SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Apolarity Theory and Macaulay's Theorem

av

George Mouselli

2015 - No 9

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Apolarity Theory and Macaulay's Theorem

George Mouselli

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Alessandro Oneto

2015

Introduction

In this thesis, we introduce Apolarity Theory and examine its unexpected applications in the study of Hilbert functions and Hilbert series, two key words in the area of commutative algebra known as dimension theory. The idea is simple; let k be a field and consider the two polynomial rings $R = k[x_0, ..., x_n]$ and $S = k[y_0, ..., y_n]$. Now, we want to think of the polynomials in R to act like partial differential operators on the polynomials in S, this is what we formally define as the *apolarity action*. From this innocent definition, we establish powerful tools that can be used to compute Hilbert functions of quotient rings R/I. Later, we study the idea of *Artinian Gorenstein* rings and state Macaulay's theorem which gives a complete characterization of these rings, we also give a proof for this theorem, using the developed tools from Apolarity Theory.

In the first chapter of this thesis we define the meaning of Hilbert functions and series for graded R-modules, and show how one can compute the Hilbert series of R/I in specific cases without using Apolarity Theory.

In the first section of Chapter 2, we define the apolarity action, we then explain the idea of nonsingular bilinear maps which is an an essential part of the thesis. In Section 2.3 we introduce the "perp", an definition involving the apolarity action which is important in our proof for Macaulay's theorem. In Section 2.4, we define the idea of Invere Systems and establish the remarkable connection between Apolarity Theory and the computation of Hilbert funtions. After developing all theory needed, we move on to Section 2.5 where we give some general theory on Artinian rings, needed in order to understand the idea of Gorenstein rings. In the last section, we state and prove Macaulay's theorem.

Contents

Chapter 1

Background

1.1 Graded rings and modules

Let R be a commutative ring with unity. We say that a ring R is **graded** if we can write $R = \bigoplus_{d\in\mathbb{Z}} R_d$, where each R_d is an abelian subgroup of R, and $R_iR_j \subseteq R_{i+j}$ for all $i, j \in \mathbb{Z}$.

Likewise whenever having a graded ring R, we say that an R-module M is graded if we can write $M = \bigoplus_{d\in\mathbb{Z}} M_d$, where each M_d is an abelian subgroup of M, and $R_iM_j \subseteq M_{i+j}$ for all $i, j \in \mathbb{Z}$.

Example 1.1. Let k be a field and consider the ring $S = k[x_1, ..., x_n]$ of polynomials in n variables with coefficients in k. Then, S is a graded ring where S_d denotes the k-vector space consisting of homogeneous polynomials in degree d. Such a grading, where $\deg x_i = 1$, is called the **standard grading**. Note that $\deg x_i = 1$ implies that $S_d = 0$ for all $d < 0$.

Remark 1.2. (i) If R is a graded ring and I is a homogeneous ideal of R, then

$$R/I = \bigoplus_{d\in\mathbb{Z}} (R_d + I)/I.$$

However, by the second isomorphism theorem we have $R_d/(I \cap R_d) \cong (R_d + I)/I$, we will use this remark later.

(ii) Every R_d is an R_0-module and R_0 is a subring of R. Indeed it is enough to observe that $1 \in R_0$; to see this write $1 = \sum_{d\in\mathbb{Z}} x_d$ where $x_d \in R_d$ then, for all n, we have $x_n = 1 \cdot x_n = \sum_d x_d x_n$ and, comparing degree by degree we see that $x_n = x_n x_0$ for all n. Thus we have

$$x_0 = 1 \cdot x_0 = \sum_{d\in\mathbb{Z}} x_d x_0 = \sum_{d\in\mathbb{Z}} x_d = 1,$$

and hence $1 \in R_0$.

1.2 Hilbert functions and Hilbert series

Let $S = k[x_0, ..., x_n]$ with the standard grading described in Example 1.1. Note that, for any graded S-module $M = \oplus_{d\in\mathbb{Z}} M_d$, M_d is a k-vector space for all $d \in \mathbb{Z}$. This allows us to give the next definition.

Definition 1.3. Let $M = \bigoplus_{d\in\mathbb{N}} M_d$ [1] be a graded S-module. We define the **Hilbert function** of M, $\mathrm{HF}(M, -) : \mathbb{N} \to \mathbb{N}$, to be

$$\mathrm{HF}(M, d) := \dim_k M_d, \text{ for all } d \in \mathbb{N}.$$

Furthermore, we define the **Hilbert series** of M as

$$\mathrm{HS}(M, t) := \sum_{d\in\mathbb{N}} \mathrm{HF}(M, d)t^d.$$

Example 1.4. Consider $R = k[x_0, x_1, x_2]$ and $I = (x_0, x_1x_2, x_2^2, x_1^3)$. Then,

$$R/I = k + I \oplus (kx_1 + I \oplus kx_2 + I) \oplus (kx_1^2 + I).$$

Hence,

$$\mathrm{HF}(R/I, 0) = 1, \ \mathrm{HF}(R/I, 1) = 2, \ \mathrm{HF}(R/I, 2) = 1;$$

thus we have

$$\mathrm{HS}(R/I, t) = 1 + 2t + t^2.$$

[1] When we use $\mathbb{N}$ as index instead of $\mathbb{Z}$ we simply mean that $M_d = 0$ for all $d < 0$.

Example 1.5. If we consider the polynomial ring $S = \bigoplus_{d\in\mathbb{N}} S_d$, we see that HF$(S, d)$= $\binom{d+n}{d}$, since $\binom{d+n}{d}$ is the number of monomials needed to span S_d as a k-vector space. With this we can compute the Hilbert series of S,

$$\mathrm{HS}(S,t) = \sum_{d\in\mathbb{N}} \binom{d+n}{d} t^d = \frac{1}{(1-t)^{n+1}} \in \mathbb{N}[\![t]\!].$$

Definition 1.6. Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a graded S-module. The **shifting** of M of degree e is the graded S-module $M(e) = \bigoplus_{d\in\mathbb{Z}}[M(e)]_d$ with the graded structure given by

$$[M(e)]_d := M_{d+e}, \quad \text{for any } d \in \mathbb{Z}$$

If e is a positive integer, there is a relationship between the Hilbert series for $S(-e)$ and the Hilbert series for S, here we make the convention that $\binom{n}{k} = 0$ whenever $k < 0$,

$$\mathrm{HS}(S(-e),t) = \sum_{d\geq 0} \binom{n+d-e}{d-e} t^d = \sum_{d\geq 0} \binom{n+d-e}{d-e} t^{d-e}t^e = t^e \sum_{k\geq 0} \binom{n+k}{k} t^k$$
$$= t^e \mathrm{HS}(S,t).$$

Note that the same conclusion, i.e $\mathrm{HS}(M(-e),t) = t^e\mathrm{HS}(M,t)$, holds for any graded S-module M.

If M and N are graded S-modules and $\phi : M \to N$ is a homomorphism of modules, we say that the map ϕ is **graded** if $\phi(M_i) \subset N_{i+j}$ for some $j \in \mathbb{Z}$. We call such an integer j the **degree** of ϕ, denoted by $\deg(\phi)$.

Lemma 1.7. Let M, N and P be graded S-modules. If

$$0 \to M \to N \to P \to 0$$

is a short exact sequence (s.e.s) of graded S-modules with degree-0 maps, then

$$\mathrm{HS}(N,t) = \mathrm{HS}(M,t) + \mathrm{HS}(P,t).$$

Proof. The s.e.s

$$0 \to M \to N \to P \to 0$$

induces a new s.e.s of k-vector spaces,

$$0 \to M_d \xrightarrow{\alpha} N_d \xrightarrow{\beta} P_d \to 0, \quad \text{for any } d \in \mathbb{N}.$$

Hence we have $\dim_k N_d = \dim_k M_d + \dim_k P_d$ for all $d \in \mathbb{N}$, and so

$$\sum_{d\in\mathbb{N}} (\dim_k N_d)t^d = \sum_{d\in\mathbb{N}} (\dim_k M_d)t^d + \sum_{d\in\mathbb{N}} (\dim_k P_d)t^d.$$

□

Now we want to give an example of how Lemma 1.7 can be applied in order to compute Hilbert series.

Definition 1.8. Let R be a commutative ring and $f_1, ..., f_g \in R$. We say that $\{f_1, ..., f_g\}$ is a **regular sequence** if

- f_1 is a NZD in R;
- f_i is a NZD in $R/(f_1, ..., f_{i-1})$ for $i = 2, .., g$.

Furthermore the quotient ring S/I is called a **complete intersection** if the generators of I form a regular sequence.

Theorem 1.9. Let S/I be a complete intersection where $I = (f_1, ..., f_g)$. Then

$$\mathrm{HS}(S/I, t) = \frac{\prod_{i=1}^{g}(1 - t^{e_i})}{(1 - t)^{n+1}}$$

where $e_i := \deg(f_i)$.

Proof. We proceed by induction on the number of generators for I.

If $I = (f_1)$, consider the s.e.s

$$0 \longrightarrow S(-\deg(f_1)) \xrightarrow{\phi_1} S \xrightarrow{\phi_2} S/(f_1) \longrightarrow 0, \tag{1.1}$$

where ϕ_1 is the map of multiplication by f_1 and ϕ_2 is the natural surjection. Note that the shifting of S in (1.1) makes ϕ_1 into a degree-0 map so that we can apply Lemma 1.7,

$$\mathrm{HS}(S/(f_1), t) = \sum_{d\geq 0} \binom{n+d}{d} t^d - t^{e_1} \sum_{d\geq 0} \binom{n+d}{d} t^d = \frac{1 - t^{e_1}}{1 - t^{n+1}}.$$

Now, assume that

$$\mathrm{HS}(S/(f_1, ..., f_k), t) = \frac{\prod_{i=1}^{k}(1 - t^{e_i})}{(1 - t)^{n+1}},$$

set $R' := S/(f_1, ..., f_k)$ and consider the s.e.s of degree-0 maps,

$$0 \longrightarrow R'(-\deg(f_{k+1})) \xrightarrow{\alpha} R' \xrightarrow{\beta} R'/(f_{k+1}) \longrightarrow 0 \tag{1.2}$$

where α is the multiplication by f_{k+1} and β is the natural surjection. The assumption that S/I is a complete intersection is crucial for (1.2) to be a s.e.s, since it gives us the injectivity of α. Applying Lemma 1.7 again we get

$$\mathrm{HS}(R'/(f_{k+1}), t) = (1 - t^{e_{k+1}})\mathrm{HS}(R', t) = (1 - t^{e_{k+1}})\frac{\prod_{i=1}^{k}(1 - t^{e_i})}{(1 - t)^{n+1}}.$$

$\square$

Remark 1.10. In the above proof we have used that $R'/(f_{k+1}) \cong S/(f_1, ..., f_{k+1})$ which follows directly from the third isomorphism theorem for rings.

Chapter 2

Apolarity Theory and Macaulay's Theorem

2.1 Apolarity action

Let k be a field field with $\mathrm{char}(k) = 0$. In this chapter we consider two polynomial rings with the standard gradation, namely $S = k[y_0, ..., y_n] = \bigoplus_{d \geq 0} S_d$ and $R = k[x_0, ..., x_n] = \bigoplus_{d \geq 0} R_d$, where S_j denotes the subset of S consisting of homogeneous polynomials in degree j, we think of R_j similarly.

Here, we mainly follow the notes of A.V. Geramita [Ger96].

We want to think of the polynomials in R as partial differential operators acting on the polynomials in S, which motivates the next definition.

Definition 2.1. The **apolarity action**

$$\circ : R_1 \times S_1 \to k$$

of R_1 on S_1 is defined as,

$$(a_0x_0 + \cdots + a_nx_n) \circ (b_0y_0 + \cdots + b_ny_n) := \sum_{i=0}^{n} a_i \frac{\partial}{\partial y_i}(b_0y_0 + \cdots + b_ny_n).$$

Example 2.2.

Consider $f = 5x_0 + x_3 \in R_1$ and $g = y_0 + 2y_3 + y_5 \in S_1$, then

$$f \circ g = 5 + 2 \in k.$$

If $\alpha = (\alpha_0, ..., \alpha_n)$ with $\alpha_i \in \mathbb{N}$ then we will denote the monomial $x_0^{\alpha_0} \cdot ... \cdot x_n^{\alpha_n}$ by $\mathbf{x}^\alpha$, we define $\mathbf{y}^\beta$ similarly. If $f = a_0\mathbf{x}^{\alpha_0} + \cdots + a_n\mathbf{x}^{\alpha_n} \in R_i$ and $g = b_0\mathbf{y}^{\beta_0} + \cdots + b_k\mathbf{y}^{\beta_k} \in S_j$, we can extend the action of R_1 on S_1 by using the usual properties of differentiation; namely, by considering

$$\circ : R_i \times S_j \longrightarrow S_{j-i}$$

where

$$f \circ g = \sum_{i=0}^{n} a_i \frac{\partial}{\partial \mathbf{y}^{\alpha_i}}(g).$$

We will give an example to better illustrate the definition.

Example 2.3.

Let $f = x_3x_5 + x_1^2 \in R_2$ and $g = y_1^3 \in S_3$, then $f \circ g = 6y_1 \in S_1$.

Remark 2.4. (i) In Example 2.3, note the importance of $\mathrm{char}(k) = 0$, for instance if $\mathrm{char}(k) = 2$, then we would have $f \circ g \in k$.

(ii) The apolarity action of R on S makes S into an R-module, namely for $r, r_1, r_2 \in R$ and $s, s_1, s_2 \in S$ we have,

(1) $r \circ (s_1 + s_2) = r \circ s_1 + r \circ s_2$

(2) $(r_1r_2) \circ s = r_1 \circ (r_2 \circ s)$

(3) $(r_1 + r_2) \circ s = r_1 \circ s + r_2 \circ s$

(4) $1_R \circ s = s$.

However, S is not a finitely generated R-module, because if we assume S to be generated by $f_1, ..., f_k$, then any polynomial $f \in S$ with

$$\deg(f) > \max\{\deg(f_i) : i = 1, ..., k\}$$

can never be obtained since the apolarity action lowers degree.

Definition 2.5. Let R be a commutative ring and let M, N and P be R-modules. An R-**bilinear map** is a function

$$f : M \times N \longrightarrow P$$

such that for any $r \in R$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$ satisfies

(i) $f(rm, n) = f(m, rn) = r \cdot f(m, n)$

(ii) $f(m_1 + m_2, n) = f(m_1, n) + f(m_2, n)$

(iii) $f(m, n_1 + n_2) = f(m, n_1) + f(m, n_2)$.

Now, if $e \in k$, $r \in R$ and $s \in S$, we have that

$$(er) \circ s = r \circ (es) = e(r \circ s).$$

Furthermore, we have that S is an R-module so that for any $j \in \mathbb{N}$ the apolarity action gives a k-bilinear map,

$$R_j \times S_j \longrightarrow k. \tag{2.1}$$

If V and W are two k-vector spaces then, whenever having a k-bilinear map $\circ : V \times W \to k$ given by $(v, w) \mapsto v \circ w$, we will have two induced k-linear maps

$$\phi : V \longrightarrow \mathrm{Hom}_k(W, k) \quad \text{and} \quad \psi : W \longrightarrow \mathrm{Hom}_k(V, k),$$

where $\phi(v) := \phi_v$ with $\phi_v(w) = v \circ w$, similarly we define $\psi(w) := \psi_w$ with $\psi_w(v) = v \circ w$.

With this, we are ready to state the definition of the "perp", but first let us give the definition of *nonsingular* bilinear pairings and some basic propositions that will be useful later.

2.2 Nonsingular bilinear map

Definition 2.6. If the maps ϕ and ψ are isomorphisms then the bilinear map $V \times W \longrightarrow k$ is called **nonsingular**.

We recall that if W and V are k-vector spaces and $T : V \to W$ is a linear map, then

$$\dim(\mathrm{Im}(T)) + \dim(\ker(T)) = \dim V.$$

In particular if $\dim V = \dim W = n$, in order to prove that T is an isomorphism, it is enough to prove either injectivity or surjectivity and the one will imply the other. We use this idea to prove the following proposition.

Proposition 2.7. The bilinear map $V \times W \longrightarrow k$ is nonsingular iff for any basis $\{v_1, ..., v_n\}$ of V and $\{w_1, ..., w_n\}$ of W the $n \times n$ matrix $(b_{ij} = v_i \circ w_j)$ is invertible.

Proof.

$V \to \mathrm{Hom}_k(W, k)$ is an isomorphism.

$\iff$ It has trivial kernel.

$\iff$ The only vector v satisfying $\phi_v(w) = 0$ for all w is $v = 0$.

$\iff$ The only vector v satisfying $\phi_v(w_j) = 0$ for all j is $v = 0$.

$\iff$ The only α_i satisfying $\phi_{a_1v_1+...+a_nv_n}(w_j) = 0$ for all j are $\alpha_i = 0$.

$\iff$ The only α_i satisfying $\sum \alpha_i b_{ij} = 0$ for all j are $\alpha_i = 0$.

$\iff$ The matrix $(b_{ij} = v_i \circ w_j)$ has trivial left null space.

$\iff$ The matrix $(b_{ij} = v_i \circ w_j)$ is invertible.

By a similar proof, we can conclude that $W \to \mathrm{Hom}_k(V, k)$ is an isomorphism iff the matrix $(b_{ij} = v_i \circ w_j)$ is invertible. □

With Proposition 2.7 in mind, let us state the next proposition.

Proposition 2.8. The bilinear map

$$R_j \times S_j \longrightarrow k$$

induced by the apolarity action, is nonsingular.

Proof. Let $\{\mathbf{x}^{\alpha_1}, ..., \mathbf{x}^{\alpha_n}\}$ be the monomials of R_j and $\{\mathbf{y}^{\alpha_1}, ..., \mathbf{y}^{\alpha_n}\}$ the monomials of S_j, then the $n \times n$ matrix $(b_{ij} = \mathbf{x}^{\alpha_i} \circ \mathbf{y}^{\alpha_j})$ is a diagonal matrix whose determinant is different from 0, thus the matrix is invertible.

□

2.3 Perp ideal

Definition 2.9. If $V \times W \longrightarrow k$ is a k-bilinear map and $V_1 \subseteq V$ is a subvector space, we define the **perp** of V_1, denoted ${V_1}^{\perp}$, as

$${V_1}^{\perp} := \{w \in W : \psi_w(V_1) = 0\} \subseteq W$$

Likewise, if $W_1 \subseteq W$, we define

$$W_1{}^{\perp} := \{v \in V : \phi_v(W_1) = 0\}.$$

If $F \in S_j$, we define the **annihilator** of F as

$$\operatorname{Ann}(F) := \{G \in R : G \circ F = 0\}.$$

With abuse of language, we occasionally write $\operatorname{Ann}(F) = F^{\perp}$, but this has nothing to do with Definition 2.9.

Example 2.10. Let $F = y_0^{a_0} \cdot ... \cdot y_n^{a_n} \in S_j$ be a monomial, then we have $F^{\perp} = (x_0^{a_0+1}, x_1^{a_1+1}, ..., x_n^{a_n+1})$.

Example 2.11. Let $F \in S_j$, and let $\partial \in R_1$. Then,

$$(\partial F)^{\perp} = \{G \in R : G \circ (\partial \circ F) = 0\}.$$

However, by (ii) in Remark 2.4, we have $G \circ (\partial \circ F) = (G\partial) \circ F$. Hence, $(\partial F)^{\perp}$ consist of all $G \in R$ such that $G\partial \in F^{\perp}$, and this set can be constructed by considering all elements in $F^{\perp}$ that is divisible by ∂.

Proposition 2.12. Let $V \times W \longrightarrow k$ be a nonsingular k-bilinear map where $n = \dim_k V = \dim_k W$, if $V_1 \subseteq V$ with $\dim_k V_1 = t$ then,

$$\dim_k V_1{}^{\perp} = n - t.$$

Proof. Let $\{v_1, ..., v_t\}$ be a basis for V_1. We extend this basis for V_1 to a basis for V by $\Lambda = \{v_1, ..., v_t, v_{t+1}, ..., v_n\}$. Since

$$\phi : V \to \operatorname{Hom}_k(W, k)$$

is an isomorphism, the basis $\{v_1, ..., v_n\}$ for V will correspond to the basis $\{\phi_{v_1}, ..., \phi_{v_n}\}$ for $\operatorname{Hom}_k(W, k)$. Now, for $\phi_{v_1} : W \to k$, it exist a $w_1 \in W$ such that $\phi_{v_1}(w_1) = 1$, and since

$$\dim \operatorname{Im}(\phi_{v_1}) + \dim \ker(\phi_{v_1}) = \dim W,$$

we must have $\phi_{v_i}(w) = 0$ for any other $w \in W$. Similary, for ϕ_{v_2}, it exist a $w_2 \in W$ such that $\phi_{v_2}(w_2) = 1$, and $\phi_{v_2}(w) = 0$ for all other $w \in W$, furthermore

$w_1 \neq w_2$ because otherwise ϕ_{v_1} and ϕ_{v_2} would not be linearly independent. Hence, continuing this way , we construct the set $\Lambda^* = \{w_1, ..., w_n\}$ of elements in W with the property that $v_i \circ w_j = \delta_{ij}$. This set is a basis for W; to see this note that the elements are linearly independent, because if we can write

$$w_i = a_1 w_1 + ... + a_{i-1} w_{i-1} + a_{i+1} w_{i+1} + ... + a_n w_n,$$

for some w_i, then we would have

$$\phi_{v_i}(w_i) = \phi_{v_i}(a_1 w_1 + ... + a_{i-1} w_{i-1} + a_{i+1} w_{i+1} + ... + a_n w_n),$$

however $\phi_{v_i}(w_j) = 0$ whenever $i \neq j$, contradiction.

We claim that

$$V_1{}^{\perp} = \langle w_{t+1}, ..., w_n \rangle.$$

Obviously $w_{t+1}, ..., w_t \in V_1{}^{\perp}$. Now let $w = a_1 w_1 + ... + a_n w_n$ be an element of $V_1{}^{\perp}$ where $a_i \in k$, then we have

$$v_i \circ w = a_i \quad \text{and} \quad v_i \circ w = 0, \quad \text{for } i = 1, .., t.$$

Hence, $a_1 = a_2 = ... = a_t = 0$, in other words $w \in \langle w_{t+1}, ..., w_n \rangle$. We conclude that $\dim_k V_1{}^{\perp} = n - t$.

□

2.4 Inverse System

We will now give the definition of Inverse Systems.

Definition 2.13. Let I be a homogeneous ideal of the ring R. The **inverse system** of I, denoted I^{-1}, is the R-submodule of S consisting of all elements of S which are annihilated by I, i.e

$$I^{-1} = \{G \in S : F \circ G = 0, \forall F \in I\}.$$

The inverse system I^{-1} is not generally an ideal of S. For instance consider $I = (x_1^2)$, then $y_1 \in I^{-1}$ but, $y_1^2 \notin I^{-1}$ since $x_1^2 \circ y_1^2 = 2$.

Remark 2.14. (i) If I is a homogeneous ideal of R then we can look at

$$[I^{-1}]_j := I^{-1} \cap S_j.$$

Whenever saying that I^{-1} is graded we simply mean that I^{-1} can be written as a direct sum

$$I^{-1} = \bigoplus_{j \in \mathbb{N}} [I^{-1}]_j,$$

but this does not mean that it is graded as an R-submodule of S.

This allows us to describe the inverse system of a homogeneous ideal degree by degree, as the following example wants to illustrate.

Example 2.15. Let $I = (x^2) \subset k[x]$ then by definition we have

$$I^{-1} = \{s \in S : x^2 \circ s = 0\}$$

Now I^{-1} is graded so we can look at it in each degree. If $ay \in S_1$, we have $x^2 \circ ay = 0$; instead, for $ay^2 \in S_2$ we have $x^2 \circ ay^2 \neq 0$, continuing this way we see that

$$I^{-1} = k \oplus \langle y \rangle,$$

where $\langle y \rangle$ is the k-vectorspace generated by y.

It is not always as easy as in Example 2.15 to describe I^{-1}. The following provides a tool to compute I^{-1} in general . Since

$$R_j \times S_j \longrightarrow k$$

is a pairing and I_j is a k-vector subspace of R_j, it makes sense to talk about $I_j^{\perp}$. Now by definition we have,

$$I_j \times I_j^{\perp} \longrightarrow 0$$

which gives

$$(I^{-1})_j \subseteq I_j^{\perp}.$$

Actually, the following proposition holds.

Proposition 2.16. Let I be a homogeneous ideal of R, then

$$(I^{-1})_j = I_j^{\perp}.$$

Proof. The inclusion $(I^{-1})_j \subseteq I_j^{\perp}$ is given above. Suppose that $G \in I_j^{\perp}$, we want to prove that $F \circ G = 0$ for all $F \in I$. The following three cases covers the proof.

Case 1: If $\deg(F) = j$ then, since $G \in I_j^{\perp}$, we have $F \circ G = 0$ for all $F \in I_j$.

Case 2: If $\deg(F) > j$ then, $F \circ G = 0$ because the apolarity action lowers degree and $\deg(G) = j$.

Case 3: If $\deg(F) < j$, choose $a_1, ..., a_n$ such that

$$\sum_{i=1}^{n} a_i = j - \deg(F).$$

It follows that $\deg(\prod_{i=1}^{n} x_i^{a_i} F) = j$ and so we have

$$(\prod_{i=1}^{n} x_i^{a_i} F) \circ G = 0 \Longleftrightarrow \prod_{i=1}^{n} x_i^{a_i} \circ (F \circ G) = 0$$

which implies that $F \circ G$ is annihilated by any monomial of degree $j - \deg(F)$. Note that $\deg(\prod_{i=1}^{n} x_i^{a_i}) = j - \deg(F)$ and $F \circ G \in S_{j-\deg(F)}$, since the bilinear pairing

$$R_{j-\deg(F)} \times S_{j-\deg(F)} \longrightarrow k$$

is nonsingular, we have $F \circ G = 0$.

$\square$

Example 2.17. If I is a monomial ideal, i.e an ideal generated by a finite set of monomials, Proposition 2.16 allows us to describe the inverse system of I in the following way. In each degree j, I_j is the k-vector space generated by monomials of degree j. If we let A be the set of all monomials in S_j, we define

$$B := \{f \in A : f(x_0, ..., x_n) \notin I_j\},$$

then $I_j^{\perp}$ is generated by all elements in B as a k-vector space.

Remark 2.18. Proposition 2.16 allows us also to obtain information about the Hilbert series of R/I as follows; in any degree j , we have

$$\mathrm{HF}(R/I, j) = \dim_k(R_j + I)/I = \dim_k(R_j/I_j),$$

which follows by Remark 1.2. Now consider the bilinear pairing

$$R_j \times S_j \longrightarrow k,$$

since $I_j \subset R_j$, we have by Proposition 2.12, $\dim_k I_j^{\perp} = \dim_k R_j - \dim_k I_j$. Furthermore, since $(I^{-1})_j = I_j^{\perp}$, it follows that

$$\mathrm{HF}(R/I, j) = \dim_k(I^{-1})_j, \text{ for any j.}$$

At first sight it is not clear that Apolarity theory has anything to do with Hilbert functions, the connection described above is quite remarkable, in particular we have a new tool to study Hilbert functions which might give us new valuable information. After all the study of mathematical problems is the study of describing them with different words.

We will now give a tool for computing the inverse system of $I \cap J$ where I and J are homogeneous ideals of R.

Lemma 2.19. Let $V \times W \longrightarrow k$ be a nonsingular k-bilinear pairing with $\dim_k V = \dim_k W = n$. If V_1 and V_2 are subspaces of V, then

$$(V_1 \cap V_2)^{\perp} = V_1^{\perp} + V_2^{\perp}.$$

Proof. $V_1 \cap V_2 \subseteq V_i$, so that if $w \in V_i^{\perp}$ then $w \in (V_1 \cap V_2)^{\perp}$ for $i = 1, 2$, thus we have $V_i^{\perp} \subseteq (V_1 \cap V_2)^{\perp}$ for $i = 1, 2$ which implies that $V_1^{\perp} + V_2^{\perp} \subseteq (V_1 \cap V_2)^{\perp}$.

For the other inclusion: Note that $V_1^{\perp} \cap V_2^{\perp} = (V_1 + V_2)^{\perp}$. Since $V_1^{\perp}$ and $V_2^{\perp}$ are k-vector spaces and the pairing in nonsingular we have

$$\dim_k(V_1^{\perp} + V_2^{\perp}) = \dim_k V_2^{\perp} + \dim_k V_1^{\perp} - \dim_k(V_1^{\perp} \cap V_2^{\perp})$$

$$= (n - \dim_k V_1) + (n - \dim_k V_2) - \dim_k(V_1 + V_2)^{\perp}$$

$$= (n - \dim_k V_1) + (n - \dim_k V_2) - (n - \dim_k(V_1 + V_2))$$

$$= n - \dim_k(V_1 \cap V_2) = \dim_k(V_1 \cap V_2)^{\perp}.$$

Since we have proven one inclusion and the two k-vector spaces have the same dimension, they must be equal.

□

Proposition 2.20. Let I and J be homogeneous ideals of R. Then

$$(I \cap J)^{-1} = I^{-1} + J^{-1}.$$

Proof. The inverse system is graded thus by Lemma 2.19 and Proposition 2.16 the result follows immediately □

Remark 2.21. Let $R = k[x_0, ..., x_n]$. We recall that if $I = (f_1, ..., f_k)$ and $J = (g_1, ..., g_t)$ are two monomial ideals of R, then $I \cap J$ is also a monomial ideal generated by the elements $h_{ij} = \text{lcm}(f_i, g_j)$. To see this, note that if $G \in I \cap J$, then every summand in G is divisible by some generator f_i of I and some generator g_j of J; thus every summand in G is divisible by some $h_{ij} = \text{lcm}(f_i, g_j)$. Conversely, if every summand in G is disvisible by some h_{ij} then it most certainly is disvisible by some g_j and f_i. We use this idea in the next example.

Example 2.22. Let $R = k[x_0, x_1]$, and consider the monomial ideals $I = (x_0, x_1^2x_0^2, x_1^3)$ and $J = (x_1^2, x_0^2)$. By Remark 2.21, we obtain

$$I \cap J = (x_0^2, x_0x_1^2, x_1^3).$$

As in Example 2.15, we can construct $(I \cap J)^{-1}$ piece by piece in each degree. Let $ay_0 + by_1 \in S_1$, then $ay_0 + by_1$ is annihilated by every generator of $I \cap J$. Instead, for $ay_0^2 + by_0y_1 + cy_1^2 \in S_2$, only by_0y_1 and cy_1^2 are annihilated by every generator of $I \cap J$. Continuing this way, we see that

$$(I \cap J)^{-1} = k \oplus \langle y_0, y_1 \rangle \oplus \langle y_0y_1, y_1^2 \rangle.$$

By using the same method as above, we see that

$$I^{-1} = k \oplus \langle y_1 \rangle \oplus \langle y_1^2 \rangle \text{ and } J^{-1} = k \oplus \langle y_0, y_1 \rangle \oplus \langle y_0y_1 \rangle.$$

From here, it is clear that

$$(I \cap J)^{-1} = I^{-1} + J^{-1}.$$

2.5 Artinian rings

In the same notation as in the previous sections, we consider the polynomial rings $R = k[x_0, ..., x_n]$ and $S = k[y_0, ..., y_n]$, together with the standard gradation. Throughout, we will assume all ideals to be homogeneous.

Up to Remark 2.29, R is assumed to be a general commutative ring.

Definition 2.23. A commutative ring R is an **Artinian** ring if every descending chain of ideals

$$I_1 \supseteq I_2 \supseteq ... \supseteq I_k \supseteq ...$$

eventually stabilizes, i.e for some k, $I_k = I_{k+h}$, $\forall h \geq 0$.

Likewise an R-module M is an **Artinian** module if every descending chain of submodules eventually stabilizes.

Remark 2.24. Note that commutative ring R is said to be **Noetherian** if every ascending chain of ideals

$$I_1 \subseteq I_2 \subseteq I_3 ...$$

eventually stabilizes, i.e for some k, $I_k = I_{k+h}$, $\forall h \geq 0$. For example every field k is Noetherian, since a field only has two ideals. Furthermore, the polynomial ring $k[x_1, ..., x_n]$ in n variables is Noetherian, which follows by Hilbert's basis theorem [MR95, Theorem 3.6].

If R is an Artinian ring then it is also Noetherian, however the converse is not true; for instance consider the ring of integers $\mathbb{Z}$.

The ring R can be seen as a k-vector space, so can R/I. It should be noted that a k-vector space and a k-module are the same, the definitions are word by word identical. For the next proposition we will use two well-known theorems in commutative algebra.

Theorem 2.25. [AM69, Theorem 8.7] *(Structure theorem for Artinian rings)* An Artinian ring R is (up to isomorphism) a finite direct sum of Artinian local rings.

Theorem 2.26. [AM69, Corollary 5.24] *(Hilbert's Nullstellensatz)* Let k be a field and B a finitely generated k-algebra. If B is a field, then it is a finite extension of k.

Remark 2.27. (i) We recall that an R-module M that does not have any nonzero proper submodules is called a **simple** module.

(ii) Let R be a k-algebra, then R is a **finite k-algebra** if it is finite as a k-module. Furthermore, R is said to be **finitely generated** if it exists a finite number of elements $a_1, .., a_n \in R$ such that $R = k[a_1, ..., a_n]$. Note that if R is finite then it is also finitely generated, but the converse is not true in general.

Proposition 2.28. Let k be a field and R a finitely generated k-algebra, then R is Artinian if and only if R is a finite k-algebra.

Proof. First, assume that R is a finite k-algebra, in other words $\dim_k R < \infty$. Note that an ideal I of R is a k-vector subspace of R. Now, letting

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq ...$$

be a descending chain of ideals, we see that it must eventually stabilize, because

$$\dim_k I_1 > \dim_k I_2 > ...$$

and $\dim_k R < \infty$.

Conversely, assume that R is Artinian. Then, by Theorem 2.25 , we can write R as a finite direct sum of Artinian local rings, say

$$R = \bigoplus_{i=1}^{n} R_i.$$

Now, pick any R_t from the finite set $\{R_1, ..., R_n\}$ and let $\mathfrak{m}$ be the maximal ideal of R_t. Since R is a finitely generated k-algebra, i.e $R = k[a_1, ..., a_n]$ with $a_i \in R$, it follows that so is R_t. Considering the natural map

$$k \to R_t/\mathfrak{m},$$

given by $a \mapsto a + \mathfrak{m}$ we see that $R_t/\mathfrak{m}$ is a finitely generated k-algebra, furthermore it is a field since $\mathfrak{m}$ is a maximal ideal; thus, by Hilbert's Nullstellensatz, it follows that $R_t/\mathfrak{m}$ is a finite (algebraic) extension of k, in otherwords $\dim_k R_t/\mathfrak{m} < \infty$.

Now, consider R_t as a module over itself, since it is Artinian, let

$$0 = m_0 \subset m_1 ... \subset m_n = \mathfrak{m} \subset R_t \tag{2.2}$$

be the longest possible descending chain of submodules of R_t.

Now, fix $i \in \{1, ..., n\}$, and set $N := m_i/m_{i-1}$, note that then N is a simple module, otherwise (2.2) would not have been the longest possible descending chain of submodules. We claim that $N \cong R_t/\mathfrak{m}$. Fix $0 \neq n \in N$ and consider the homomorphism

$$\phi : R_t \to N$$

given by $r \mapsto rn$. Then, since ϕ is surjective, we have

$$R_t/\ker\phi \cong N.$$

If $\ker\phi$ is a maximal ideal of R_t and hence equals $\mathfrak{m}$, we are done. If not, it exist a proper ideal J of R_t with $\ker\phi \subset J$ and so $J/\ker\phi$ is a proper ideal of $R_t/\ker\phi$; thus N contains a proper submodule, contradicting the definition of N as a simple module and this proves the claim.

Consider the s.e.s

$$0 \to m_n \hookrightarrow R_t \twoheadrightarrow R_t/m_n \to 0,$$

then we have

$$\dim_k R_t = \dim_k m_n + \dim_k R_t/m_n,$$

and since we know that $\dim_k R_t/m_n$ is finite, we need to calculate $\dim_k m_n$. Consider the s.e.s

$$0 \to m_{n-1} \hookrightarrow m_n \twoheadrightarrow m_n/m_{n-1} \to 0,$$

then, we have $\dim_k m_n = \dim_k m_{n-1} + \dim_k m_n/m_{n-1}$ and since $\dim_k m_n/m_{n-1} = \dim_k R_t/m_n$, we need to calculate $\dim_k m_{n-1}$. So, consider the s.e.s

$$0 \to m_{n-2} \hookrightarrow m_{n-1} \twoheadrightarrow m_{n-1}/m_{n-2} \to 0,$$

then, $\dim_k m_{n-1} = \dim_k m_{n-2} + \dim_k m_{n-1}/m_{n-2}$, and now we have to calculate $\dim_k m_{n-2}$. Continuing this way, we see that $\dim_k R_t < \infty$, for any R_t in the finite direct sum of R. Hence, $\dim_k R < \infty$. □

Remark 2.29. As a direct consequence of Proposition 2.28 we see that the ring R/I is Artinian if and only if $\dim_k R/I < \infty$ which occurs if and only if $I_j = R_j$ for some $j \in \mathbb{N}$.

2.6 Gorenstein rings

In the following, we denote the image of $x \in R$ under the natural map $R \twoheadrightarrow R/I$ by $\tilde{x}$, furthermore we let m be the maximal ideal $m = (x_0, ..., x_n)$ of R and A be a homogeneous quotient of R, i.e $A = R/I$ for some homogeneous ideal I.

Definition 2.30. The **socle** of A, denoted $\mathrm{Soc}(A)$ is the subset of A defined by

$$\mathrm{Soc}(A) := (0 : m) = \{g \in A : g\tilde{m} = 0\}$$

Let us give example in order to get confident with Definition 2.30.

Example 2.31. Consider $A = k[x_0, x_1]/(x_0^3, x_1)$, then

$$A = k \oplus (k\tilde{x}_0) \oplus (k\tilde{x}_0^2).$$

It is clear that $\tilde{x}_0^2 \in \mathrm{Soc}(A)$ and in fact we have $\mathrm{Soc}(A) = (\tilde{x}_0^2)$.

Example 2.32. Consider $A = k[x_0, x_1]/(x_0^2, x_1^2, x_0x_1)$, then

$$A = k \oplus (k\tilde{x}_0 \oplus k\tilde{x}_1).$$

Hence, it follows that $\mathrm{Soc}(A) = (\tilde{x}_0, \tilde{x}_1)$.

If $A = k[x_0, ..., x_n]/I$, then for a homogeneous element $f \in A$, we have $f \in \mathrm{Soc}(A) \implies f\tilde{x}_i = 0$ for $i = 0, ..., n$, conversely if f is annihilated by $\tilde{x}_i$ for $i = 0, ..., n$ then it is annihilated by every element of $\tilde{m}$, thus $f \in \mathrm{Soc}(A)$. Note that if A is an Artinian ring, we can write

$$A = k \oplus A_1 \oplus ... \oplus A_\varrho,$$

where $A_\varrho \neq 0$, then $A_\varrho \subseteq \mathrm{Soc}(A)$. If this was not the case then we would have an element $0 \neq \tilde{g} \in A$ with $\deg g > \varrho$ so that $A_{\deg \varrho} \neq 0$.

Definition 2.33. Let A be Artinian as above, i.e.

$$A = k[x_0, ..., x_n]/I = k \oplus A_1 \oplus ... \oplus A_\varrho,$$

with $A_\varrho \neq 0$. The natural number ϱ is called the **socle degree** of A, denoted $\overline{\mathrm{Soc}}(A)$.

Remark 2.34. The socle degree of A is the least postive integer such that $m^{\varrho+1} \subseteq I$. It is clear that any element of $m^{\varrho+1}$ is in I, otherwise this would contradict the fact $A_{\varrho+1} = 0$, and since $A_k \neq 0$ for $k < \varrho$, it follows that ϱ is the least such integer.

Definition 2.35. The graded Artinian ring A is called a **Gorenstein ring** if $\dim_k \mathrm{Soc}(A) = 1$.

In particular, if A is Artinian with $\overline{\mathrm{Soc}}(A) = \varrho$, then we see that A is Gorenstein if and only if $\mathrm{Soc}(A) = A_\varrho$ and $\dim A_\varrho = 1$. For example, the Artinian ring A in Example 2.31 is Gorenstein, but the Artinian ring in Example 2.32 is not.

Proposition 2.36. Let A be an Artinian Gorenstein ring with $\overline{\mathrm{Soc}}(A) = \varrho$. Then

$$\mathrm{HF}(A, d) = \mathrm{HF}(A, \varrho - d),$$

for $d \in \mathbb{Z}$.

Proof. First note that $A_\varrho \cong k$. So, for $t \neq \varrho$, consider the pairing

$$A_t \times A_{\varrho-t} \longrightarrow A_\varrho \tag{2.3}$$

induced by the multiplication of the ring A. Since

$$\dim_k[\mathrm{Hom}_k(A_{\varrho-t}, k)] = \dim_k A_{\varrho-t},$$

which holds because $A_{\varrho-t}$ is finite dimensional, the result follows if we can prove that the pairing (2.3) is nonsingular; thus we have $\mathrm{Hom}_k(A_{\varrho-t}, k) \cong A_t$, and this would conclude our proof.

We proceed by proving that if $a \in A_t$ and $ab = 0$ for all $b \in A_{\varrho-t}$ then $a = 0$. The set $A_{\varrho-t}$ is generated by a finite set of monomials $\tilde{\mathbf{x}}^{\alpha}$, where $\alpha = (a_0, ..., a_n)$ such that $\sum_{i=0}^{n} a_i = \varrho - t$, by assumption we have $a\tilde{\mathbf{x}}^{\alpha} = 0$; thus

$$(a\tilde{\mathbf{x}}^{\alpha'})\tilde{x}_i = 0,$$

for all $i = 0, ..., n$, where $\deg \alpha' = \varrho - t - 1$, i.e. $a\tilde{\mathbf{x}}^{\alpha'} \in \mathrm{Soc}(A)$. However, $\deg a\tilde{\mathbf{x}}^{\alpha'} = t + (\varrho - t - 1) = \varrho - 1$, so that we must have $a\tilde{\mathbf{x}}^{\alpha'} = 0$. If we continue the process illustrated above we may step by step lower the degree of $a\tilde{\mathbf{x}}^{\alpha'}$ and hence obtain $a\tilde{x}_i = 0$ for $i = 0, ..., n$, and so $a \in \mathrm{Soc}(A)$. But, since $\deg a = t \neq \varrho$, we have $a = 0$ and this completes the proof.

□

Remark 2.37. Let A be an Artinian ring, then A is Gorenstein if and only if the pairing

$$A_t \times A_{\varrho-t} \longrightarrow A_\varrho$$

is nonsingular for $0 \leq t \leq \varrho$. Indeed from Proposition 2.36 it follows that if A is an Artinian ring with $\overline{\mathrm{Soc}}(A) = \varrho$ and $\dim A_\varrho = 1$, then if A is Gorenstein, the pairing

$$A_t \times A_{\varrho-t} \longrightarrow A_\varrho$$

is nonsingular for $0 \leq t \leq \varrho$. Conversely if the pairing is nonsingular then A is Gorenstein, to see this assume that A is not Gorenstein then it exist an element $0 \neq a \in A_t$, for some $t < \varrho$, such that $a \in \mathrm{Soc}(A)$ and so every element in $A_{\varrho-t}$ is annihilated by a, contradicting the fact that the pairing is nonsingular.

We will soon state Macaulay's theorem, but first let us give one remark on the inverse system.

Remark 2.38. If $R = k[x_0, ..., x_n]$ and $S = k[y_0, ..., y_n]$ then for an ideal I of R we have

$$I^{-1} \text{ is finitely genetaed } R\text{-submodule} \iff I \text{ is an Artinian ideal.}$$

To see this note that I^{-1} is finitely generated $\iff$ $(I^{-1})_j = 0$ for all but finitely many j however this happens if and only if $\mathrm{HF}(R/I, j) = 0$ for all but finitely many j and this occurs if and only if I is an Artinian ideal.

2.7 Macaulay's theorem

Theorem 2.39. (Macaulay) Let $R = k[x_0, ..., x_n]$ and let $A = R/I$ be Artinian, then

$$A \text{ is Gorenstein with } \overline{\mathrm{Soc}}(A) = \varrho \iff I = \mathrm{Ann}(F) \text{ for some } F \in S_\varrho.$$

The theorem tells us that whenever having an Artinian Gorenstein ring R/I we see that $I = F^\perp$ for some homogeneous $F \in S_j$ and conversely taking any homogeneous element $F \in S_j$ we may construct the Artinian Gorenstein ring $A = R/F^\perp$; thus we have obtained a useful $1-1$ correspondence between the Artinian Gorenstein rings and the perp of homogeneous elements in S.

In order to prove Macaulay's theorem, we will follow the notes of A.V. Geramita [Ger96].

2.7.1 Ancestor ideal

If $R = k[x_0, ..., x_n]$ and $V \subseteq R_j$, we will define the set $V : R_i$ as

$$V : R_i := \{g \in R_{j-i} : gR_i \subseteq V\},$$

which is a k-vector subspace of R_{j-i}.

Definition 2.40. Let $R = k[x_0, ..., x_n]$ and $V \subseteq R_j$. We define the set $\overline{V}$ as

$$\overline{V} := \Big[\sum_{i=j}^{1} V : R_i\Big] \oplus (V)$$

where

$$\Big[\sum_{i=j}^{1} V : R_i\Big] = \langle V : R_j\rangle \oplus \langle V : R_{j-1}\rangle \oplus ... \oplus \langle V : R_1\rangle$$

and $(V) = V \oplus R_1V \oplus R_2V \oplus ...$

We will now give the first proposition needed in order to prove Macaulay's theorem.

Proposition 2.41. The set $\overline{V}$ described in Definition 2.40 is a homogeneous ideal of R, and it is the largest ideal J of R for which

$$J_{j+t} = (V)_{j+t},$$

for all $t \in \mathbb{N}$.

Proof. We will first prove that $\overline{V}$ is an ideal, recall that an ideal I is homogeneous if and only if whenever $a = a_1 + a_2 + ... + a_n \in I$ with a_i homogeneous, then $a_i \in I$. Now if $A, B \in \overline{V}$, then we may carry out the addition $A + B$ componentwise and we will obtain a new element in $\overline{V}$, in other words $\overline{V}$ is closed under addition.

If $B \in \overline{V}$ is an element of degree $\geq j$, then $B \in (V)$; thus $AB \in (V)$ for any $A \in R$. The only multiplication left to consider is whenever

$$B \in R_t \ \ (t \in \mathbb{N}) \ \text{ and } \ H \in \langle V : R_i \rangle,$$

for $1 \leq i \leq j$.

Case 1: If $t \geq i$ then we may split every summand in B so that

$$B = \sum_k F_k G_k,$$

where $\deg G_k = i$ and $\deg F_k = t - i$. Then

$$BH = (\sum_k F_k G_k) H = \sum_k F_k (G_k H),$$

but $G_k \in R_i$ and $H \in \langle V : R_i \rangle$; so that by definition we must have $HG_k \in V$. Hence, $F_k(HG_k) \in (V)$, and this completes the proof for the first case.

Case 2: If $t < i$, then $\deg BH = t + (j - i) = (t - i) + j < j$, and $BH \in R_{t+j-i}$. We will prove that $BH \in \left[\sum_{i=j}^{1} V : R_i\right]$, more precisly we will prove that

$$BH \in \langle V : R_{i-t} \rangle,$$

i.e we want to prove that $(BH)R_{i-t} \subseteq V$. We have

$$(BH)R_{i-t} = H(BR_{i-t}),$$

where $\deg B = t$; thus we are multiplying H by an element of R_i, and we have chosen H such that $HR_i \subseteq V$, so that $(BH)R_{i-t} \subseteq V$ and this completes the proof for the second case. To see why $\overline{V}$ is the biggest homogeneous ideal J of R such that $J_{j+t} = (V)_{j+t}$, for all $t \in \mathbb{N}$, we will give a proof by contradiction. Suppose that $\overline{V} \subseteq J$, and that $\overline{V}_i \subset J_i$, for some $i < j$. Then, it exist an element $T \in J_i$ such that $T \notin \langle V : R_{j-i} \rangle$, in other words, it exists an element $H \in R_{j-i}$ such that $TH \notin V$. However, since $H \in R_{j-i}$ and $T \in J_i$, we must have $TH \in J_j = V$, contradiction.

□

The ideal $\overline{V}$ is called the **ancestor ideal** of V. Two more propositions are needed before we can give a proof for Macaulay's theorem.

Proposition 2.42. If $F \in S_j$ and $I = \mathrm{Ann}(F)$, then

(i) $I_j = \langle F \rangle^{\perp}$ in the pairing $R_j \times S_j \longrightarrow k$.

(ii) $I = \overline{\langle F \rangle^{\perp}} + m^{j+1}$.

Proof. *i)* I consist of all elements in R that annihilates F, and I_j is the subset of R_j that annihilates F, which is precisely $\langle F \rangle^{\perp}$ by the definition of the perp.

ii) We will start with the inclusion $\overline{\langle F \rangle^{\perp}} + m^{j+1} \subseteq I$. The elements in m^{j+1} are at least of degree $j+1$, so they all annihilate $F \in S_j$, furthermore $\langle F \rangle^{\perp} = I_j$ by *i)*; thus we may only consider the elements in $\overline{\langle F \rangle^{\perp}}$ of degree $< j$. Let $G \in \overline{\langle F \rangle^{\perp}}$ with $\deg G = t < j$, then $G \in \langle \langle F \rangle^{\perp} : R_{j-t} \rangle$, and we want to prove that $G \circ F = 0$. By definition we have $GR_{j-t} \subseteq \langle F \rangle^{\perp}$, so that $G\mathbf{x}^{\alpha} \in I_J = \langle F \rangle^{\perp}$ for every monomial $\mathbf{x}^{\alpha} \in R_{j-t}$, it follows that

$$(\mathbf{x}^{\alpha} G) \circ F = 0 \iff \mathbf{x}^{\alpha} \circ (G \circ F) = 0,$$

which holds for every monomial $\mathbf{x}^{\alpha} \in R_{j-t}$. By Proposition 2.8 the pairing

$$R_{j-t} \times S_{j-t} \longrightarrow k$$

is nonsingular, and since $G \circ F \in S_{j-t}$, we have $G \circ F = 0$.

For the inclusion $I \subseteq \overline{\langle F\rangle^{\perp}} + m^{j+1}$, we have three different cases. If $G \in I$ and $\deg G = j$, then $G \in I_j = \langle F\rangle^{\perp}$. If $\deg G > j$, then $G \in m^{j+1}$, and so the last case is when $\deg G = t < j$. Let $H \in R_{j-t}$, so that $GH \in I_j = \langle F\rangle^{\perp}$, but since H is arbitrary we have $GR_{j-t} \subseteq \langle F\rangle^{\perp}$. Hence, $G \in \langle\langle F\rangle^{\perp} : R_{j-t}\rangle$, it follows that $G \in \overline{\langle F\rangle^{\perp}}$.

□

Proposition 2.43. Let $A = R/I$ be an Artinian graded ring with $\overline{\mathrm{Soc}}(A) = j$ and $\dim_k A_j = 1$, then

$$A \text{ is Gorenstein} \iff I = \overline{I_j} + m^{j+1}.$$

Proof. Assume A to be Gorenstein. We start by proving the inclusion $I \subseteq \overline{I_j} + m^{j+1}$. We have $\overline{\mathrm{Soc}}(A) = j$, which simply means that every element $G \in R$ of degree $\geq j+1$ is in I, so that $I_k = (m^{j+1})_k$ for $k \geq j+1$. In degree j, we have $I_j \subseteq \overline{I_j} \subseteq \overline{I_j} + m^{j+1}$ and of course $\overline{I_j} + m^{j+1} \subseteq I$ (in degree j); thus we get the equality in degree j. For the case where $G \in I$ with $\deg G = t < j$ we have $GR_{j-t} \subseteq I_j$, hence $G \in (\overline{I_j})_t$, by the definition of an ancestor ideal.

In order to prove $\overline{I_j} + m^{j+1} \subseteq I$, we first note that since $\overline{\mathrm{Soc}}(A) = j$ we must have $m^{j+1} \subseteq I$ and we have just shown that $\overline{I} + m^{j+1} = I_j$ in degree j; thus what is left to prove is $(\overline{I_j})_t \subseteq I_t$ for $t < j$. Consider the pairing

$$R_t/I_t \times R_{j-t}/I_{j-t} \longrightarrow R_j/I_j$$

given by $(a+I_t, b+I_{j-t}) \mapsto ab+I_j$, and choose $G \in (\overline{I_j})_t$, then $GR_{j-t} \subseteq I_j$ which implies that $\tilde{G}\tilde{\mathbf{x}}^{\alpha} = 0$ in the pairing above for every $\tilde{\mathbf{x}}^{\alpha}$ with $\deg \mathbf{x}^{\alpha} = j - t$. Since the pairing is nonsingular (Remark 2.37) we have $\tilde{G} = 0$ and so $G \in I_t$ which completes the proof.

Conversely, let us assume that $I = \overline{I_j} + m^{j+1}$. In order to prove that R/I is Gorenstein, it will be sufficient to prove that the pairing

$$R_t/I_t \times R_{j-t}/I_{j-t} \longrightarrow R_j/I_j$$

is nonsingular for every $0 \leq t \leq j$ (Remark 2.37). Let $\tilde{H} \in R_t/I_t$ and suppose that $\tilde{H}\tilde{\mathbf{x}}^{\alpha} = 0$ for every $\tilde{\mathbf{x}}^{\alpha}$ with $\deg \tilde{\mathbf{x}}^{\alpha} = j - t$, so we have

$$HR_{j-t} \subseteq I_j \implies H \in (\overline{I_j})_t.$$

However, by assumption, we have $I = \overline{I_j} + m^{j+1}$ so that $H \in I_t$, in other words, $\tilde{H} = 0$ and this completes the proof. $\square$

We are now ready to prove Macaulay's theorem (Theorem 2.39).

Proof. (Macaulay's theorem) Let $A = R/I$ and suppose that $I = \mathrm{Ann}(F)$ with $F \in S_j$. The apolarity pairing

$$R_j \times S_j \longrightarrow k$$

is perfect, since $I_j = \langle F \rangle^\perp$ by Proposition 2.42, it follows by Proposition 2.12 that $1 = \dim_k I_j^\perp = \dim_k R_j - \dim_k I_j$; in other words $\dim_k (R_j/I_j) = 1$. Now we have $F \in S_j$ so that all elements of degree $\geq j+1$ annihilates F, that is, $m^{j+1} \subseteq I$; hence A is an Artinian ring of socle degree j and $\dim_k A_j = 1$. By Proposition 2.43 we have

$$A \text{ is Gorenstein } \iff I = \overline{I_j} + m^{j+1}.$$

However, $I_j = \langle F \rangle^\perp$, and by Proposition 2.42 we have $I = \overline{\langle F \rangle^\perp} + m^{j+1}$, this completes the proof for one implication.

Conversely, let us assume that $A = R/I$ is Gorenstein with $\overline{\mathrm{Soc}}(A) = j$. By Proposition 2.43, we have

$$I = \overline{I_j} + m^{j+1}.$$

Now, since A is Gorenstein, we must have $\dim_k(R_j/I_j) = 1$; thus it exist an $F \in S_j$ such that $I_j = \langle F \rangle^\perp$.What remains to prove is that $I = \mathrm{Ann}(F)$. Let $J = \mathrm{Ann}(F)$, then $J_j = I_j$, however by Proposition 2.42 we have

$$J = \mathrm{Ann}(F) = \overline{J_j} + m^{j+1};$$

but since $I_j = J_j$, it follows that $\mathrm{Ann}(F) = \overline{I_j} + m^{j+1} = I$.

$\square$

Remark 2.44. (i) Let $F = y_0^{a_0} \cdot ... \cdot y_n^{a_n} \in S_j$ be a monomial, by Example 2.10, we have $F^\perp = (x_0^{a_0+1}, ..., x_n^{a_n+1})$; thus $R/F^\perp$ is Artinian. Now, by Macaulay's theorem, $A = R/F^\perp$ is Gorenstein with $\overline{\mathrm{Soc}}(A) = \deg F = j$, furthermore by Proposition 2.36, $\mathrm{HF}(R/F^\perp, d)$ is symmetric.

(ii) Let $F = y_0^{a_0} \cdot ... \cdot y_n^{a_n}$. Then, $\mathrm{HF}(R/F^{\perp}, d)$ is the number of nonzero elements of form $\tilde{x_0}^{b_0} \cdot ... \cdot \tilde{x_n}^{b_n}$, with $\sum_{i=0}^{n} b_i = d$. Now, $\tilde{x_0}^{b_0} \cdot ... \cdot \tilde{x_n}^{b_n}$ is nonzero if and only if $0 \leq b_i < a_i + 1$ for $i = 0, ..., n$, which follows since $F^{\perp} = (x_0^{a_0+1}, ..., x_n^{a_n+1})$. Thus, $\mathrm{HF}(R/F^{\perp}, d)$ is the coefficient of x^d in the generating function

$$(1+x+\cdots+x^{a_0})(1+x+\cdots+x^{a_1})\cdot ... \cdot(1+x+\cdots+x^{a_n}) = \frac{\prod_{i=0}^{n}(1-x^{a_i+1})}{(1-x)^{n+1}}.$$

Since the generators of $F^{\perp}$ is a regular sequence, by Theorem 1.9, the formula above for finding $\mathrm{HF}(R/F^{\perp}, d)$ was expected. We give an example to illustrate how *(ii)* in Remark 2.44 can be used.

Example 2.45. Consider $R = k[x_0, x_1]$, and let $F = y_0^2 y_1 \in S_3$. Then, we have $F^{\perp} = (x_0^3, x_1^2)$, and our generating function is

$$f(x) = (1 + x + x^2)(1 + x) = x^3 + 2x^2 + 2x + 1.$$

Thus,

$$\mathrm{HF}(R/F^{\perp}, 1) = 2,\ \mathrm{HF}(R/F^{\perp}, 2) = 2,\ \mathrm{HF}(R/F^{\perp}, 3) = 1;$$

and it is easily seen that we actually have

$$R/F^{\perp} = k \oplus (k\tilde{x_0} \oplus k\tilde{x_1}) \oplus (k\tilde{x_0}^2 \oplus k\tilde{x_0}\tilde{x_1}) \oplus (k\tilde{x_0}^2\tilde{x_1}).$$

Note that in Example 2.45, we have that $\mathrm{HF}(R/F^{\perp}, d)$ is symmetric and furthermore $R/F^{\perp}$ is Gorenstein with $\overline{\mathrm{Soc}}(R/F^{\perp}) = \deg F = 3$, as expected by *(i)* in Remark 2.44.

Bibliography

[AM69] Atiyah, Michael and Macdonald, Ian, Introduction to Commutative Algebra, Addison-Wesley series in mathematics, 1969.

[Ger96] Geramita, Anthony V., Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals, The Curves Seminar at Queens, Vol. 10, 1996.

[MR95] Reid, Miles. Undergraduate commutative algebra. Vol. 29. Cambridge University Press, 1995.