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# Dynamic Programming and Applications in Economics

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## Abstract

The thesis investigates how Dynamic Programming can be applied to Economics. Theory of time discrete Dynamic Programming is described and some example problems are examined. Applications to economic theory are thereafter studied, with focus on three different problems with relevance to micro-, macro- and financial economics. Finally, the general validity of mathematics in economic theory is discussed.

# Contents

# Chapter 1

# Background

The process of Dynamic Programming was established in the 1950's by the American scientist Richard Bellman. He was interested in developing a general model for producing optimal solutions to problems involving some sequence of controls. At the time, though, Bellman worked at a government institution which head, the Secretary of Defence Charles Erwin Wilson, despised research and particularly mathematical research [1]. He therefore struggled with the idea that somehow find a way to disguise the actual intentions of his work. He thus chose to name the project in a way that would stay clear of Wilson's suspicion and at the same time capture the essence of what the project involved. The result was Dynamic Programming. Dynamic, for the focus on time as an essential component and Programming, a term that was mainly recognized as the process of finding an optimal program for the scheduling of military training and industrial production.

Dynamic Programming presents a powerful tool for finding an optimal solution to problems that involve repeated controls. Since Bellman, it has gained success in a broad spectrum of applications, among them bioinformatics, computer programming and economics. This thesis will focus on the last one of these, namely - Economical applications of the Dynamic Programming algorithm. With the division of large problems into a sequence of sub-problems, Dynamic Programming gives an intuitive tool to guide economic choices, like what quantities a firm should keep in its stock at the end of each day or how much of the salary one should save versus consume each month, in order to maximize some utility function over a given time period.

### Aims of the Present Thesis

The present work aims to (1) give an expository study to the theory of Dynamic Programming and (2) analyze its applications to economics. A delimitation, with regard to the scope of the thisis has been made, such that the content focus on discrete-time problems. As a complement to the theoretical overview, some example problems will be examined. The main purpose of this is however to lay a foundation that enables the thesis to derive a number of economic applications. Finally, a general discussion of how mathematics in economical analysis should be regarded will be presented.

# Chapter 2

# Introduction to Dynamic Programming

> Life must be lived forward and
> understood backwards
>
> *Soren Kierkegaard*

Dynamic programming (DP) is an optimization technique, captured in the words of the Danish philosopher Kierkegaard; we investigate a problem by examining it backwards. DP utilizes that a basic problem with discrete time periods can be divided into N subsequent subproblems and has an additive cost function in the form of

$$G_N(x_N) + \sum_{t=0}^{N-1} g_t(x_t, u_t, w_t),$$

where $G_N(x_N)$ describes the terminal cost at the end of the time period and $g_t(x_t, u_t, w_t)$ is the cost function for each subproblem. $x_t$ is defined as the state variable which summarizes past information relevant for future optimization. Further, the state variables interdependency is expressed as

$$x_{t+1} = f_t(x_t, u_t, w_t), t = 0, 1..., N-1 \tag{1}$$

where $t$ indexes discrete time,
$u_t$ is the control or decision variable to be selected at time $t$,
$w_t$ is a random parameter or disturbance/noise variable,
$N$ is the horizon or number of times control is applied, and lastly
$f_t$ is a function that describes the system and in particuar the mechanism by which the state is updated.

However, from the randomness of the disturbance variable $w_k$ the basic problem generally is in the form of optimizing an expected cost function. Hence, we have the following objective function:

$$E\Bigg\{G_N(x_N) + \sum_{t=0}^{N-1} g_t(x_t, u_t, w_t)\Bigg\}. \tag{2}$$

The aim of the DP algorithm is to find an optimal set of controls, $u_k$, in order to minimize or maximize the cost function, although in the case of a maximization problem the cost function is usually regarded as a reward function. In general, the controls are determined in two different ways; in a closed loop form, where each control is chosen to enable the interpretation of additional information from earlier controls, or in an open loop form, where all the controls are chosen at time $t = 0$. However, the DP algorithm has the advantage of producing a closed-loop control mapping for all problems applied. We denote this as follows:

Let

$$\pi = (\mu_0, \mu_1 ... \mu_{N-1}),$$

be a policy for the basic problem. Then $\mu_t$ is the chosen policy function at time $t$ which maps the state $x_t$ into controls $u_t = \mu_t(x_t)$, thus a feedback control. The objective is thus to find an optimal control function $\mu_t^*$ for all times $t$.

## 2.1 Principle of Optimality

Bellman stated that an optimal solution has the property of being optimal from any point forth on the trajectory course. This is a vital part of the DP algorithm and may be demonstrated as following: We assume that we are interested in finding the optimal travel route across Sweden, from Stockholm to Gothenburg. Then if the optimal path is through Jönköping then the chosen trajectory between Jönköping and Gothenburg is also the optimal choice for this particular subproblem and so forth for every given point on the optimal trajectory. The sum of the optimal solutions to the subproblems defines an optimal policy and hence we formulate the Bellman Principle of Optimality [2]:

Let $\pi_t = (\mu_0, \mu_1 ... \mu_{N-1})$ be a policy for the basic problem. From (1) and (2) we can write that this policy leads to the following cost-to-go function from time $s \geq 0$ onward:

$$J_\pi(x_s) = E\Bigg\{G_N(x_N) + \sum_{t=s}^{N-1} g_t(x_t, \pi_t, w_t)\Bigg\}, s \geq 0. \tag{2.1}$$

Denote

$$J_{\pi^*}(x_s) = \inf_{\pi \in \Pi} J_\pi(x_0),$$

where $\Pi$ is the set of all allowable policies. Then an optimal policy leads to

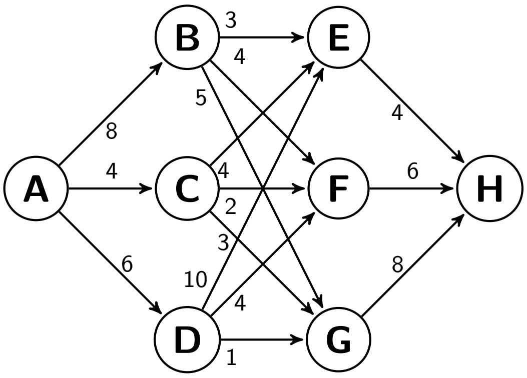
$$J^*(x_0) \overset{\text{def}}{=\!=} J_{\pi^*}(x_0), \tag{2.2}$$

which contain a truncated policy from any point $i \in [0, N-1]$ such that $\pi_i = \{\mu_i^*, \mu_{i+1}^*, ..., \mu_{N-1}^*\}$.

**Example I: Shortest path**

We demonstrate the principle of optimality by using the above to solve a basic problem. A widely used application of DP is as an algorithm for solving so called shortest path (SP) problems, which revolves around question like the already mentioned task of choosing an optimal travelling path from Stockholm to Gothenburg. More formally expressed, SP problems are on the form of finding a path between nodes in a graph, such that the sum of the constituent edges are of minimal weight (cost).

We define a graph as a set of nodes $I = \{A, B, C, D, E, F, G, H\}$ and a set of corresponding Edges $E = (i, j)$, where $i, j \in I$, that define connections between two nodes from $i$ to $j$. Let us assume the following graph, where each arrow define an edge:



The objective is to find the best possible route from A to H, with respect taken to the costs associated with each edge and we may formulate this in the DP form. Our objective is to minimize the deterministic cost function

$$\min J(x_0) = G_3(x_3) + \sum_{t=0}^{2} g_t(x_t, u_t),$$

from the starting state $x_0 = A$ to the end state $x_3 = H$, with terminating cost $G_3(x_3) = 0$ and where the function $g_t$ represents the cost moving from $x_t$ to an adjacent node according to the decision $u_t$. Hence the system is subject to the following state evolution:

$$x_{t+1} = u_t(x_t),$$

where $u_t \in \Pi$ and $\Pi$ is the subset of edges $(i, j)$ such that $i = x_t$. The control $u_t$ then represents the decision to go from the present node $x_t$ to an adjacent node $x_{t+1}$.

We may apply the principle of optimality as stated in the above, by using a backward recursive approach. If an optimal cost function $J^*(X)$ is set to represent the minimal cost when moving from X to H, we instantly find that $J^*(H) = 0$, $J^*(E) = 4$, $J^*(F) = 6$ and $J^*(G) = 8$. Keeping this in memory, we continue with the problem recursively for each time $t$ toward $t = 0$ and get

$$
\begin{aligned}
J^*(B) =& \min\big(3 + J^*(E), 4 + J(F), 5 + J^*(G)\big)\\
J^*(C) =& \min\big(4 + J^*(E), 2 + J(F), 3 + J^*(G)\big)\\
J^*(D) =& \min\big(10 + J^*(E), 4 + J(F), 1 + J^*(G)\big).
\end{aligned}
$$

Optimizing this we find that

$$
\begin{aligned}
J^*(B) = 3 + J^*(E) = 7\\
J^*(C) = 4 + J^*(E) = 2 + J*(F) = 8\\
J^*(D) = 1 + J^*(G) = 9,
\end{aligned}
$$

which lead us to the final step

$$J^*(A) = \min\big(8 + J^*(B), 4 + J^*(C), 6 + J^*(D)\big)$$

which, finally, is optimized by

$$J^*(A) = 4 + J^*(C) = 12.$$

Hence, we have found the shortest path from A to H and it goes through the nodes C and either E or F.

## 2.2 The Dynamic Programming Algorithm

With the fundamentals of DP, the principle of optimality and the Shortest Path example we are now ready to formulate a formal description of the DP Algorithm. For every initial state $x_0$, the optimal cost $J^*(x_0)$, as seen in (2.2), of the basic problem is equal to $V(x, t)$, where x is the state at time t, given by the final step of the following algorithm.

From the optimality criteria, we thus have that

$$V(x_t, t) = \inf_{u_t,\ldots,u_{N-1}} J_{u_t}(x_t)$$

and the terminal cost $G_N(x_N)$ is defined as

$$J_{u_N}(x_N) = G_N(x_N).$$

Thereafter, as examplified in the shortest path problem, the algorithm uses the information from previous states (1) and proceeds backward in time from period $t = N - 1$ to period 0. For $t = 0, 1, ..., N - 1$ it calculates

$$V(x_t, t) = \inf_u E\bigg\{g_t(x_t, u_t, w_t) + V\big(f_t(x_t, u_t, w_t), t + 1\big)\bigg\}, \tag{2.3}$$

where $g_t(x_t, u_t, w_t)$ is the cost function for the present subproblem at time $t$ and where the expectation is taken with respect to the distribution of $w_t$, which depends on $x_t$ and $u_t$. In the previous shortest path problem though, there were no stochastic variable. This makes for a special case where the final answer is not a minimized expectancy but an absolute value. For problem as such, we can exclude the stochastic disturbance variable $w_t$ from the algorithm and get a deterministic cost function

$$V(x_t, t) = \inf_u \Big[ g(x, u, t) + V\big(f(x, u, t), t + 1\big) \Big]. \tag{2.4}$$

And, of course, the minimizing could analogously be changed to maximizing, if that is desired of the basic problem. For $t = N$, we have that $V(x, N) = J_N^*(N) = G_N(x_N)$ and from induction we then have that the optimal cost function $J_\pi^*(x_t)$ are equal to the functions $V(x, t)$ generated by the DP algorithm. Specifically we have that

$$V(x_t, 0) = J_\pi^*(x_0),$$

and hence, this algorithm will lead to an optimal policy as stated in (2.2).

**Example II: Knapsack**

The DP algorithm may now be used to demonstrate another common example - The knapsack problem. This can be applied to, for example, questions that regard the distribution of limited resources, a problem of significant importance in economics. From its name it is possible to derive the intuitive explanation: How much can be placed inside the constraints of a knapsack?

Let us imagine that a burglar breaks into a house. With him he has a knapsack and inside the house he finds three particularly valuable items, a statue, a bowl and a plate. He is now faced with the problem of choosing which of these to fill his bag with. The items weigh three, two and one units each and are valued at five, three and four units respectively. The bag can carry a maximum of five units and for explanatory reasons he may not carry anything outside the knapsack.

This problem is also called a 0/1-Knapsack problem, as it can be broken down to the binary question of going through the items one by one and for each decide whether to take it or not. Hence this particular problem can be formulated as three questions given to the burglar, which he may answer either by putting the object into the knapsack or not, effecting both the value contained in the knapsack and the remaining capacity.

To formalize this problem as a optimal control problem, which we may apply the DP algorithm upon, we state the objective function as a deterministic maximization of the reward function:

$$\max J(x_0) = \sum_{t=1}^{3} g_t(x_t, u_t),$$

subject to the state evolution

$$x_{t+1} = x_t + v_t u_t$$

and the knapsack capacity constraint

$$\sum_{t=1}^{3} u_t w_t \leq 5.$$

The decision $u_t$ at each time $t = 1, 2, 3$ represent the decision to either place the item in the bag $u_t = 1$ or not $u_t = 0$. The reward function $g_t$ is the value change in the knapsack, based on the total value of the items chosen thus far $x_t$ and the decision regarding the item at hand $u_t$. Lastly, $w_t$ and $v_t$ represent the weight and value of each item respectively.

We solve this problem by first introducing a new variable $d_{ij}$, where $i \in [0, 3]$ is the time step, representing the burglars decision regarding each item, and $j \in [0, 5]$ is a weight restriction. Using the DP algorithm, we recursively solve for the decision that maximizes the value in the knapsack for each time step $i$ and weight restriction $j$. This give us the following function:

$$d_{ij} = \max\{d(i-1, j), v_i + d(i-1, j-w_i)\},$$

which describes the recursion relation of maximization of the value for $d(i, j)$ with respect to the two possible controls of keeping the knapsack as it is (left term in the right hand side) or adding the current item $i$ (right term). We also define initial values for $d_{ij} = 0$ when either $i = 0$ or $j = 0$.

A commonly used way to present the computations of this kind of problem is by a table set up. In this particular case, the columns represent choices at each time step $t$ and the rows represent the used capacity of the knapsack. We begin in the top left box, work our way down each column whilst solving for the best possible value according to the system set up above. After doing so, we get the following value table:

| Weight/Item | Statue (5) | Bowl (3) | Plate (4) |
|---|---|---|---|
| 1 Unit: | 0 | 0 | 4 |
| 2 Units: | 0 | 3 | 4 |
| 3 Units: | 5 | 5 | 7 |
| 4 Units: | 5 | 5 | 9 |
| 5 Units: | 5 | 8 | 9 |

.

As stated, we've begun with the top left box and gone down the column before continuing with the next and so forth. In the first column the question is whether to take the statue, weighing three units with the value of five. Using only one or two weight units in the knapsack the burgler cannot place the statue in it, hence the zeros on row one and two, but using three, four or five units he can choose it, hence the fives. Continuing with the bowl, the burgler now have the information from the statue column in mind which makes each row contain the question of choosing a new combination, with the bowl, or an old one, with only the statue. In the first row he still cannot choose any combination, in the second he can choose the bowl and does so, in the third he can choose either the bowl or a better combination from earlier and does so by still choosing the statue, the same goes for row four but when he has five units to use he can make a new best combination with both the bowl and the statue.

Summarizing this problem, we have an optimal value of $J^*(x_0) = 9$ as seen in the plate column row four and five where the knapsack contain the plate and the statue. We have therefore found an optimal set of controls for the current problem; $u_1 = 0, u_2 = 1$ and $u_3 = 1$.

## 2.3 Formulating a Dynamic Programming Model

The tableau described in the knapsack problem above, is a typical representation of computations done with the DP algorithm. We now formulate a general step-by-step method for the formulation of a general DP model, where we assume a minimization problem and an additive objective function.

1. **Define the Stages**
   t=1..., T.

2. **Define the Control Variables**
   $u_t$. t= 1...,T.

3. **Define the States**
   $x_t$, t=1...,T.

4. **Define the State Evolution**
   $x_{t+1} = f_t(x_t, u_t, w_t)$, t= 1..., N-1, where $f_k$ represent the transformation of the current state $x_t$ into the new state $x_{t+1}$ with respect to a control $u_t$ and some random disturbance $w_t$.

5. **Define Recursive Value Function**
   Expressing the optimal objective function value from time $t = i$ and onward:

   $$J^*(x_i) = G_N(x_N) + \inf \sum_{t=0}^{N-1} g_t(x_t, u_t, w_t),$$

   where $g_t$ is the cost function for time $t = 1, ..N-1$. and $G_N$ is the terminal cost function for state $x_t$.

6. **Define Initial Conditions**
   $x_N$ for backward recursion and $x_0$ in the case of forward recursion.

7. **Specify Restrictions**
   For the control and state variables, by defining their space: $u_t \in U$ and $x_t \in X$

8. **Make Recursive Computations**
   As specified in the earlier steps, compute the value function; either for $T, ..., 1$ in the backward recursion case or $1, ..., T$ in the forward recursive case.

9. **Backtrack**
   As exemplified in the DP tableu of the Knapsack example, make a backtracking table through the computed values in order to find the optimal solution path.

## 2.4 Linear Quadratic Stochastic Control

A common category of problems for DP examines a linear system with a quadratic cost function, a combination of properties that defines the Linear Quadratic (LQ) regulator. Further, these systems can either be stochastic, as will be described below, or deterministic, where the disturbance variable is excluded.

Systems in this form may, for example, contain an objective to minimize the distance of the state of the system from the origin, i.e. $\min \sum_{t=1}^{N} x_t^2$, or from a specific trajectory path, i.e. $\min \sum_{t=1}^{N} (x_t - \bar{x_t})^2$. Further, the system have a linear state evolution described on the form of

$$x_{t+1} = A_t x_t + B_t u_t + w_t, t = 0, ..., N-1,$$

where $A_t$and $B_t$ are nxn-matrices and $x_t, u_t$ and $w_t$ are vectors representing the state, control and disturbance variables over a given time horizon, finite or infinite. Further, $w_k$ has a zero mean and a finite second moment. The objective function is in the following form:

$$J_\pi(x_t) = E\Bigg( x_N^T Q_f x_N + \sum_{t=0}^{N-1} (x_t^T Q_t x_t + u_t^T R_t u_t) \Bigg), \tag{2.5}$$

where $Q_t$ and $R_t$ are symmetric nxn-matrices, with $Q_t$ being positively semidefinite and $R_t$ positively definite. The value of $J_t(x_t)$ is understood as the expected cost from time $t$ and state $x_t$ until time $N$ and, as in the previous cases, the objective is to find an optimal mapping policy function $\phi : x_t \to u_t$ that minimizes (or maximizes) the given objective function.

Using the DP algorithm, we have that

$$J_\pi^*(x_t) = \min E\Bigg( (x_t^T Q_t x_t + u_t^T R_k u_t) + J_{t+1}(A_t x_t + B_t u_t + w_t) \Bigg) \tag{2.6}$$

and

$$J_N(x_N) = x_N^T Q_N x_N.$$

It is possible to solve this problem with the DP algorithm, $t = N, ..., 1$, and by doing so there exists an optimal control law for every t in the form:

$$u_t^* = L_t x_t,$$

where the matrices $L_t$ are given by the equation

$$L_t = -(B_t' K_{t+1} B_k + R_k)^{-1} B_t' K_{t+1} A_t.$$

Here, we have that the matrix $K_t$ is the solution to the Riccati equation (2.8) [3], which is given recursively by the following:

$$K_N = Q_N, \tag{2.7}$$

$$K_t = A_t'(K_{t+1} - K_{t+1} B_t (B_t' K_{t+1} B_t + R_t)^{-1} B_t' K_{t+1}) A_t + Q_t. \tag{2.8}$$

**The Riccati Equation and infinite horizon problems** The Riccati equation stated in (2.8) has a very important property when applied to mathematical control problems: The solution $K_t$ to the Riccati equation converges as $t \to -\infty$, given the following conditions: The matrices $A_t, B_t, Q_t$ and $R_t$ are constant and thus equal to A, B, Q and R, the pair (A,B) is controllable and Q may be written as C'C, where the pair (A,C) is observable.

Controllable and observable pairs are defined as the following [3]: A pair (A,B) where A is a nxn-matrix and B a nxm-matrix, is said to be controllable if the nxnm-matrix

$$[A, AB, A^2B, ..., A^{n-1}B]$$

has full rank (i.e. the rows are linearly independent). Further, a pair (A,C), where A is a nxn-matrix and C is a mxn-matrix, is said to be observable if the pair (A',C') is controllable with A' and C' denoting the transposes of matrices A and C.

This transforms the solution of the Riccati equation into a steady-state K satisfying

$$K = A'(K - KB(B'KB + R)^{-1}B'K)A + Q, \tag{2.9}$$

which is the algebraic Riccati equation. This indicates that a control mapping $L_t x_t = u_t^*$ for a system

$$x_{t+1} = A_t x_t + B_t u_t + w_t, t = 0, ..., N - 1,$$

and a large number of stages, N, may be approximated through

$$\phi^*(x) = Lx,$$

where

$$L = -(B'KB + R)^{-1}B'KA.$$

This is a very useful property when solving LQ-regulator problems with an infinite horizon problem, i.e. $N = \infty$. Later, we examine a problem in this form as we turn to macroeconomics and how a central bank may find an optimal monetary policy using the DP algorithm for the LQ- regulator and the Algebraic Riccati equation.

## 2.5 An Intriguing Application to Bioinformatics

DP has, as mentioned before, proven to be useful in many applications. However, before focus is turned specifically to the one application of primary interest for the present thesis, we give a brief overview of another important application.

In the field of bioinformatics the DP algorithm is extensively used in different applications and has been the most popular method in computational molecular biology [4]. Particularly in sequencing problems that involve the assembly of DNA or RNA fragments in order to determine the degree of similarity between two different strings, the DP algorithm represents an efficient tool. Through this analysis it is possible to examine, for example, the degree of kinship between two species which led Needleman and Wunsch [5] to derive a particular DP algorithm to solve for an optimal alignment when determining the sequences of nucleotides or protein. Without any deeper review of the bioinformatics involved, we briefly examine the Needleman-Wunsch algorithm, which is solved in the same manner as in the knapsack problem.

We will examine two small DNA-strings, AATCGG and ATTCG, and the problem of aligning these two can be broken down into a sequence of subproblems of either keeping each string as they are or inserting a gap between adjacent nucleotides in one of them. This is somewhat similar to the possible controls of the knapsack problem, where for each item there was a binary set of possible controls; to place the current item in the bag or to skip it. In this example, the possible controls are three; (1) inserting a gap to string A, (2) inserting a gap to string B or (3) keeping both strings as they are.

The basic problem consists of a cost function, which will be dependent upon the particular biological context. However, for the convenience of this example, we consider a cost/reward function with $-1$ when inserting a gap, $-1$ for a mismatch and $+1$ for a match between the two strings. Thus, we have an objective function that we wish to maximize:

$$\max \sum_{t=0}^{N} g_t(x_t, u_t),$$

where $g_t$ represents the alignment value change of the system with respect to the state variable $x_t$ representing the alignment of both strings at time $t$, depending on the choice at time $t = 1, .., t-1$ to either insert a gap to one of the strings or to let them be as they are. The policy $\pi$ is therefore the set of these controls for each place in each string. We thus have the cost function

$$\begin{cases} g_t(x_t, u_t) = 1 & \text{if inserting no gap and strings match} \\ g_t(x_t, u_t) = -1 & \text{otherwise.} \end{cases}$$

As in the knapsack problem, we introduce a variable d(i,j) to recursively solve for an optimal alignment using the DP algorithm. In this case we have

$$d_{ij} = \max\{d(i-1, j-1) + g_t, (i-1, j) - 1, (i, j-1) - 1\},$$

which describes the recursion relation of maximization of the value for $d(i, j)$ with respect to the possible controls; keeping the strings as they are (left term in right hand side), insert a gap to the column-string (middle term) or insert a gap to the row-string (right term). As described in the definition of this variable, a reward from a string match is only possible when inserting no gap. We also define initial values $d_{ij} = 0$ for $i = 0, j = 0$ or $i = j = 1$. In the following, we describe the computations of this variable d(i,j).

As in the knapsack, we present these computations using a table. We place the shorter string ATTCG in the top row, defining this as the row string, and the AATCGG in the first column, defining this as the column string. This gives us the following table for values of d(i,j):

|  |  | A | T | T | C | G |
|---|---|---|---|---|---|---|
|  | 0 | -1 | -2 | -3 | -4 | -5 |
| A | -1 |  |  |  |  |  |
| A | -2 |  |  |  |  |  |
| T | -3 |  |  |  |  |  |
| C | -4 |  |  |  |  |  |
| G | -5 |  |  |  |  |  |
| G | -6 |  |  |  |  |  |

.

The second row and column can instantly be determined as above, because they represent only adding gaps. We then move to the box at row three and column three. Moving here has three possible origins; the top left box (0), the one above (-1) and the one to the left (-1). The box above represents starting with a gap in the column string and keeping the row string, resulting in a $-1$ penalty. Moving from here and downward, represents inserting a gap in the row string while keeping the column string as it is, hence resulting in another $-1$ penalty. This gives a possible score of $-2$ in this box, which is analogously true for the move from left. Originating in the top left box, though, represents keeping the first element in both strings as they are, generating a reward of $+1$. Therefore we assign 1 to the present box, since this is the best value possible. While doing this we also memorize from where we came to this box, in this case by adding an arrow

|  |  | A |
|---|---|---|
|  | 0 | -1 |
| A | -1 | ↖ 1 |

.

From here we move down the rows, which represent the inserting additional gaps to the row string while keeping the column string. Analogously moving along the third row and down the columns, we keep the row string while inserting additional gaps to the column one. Thereafter we continue to determine the optimal value for each box, by comparing possible scores when arriving from the box above, left or top left. Doing this we get the following table

|  |  | A | T | T | C | G |
|---|---|---|---|---|---|---|
|  | 0 | -1 | -2 | -3 | -4 | -5 |
| A | -1 | 1 | 0 | -1 | -2 | -3 |
| A | -2 | 0 | 0 | -1 | -2 | -3 |
| T | -3 | -1 | 1 | 1 | 0 | -1 |
| C | -4 | -2 | 0 | 0 | 2 | 1 |
| G | -5 | -3 | -1 | -1 | 1 | 3 |
| G | -6 | -4 | -2 | -2 | 0 | 2 |

,

from which we get that the optimal score after aligning the two strings is 2, interpreted as some value defining, for example, the degree of kinship between the strings. We also get that there are two different alignments that both sum up to that optimal score. Starting at the last, bottom right, box we find the ways to sequence optimal by continuing along the path back to the origin, we find the two following sequences:

| I | II |
|---|---|
| ATTC-G | ATTCG- |
| AATCGG | AATCGG |

,

which hence are optimal solutions to the basic sequencing problem.

# Chapter 3

# Economical Applications of Dynamic Programming

From now on, we turn to the main topic of this thesis - applications in economics. To cover this subject the thesis will present three areas of economic theory and supplement these with example problems for DP. With regard to the scope of the present thesis, economic theory will only be covered briefly and focus will be held on the DP example. We begin with microeconomics, the study of how actors and firms behave on markets, and then turn to financial economics, which covers theory of the allocation of resources, e.g. investment theory, and finally we discuss DP applications to macroeconomics, the study of behaviour of the aggreregate economy which cover topics such as monetary and fiscal policy, inflation, unemployment and growth.
In general, the management of an asset or decisions regarding the allocation of resources highlights an intuitive advantage of DP in economics: What policy should be used in order to maximize some utility function over a given time period? Accordingly, DP has been much appreciated in economic theoryand play a central part of a theoretical field known as recursive macroeconomics [6]. This is a relatively young field of growing importance, in which Lars Ljungqvist together with Nobel laureate Thomas Sargent might be the most prominent researchers [7].

## 3.1 Microeconomics: Asset Management

As mentioned, microeconomics is the study of how actors behave inside an economy. This is also the intuitive reason for covering this application first, microeconomics from the aggregate perspective also have direct bearing for the financial and macroeconomic cases. Therefore, we begin with what can be regarded as the smallest unit and continue on from there.

In demonstrating the applications of DP in this particular field of economics, we focus on an optimal stopping problem [3]. The optimal stopping problem consists of one specific control at each state, that breaks the evolution of a certain process. This can be exemplified by a factory manager that each month is required to decide whether to give service to a machine or not; taking it out will stop the production for some time, with inevitable loss of profit, but not maintaining the machine will decrease production and risk serious damage. The evolution of this system defines the function for the state variable, as seen in (1), where the noise variable, $w$, may represent a variation in the efficiency decrease and/or a possible break down damage.

The example above represents one interpretation of an optimal stopping problem in microeconomics, which could be solved by using DP. Another problem of basically the same kind, is when a private house owner decides whether to sell at a given price or keep it and hope for a better price at a later time. This section will formulate and solve this particular problem.

### 3.1.1 Problem formulation

Suppose that a person, let us call him Johan, owns a house which he is interested in managing in the best possible way, such that at the time of retirement he gets as much value out of it as possible. He therefore wants to figure out an optimal strategy for decisions regarding whether to accept or reject offers given on the house during this time. If he chooses to sell he will put all the money into a bank account, which will earn him a fixed rate of interest for the remaining years to his retirement.

We assume, in order to delimit the mathematics, that he is given one offer each year and that they are random and independent, i.e. no influence from house improvements or earlier offers). We also assume that there is no inflation in the economy, value of money stays the same throughout, and that the last offer, at time $N-1$, must be accepted. We thus have the following:

| | |
|---|---|
| N | The total number of years (controls) until retirement, |
| t | Current time, |
| $r > 0$ | The fixed rate of interest, |
| $u_t \in \{u^a, u^r\}$ | Control variable at time $t$ in the set of the two possiblities, accept, $u^a$, or reject, $u^r$, the offer, |
| $v_t$ | Offer given at time $t$. |

We can define the function for the state variable as

$$x_{t+1} = \begin{cases} T & \text{if } u_t = u^a \text{ (sell) or } x_t = T, \\ v_t & \text{otherwise,} \end{cases}$$

where the state T is a terminal state, which means that an offer has been accepted and no more controls are possible. As stated, when this happens Johan will take the money and put it into a bank account which will provide him with a safe return each year until retirement, summing up to the total value $x_t(1+r)^{N-t}$ at that time $t = N$. From the above, we formulate a reward function (2) as demonstrated in the introduction, which our objective is to maximize

$$E\left\{ g_N(x_N) + \sum_{t=0}^{N-1} g_t(x_t, u_t, v_t) \right\},$$

where we define

$$g_N = \begin{cases} x_N & \text{if } x_N \neq T \\ 0 & \text{if} x_N = T, \end{cases}$$

$$g_t(x_t, u_t, v_t) = \begin{cases} x_t(1+r)^{N-t} & \text{if } u_t = u^a \\ 0 & \text{if } x_t = T \\ 0 & \text{otherwise.} \end{cases}$$

### 3.1.2 Solution

The decision that Johan is faced at each time $t$ is hence whether to accept the offer given or not. The intuitive solution is easy: if Johan expects to earn more from selling to a later offer he should reject the current one, otherwise he should accept. From the cost function above, we may therefore formulate a recursive algorithm that solve for a reward function when using an optimal policy $\pi$. Starting with the last period $t = N$, the DP algorithm gives the following formulation:

$$J_\pi(x_N) = \begin{cases} x_N & \text{if } x_N \neq T, \\ 0 & \text{if } x_N = T. \end{cases} \tag{3.1}$$

With the recursion

$$J_\pi(x_t) = \begin{cases} \max\left[x_t(1+r)^{N-t}, E\{J_\pi(v_t)\}\right] & \text{if } x_t \neq T, \\ 0 & \text{if } x_t = T. \end{cases} \tag{3.2}$$

From this formulation, we discount the expected future revenue to the present time $t$ and from this introduce the new variable $e_t$

$$e_t = \frac{E\{J_\pi(v_t\}}{(1+r)^{N-t}}.$$

This variable $e_t$ can thus be interpreted as the expected value today of a later accepted offer. Hence we have that the optimal value of the objective function $J_\pi^*$ imply that Johan

accept the offer $x_t$ if $x_t > e_t$
reject the offer $x_t$ if $x_t < e_t$.

If we, however, want to solve this numerically there is a need for additional information. We therefore assume that the interest rate given at all time $t$ is 0.03. The disturbance variable, $v_t$, represent the probability distribution by which the offers are given and if we assume that the offers given are randomly distributed inside a specific range, we can use this to extract a recursive function that will solve this problem numerical. Let us therefore assume that offers can take values in the range $[0, 1]$, where the 0 is regarded as no offer at all and 1 represent some arbitrary roof value. Within this range, the offers are randomly distributed. This information give us that the expected value at time $t = N$, if no earlier offer has been accepted, is $x_N = \frac{1}{2}$. Hence,

$$E\{g(x_N)\} = \begin{cases} \frac{1}{2} & \text{if } x_N \neq T, \\ 0 & \text{if } x_N = T. \end{cases}$$

Therefore at time $t = N-1$, according to the recursive function (3.2), he should only accept an offer given that exceeds the expected value, $E\{g(x_N)\}$, for the subsequent time $t = N$. Offers accepted at $t = N-1$ will therefore have a lower bound $a_{N-1}$ given by the following

$$a_{N-1}(1.03)^{N-(N-1)} = \frac{1}{2},$$

which give us that

$$a_{N-1} = \frac{1}{2.06}.$$

This lower bound represents an offer that is equally well to accept as to reject, expected value is equal for both controls. Accordingly, the expected value at time $t = N-1$, if no earlier offer has been accepted, is the mean of its range and because the upper bound $b_t$ is

$$b_t = (1.03)^{N-(N-1)} \iff b_t = (1.03)$$

we have

$$E\{g(x_{N-1})\} = \frac{\frac{1}{2.06} + (1 - \frac{1}{2.06})(\frac{1}{2} + 1.03)}{2} \approx 0.636.$$

In the same manner, we can continue and solve for which offers to accept at each time $t = N-2, .., 1$.

### 3.1.3 Further applications of this kind

We have thus described a way to optimally guide Johan's choices toward retirement with the DP algorithm. Problems in the form of optimal stopping is found in many other kinds of microeconomic issues [8]. The house in the above example, could just as well be interpreted as some investment opportunity where the investor decides whether to take the opportunity or hope for a better one later. The Optimal stopping frame work has also been used for interpreting job searching choices [9]. When an available job is found, salary and other conditions are announced. Accordingly, a decision then has to be made whether to accept the job or to hope that a later opportunity will be even better.

## 3.2 Financial Economics: Utility Maximization

Financial economics could in some sense be seen as a link between what is traditionally seen as the sphere of microeconomics, behaviour of actors in a market, and macroeconomics, development and policy governing markets. Therefore, the chapter structure of this thesis is somewhat arbitrarily chosen. However, when we now turn toward an application to finance the primary focus will be on how scarce resources can be optimally allocated with the help of DP. Many problems in economics [7] regard maximizations in the form of

$$\max \sum_{t=0}^{T} \beta^t U(x_t, u_t),$$

where

| | |
|---|---|
| $T$ | scope of time, |
| $t$ | current time, |
| $\beta \in (0,1)$ | time discount factor, |
| $U$ | function of Utility for concerned actor, |
| $u_t$ | control variable, i.e. changes in allocation, |
| $x_t$ | state variable, i.e. portfolio wealth. |

The time discount factor captures impatience of an actor and the Utility function is strictly increasing and concave, i.e. $u' > 0$ and $u'' < 0$, capturing the phenomena of diminishing marginal utility. We see that it resembles the standard form (2) of DP and use this in an example.

### 3.2.1 Problem formulation

Suppose that a person want to optimize the allocation of assets between investing and consumption in order to maximize utility over a given time period $T$. In an example of this kind we need to assume some utility function and psychological discount factor to fit the model above. However, to simplify the example and delimit the mathematics, we assume a constant utility and no impatience. This turns the problem into the form of

$$J_\pi = \sup \sum_{t=0}^{T} u_t,$$

where $0 \leq u_t \leq x_t$ is the amount consumed at every time $t$ and $\pi$ hence denotes the chosen sequence of consumption $\pi = u_0, u_1, ..., u_T$. Further, at each time $t$ he receives a salary of $x_t$ which he may place in $n$ different forms of investments, each with the rate of return defined as $\theta_{ti}$, his capital therefore increases by the state evolution

$$x_{t+1} = x_t - u_t + \sum_{j=0}^{n} \theta_{ti} c_{ti}.$$

However, Johan decides that he will only save his money in a bank account with a constant rate of return such that he does not expose himself to any financial risk. Hence we rewrite the state evolution to

$$x_{t+1} = x_t + \theta(x_t - u_t),$$

where $\theta$ is the rate of return and $0 \leq u_t \leq x_t$. Since there is no concave utility function or factor of impatience to discount for, the problem is time invariant. We may therefore write the recursive value function for each state $s = N - t$, transforming backward induction into forward, as

$$V_s(x) = \max_{0 \leq u \leq x} [x + V_{s-1}(x + \theta(x-u)],$$

with the terminal condition $V_0(x) = 0$ since no more can be consumed after time N is reached. We now maximize the consumption for each time $s = 1, 2...N$ and thus have

$$V_1(x) = \max_{0 \leq u \leq x} [x + V_{s-1}(x + \theta(x-u)] = \max_{0 \leq u \leq x} [u + 0] = x,$$

$$V_2(x) = \max_{0 \leq u \leq x} [x + V_{s-1}(x + \theta(x-u)] = \max_{0 \leq u \leq x} [u + x + \theta(x-u)]$$

and so forth. Since both the value function and state evolution are linear we have that maximum will be obtained at either $u = 0$ or $u = x$ and of this property problems like this are sometimes called Bang Bang control. We have that

$$V_2(x) = \max[(1+\theta)x, 2x] = \max[1+\theta, 2]x = cx,$$

where $c$ is a constant. We may therefore guess that the maximized reward function is on the form of $V_s(x) = c_s x$. Proving this by induction, we use that this is valid for $V_2(x)$ and control if it is valid also for $V_{s+1}(x)$. We have that

$$V_{s+1}(x) = \max_{0 \leq u \leq x} [u + c_s(x + \theta(x-u))] = \max[(1+\theta)c_{s+1}, 1 + c_s]x = c_s x,$$

thus it is possible to conclude that $V_s(x) = c_s x$. Further we have that

$$c_s = c_{s-1} + \max[1, \theta c_{s-1}]$$

which lead us to the conclusion that at a certain point $s^*$ on the trajectory path we have that $\theta c_{s^*+m} < 1$, for all $m = 1, .., N - s^*$ and thus

$$c_s = \begin{cases} s & s \leq s^* \\ (1+\theta)^{s-s^*} s^* & s^* \leq s. \end{cases}$$

$s^*$ should be understood as the least integer where $s^* > \frac{1}{\theta}$ and the optimal consumption policy as building up a capital by saving all of the income in the years $s \leq s^*$ and thereafter consuming the whole income.

### 3.2.2 Further applications of this kind

The model used in the problem above may, if desired, be revised to include additional variables, such as more investment alternatives. Doing so complicates the economics and mathematics, dependening on the properties of the additional alternatives. In the cases of microeconomics and finance, problems of this kind often go under the title ”cake eating problem” [10] as they comprise the optimal usage of a scarce resource. As in the above, the model

$$\max \sum_{t=0}^{T} \beta^t U(x_t, u_t),$$

is the standard one and if we aggregate it, i.e. to the level of households in a market, we can understand this as an optimal growth model [7], providing a tool for analysis of optimal consumption patterns at the macro level.

## 3.3 Macroeconomics: Monetary Policy

At the level of macroeconomics, recursive methods such as DP has gained a lot of influence (Ljungqvist and Sargent, 2004). As mentioned in the previous section, theory regarding optimal growth can be translated into a DP problem in the form of the cake eating problem. The formulation would in this case be in the form of:

$$\max \sum_{t=0}^{T} \beta^t L(x_t, u_t),$$

where $L$ captures some cost function for a government and $\beta$ is a time discount factor. The cost function could in this case be understood as some undesired outcome that a government faces when implementing a policy $\pi$ that leads to the decisions $u_t$. This could, for example, be a question regarding how to stabilize the economy during different times in a business cycle. If the cost function then is set to some penalty dependent on the employment rate $x_t$ which in turn is dependent on stimulations $u_{t-1}$, this could provide an interesting tool in policy development of governments.

### 3.3.1 Problem formulation

In a manner like this, Kato and Nishiyama [11] uses DP when examining monetary policy in the low inflation economy of Japan during the early 1990's. Their primary concern was with evaluating a zero-bound on the nominal interest rate, the main tool for central banks control mechanism of inflation rates. In the section below, the basic DP model that they used for deriving an optimal policy for the control of the interest rate is described.

With regard to the most common goals assigned to a central bank, low and steady inflation and a low unemployment rate, we suppose that a cost function can be written in the form of

$$L_t = \frac{1}{2}\bigg((y_t - y^*)^2 + \lambda(z_t - z^*)^2\bigg),$$

where the first factor denotes the output gap between current, $y_t$, and potential, $y^*$, GDP; caused either by an unemployment rate too high, causing a lower than potential growth, or too low, leading to an over heated market. The second factor denotes the inflationary goal $z^*$ set by the central bank subtracted from the inflation $z_t$ present at the current time $t$. Further, $\lambda \in \Re$ is some weighting factor which represents a preference of the central bank. The model thus contain two state variables which the cost function depend upon and the economy that govern the evolution of these can be described as

$$y_{t+1} = p(y_t - y^*) - \sigma(i_t - E\{z_{t+1}\}) + v_{t+1} \tag{3.3}$$

$$z_{t+1} = z_t + \alpha(y_t - y^*) + \epsilon_{t+1}, \tag{3.4}$$

where $v$ and $\epsilon$ are assumed to be disturbance variables, $p$, $\sigma$ and $\alpha$ constants and $i_t$ are the nominal interest rate set by the central bank at time $t$, thus the control variable. In (3.3) we find a variable that is forward-looking, in the sense that it is concerned with a development that is not directly available at time $t$; $E\{z_{t+1}\}$ represents the expected rate of inflation for time $t+1$ at time $t$. This can, however, be substituted as a function of the relationship between the expected inflation rate and the current inflation rate and output gap, which is given by (3.4) when we exclude the disturbance variable,

$$E\{z_{t+1}\} = z_t + \alpha(y_t - y^*).$$

The objective function, which a policy for the central bank wishes to minimize over some time period, can thus be formulated as

$$\min_{i_t} E \sum_{t=0}^{N} \frac{1}{2}\bigg( (y_t - y^*)^2 + \lambda(z_t - z^*)^2 \bigg) \tag{3.5}$$

subject to

$$z_{t+1} = z_t + \alpha(y_t - y^*)^2 + \epsilon_{t+1}.$$
$$y_{t+1} = (p + \alpha\sigma)(y_t - y^*)^2 - \sigma(i_t - z_t) + v_{t+1},$$

From the above, we recognize that this monetary optimization system contains a quadratic output function and a linear input. It thus fulfil the properties of a Linear Quadratic regulator (LQ) problem, which we therefore introduce to our analysis.

**Monetary policy example as a LQ regulator** The system described in the above, governing an optimal monetary policy, can be rewritten as a LQ regulator problem, described in section 2.4, and we thus have the following:

$$E\bigg( (x_N - \bar{x_N})^T Q_N (x_N - \bar{x_N}) + \sum_{t=0}^{N-1} (x_k - \bar{x_k})^T Q (x_k - \bar{x_k}) + u_t^T R u_t) \bigg), \tag{3.6}$$

where $x_k = (y_t, z_t)$, contain the state variables and $\bar{x_k} = (y_t^*, z_t^*)$ are the desired values for the state variables at each time $t$. We assume that there is no cost associated with the decision to alter the nominal interest rate and therefore the matrix R is the null matrix $R = 0$. Furthermore, if the priority between the target for inflation and output is equal (i.e. $\lambda = 1$), we have that $Q_t = I, t = 1, .., N$.

The state evolution system is rewritten as

$$x_{t+1} = A_t x_t + B_t u_t + w_t,$$

where

$$A_t = \begin{pmatrix} 1 & \alpha \\ 0 & p + \sigma(1+\alpha) \end{pmatrix}, B_t = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma \end{pmatrix}, w_t = \begin{pmatrix} \epsilon \\ v \end{pmatrix}.$$

The system is thus time-invariant because $A_t, B_t$ and $w_t$ consist only of constants. We know from the introduction of the LQ-regulator problem in section 2.4 that a problem of this kind has a solution on the form of

$$u_t^* = \phi_t^*(x_t) = L_t x_t,$$

where the matrices $L_t$ are given by

$$L_t = -(B_t')K_{t+1}B_k + R_k)^{-1}B_t'K_{t+1}A_t$$

and where $K_t$ solves the corresponding Riccati equation, described in (2.6).

However, from the description of the problem, we have that the matrices $A_t, B_t, Q_t$ and $R_t$ have constant coefficients and thus independent of time. Therefore they may be transformed into the constant matrices A, B, Q and R. Further, analyzing the problem formulation in the above, it is a highly plausible assumption that the objective of a central bank to control the inflation and output gap should be regarded as an infinite horizon problem, i.e. $N = \infty$ in the equations above. Thus we may apply the algebraic Riccati euqation (2.7) to solve for the optimal policy mapping $L_t$ above.

An optimal interest rate policy for the central bank may therefore be approximated through solving for L in the system of equations

$$\begin{cases} L = -(B'KB + R)^{-1}B'KA \\ K = A'(K - KB(B'KB + R)^{-1}B'K)A + Q. \end{cases}$$

### 3.3.2 Further applications of this kind

This approach by Kato and Nishiyama [11] represents an intriguing example of determining and evaluating monetary policies. Macroeconomics in general has been a major field of DP applications [7] but even though most examples of these are outside the mathematical boundaries of this thesis, it is probably a necessity to mention Nobel laureates Finn E. Kydland and Edward C. Prescott. In 2004 they received the Swedish central banks economic price in memory of Alfred Nobel "for their contributions to dynamic macroeconomics: the time consistency of economic policy and the driving forces behind business cycles" [12]. Studying the ability of government to implement desirable economic policies and how business cycles fluctuate from technological development they used recursive models of optimal monetary policy, similar to the one exemplified in the above, and utility maximization of individual households, much like the examples above [12].

# Chapter 4

# Summary and Analysis

In the thesis, we have examined how the algorithm of dynamic programming may be applied to economics. After a review of the theory and the example of the Needleman-Wunsch algorithm, the focus were shifted toward economical applications. A couple of examples were investigated and some numerical results were presented, our focus was however to supply a broad interpretation of possible applications of dynamic programming in economics. Therefore, micro- and macroeconomics as well as financial economics each received a section of brief analysis. In general, it seems reasonable to conclude that the dynamic programming algorithm represents a powerful tool in the interpretation of a broad range of economic phenomena: From the asset management and utility maximization of an individual actor to the possible guidance of a central banks' monetary policy.

However, the above can only be assumed to represent a rather simplified interpretation of the different topics covered. For one, with regard to the expository aim of the thesis and the resources available to the author, the above should be considered to lack the necessary depth of mathematical analysis to make justice to the subject. Further, the lack of validity analysis of the economic models above may effectively disqualify any conclusion made. Therefore, the remainder of this thesis will be devoted to the general topic of mathematics in economics, with the purpose of complementing the previous sections with a discussion of its general validity.

## 4.1 The Validity of Mathematical Economics

Mathematical theory and modelling has since the middle of the twentieth century been at the very center of theoretical advances of the science of economics [13]. Further, some remark that the advances of mathematical economics has been to such an extent that it more or less has transformed economics into a field of mathematics [14]. This raises some important questions however: What kind of problems do arise when we draw economic inferences from mathematical deduction? And how may the validity of a mathematical model be understood, such that its consequences can be interpreted accordingly? These are, of obvious reasons, too grand questions for a proper examination here but, with this reservation made, we will briefly discuss the question of how mathematics in economic theory should be interpreted.

In an article of Beed and Kane [13] seven points of critique against the "mathematization of economics" are raised and analyzed, among them:

- The axioms of mathematical economics do not correspond to real world behaviour.
- Some/much of economics is not naturally quantitative and therefore does not lend itself to mathematical exposition.
- The number of empirically testable hypotheses generated by mathematical economics is small compared with the volume of mathematical analysis.

The first two is somewhat similar to each other; both pinpoint the question of whether it is reasonable to assume that reality may be captured in a mathematical model. In this regard however, it might be necessary to distinguish economics that involve solely deterministic processes, for example the maintenance scheduling of a machine, from economics of a stochastic nature, for example involving human behaviour. It is a plausible assumption that the modelling of a phenomena of the latter kind is more problematic than the modelling of a phenomena of the first, therefore the discussion will focus accordingly.

One example of how axioms may diverge from reality is the concept of utility; it is often regarded as a strictly concave and continuous function ($u' > 0$ and $u'' < 0$), i.e. capturing the concept of diminishing marginal utility and the idea that more is always better. Much of economic theory, including the examples of finance and microeconomics examined in the above, take this concept for granted. Furhter, the foundation for this formulation is the assumption that economic decision making is based on a rational analysis of the information at hand and, accordingly, the dominating paradigm in microeconomics is the theory of rational choice [15]. However, the correctness of this assumption has for example been challenged by prospect theory [16, 17] which promote a more psychological perspective on decision making. Regardless of the conflicting details of these theories, they highlight the critical point of the first and second critique of Beed and Kane [13] described above. Every mathematical model is dependent of the set of axioms which it is built upon, which consequently determines how well the theory fits with reality. Further, Beed and Kane[13] argues that no set of axioms may capture the reality in a completely accurate way. This is intuitively a rather reasonable assumption, which ultimately leads to the conclusion that mathematical economics are per definition incomplete.

The third critique by Beed and Kane is presumably even worse; if the gap between theory and reality is extended to include the hypothesis, the theory will not be possible to confirm. And if the mathematical analysis produce conclusions that are impossible to validate, it is also impossible to confirm the correctness of the set of assumptions that govern the analysis. This is effectively described by the scientific philosopher Karl Popper who said that good science should make bold conjectures, meaning that a theory must comprise conditions for how it would be falsified and thereafter be tested accordingly. Therefore, if mathematical economics fail to present testable predictions, the consequence might be that we get deluded into interpreting the simplicity of its mathematical deduction as an accurate analysis of reality.

In future research of mathematical applications to economics, it should therefore be considered as fundamental that theories state criteria for falsification explicitly and that experimental economics test and challenge these criteria repeatedly. In this way it might be possible to ensure that theories do not gain influence solely from mathematical rigour but from proved applicability.

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