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Gamma function related to Pick functions

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Abstract

This thesis is divided into two parts. In the first part we will study some properties of the Gamma function, $\Gamma(z)$, which can be viewed as an extension of the factorial function $(n+1) \mapsto n!$ to a subset of the complex plane (more precisely to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$). The Gamma function has several representations and we will represent some of them.

The second part of this thesis is about Pick functions and mainly follows the paper *Pick Functions Related to the Gamma Function* by C. Berg and H.L. Pedersen (see [BP]). Pick functions are holomorphic functions from the open upper complex half plane to the closed upper complex half plane. We will prove that a special class of maps are Pick functions. We end up with proving that $\frac{\mathrm{Log}(\Gamma(z+1))}{z}$ is a Pick function which connects the two subjects of my thesis.

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Chapter 1

Introduction

The aim of this thesis is to give a brief introduction to the Gamma function and Pick functions and prove some of the relations between the Gamma function and Pick functions.

In the second chapter, the definition of the Gamma function is recalled. Some properties of the Gamma function are shown and several representations of the Gamma function are presented.

In the third chapter we deal with Pick functions and prove that a certain class of functions are Pick functions. We end the chapter with proving that $\frac{\operatorname{Log}\Gamma(z+1)}{z}$ is a Pick function, which connects one of the relations between the Gamma function and Pick functions.

The Gamma function was first introduced by the famous mathematician Leonhard Euler (1707-1783) in 1729, as a natural extension of the factorial function $n \mapsto (n-1)!$ on positive integers $n \in \mathbb{N}$ to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Later on, the Gamma function was studied by other famous mathematicians such as Adrien-Marie Legendre, Carl Friedrich Gauss, Karl Weierstrass, Charles Hermite and many others mathematicians.

The Gamma function appears in various mathematical areas and has connections to special transcendental functions, asymptotic series, definite integration, number theory, Riemann zeta function and has also applications in other sciences such as physics and programming.

In this thesis we also follow [BP] and introduce Pick functions which are holomorphic functions from the open upper half-plane $H = \{z \in \mathbb{C} \,:\, \operatorname{Im}(z) > 0\}$

to the closed upper half-plane $H \cup \mathbb{R} = \{z \in \mathbb{C} \ : \ \text{Im}(z) \geq 0\}$. Pick functions have been studied for a long time and under various names like Nevanlinna functions, Herglotz functions and R-functions.

We will show that functions on the form

$$F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{1}{1+t^2} \right) h(t) dt$$

where $\alpha \in \mathbb{R}, \ \beta \geq 0, \ h(t) \geq 0$ and $\int_{-\infty}^{\infty} \frac{h(t)}{1+t^2} dt < \infty$, are Pick functions.

At the end we prove that $\text{Log}(\Gamma(z+1))/z$ is a Pick function, which connects the two subjects of this thesis.

Chapter 2

The Gamma function

2.1 Introduction to the Gamma function

In this chapter the definition of the Gamma function is recalled and some of its properties is proven. The author's main references for this chapter is [An],[Ca],[PB1],[Ra].

The Gamma function is a function $\Gamma : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$ which can be defined in many ways. One way of expressing it is by means of an improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2.1}$$

for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. This defintion may be extended it to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ by methods we will present later. In the first two subsections we will show that (2.1) is well-defined and extends the factorial function $n \mapsto (n-1)!$. The last two subsections will give alternative representations of the Gamma function.

Lemma 2.1.1 *Let* $0 < a < \infty$. *Then*

I. *The integral* $\int_a^\infty x^{-p} dx$ *is equal to* $\frac{a^{1-p}}{p-1}$ *if* $p > 1$ *and is divergent if* $p \leq 1$.

II. *The integral* $\int_0^a x^{-p} dx$ *is equal to* $\frac{a^{1-p}}{1-p}$ *if* $p < 1$ *and is divergent if* $p \geq 1$.

Proof. We only prove II, because the proof of I is similar.

If $p < 1$, we have that

$$\int_0^a x^{-p} dx = \lim_{c \to 0^+} \int_c^a x^{-p} = \lim_{c \to 0^+} \frac{x^{1-p}}{1-p}\bigg|_c^a = \lim_{c \to 0^+} \frac{a^{1-p} - c^{1-p}}{1-p} = \frac{a^{1-p}}{1-p}.$$

If $p > 1$, we have that

$$\int_0^a x^{-p} dx = \lim_{c \to 0^+} \int_c^a x^{-p} = \lim_{c \to 0^+} \frac{x^{1-p}}{1-p}\Big|_c^a = \lim_{c \to 0^+} \frac{a^{1-p} - c^{1-p}}{1-p} = \infty.$$

If $p = 1$,

$$\int_0^a x^{-p} dx = \lim_{c \to 0^+} \int_c^a x^{-1} = \lim_{c \to 0^+} \operatorname{Log}(x)|_c^a = \lim_{c \to 0^+} (\operatorname{Log}(a) - \operatorname{Log}(c)) = \infty.$$

□

The integrals in Lemma 2.1.1 are called p-integrals, and the theorem above can be used in order to prove that some improper integrals are convergent.

Lemma 2.1.2 *Assume $0 \leq f(x) \leq g(x)$ for all $x \in (a, b)$ where $-\infty \leq a < b \leq \infty$. If $\int_a^b g(x)dx$ is convergent, then $\int_a^b f(x)dx$ is also convergent. If $\int_a^b f(x)dx$ is divergent, then $\int_a^b g(x)dx$ is also divergent.*

Proof. See Theorem 11, p.306 in [PB1].

Remark. The improper integral $\int_{-\infty}^{\infty} f(x)dx$ is defined as the sum

$$\lim_{R \to \infty} \int_{-R}^a f(x)dx + \lim_{R \to \infty} \int_a^R f(x)dx$$

where a is any real number. We say that the improper integral converges when both $\lim_{R \to \infty} \int_{-R}^a f(x)dx$ and $\lim_{R \to \infty} \int_a^R f(x)dx$ converge.

2.2 Gamma function on the real line

Recall that the Gamma function restricted to $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ is given by the improper integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

This is of course well-defined only if the integral above is convergent for all points in $\{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$. In this section we just prove that the integral is convergent for all $z \in (0, \infty)$. Thereafter we prove some of the properties of Γ on the real line.

Proposition 2.2.1 *The improper integral*

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

is convergent for $x > 0$.

Proof. We split the integral as follows;

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \int_0^1 e^{-t} t^{x-1} dt + \int_1^\infty e^{-t} t^{x-1} dt \ .$$

In order to show that whole integral converges we show each integral in the right hand side is convergent (as both are generalized).

We have that $\max_{t \in [0,1]} e^{-t} = 1$, implying that $0 \leq e^{-t} t^{x-1} \leq t^{x-1}$ for all $t \in [0, 1]$. Hence

$$0 \leq \int_0^1 e^{-t} t^{x-1} dt \leq \int_0^1 t^{x-1} dt = \lim_{a \to 0} \int_a^1 t^{x-1} dt = \lim_{a \to 0} \left. \frac{t^x}{x} \right|_{t=a}^1 = \frac{1}{x} < \infty$$

From Lemma 2.1.2 it follows that $\int_0^1 e^{-t} t^{x-1} dt$ is convergent.

Now we prove that $\int_1^\infty e^{-t} t^{x-1} dt$ is convergent. Since the $e^{t/2}$ grows faster than t^{x-1} for any $x > 0$, given a fixed $x > 0$ there is some $k_x \in \mathbb{R}_+$ such that for every $t > k_x$ it holds that

$$e^{t/2} > t^{x-1}.$$

This gives that

$$0 \leq \int_1^\infty e^{-t} t^{x-1} dt = \int_1^{k_x} e^{-t} t^{x-1} dt + \int_{k_x}^\infty e^{-t} \underbrace{t^{x-1}}_{\leq e^{t/2}} dt$$

$$\leq \int_1^{k_x} e^{-t} t^{x-1} dt + \int_{k_x}^\infty e^{-t} e^{t/2} dt$$

Obviously $\int_1^{k_x} e^{-t} t^{x-1} dt$ is finite. If we can show that $\int_{k_x}^\infty e^{-t} e^{t/2} dt$ is convergent, then the inequality above together with Lemma 2.1.2 will give that $\int_1^\infty e^{-t} t^{x-1} dt$ is convergent. We have

$$\int_{k_x}^\infty e^{-t} e^{t/2} dt = \lim_{b \to \infty} -2e^{-t/2}|_{k_x}^b = 2e^{-k_x/2} < \infty$$

This completes the proof. $\square$

Next we prove some properties of the Gamma function which show how it is related to the factorial map $(n+1) \mapsto n!$.

Proposition 2.2.2 $\Gamma(x+1) = x\Gamma(x)$ *for all* $x > 0$.

Proof. Integrating by parts gives

$$\Gamma(x+1) = \int_0^\infty e^{-t}t^x dt = \underbrace{\lim_{\substack{R\to\infty \\ c\to 0}} (-e^{-t}t^x|_{t=c}^R)}_{=0} + x\int_0^\infty e^{-t}t^{x-1}dt = x\Gamma(x)$$

$\square$

Proposition 2.2.3 *For any non-negative integer* n, $\Gamma(n+1) = n!$

Proof. We prove the assertion by induction on n. For $n = 0$ we have that

$$\Gamma(0+1) = \int_0^\infty e^{-t}t^{1-1} = \int_0^\infty e^{-t}dt = \lim_{R\to\infty} -e^{-t}|_{t=0}^R = 1.$$

Assuming that the assertion is true for $n-1$, Proposition 2.2.2 yields

$$\Gamma(n+1) = n\Gamma(n) = n \cdot (n-1)! = n!\ .$$

$\square$

Proposition 2.2.4 *The improper integral*

$$\int_0^\infty e^{-t}t^{x-1}dt$$

diverges for $x < 0$.

Proof. We have that $e^{-t} \geq e^{-1}$ for any $t \in [0,1]$. Hence

$$\int_0^1 e^{-t}t^{x-1}dt \geq \int_0^1 e^{-1}t^{x-1}dt = \infty$$

(last equality holds by Lemma 2.1.1). By Lemma 2.1.2, $\int_0^\infty e^{-t}t^{x-1}dt$ diverges. $\square$

Proposition (2.2.4) shows that $\Gamma(x)$ cannot be defined as the improper integral $\int_0^\infty e^{-t}t^{x-1}dt$ for $x < 0$. However we can extend the relation $\Gamma(x+1) = x\Gamma(x)$ (see Proposition 2.2.2) for negative numbers which makes Γ defined for $\mathbb{R} \setminus \mathbb{Z}_{\leq 0}$.

2.3 The Gamma function on the complex plane

In this section we extend the domain of Γ to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. By the previous section we had that Γ is given by the improper integral $\int_0^\infty e^{-t}t^{x-1}dt$ on the positive half line. We will show that this integral is convergent even for complex number with strictly positive realparts.

Proposition 2.3.1 $\int_0^\infty e^{-t}t^{z-1}dt$ *converges for all complex numbers* z *with* $\mathrm{Re}(z) > 0$.

Proof. Let $z = x + iy$ where $x > 0$ and let $t \geq 0$. We have that

$$|e^{-t}t^{z-1}| = |e^{-t}t^{x+iy-1}| = |e^{-t}t^{x-1}t^{iy}| = e^{-t}t^{x-1}|t^{iy}|$$
$$= e^{-t}t^{x-1}|e^{i\cdot\ln(t)y}| = e^{-t}t^{x-1}$$

Since $x = \mathrm{Re}(z) > 0$, it follows from Proposition 2.2.1 that $\int_0^\infty e^{-t}t^{x-1}$ is convergent. This means that $\int_0^\infty e^{-t}t^{z-1}dt$ is absolute convergent, implying the convergence of the integral itself. □

Now we give a more general version of Proposition 2.2.2. Its proof is similar to that of Proposition 2.2.2 and is therfore omitted.

Proposition 2.3.2 *For any* $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0$*, it holds that* $\Gamma(z+1) = z\Gamma(z)$. □

So far we have a function $\Gamma(z)$ whose domain is $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$, and which has the property that $\Gamma(z+1) = z\Gamma(z)$. If we want to extend this function so that the property $\Gamma(z+1) = z\Gamma(z)$ still holds on a larger domain, there is only one possible way of doing that. Given any $z \in \mathbb{C}$ with $-n < \mathrm{Re}(z) < -(n-1)$, we define $\Gamma(z)$ by

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{(z+1)z} = \cdots = \frac{\Gamma(z+n)}{(z+n)(z+n-1)\cdots z}$$
$$= \frac{\int_0^\infty e^{-t}t^{z+n-1}dt}{(z+n)(z+n-1)\cdots z}. \tag{2.2}$$

Note that if $z \in \mathbb{Z}_{n\leq 0}$, it is impossible to define $\Gamma(z)$ as above, since we will have zero in the denominator in the last equality.

Now we are ready to define the Gamma function as a function on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Definition 2.3.3 *The Gamma function* $\Gamma : \mathbb{C} \setminus \mathbb{Z}_{\leq 0} \to \mathbb{C}$ *is the function whose restriction to* $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ *is given by* $\int_0^\infty e^{-t} t^{z-1} dt$*, and which is extended to the rest of its domain by the relation* $\Gamma(z) = \frac{\Gamma(z+1)}{z}$ *as in* (2.2)*.*

2.4 The Gauss representation

In this section we will present another representation of the Gamma function, which is called the Gauss representation of the Gamma function. This representation is needed in order to introduce a third representation of the Gamma function, called the Weierstrass representation.

Recall that $e^t = \lim_{n\to\infty} \left(1 + \frac{t}{n}\right)^n$.

Lemma 2.4.1 *The following holds*

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} dt$$

for $z \in \mathbb{C}$ *with* $\mathrm{Re}(z) > 0$*.*

Proof. See last paragraf in p.17 in [Ra]. □

Integrating $\int_0^n t^{z-1}(1 - t/n)^n dt$ by parts repeatedly gives

$$\begin{aligned}
\int_0^n t^{z-1}(1-t/n)^n dt &= \underbrace{\frac{t^z}{z}\left(1 - \frac{t}{n}\right)\Big|_{t=0}^{n}}_{=0} + \int_0^n \frac{n}{nz} t^z \left(1 - \frac{t}{n}\right)^{n-1} dt \\
&= \frac{n}{zn}\frac{n-1}{(z+1)n} \int_0^n t^{z+1}\left(1 - \frac{t}{n}\right)^{n-2} dt \\
&\ \vdots \\
&= \frac{n}{zn}\frac{n-1}{(z+1)n}\cdots\frac{1}{(z+n-1)n}\int_0^n t^{z+n-1} dt \\
&= \frac{n}{zn}\frac{n-1}{(z+1)n}\cdots\frac{1}{(z+n-1)n}\frac{t^{z+n}}{z+n}\Big|_{t=0}^{n} \\
&= \frac{n!}{n^n} n^{z+n} \prod_{k=0}^{n} \frac{1}{z+k} \\
&= \frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k}
\end{aligned}$$

By letting n go to infintity, we get $\Gamma(z)$ for $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$. Of course, this representation is defined on $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, but we have not shown that it will agree with Γ on $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0\}$. Let

$$\tilde{\Gamma}(z) = \lim_{n\to\infty} \frac{n^z}{z} \prod_{k=1}^{n} \frac{k}{z+k} \ . \tag{2.3}$$

If we can show that $\tilde{\Gamma}(z+1) = z\tilde{\Gamma}(z)$, then Γ and $\tilde{\Gamma}$ will agree on all of $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$ We have that

$$\begin{aligned}
\tilde{\Gamma}(z+1) &= \lim_{n\to\infty} \frac{n^{z+1}}{z+1} \prod_{k=1}^{n} \frac{k}{z+1+k} = \lim_{n\to\infty} \frac{n}{z+1} \frac{zn^z}{z} \prod_{k=1}^{n} \frac{k}{z+1+k} \\
&= \lim_{n\to\infty} \frac{n}{z+1} \frac{zn^z}{z} \cdot \frac{n!}{\prod_{k=2}^{n+1} z+k} = \lim_{n\to\infty} \underbrace{\frac{z+1}{z+n+1} \frac{n}{z+1}}_{\overset{n\to\infty}{\longrightarrow} 1} \frac{zn^z}{z} \prod_{k=1}^{n} \frac{k}{z+k} \\
&= z \lim_{n\to\infty} \frac{n^z}{z} \prod_{k=1}^{n} \frac{k}{z+k} = z\tilde{\Gamma}(z)
\end{aligned}$$

Hence, $\Gamma = \tilde{\Gamma}$. The product representation (2.3) of the Gamma function is called the Gauss representation.

2.5 The Weierstrass representation

In this section we present another representation of the Gamma function, which will be used in the next chapter. We start by following lemma and definition.

Definition/Lemma 2.5.1 *The limit*

$$\gamma = -\lim_{n\to\infty}\left(\ln(n) - \sum_{k=0}^{n}\frac{1}{k}\right)$$

exists and is finite. The constant γ is called the Euler–Mascheroni constant.

Proof. See §7.9 in [PB1] □

Now we manipulate the Gauss representation of the Gamma function until it attains the desired form, which is called the Weierstrass representation of the Gamma function.

Recall the Gauss representation

$$\Gamma(z) = \lim_{n\to\infty}\frac{n^z}{z}\prod_{k=1}^{n}\frac{k}{z+k}$$

defined for $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. For $z \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we rewrite n^z as $\exp(z\,\mathrm{Log}\,n)$ and we rewrite $k/(z+k)$ as $(1+z/k)^{-1}$, giving that

$$\Gamma(z) = \lim_{n\to\infty}\frac{\exp(z\,\mathrm{Log}\,n)}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}.$$

We add $0 = \sum_{k=1}^{n} z/k - \sum_{k=1}^{n} z/k$ to the argument of exp gives

$$\Gamma(z) = \lim_{n\to\infty}\frac{\exp(\sum_{k=1}^{n}\frac{z}{k} - \sum_{k=1}^{n}\frac{z}{k} + z\,\mathrm{Log}\,n)}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}$$

$$= \lim_{n\to\infty}\frac{\exp(\sum_{k=1}^{n}\frac{z}{k})\cdot\exp(z\,\mathrm{Log}\,n - \sum_{k=0}^{n}\frac{z}{k})}{z}\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)^{-1}$$

We know that $\exp(\sum_{k=1}^{n}\frac{z}{k}) = \prod_{k=1}^{n} e^{z/k}$ and that $\lim_{n\to\infty} z\,\mathrm{Log}\,n - \sum_{k=0}^{n}\frac{z}{k} = -\gamma z$. Hence the expression above becomes

$$\Gamma(z) = \lim_{n\to\infty}\frac{e^{-\gamma z}}{z}\prod_{k=1}^{n} e^{\frac{z}{k}}\left(1+\frac{z}{k}\right)^{-1}.$$

The last expression is called the Weierstrass representation of the Gamma function.

Chapter 3

Relations of the Gamma function to Pick functions

In this section we mainly follow [BP] where some relations between Pick functions and the Gamma function are shown.

We start by defining Pick functions.

Definition 3.0.2 *A Pick function is a holomorphic function F from the open upper half-plane $H = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) > 0\}$ to the closed upper half-plane $H \cup \mathbb{R} = \{z \in \mathbb{C} \ : \ \mathrm{Im}(z) \geq 0\}$.*

In the first section we prove that functions on the form

$$F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$$

where $\alpha \in \mathbb{R}, \ \beta \geq 0, \ h(t) \geq 0$ and $\int_{-\infty}^{\infty} \frac{h(t)}{1+t^2} dt < \infty$.
In the second and the third section, we prove that $-\mathrm{Log}(1+z)/z$ and $-\mathrm{Log}(\Gamma(1+z))/z$ are Pick functions on the form above.

3.1 A class of Pick functions

The goal of this section is to prove that functions on the form

$$F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt \tag{3.1}$$

where $\alpha \in \mathbb{R}, \ \beta \geq 0, \ h(t) \geq 0$ and $\int_{-\infty}^{\infty} \frac{h(t)}{1+t^2} dt < \infty$, are Pick functions.

In order to do that we need following lemma

Lemma 3.1.1 *Let $f : \Omega \times \mathbb{R} \to \mathbb{C}$, where $\Omega \subseteq \mathbb{C}$. Assume that $f(z,t)$ and $f'_z(z,t)$ are continuous for all $z \in \mathbb{C}$ and all $t > t_0$ for some $t_0 \in \mathbb{R}$. Moreover assume that the integral*

$$F(z) = \int_{t_0}^{\infty} f(z,t)dt$$

is convergent for all $z \in \mathbb{C}$ and such that there is a real valued function $g : \mathbb{R} \to \mathbb{R}$ such that
(a) $|f'_z(z,t)| < g(t)$ for all $t > t_0$,
(b) $\int_{t_0}^{\infty} g(t)dt$ is convergent.

Then $F(z)$ is differentiable and the derivative is given by

$$F'(z) = \int_{t_0}^{\infty} f'_z(z,t)dt$$

Proof. See Theorem 3, p.189 in [PB2]. □

Now we prove the main theorem of this section.

Theorem 3.1.2 *Functions on the form*

$$F(z) = \alpha + \beta z + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$$

where $\alpha \in \mathbb{R}$, $\beta \geq 0$, $h(t) \geq 0$ and $\int_{-\infty}^{\infty} \frac{h(t)}{1+t^2}dt < \infty$, are Pick functions.

Proof. We need to prove for all $z \in H$ that i) $\int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$ is convergent, ii) $F(z)$ is holomorphic, and iii) Im $F(z) \geq 0$. In order to do that we fix an arbitrary compact set K in the open upper half plane. If we can show that i),ii),iii) holds on K then this holds on the open upper half plane (since every point in the upper half plane is a member of the interior of some compact set K in the open upper half plane).

i) We prove that each of $\int_{-\infty}^{0} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$ and $\int_{0}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$ are convergent for each $z \in K$ (in order to prove that their sum, i.e. $\int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t)dt$ is convergent). We start by rewriting the integrand as follows;

$$\left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t) = \frac{1+tz}{t-z} \cdot \frac{h(t)}{1+t^2} \ .$$

Since K is compact set in the upper half plane, we have that $d(K, \mathbb{R}) = \varepsilon > 0$. Hence for each $t \in \mathbb{R}$ and $z \in K$ we have that $|t - z| \geq \varepsilon$. In particular that means $\int_a^b \frac{1+tz}{t-z} \cdot \frac{h(t)}{1+t^2} dt$ is convergent for all $z \in K$ and all finite intervals (a, b).

Now we investigate the nature of the integrand as t gets big. Taking absolute value of the integrand and writing z on the form $x + iy$ gives

$$\left| \frac{1+tz}{t-z} \cdot \frac{h(t)}{1+t^2} \right| = \left| \frac{1+t(x+iy)}{t-(x+iy)} \right| \frac{h(t)}{1+t^2} = \sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} \frac{h(t)}{1+t^2}$$

We have

$$\lim_{t \to \infty} \sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} = \lim_{t \to \infty} \sqrt{\frac{t^2(\frac{1}{t^2} + \frac{2x}{t} + x^2 + y^2)}{t^2(1 - \frac{2x}{t} + \frac{x^2}{t^2} + \frac{y^2}{t^2})}} = \sqrt{x^2 + y^2}$$

In particular this means that there exsits some $t_0 > 0$ such that for any $t > t_0$ we have that

$$\sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} \leq 1 + \sqrt{x^2 + y^2} = 1 + |z|$$

This inequality can be used in order to prove that $\int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t) dt$ is absolute convergent:

$$\begin{aligned}
\int_0^\infty \left| \frac{1}{t-z} - \frac{1}{1+t^2} \right| h(t) dt &= \int_0^\infty \sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} \cdot \frac{h(t)}{1+t^2} dt \\
= \int_0^{t_0} \sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} \cdot \frac{h(t)}{1+t^2} dt \quad &+ \int_{t_0}^\infty \sqrt{\frac{(1+tx)^2 + (ty)^2}{(t-x)^2 + y^2}} \cdot \frac{h(t)}{1+t^2} dt \\
\leq \int_0^{t_0} \sqrt{\frac{1 + 2tx + t^2x^2 + t^2y^2}{t^2 - 2tx + x^2 + y^2}} \cdot \frac{h(t)}{1+t^2} dt &+ \int_{t_0}^\infty (1+|z|) \frac{h(t)}{1+t^2} dt \\
= \int_0^{t_0} \sqrt{\frac{1 + 2tx + t^2x^2 + t^2y^2}{t^2 - 2tx + x^2 + y^2}} \cdot \frac{h(t)}{1+t^2} dt &+ (1+|z|) \int_{t_0}^\infty \frac{h(t)}{1+t^2} dt
\end{aligned}$$

The last expression is finite as $\int_0^{t_0} \sqrt{\frac{(1+tx)^2+(ty)^2}{(t-x)^2+y^2}} \cdot \frac{h(t)}{1+t^2} dt$ is a proper integral, and $(1+|z|) \int_{t_0}^\infty \frac{h(t)}{1+t^2} dt$ is convergent. This implies the (absolute) convergence of $\int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t) dt$.

Similar arguments shows that $\int_{-\infty}^0 \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t) dt$ is convergent.

ii) We will prove that $F(z)$ is analytic in the interior of K, which is equivalent to prove that $F(z)$ is differentiable there. Let $f(z,t) = \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) h(t)$. Then $f_z'(z,t) = \frac{h(t)}{(t-z)^2}$ and

$$|f_z'(z,t)| = \left|\frac{h(t)}{(t-z)^2}\right| = \frac{h(t)}{(t-x)^2+y^2}$$

Now it is easily seen that

$$\lim_{t\to\infty} \frac{\frac{h(t)}{(t-x)^2+y^2}}{\frac{h(t)}{1+t^2}} = 1$$

Hence there exists some $t_z > 0$ such that for any $t > t_z$ we have that

$$\frac{\frac{h(t)}{(t-x)^2+y^2}}{\frac{h(t)}{1+t^2}} < 2 \quad \Longleftrightarrow \quad \frac{h(t)}{(t-x)^2+y^2} < 2\frac{h(t)}{1+t^2}$$

$$\Longleftrightarrow \quad \left|\frac{h(t)}{(t-z)^2}\right| < 2\frac{h(t)}{1+t^2}\ .$$

Now let $t_0 := \max_{z\in K} t_z$ (this value exists as K is compact).

Now we prove that $G(z) = \int_0^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) h(t)dt$ is differentiable by proving that each of $G_1(z) = \int_0^{t_0} \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) h(t)dt$ and $G_2(z) = \int_{t_0}^\infty \left(\frac{1}{t-z} - \frac{t}{1+t^2}\right) h(t)dt$ are differentiable.
From Lemma 3.1.1 it follows directly that $G_1(z)$ is differentiable.
Since $\left|\frac{h(t)}{(t-z)^2}\right| < 2\frac{h(t)}{1+t^2}$ for all $t > t_0$, and since $\int_{t_0}^\infty 2\frac{h(t)}{1+t^2}$ is convergent, it follows from Lemma 3.1.1 that $G_2(z)$ is differentiable. Hence $G(z) = G_1(z) + G_2(z)$ is differentiable, and also $F(z) = \alpha + \beta z + G(z)$ is differentiable. Thus condition ii) is satisfied.

iii) One can easily compute that the imaginary part of the integrand is given by

$$\frac{y}{(t-x)^2+y^2}h(t)$$

(where $x + iy = z$) and hence the imaginary part of $F(z)$ is given by

$$\beta y + \int_{-\infty}^{\infty} \frac{y}{(t-x)^2+y^2}h(t)dt$$

which is positive (as $y = \text{Im}(z) > 0$, $h(t) \geq 0$, and $\beta \geq 0$). Now it follows that $\text{Im}(F(z)) \geq 0$. $\square$

3.2 The function $-\operatorname{Log}(1+z)/z$

Let Log denote the principal logarithm function, which is analytic on $\mathbb{C} \setminus (-\infty, 0]$ (readers who are unfamiliar with complex logarthims are refered to the appendix). We want to show that

$$-\frac{\operatorname{Log}(1+z)}{z} = -\frac{\pi}{4} + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \frac{dt}{-t} \, .$$

The results from the previous section yields that $-\operatorname{Log}(1+z)/z$ is a Pick function.

Theorem 3.2.1 *The function $-\operatorname{Log}(1+z)/z$ restricted to the open upper half plane is a Pick function whose integral representation is given by*

$$-\frac{\operatorname{Log}(1+z)}{z} = -\frac{\pi}{4} + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \frac{dt}{-t} \tag{3.2}$$

Proof. The right-hand side of (3.2) is a Pick function by Theorem 3.1.2, so we only need to prove the equality (3.2). By differentiating

$$F(t) = \arctan(t) + \frac{1}{z} \operatorname{Log}\left(\frac{t}{t-z} \right)$$

one can easily check that $F(t)$ is a primitive function to the integrand of the integral in (3.2). Note that $F(t)$ is analytic for $t \in (-\infty, 1)$ and $z \in H$ (since $t/(t-z) \notin (-\infty, 0]$). Hence

$$-\frac{\pi}{4} + \int_{-\infty}^{-1} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) \frac{dt}{-t} = -\frac{\pi}{4} + \lim_{a \to -\infty} (F(-1) - F(a))$$

$$= -\frac{\pi}{4} + \lim_{a \to -\infty} \left(\arctan(-1) + \frac{1}{z} \operatorname{Log}\left(\frac{-1}{-1-z} \right) - \arctan(a) - \operatorname{Log}\left(\frac{a}{a-z} \right) \right)$$

$$= -\frac{\operatorname{Log}(1+z)}{z} \, .$$

The equality $\operatorname{Log}(1/(1+z)) = -\operatorname{Log}(1+z)$ may need a justification. We have that $\operatorname{Im}(z) > 0$ and hence $Arg(1+z) \in (0, \pi)$ and we have that $1/(1+z) = (1-z)/|1+z|^2$ giving that $Arg(1/(1+z)) \in (-\pi, 0)$. Moreover, we have that $Arg(1/(1+z)) = -Arg(1+z) + ik2\pi$ for some $k \in \mathbb{Z}$, but since both $Arg(1/(1+z))$ and $-Arg(1+z)$ are in $(-\pi, 0)$ the equality $\operatorname{Log}(1/(1+z)) = -\operatorname{Log}(1+z)$ follows. This completes the proof. □

This result will be used in next section where we connect the subject of the Gamma function with the subject of Pick functions.

3.3 The function $\frac{\mathrm{Log}_\alpha \Gamma(1+z)}{z}$

We prove the main theorem of this thesis.

Theorem 3.3.1 *There is some branch* Log_α *of the multivalued logarithmic function* log *such that* $\mathrm{Log}_\alpha(\Gamma(1+z))/z$ *is a Pick function and is of the form*

$$\alpha + \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) h(t) dt \tag{3.3}$$

where $\alpha \in \mathbb{R},\ h(t) \geq 0$ *and* $\int_{-\infty}^{\infty} \frac{h(t)}{1+t^2} dt < \infty$*,*

Proof. Let log denote the multivalued logarithmic function. Recall that $\Gamma(z+1) = z\Gamma(z)$ (see Lemma 2.3.2) and recall the Weierstrass representation of the Gamma function (see section 2.5). That gives

$$\frac{\log(\Gamma(z+1))}{z} = \frac{\log(z) + \log(\Gamma(z))}{z} = \frac{\log(z) + \log\left(\frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} e^{z/k} \left(1+\frac{z}{k}\right)^{-1}\right)}{z}$$

$$= \frac{\log(z) - \gamma z - \log(z) + \sum_{k=1}^{\infty} \left(\frac{z}{k} - \log(1+\frac{z}{k})\right)}{z}$$

$$= -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{\log(1+\frac{z}{k})}{z} \right)$$

If we can show that $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{\mathrm{Log}(1+\frac{z}{k})}{z} \right)$ is convergent for all $z \in H$ (where Log is the principal logarithm), it follows from the equalities above that there is some branch Log_α of log such that

$$\frac{\mathrm{Log}_\alpha(\Gamma(1+z))}{z} = -\gamma + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{\mathrm{Log}(1+\frac{z}{k})}{z} \right) \tag{3.4}$$

for all $z \in H$.
We have that $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{\mathrm{Log}(1+\frac{z}{k})}{z} \right)$ is convergent if and only if $\int_1^{\infty} \left(\frac{1}{t} - \frac{\mathrm{Log}(1+\frac{z}{t})}{z} \right) dt$ is convergent.
Basic computations show that

$$\int_1^{\infty} \left(\frac{1}{t} - \frac{\mathrm{Log}(1+\frac{z}{t})}{z} \right) dt = \left[-\frac{t\,\mathrm{Log}\left(\frac{t+z}{t}\right)}{z} + \mathrm{Log}\left(\frac{t}{t+z}\right) \right]_{t=1}^{\infty}$$

$$= \left(\frac{1}{z} + 1\right) \mathrm{Log}(z+1) - 1$$

which has a finite value for all $z \in H$. Hence, there is some branch Log_α such that equality (3.4) holds. If we rewrite $\mathrm{Log}_\alpha(\Gamma(z+1))/z$ as follows

$$\frac{\mathrm{Log}_\alpha(\Gamma(1+z))}{z} = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{\mathrm{Log}(1+\frac{z}{k})}{z}\right) \tag{3.5}$$

$$= -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k}\left(1 - \frac{\mathrm{Log}(1+\frac{z}{k})}{\frac{z}{k}}\right)\right)$$

By the previous section we have that $1 - \frac{\mathrm{Log}(1+\frac{z}{k})}{z/k}$ is a Pick function for all k. Hence it follows that $\mathrm{Log}_\alpha(\Gamma(z+1))/z$ is a sum of Pick functions, and will therefore be a Pick function.

Now we want show that $\mathrm{Log}_\alpha(\Gamma(z+1))/z$ can be expressed on the form (3.3).

If we can show that $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{\mathrm{Log}(1+\frac{z}{k})}{z}\right)$ is a Pick function of the form (3.1), it will follow from equality (3.5) that $\mathrm{Log}_\alpha(\Gamma(z+1))/z$ is a Pick function of the form (3.3).

By the previous section we had that

$$-\frac{\mathrm{Log}(1+\frac{z}{k})}{z/k} = -\frac{\pi}{4} + \int_{-\infty}^{-1}\left(\frac{1}{t-z/k} - \frac{t}{1+t^2}\right)\frac{dt}{-t}$$

Multiplying both sides with $1/k$ and doing the change of variables $t = s/k$ gives

$$\begin{aligned}
&-\frac{\mathrm{Log}(1+\frac{z}{k})}{z} = -\frac{\pi}{4k} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+k^2}\right)\frac{ds}{-s}\\
&= -\frac{\pi}{4k} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1} + \frac{s}{s^2+1} - \frac{s}{s^2+k^2}\right)\frac{ds}{-s}\\
&= -\frac{\pi}{4k} + \int_{-\infty}^{-k}\left(\frac{s}{s^2+1} - \frac{s}{s^2+k^2}\right)\frac{ds}{-s} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\frac{ds}{-s}\\
&= -\frac{\pi}{4k} + \left(-\arctan(s) + \frac{1}{k}\arctan(s/k)\right)\Big|_{-\infty}^{-k} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\frac{ds}{-s}\\
&= \arctan(k) - \frac{\pi}{2} + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\frac{ds}{-s}\\
&= -\arctan(1/k) + \int_{-\infty}^{-k}\left(\frac{1}{s-z} - \frac{s}{s^2+1}\right)\frac{ds}{-s}
\end{aligned}$$

Hence

$$\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{\mathrm{Log}(1+\frac{z}{k})}{z}\right)=\sum_{k=1}^{n}\left(\frac{1}{k}-\arctan(1/k)+\int_{-\infty}^{-k}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)\frac{ds}{-s}\right)$$

$$=\sum_{k=1}^{n}\left(\frac{1}{k}-\arctan(1/k)\right)+\sum_{k=1}^{n}\left(\int_{-\infty}^{-k}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)\frac{ds}{-s}\right)$$

$$=\sum_{k=1}^{n}\left(\frac{1}{k}-\arctan(1/k)\right)+\int_{-\infty}^{-1}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)\varphi_n(s)ds \qquad (3.6)$$

where

$$\varphi_n(s)=\begin{cases}-(k-1)/s & \text{for } s\in[-k,-k+1[,\ k=2,...,n,\\ -n/s & \text{for } s<-n.\end{cases}$$

Note that $0<\varphi_n(s)\leq 1$ for all $n\in\mathbb{Z}_{\geq 2}$ and all $s\in(-\infty,-1)$, so it follows that $\int_{-\infty}^{-1}\frac{\varphi_n(s)}{1+s^2}ds$ is convergent and hence $\int_{-\infty}^{-1}\left(\frac{1}{s-z}-\frac{s}{s^2+1}\right)\varphi_n(s)ds$ is a Pick function. Thus, $\sum_{k=1}^{n}\left(\frac{1}{k}-\arctan(1/k)\right)$ converges to some real number as $n\to\infty$ (since the limit of the expression in (3.6) is equal $\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{\mathrm{Log}(1+\frac{z}{k})}{z}\right)$, which we have shown is convergent). In other words, $\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{\mathrm{Log}(1+\frac{z}{k})}{z}\right)$ is a Pick function of the form (3.3), and hence $\mathrm{Log}_\alpha(\Gamma(1+z))/z$ is a Pick function of the form (3.3). This completes the proof.

Appendix A

Logarthmic functions

In this appendix we recall the notion of logarithms and arguments of complex numbers. We mainly follow [SS].

A.1 Arguments of complex numbers

An argument of a non-zero complex number z, is a real number θ that equals one of the angles between the line joining the point z to the origin and the positive real axis. Arguments of complex numbers are not unique, and any two possible values for the argument differs by an integer multiple of 2π. We define $\arg(z)$ to be the set of all possible arguments of z.

It is convenient to have a notation for some definite value of $\arg(z)$. Notice that any half-open interval of length 2π will contain just one value of the argument of z. By choosing such an interval for possible values on the argument, we say that we have selected a particular branch of $\arg(z)$.

The branch of $\arg(z)$ specified from the interval $(-\pi, \pi]$ is called the principal value of the argument and is denoted by $Arg(z)$. The relation between $\arg(z)$ and $Arg(z)$ is given by $\arg(z) = Arg(z) + 2k\pi$, where $k \in \mathbb{Z}$. Note that $Arg(z)$ is discontinuous as z crosses the negative real line. The line of discontinuities is called the branch cut.

A.2 Logarithms of complex numbers

Recall that the logarithm of a non-zero complex number z, is a number w such that $e^w = z$. Note that there are infinitely many such numbers since $e^w = z$ if and only if $e^{w+ik2\pi} = z$.

In particular we must have that $\mathrm{Re}(w) = \ln(|z|)$ (where ln is the real logarithmic function) and $\mathrm{Im}(w) = Arg_\alpha(z)$ for some branch Arg_α of arg. For any non-zero complex number z let $\log(z)$ be the set of all logarithms of z. Sometimes one is refering to $\log(z)$ as a multiple-valued function;

$$\log(z) = \ln|z| + i\arg(z) = \ln|z| + iArg(z) + i2k\pi \ .$$

In particular this multiple-valued function inherits the familiar properties of the real logarithmic function, namely

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

and

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2) \ .$$

It is convenient to have a notation for some definite value of $\log(z)$. Notice that for any half-open interval of length 2π, say $(a-\pi, a+\pi]$ we have that $\log(z) \cap \{w \in \mathbb{C} \mid \mathrm{Im}(w) \in (a-\pi, a+\pi]\}$ contains exactly one element. By choosing such an interval for possible values on the imaginary part of the logarithm, we say that we have selected a particular branch of $\log(z)$.

The branch of $\log(z)$ specified from the interval $(-\pi, \pi]$ is called the principal value of the logarithm and is denoted by $\mathrm{Log}(z)$ and is given by

$$\mathrm{Log}(z) = \ln(|z|) + iArg(z) \ .$$

The relation between $\log(z)$ and $\mathrm{Log}(z)$ is given by

$$\log(z) = \mathrm{Log}(z) + i2k\pi = \ln(|z|) + iArg(z) + i2k\pi \ ,$$

where $k \in \mathbb{Z}$. It is discontinuous and as z crosses the negative real line. The line of discontinuities is called the branch cut.

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