a_4a_5a_1$, or in the path $a_2a_3a_1$, and the pebbling number with respect to a_1 for both these paths is 4 (see Theorem 3.1)).

The longest path to some vertex in one of the copies of $\overrightarrow{FR}_2$ is of length 4, such as the path from a_2 to a_5. One needs to pass a_3, a_1 and a_4 to arrive at a_5. Ignoring a_5, if we place 16 pebbles on the vertices a_1 to a_4, we will have 16 pebbles on a path of length 4 including all vertices. This configuration is solvable by Theorem 3.1. Symmetrically, the same argument may be applied to the path $a_4a_5a_1a_2a_3$ of length 4 with a_3 as the end vertex.[73]

[72]Note: One should thus reformulate Theorem 5.1, the cover pebbling theorem, and only allow strongly connected directed graphs when mentioning directed graphs.

[73]When pebbling a_4 or a_2 in $\overrightarrow{FR}_2$ the longest path is of length 3 (the path $a_2a_3a_1a_4$ for

The above shows that $\pi(\overrightarrow{FR}_2) = 16$ since the configuration placing 15 pebbles at a_2 is a_5-unsolvable.

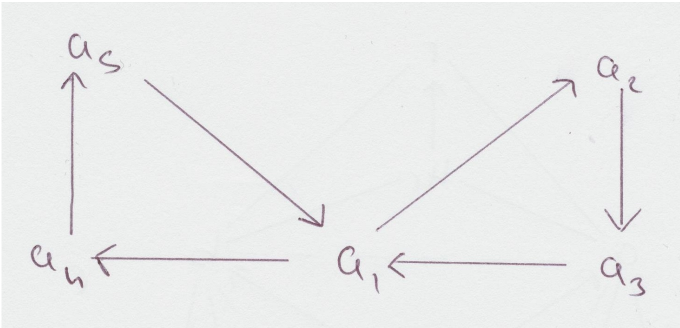

Figure 20: The strongly connected directed friendship graph $\overrightarrow{FR}_2$.

An open problem presented in (Gundam & Higgins, 2004)[12] is

Open Problem 1. *If $\overrightarrow{G}$ and $\overrightarrow{H}$ are two strongly connected directed graphs, does Graham's conjecture hold? That is: Is it true that $\pi(\overrightarrow{G} \times \overrightarrow{H}) \leq \pi(\overrightarrow{G})\pi(\overrightarrow{H})$?*

Additionally we formulate two open problems:

Open Problem 2. *Find the pebbling number $\pi(\overrightarrow{FR}_n)$ for the strongly connected directed friendship graph $\overrightarrow{FR}_n$ on n vertices.*

Open Problem 3. *Find the pebbling number $\pi(\overrightarrow{Q}_n)$ for the strongly connected directed n-dimensional cube $\overrightarrow{Q}_n$.*

I suggest that future research will use more general notions of graph pebbling. Allowing configurations to be real valued, and to let them be dependent on some price function. t-solvability should be investigated rather than solvability only. Directed graphs is an important study, since there are reasons that studying them will raise the probability of finding new possible applications. Also, I want to suggest that one provides a clear example when graph pebbling was successfully, and not trivially,

a_4, and the path $a_4a_5a_1a_2$ for a_2). One may show that 9 pebbles is sufficient to pebble a_2 or a_4.

used in applications; because I have not yet seen such an example, and if one wants graph pebbling to be viewed as part applied mathematics, one must point to an application which is not trivial.[74] It would definitely be even more interesting if one could somehow apply the probabilistic model to directed graphs, since many empirical problems involve not only networks which are best described by directed graphs, but also degrees of uncertainty and thus probability.

[74]The following may be an example of a trivial application of mathematics: providing mathematics for the pleasure of those who work with mathematics. I will not discuss what constitutes trivial mathematics, so I hope that the reader's intuitive understanding of this concept is similar to mine. One may certainly philosophize about this, the value of mathematics and so on.

6 Summary and conclusion

Foundational theorems in graph pebbling have been proven in section 2 to 4.

In section 2.2.1, Theorem 2.5, we derived lower and upper bounds for pebbling numbers.

In section 3.1 we proved the pebbling numbers for several types of graphs, such as the path, cycle and the n-dimensional cube. The results are summarized in Table 1 in appendix A.

In section 3.2 we characterized diameter 2 graphs. When G is a graph of diameter 2, it is either of Class 0 or of Class 1. This was proved in 3.2.1, Theorem 3.13. Moreover, if G is a 3-connected diameter 2 graph, then G is of Class 0. This was proved in 3.2.2, Theorem 3.18.

In section 4 the probabilistic model of graph pebbling was introduced. Preliminary results and elementary results were provided in section 4.2. Section 4.3 may be seen as the central part of section 4, where the threshold for a number of graph sequences are derived, such as the sequence of paths or the sequence of cycles. A general bound for the threshold of any graph sequence was given in section 4.3.1, Theorem 4.8. The thresholds derived in section 4.2.2 and 4.3 are found in Table 2, appendix A.

Section 5 discussed recent research. The pi-pebbling function and cover pebbling number were discussed in 5.1. In 5.1.1 the cover pebbling theorem was stated, and a corollary of this theorem was derived: Corollary 5.1.1. This corollary helped us prove the cover pebbling number for some common graphs. A summary of these results are found in Table 3 in appendix A. Some theorems and proofs were provided in 5.1.2 to illuminate how one may reason about problems regarding the pi-pebbling function.

Section 5.2 outlined some variations and possible applications on graph pebbling. In section 5.3 we made suggestions on where the research on graph pebbling should continue. More importantly we argued that graph pebbling on directed graphs should be studied more. We showed how one may reason about problems in graph pebbling modeling regarding directed graphs, and formulated three open problems. Thus one ought, and one can, increase the research on graph pebbling on directed graphs. We also asked the reader to provide a clear example of when graph pebbling modeling was successfully applied to a real world problem.

The deterministic model and the probabilistic model in graph pebbling have been studied. Generalizations have been made and possible applications have been suggested. These have given us reasons to believe that graph pebbling will have a significant impact on applications. This is the strength of our results, and answers our central question in the introduction: ″Can graph pebbling modeling, as a pure mathematical subject, be extended in such a way that it has a significant impact in the field of applied mathematics?“ The answer is ″reasonably yes“. In section 5.3 we in fact showed how such an extension may be done.

We may summarize the above by saying that graph pebbling is a *rich subject*. Its richness comes from the fact that its foundations have been stringently submitted, and it also lies in its many variations and applications, and in its possibility to have ″a significant impact in the field of applied mathematics“.

The thresholds for the sequence of paths, cycles and cubes should be proven. More graph sequences may be studied, and we encourage the reader to do this. The pi-pebbling function should be merged with the notion of a w-cover, and similar results as in section 2.2.1 and 3 should be derived for the pi-pebbling function. This, together with a more detailed account of graph pebbling on directed graphs, will benefit the search for possible applications. Furthermore, one may benefit by studying the pi-pebbling function in a probabilistic context, since uncertainty is often expected in real world problems. Also, one should try to answer Conjecture 1 and 2, and the reader is encouraged to solve Open Problem 1 to 3. Filling all these *holes* would strengthen the theory of graph pebbling significantly.

In conclusion: Graph pebbling is a rich subject, with many holes to fill.

A Tables of results

In this section three tables are presented which summarizes the results from section 3, 4.2.2, 4.3 and 5.1.

Table 1: Summary of results from section 3.

Graph	Pebbling number	Class 0	Class 1
K_n	n	Yes	No
P_n	2^{n-1}	No	No
C_{2n}	2^n	No	No
C_{2n+1}	$2\left\lfloor \frac{2^{n+1}}{3} \right\rfloor + 1$	No	No
W_n	n	Yes	No
FR_n	$2n+2$	No	Yes
F_n	n	Yes	No
$S_n, n \geq 3$	$n+1$	No	Yes
$K_{2,n}, n \geq 2$	$n+2$	Yes	No
$K_{m,n}, m, n \geq 3$	$m+n$	Yes	No
P	10	Yes	No
Q_n	2^n	No	No

Table 2: Summary of results from section 4.2.2 and 4.3.

Graph sequence	Threshold
$\mathscr{K}$	$\Theta(n^{1/2})$
$\mathscr{Q} = (Q_1, Q_2, \ldots, Q_m, \ldots)$	$th(\mathscr{Q}) \subseteq O(n(Q_m))$
$\mathscr{P}$	$th(\mathscr{P}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$ for every $\epsilon > 0$
$\mathscr{C}$	$th(\mathscr{C}) \subseteq \Omega(n) \cap o(n^{1+\epsilon})$ for every $\epsilon > 0$
$\mathscr{S}$	$\Theta(n^{1/2})$
$\mathscr{W}$	$\Theta(n^{1/2})$

Table 3: Summary of results from section 5.1.

Graph	Cover pebbling number
P_n	$2^n - 1$
C_{2n}	$3(2^n - 1)$
C_{2n+1}	$2^{n+2} - 3$
Q_n	3^n
K_n	$2n - 1$
$K_{n_1,n_2,\ldots,n_k}$, $n_1 \geq n_2 \geq \cdots \geq n_k$	$4n_1 + 2n_2 + \cdots + 2n_k - 3$
F_n	$4n - 7$
FR_n	$8n - 3$

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Index

Nomenclature

(a, b) An ordered pair, page 8

2_P The function which satisfies $2_P(v) = 2$ for any vertex v, page 78

$\binom{n}{k}$ The binomial coefficient, page 52

γ_G The cover pebbling number of G, page 78

$\gamma_G(w)$ The function $\pi(G, 2_P, n(G), w)$, page 78

$\kappa(G)$ The vertex-connectivity of G, page 11

$\mathbb{R}$ The set of real numbers, page 24

$\mathbb{R}_{\geq a}$ The set of real numbers greater than or equal to a, page 89

$\mathbb{Z}_{\geq a}$ The set of integers greater than or equal to a, page 13

$\mathcal{C}(G)$ The set of possible configurations on G, page 13

$\mathcal{G}_{n,t}$ The set of all configurations C on G_n of size t, page 58

$\mathscr{C}$ The graph sequence of cycles, page 72

$\mathscr{G}$ A graph sequence $(G_1, G_2, \ldots)$, page 60

$\mathscr{K}$ The graph sequence of complete graphs, page 63

$\mathscr{P}$ The graph sequence of paths, page 69

$\mathscr{Q}$ The graph sequence of cubes, page 66

$\mathscr{S}$ The graph sequence of stars, page 73

$\mathscr{W}$ The graph sequence of wheels, page 75

$\max_{x \in D} f(x)$ The maximum value of $f(x)$ over the set D, page 12

$\Omega(f)$ The set of functions g for which there exists constants $c, k > 0$ such that $f(n)/g(n) < c$ for all $n > k$, page 59

$\vec{FR}_n$ The strongly connected directed friendship graph on n vertices, page 91

$\vec{G}$ A strongly connected directed graph, page 89

$\vec{Q}_n$ The strongly connected directed n-dimensional cube, page 91

$\pi(G)$ The pebbling number of G, page 15

$\pi(G, P, t)$ Pi-pebbling function, page 78

$\pi(G, P, t, w)$ Generalized pi-pebbling function, page 78

$\pi(G, r)$ The pebbling number of G for (or with respect to) r, page 14

$\pi_{opt}(G)$ The optimal pebbling number of G, page 86

$\Theta(g)$ The set $O(g) \cap \Omega(g)$, page 59

$\varnothing$ The empty set, page 49

$\{x_1, x_2, \ldots, x_n\}$ A set of n elements $x_1, x_2, \ldots, x_n$, page 8

$A \cap B$ The set of elements in A and B, page 12

$A \cup B$ The set of elements in A or B or both, page 47

$A \subseteq B$ A is a subset of B, page 10

$A \times B$ The Cartesian product of A and B, page 86

$a \to^t b$ The pebbling move moving t pebbles from a to b, page 14

$a_1 \to a_2 \to \cdots \to a_n$ The pebbling move from a_1 to a_n passing the vertices a_k, $2 \leq k \leq n-1$, page 14

$C(A)$ The sum $\sum_{v \in A} C(v)$, page 13

C_n The cycle graph on n vertices, page 23

C_{2n+1} The odd cycle graph on $2n+1$ vertices, page 25

C_{2n} The even cycle graph on $2n$ vertices, page 23

$cost(u)$ The sum $\sum_{v \in V(G)} 2^{dist(u,v)}$, where u is a vertex in the vertex set $V(G)$ of G, page 80

$diam(G)$ The diameter of G, page 12

$dist(x, y)$ The distance between x and y, page 12

$E(G)$ The edge set of G, page 8

$f \lesssim g$ The limit $\lim_{n \to \infty} \sup(f(n)/g(n)) \leq 1$, page 59

$f \ll g$ The limit $\lim_{n \to \infty} f(n)/g(n) = 0$, page 59

F_n The fan graph on n vertices, page 33

FR_n The friendship graph consisting in n copies of the odd cycle graph C_3, where each vertex is adjacent to a center vertex c and where c is a vertex in each copy of C_3, page 31

$G - H$ Denotes $G - V(H)$, page 10

$G - V(H)$ The subgraph of G obtained by deleting the vertices in $V(H)$ and all edges incident with them, page 10

G_n A graph on n vertices, page 13

G_n A graph on n vertices, page 60

K_n The complete graph on n vertices, page 19

$K_{m,n}$ The complete bipartite graph on $m + n$ vertices, partitioned into two sets of vertices of size m and n where no vertex in each set is adjacent to another, page 34

$K_{n_1,n_2,\ldots,n_k}$ The complete multipartite graph on $n_1 + n_2 + \cdots + n_k$ vertices, partitioned into k sets of vertices of size $n_1, n_2, \ldots, n_k$, where no vertex in each set is adjacent to another, page 80

$N(A)$ The neighborhood of A, page 12

$n(G)$ The size of the graph G, page 8

$O(g)$ The set of functions f for which there exists constants $c, k > 0$ such that $f(n)/g(n) < c$ for all $n > k$, page 59

P Petersen's graph, page 39

P_n The path graph on n vertices, page 21

Q_m The m-dimensional cube, page 41

S_k The star graph on k vertices, page 34

$S_{u,v}$ The pebbling step from u to v, page 13

$u \to v$ The pebbling step from u to v, page 14

$V(G)$ The vertex set of G, page 8

W_n The wheel graph on n vertices, page 29

$x \in X$ x is an element in the set X, page 11