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A Riemann-Hilbert Problem Approach to Mesoscopic Fluctuations for the CUE

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A Riemann-Hilbert Problem Approach to Mesoscopic Fluctuations for the CUE

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Abstract

In this thesis we will consider a particular probability measure, the Circular Unitary Ensemble, which is a famous model within Random Matrix theory.

We will give new proofs of two central limit theorem's associated to this measure. The proofs are based on the fact that the moment generating function of a linear statistic can be written as a Fredholm determinant of an integrable operator. With a Riemann-Hilbert problem approach, it is possible to evaluate the determinant, at least asymptotically.

Acknowledgement

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Contents

1 Introduction

Consider n points $e^{i\theta_1}, \ldots, e^{i\theta_n}$ where the arguments are chosen randomly with respect to the probability measure

$$\frac{1}{n!(2\pi)^n} \prod_{1 \leq k < \ell \leq n} |e^{i\theta_k} - e^{i\theta_\ell}|^2 d\theta_1 \ldots d\theta_n \tag{1}$$

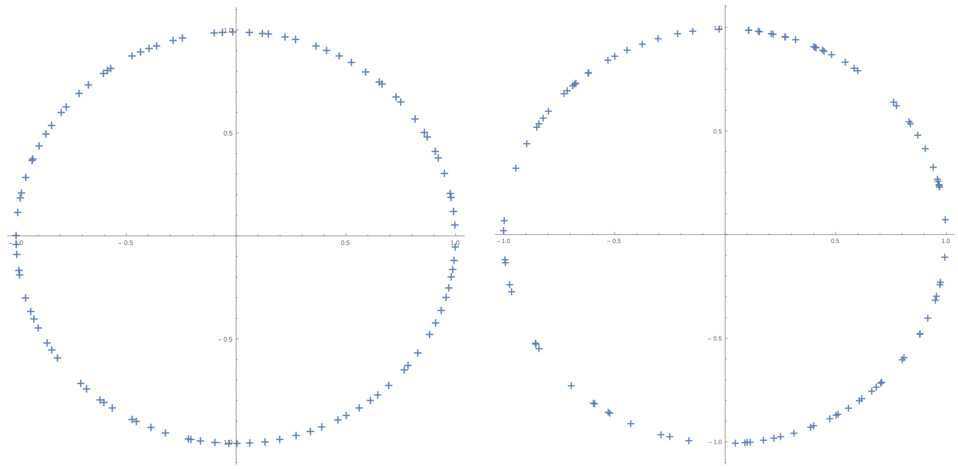
on $[-\pi, \pi)^n$. In this thesis we are interested in the behavior of these points as n tends to infinity.

From the measure we can tell that the probability that two points are close to each other is small, that is, the points appear to repel each other. A sample with respect to this measure is shown in Fig 1a. One can see that the points are random, but there is no clustering, they are more or less equidistant. In the sample from a uniform distributed measure, Fig 1b, at the other hand, there are clustering and the points are more chaotic. We know, however, that there is some kind of structure as the number of points tends to infinity, for example the classical Central Limit Theorem (CLT) for independent and identically distributed (i.i.d.) points. In our case, the classical CLT does not apply, since we do not have independent distributed points, in fact, they are strongly correlated. A natural question is, do we have a replacement of the classical CLT? The answer is yes, but significantly different. These type of questions we want to understand.

We will see that (1) has a nice structure, which allows us to compute the behavior for large n, and by that we can find new laws that are different from the laws about independent random variable. These laws are believed to be universal. They appear very often when it comes to big complex systems with some repulsion, for example the energy levels of heavy nuclei and the zeros of the Riemann-Zeta function (see [4] and [9]). Often these systems are too complicated to analyze in detail. The purpose of Random Matrix theory is to analyze models that generate the same behavior but are simple enough to analyze (toy models). The measure (1) is one of the famouse examples of such model. The $e^{i\theta}$ can be obtained as the eigenvalues of a random unitary matrix. This model is called the Circular Unitary Ensemble, CUE, in the literature. The measure (1) has also interesting mathematical properties, Lemma 3.3 gives a simple relation to Toeplitz matrices. This will be used in this thesis.

As indicated above, we want to understand the asymptotic of (1). One natural object in the study of a probability measure of this type is linear statistics.

Definition 1.1. Let f be a function on $[-\pi, \pi)$ and $(\theta_1, \ldots, \theta_n) \in [-\pi, \pi)^n$,



(a) A sample from (1) for $n = 100$.

(b) Samples from independent uniformly distributed points.

Figure 1

then

$$X_n(f) = \sum_{k=1}^{n} f(\theta_k)$$

is the *linear statistic* of f.

One of the key feature of (1) is that the moment generating function of a linear statistic can be expressed as a Fredholm determinant. Namely

$$\mathbb{E}\left[e^{\lambda X_n(f)}\right] = \det\left(I + K_n\left(e^{\lambda f} - 1\right)\right), \tag{2}$$

where K_n is defined by (16), see Lemma 3.3 and Lemma 3.4. This translates the problem of understanding the linear statistic into studying Fredholm determinants which are a part of analysis. The main proofs in this thesis are analytic in nature.

In this introduction we will mention some known results about (1) and then state the main results which we will prove. For further discussion about topics closely related to Theorem 1.1, Theorem 1.2 and Theorem 1.3 we refer to [4] and the reference therein.

A first result is given in the following theorem.

Theorem 1.1. *For a continuous function f ,*

$$\frac{1}{n}\mathbb{E}[X_n(f)] = \int_{-\pi}^{\pi} f(\theta)d\theta.$$

Proof. This follows from Lemma 3.9 and (17) $\square$

This is basically the weak law of large numbers, which we know is true for i.i.d. random variables. We recall that the fluctuation for i.i.d. random variables are given by the CLT. For (1) we also have a CLT, but it is of a different nature.

Theorem 1.2. *Assume that f is a function on $[-\pi, \pi)$ such that*

$$\sum_{k=1}^{\infty} k|\hat{f}(k)|^2 < \infty.$$

Then the random variable

$$X_n(f) - \mathbb{E}[X_n(f)] \to N(0, \sigma^2)$$

in distribution, where $N(0, \sigma^2)$ is the normal distribution with variance $\sigma^2 = 2\sum_{k=1}^{\infty} k|\hat{f}(k)|^2$ and $\hat{f}(k)$ is the Fourier coefficient.

Remark. From Lemma 3.3 this is the Strong Szegő Limit for Toeplitz determinants.

This is, as mentioned, a Central Limit Theorem, but note that we do not divide by a normalizing factor. This is a remarkable fact. Recall that the normalizing factor is $\sqrt{n}$ for i.i.d. random variables. That the sum actually converges is not clear at all. This tells us that the repulsion is powerful.

Theorem 1.1 and Theorem 1.2 are examples of results on the macroscopic scale, that is, the distribution when viewing all points at the same time. Another important result is on the microscopic scale. For the microscopic scale, one consider a part of order $\frac{1}{n}$, that is, the distance, between the eigenvalues, are of order one. To zoom in at θ_0 one can consider a function f with compact support and define $f_n(\theta) = 2\pi f(2\pi n(\theta - \theta_0))$, the constant 2π are included for simplicity. The following theorem give us a way to understand an infinite point process. Of course, to give a proper definition, more work is needed (see [7]).

Theorem 1.3. *For functions with sufficiently fast decay,*

$$\mathbb{E}\left[e^{\lambda X_n(f_n)}\right] \to \det(I + K(e^{\lambda f} - 1))$$

as $n \to \infty$. Here K is the operator defined by the sine kernel $K : \mathbb{R}^2 \to \mathbb{R}$, with

$$K(x, y) = \frac{\sin(\pi(x - y))}{\pi(x - y)}.$$

It is natural to ask what happens in between the macroscopic and microscopic scale. This is called the mesoscopic scale. This is the main topic for this thesis. The main result will tell us about the distribution when it comes to the mesoscopic scale. We will prove the following theorem.

Theorem 1.4. *Let $G \in L^2(\mathbb{R})$ be a continuous real valued function with compact support such that*

$$\int_0^\infty \xi |\mathcal{F}(G)(\xi)|^2 d\xi < \infty$$

where

$$\mathcal{F}(G)(\xi) = \int_{-\infty}^{\infty} G(x) e^{-i\xi x} dx$$

is the Fourier Transform. Fix $\alpha \in (0,1)$, $\theta_0 \in [-\pi, \pi)$ and let $G_n(\theta) = G(n^\alpha(\theta - \theta_0))$. Then

$$X_n(G_n) - \mathbb{E}[X_n(G_n)] \to N(0, \sigma^2)$$

in distribution, where

$$\sigma^2 = \frac{1}{2\pi^2} \int_0^\infty \xi |\mathcal{F}(G)(\xi)|^2 d\xi.$$

Here we are interested in a part of the unit circle of order $n^{-\alpha}$. So the number of eigenvalues in the part we are looking at, tends to infinity, that is, the expectation value of $X_n(G_n)$ tends to infinity, see (31). This is also a remarkable CLT, since we do not divide by a normalizing factor. Moreover, we can see that the limit does not depend on neither θ_0 nor α.

The functions that we consider in this theorem is a subset of a subspace of $L^2(\mathbb{R})$ equipped with a Sobolev type of seminorm,

$$\left(\int_0^\infty \xi |\mathcal{F}(G)(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

It is easy to see that this is a seminorm by the observation that

$$\left(\int_0^\infty \xi |\mathcal{F}(G)(\xi)|^2 d\xi \right)^{\frac{1}{2}} = \|1_{[0,\infty)} \sqrt{\xi} \mathcal{F}(G)\|_{L^2(\mathbb{R})}.$$

We will not go any deeper into this, but we will use that this defines a seminorm in the proof of Theorem 1.4, to be able to extend our proof to all functions stated in the theorem.

The other main theorem in this thesis concerns going from the microscopic scale to the macroscopic scale. For this we start with the right hand side in Theorem 1.3 and zoom out.

Theorem 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Schwartz function such that $\mathcal{F}(f)(\xi) \leq Ae^{-a|\xi|}$ for some positive constants a, A. Assume also that the first and second derivative of the Fourier transform of f satisfies the same condition. Further let*

$$f_n(z) = f\left(\frac{z}{n}\right).$$

Then there exists a disc around the origin such that if λ belongs to this disk, then

$$\det\left(I + K\left(e^{\lambda f_n} - 1\right)\right) = e^{n\lambda\int_{\mathbb{R}} f(\xi)d\xi + \frac{\lambda^2}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(f)(\xi)|^2 d\xi}(1 + \mathcal{O}(ne^{-2\pi n\rho}))$$

for some $\rho > 0$.

With this theorem in hand, we have the crucial part of the proof of a similar theorem as Theorem 1.4, for the sine kernel. What is left is to extend the result as mentioned in the remark after Theorem 1.4. But to do that one need a proper definition of an infinite point process, as mentioned before Theorem 1.3. We will not include this in the thesis. The limit in Theorem 1.5 can be compared with the limit in Theorem 1.4. That we get the same limit if we start from the macroscopic scale and zoom in as if we start at the microscopic scale and zoom out is remarkable.

Theorem 1.4 has been proved by Soshnikov and he mentioned that the same approach would work for Theorem 1.5, see [12]. Related work has recently been done by Johansson and Lambert, see [8]. What we will do in this thesis is a new proof, an analytic proof, using modern techniques. The proofs are inspired by [2]. Hopefully this approach will help us to get a deeper understanding about the problem. The equality (2) is true for all determinantal point process (for more on determinantal point process see [7]) for some operator. That makes it interesting to investigate if this proof apply in other situation when the operator in (2) is an integrable operator.

The outline of this thesis is the following; We will first give the necessary tools to be able to attack the two main theorems stated above. The tools we consider are the Cauchy operator, Riemann-Hilbert problem technique and Integrable operators. In section 3 and 4 we will prove the main results. Especially Theorem 3.6 and Theorem 4.5 are the crucial parts of these proofs. The proofs of these two theorem are based on a Riemann-Hilbert Problem approach and are postponed to section 5.1 and section 5.2.

2 Preliminaries

2.1 Analysis of the Cauchy operator

Here we will do analysis of the Cauchy operator for the unit circle and for the real line. It is possible to do this for more arbitrary contours (see [10]), but for the purpose of this project this will suffice.

Definition 2.1. Let Γ be a contour in $\mathbb{C}$. For $h \in L^2(\Gamma)$ the *Cauchy transform on* Γ is defined as

$$(Ch)(z) = \int_\Gamma \frac{h(w)}{w - z}\frac{dw}{2\pi i}$$

as long as the right hand side make sense.

This transform depends on the contour, but what contour we consider will be clear from the context. We will do analysis of the Cauchy transform for the unit circle and for the real line.

Definition 2.2. Let $h : \mathbb{C}\backslash\mathbb{T} \to \mathbb{C}$ and $h_+ : \mathbb{T} \to \mathbb{C}$. We say that $h \to h_+$ in $L^2(\mathbb{T})$ sense from the $+$-side if

$$\lim_{r\to 1, r<1} \int_{\mathbb{T}} |h(rz) - h_+(z)|^2 |dz| = 0.$$

In the same way we say that $h \to h_-$ from the $-$-side if the same is true for $r > 1$.

Lemma 2.1. *If $h \in L^2(\mathbb{T})$ then Ch is a well defined analytic function away from the circle. There exists bounded linear operators, C_+ and C_-, on $L^2(\mathbb{T})$ such that $Ch \to C_+h$ and $Ch \to C_-h$ in $L^2(\mathbb{T})$ sense from the $+$-side and $-$-side respectively for all $h \in L^2(\mathbb{T})$. Moreover*

$$(C_+h)(z) = \sum_{k=0}^{\infty} \hat{h}(k) z^k$$

and

$$(C_-h)(z) = -\sum_{k=-\infty}^{-1} \hat{h}(k) z^k$$

which implies the relation

$$C_+h - C_-h = h$$

for all $h \in L^2(\mathbb{T})$.

Proof. Let $|z| < 1$, then

$$\begin{aligned}
(Ch)(z) &= \int_{\mathbb{T}} \frac{h(w)}{w-z} \frac{dw}{2\pi i} \\
&= \int_{\mathbb{T}} \frac{1}{w} \frac{h(w)}{1-\frac{z}{w}} \frac{dw}{2\pi i} \\
&= \int_{\mathbb{T}} \frac{h(w)}{w} \sum_{k=0}^{\infty} \left(\frac{z}{w}\right)^k \frac{dw}{2\pi i} \\
&= \sum_{k=0}^{\infty} z^k \int_{\mathbb{T}} h(w) w^{-k} \frac{dw}{2\pi i w} \\
&= \sum_{k=0}^{\infty} \hat{h}(k) z^k.
\end{aligned}$$

Since $h \in L^2(\mathbb{T})$ the last series converges absolutely, hence it is an analytic function on the open unit disk. This suggests us to make the definition of C_+ as

$$(C_+h)(z) = \sum_{k=0}^{\infty} \hat{h}(k) z^k.$$

With a similar calculation

$$(Ch)(z) = -\sum_{k=-\infty}^{-1} \hat{h}(k) z^k. \tag{3}$$

for $|z| > 1$. Therefore we define C_- as

$$(C_-h)(z) = -\sum_{k=-\infty}^{-1} \hat{h}(k) z^k.$$

We note that C_+ and C_- are projections on $L^2(\mathbb{T})$ and we can therefore see that they are well defined linear operators.

Now, we want to show that C converges to C_+ and C_- in the correct sense. Consider $h \in L^2(\mathbb{T})$. Given $\epsilon > 0$ find an $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} |\hat{h}(k)|^2 < \frac{\epsilon}{2}$$

and find an $r < 1$ such that $|r^N - 1| < \frac{\epsilon}{2\|h\|_{L^2}^2}$. Then, from the above calculations

$$\begin{aligned}
\int_{\mathbb{T}} |Ch(rz) - C_+h(z)|^2 |dz| &= \int_{\mathbb{T}} \left| \sum_{k=0}^{\infty} \hat{h}(k)(r^k - 1) z^k \right|^2 |dz| \\
&= \sum_{k=1}^{\infty} |\hat{h}(k)|^2 |r^k - 1|^2 \\
&\le |r^N - 1| \sum_{k=1}^{N} |\hat{h}(k)|^2 + \sum_{k=N+1}^{\infty} |\hat{h}(k)|^2 \\
&< \epsilon,
\end{aligned}$$

where the second equality is by Parseval's identity. Hence $Ch \to C_+h$ in $L^2(\mathbb{T})$ sense from the $+$-side. In the same way we can see that $Ch \to C_-h$ in $L^2(\mathbb{T})$ sense from the $-$-side □

Lemma 2.2. *If $h \in C^2(\mathbb{T})$ then C_+h and C_-h are differentiable and we can differentiate termwise.*

Proof. To justify that we can differentiate C_+h and C_-h we use that $h \in C^2(\mathbb{T})$ and therefore $h'' \in L^2(\mathbb{T})$. Then

$$\sum_{k=2}^{\infty} |k\hat{h}(k)| \leq \left(\sum_{k=2}^{\infty} \frac{1}{(k-1)^2}\right)^{\frac{1}{2}} \left(\sum_{k=2}^{\infty} |k(k-1)\hat{h}(k)|^2\right)^{\frac{1}{2}}$$
$$\leq \|\hat{h}''\|_{L^2(\mathbb{T})} \left(\sum_{k=2}^{\infty} \frac{1}{(k-1)^2}\right)^{\frac{1}{2}} < \infty$$

where we have used Cauchy-Schwarz inequality, Bessel's inequality and that $k(k-1)\hat{h}(k) = \hat{h}''(k-2)$. Hence

$$(C_+h)'(z) = \sum_{k=0}^{\infty} k\hat{h}(k)z^{k-1} \tag{4}$$

and

$$(C_-h)'(z) = -\sum_{k=-\infty}^{-1} k\hat{h}(k)z^{k-1} = \sum_{k=0}^{\infty} k\hat{h}(-k)z^{-k-1}. \tag{5}$$

□

We will now continue and consider the case with the real line as contour.

Definition 2.3. Let $h : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}$ and $h_+ : \mathbb{R} \to \mathbb{C}$. We say that $h \to h_+$ in $L^2(\mathbb{R})$ sense from the $+$-side if

$$\lim_{\epsilon\to 0, \epsilon>0} \int_{\mathbb{R}} |h(x+i\epsilon) - h_+(x)|^2 dx = 0.$$

In the same way we say that $h \to h_-$ from the $-$-side if the same is true with $-\epsilon$ instead of ϵ.

Lemma 2.3. *If $h \in L^2(\mathbb{R})$ then Ch is a well defined analytic function away from the real line. There exists bounded linear operators, C_+ and C_-, on $L^2(\mathbb{R})$ such that $Ch \to C_+h$ and $Ch \to C_-h$ in $L^2(\mathbb{R})$ sense from the $+$-side and $-$-side respectively for all $h \in L^2(\mathbb{R})$. Moreover the relation*

$$C_+h - C_-h = h$$

holds.

Proof. Let $\text{Im}(z) > 0$ and let h be a Schwartz function. For any $A > 0$ we have the relation

$$\frac{1}{y-z} = i\int_0^A e^{-i\xi(y-z)} d\xi + \frac{e^{-iA(y-z)}}{y-z}.$$

If we use that in the definition of the Cauchy transform we get

$$\begin{aligned}(Ch)(z) &= \frac{1}{2\pi}\int_{\mathbb{R}}\int_0^A h(y)e^{-i\xi(y-z)}d\xi dy + \frac{e^{iAz}}{2\pi i}\int_{\mathbb{R}} h(y)\frac{e^{-iAy}}{y-z}dy \\ &= \frac{1}{2\pi}\int_0^A \mathcal{F}(h)(\xi)e^{i\xi z}d\xi + \frac{e^{iAz}}{2\pi i}\int_{\mathbb{R}} h(y)\frac{e^{-iAy}}{y-z}dy\end{aligned}$$

where the change of order of integration is valid since h is a Schwartz function. The right term above converges to zero as A growth since $\mathrm{Im}(z) > 0$. Let A go to infinity, then

$$(Ch)(z) = \frac{1}{2\pi}\int_0^\infty \mathcal{F}(h)(\xi)e^{i\xi z}d\xi. \tag{6}$$

A similar calculation leads to the relation

$$(Ch)(z) = -\frac{1}{2\pi}\int_0^\infty \mathcal{F}(h)(-\xi)e^{-i\xi z}d\xi$$

if $\mathrm{Im}(z) < 0$. Since

$$\int_0^\infty |\mathcal{F}(h)(\xi)e^{i\xi z}|d\xi \leq \left(\int_0^\infty |\mathcal{F}(h)(\xi)|^2 d\xi\right)^{1/2}\left(\int_0^\infty |e^{-\xi \mathrm{Im}(z)}|^2 d\xi\right)^{1/2},$$

by Cauchy-Schwartz inequality, and since the Schwartz functions are dense in $L^2(\mathbb{R})$ it is not difficult to see that the above equalities hold for all $h \in L^2(\mathbb{R})$. We can also see that Ch defines an analytic function.

We want to find the limiting operators as we did for the circle. For that, recall that $\mathcal{F}$ can be extended to a unitary linear operator on $L^2(\mathbb{R})$ (see e.g. [14]). Now define C_+ and C_- on $L^2(\mathbb{R})$ as

$$C_+h = \mathcal{F}^{-1}(1_{[0,\infty)}\mathcal{F}(h))$$

and

$$C_-h = -\mathcal{F}^{-1}(1_{(-\infty,0]}\mathcal{F}(h)).$$

These are clearly bounded linear operators with $\|C_+\| \leq 1$ and $\|C_-\| \leq 1$. We want to see that C converges to C_+ and C_- in the correct sense. First note that if $\epsilon > 0$, then

$$Ch(x+i\epsilon) = \mathcal{F}^{-1}(1_{[0,\infty)}e^{-x\epsilon}\mathcal{F}(h))(x).$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}} |Ch(x+i\epsilon) - C_+h(x)|^2 dx \\ &= \|\mathcal{F}^{-1}(1_{[0,\infty)}e^{-x\epsilon}\mathcal{F}(h))(x) - \mathcal{F}^{-1}(1_{[0,\infty)}\mathcal{F}(h))(x)\|^2_{L^2(\mathbb{R})} \\ &\leq \|1_{[0,\infty)}\left(e^{-x\epsilon}-1\right)\mathcal{F}(h))\|^2_{L^2(\mathbb{R})} \\ &\to 0\end{aligned}$$

as $\epsilon \to 0$ by Lebesgue Dominant Convergence Theorem. The same argument holds to see that C converges to C_- in the right sense.

The linearity of the Fourier transform and the Fourier Inversion Formula implies the relation

$$(C_+h)(x) - (C_-h)(x) = h(x).$$

$\square$

Lemma 2.4. *If h is an Schwartz function, then C_+h and C_-h are differentiable and*

$$(C_+h)'(x) + (C_-h)'(x) = \frac{-1}{2\pi i}\int_0^\infty \xi \left(\mathcal{F}(h)(\xi)e^{i\xi x} + \mathcal{F}(h)(-\xi)e^{-i\xi x}\right) d\xi.$$

Proof. This is direct by differentiating the relation

$$(C_+h)(x) + (C_-h)(x) = \frac{1}{2\pi}\int_0^\infty \mathcal{F}(h)(\xi)e^{i\xi x}d\xi - \frac{1}{2\pi}\int_0^\infty \mathcal{F}(h)(-\xi)e^{-i\xi x}d\xi$$

which we can do since $\mathcal{F}(h)$ is a Schwartz function. $\square$

Remark. To define C_+ and C_- for more arbitrary curves one can use The Plemelj Formula (see [10]).

2.2 Riemann-Hilbert problem

For the proof of the main results we will use a Riemann-Hilbert problem, RHP, approach. We will give a brief introduction with some important results. The introduction is based on [3] but adjusted to our settings.

Given a contour Γ with an orientation, let the $+$-side be to the left and the $-$-side to the right of the contour, and a jump matrix $J : \Gamma \to \mathbb{C}^{N\times N}$, we have the following definition.

Definition 2.4. A *solution to the RHP* (Γ, J) is a function $m : \mathbb{C}\backslash\Gamma \to \mathbb{C}^{N\times N}$ that fulfills the conditions

(i) m is analytic in $\mathbb{C}\backslash\Gamma$,

(ii) $m_+(z) = m_-(z)J(z)$ for $z \in \Gamma$,

(iii) $m \to I$ as $|z| \to \infty$.

Here m_+ and m_- are functions living on the contour such that $m \to m_+$ as z converges to the contour from the $+$-side and $m \to m_-$ as z converges to the contour from the $-$-side. Of course one need to specify in what sense the limit is taken, as well as in what sense $m \to I$. This can be done in different ways.

Definition 2.5. Let Γ be a finite disjoint union of oriented smooth contours with no endpoints and no self intersections. Let $h : \mathbb{C} \backslash \Gamma \to \mathbb{C}$ and $h_+ : \Gamma \to \mathbb{C}$. We say that $h \to h_+$ in locally $L^2(\Gamma)$ sense from the $+$-side if

$$\lim_{\epsilon > 0, \epsilon \to 0} \int_{t_0}^{t_1} \left| h\left(\gamma(t) + i\epsilon\gamma'(t)\right) - h_+(\gamma(t)) \right|^2 dt = 0$$

for all $z \in \Gamma$. Here γ is a regular parametrization of the contour of unit speed defined in $[t_0, t_1]$, for some t_0 and t_1, and such that $\gamma([t_0, t_1])$ is a neighborhood of z in Γ. In the same way we say that $h \to h_-$ from the $-$-side if the same is true with the natural change of ϵ to $-\epsilon$.

For this project we will consider RHP:s where the limit is taken in locally $L^2(\Gamma)$ sense. We will also say that $m \to I$ if $m(z)$ is bounded as $|z| \to \infty$ away from Γ and if

$$m(z) \to I$$

as $|z| \to \infty$ for some sequence. We will assume Γ to be a finite disjoint union of oriented smooth curves with no endpoints and no self intersections, that is, so we can use the definition of convergence in locally $L^2(\Gamma)$ sense. This is stronger than necessary but since we will basically consider the unit circle and the real line this is no restriction for us. But it is actually possible to do this for more complicated curves, even for self intersecting curves. We will view the unit circle as a contour oriented counter clockwise and the real line oriented from $-\infty$ to ∞. We will also assume that J is smooth and bounded and that $\det(J(z)) = 1$, assume further that $J - I \in L^2(\Gamma)$. These assumptions are also extra strong, but will be fulfilled in all cases within this thesis.

The solution to Riemann-Hilbert Problems turns out to be related to solutions of other type of problems. For example in the analysis of orthogonal polynomials and in differential equations. For more theory and examples see [3] and [5].

When it comes to RHP it is often the case that the existence of a solution is more problematic then the uniqueness. Especially for 2×2 RHP:s, which will be the case in this thesis, we have the following theorem.

Theorem 2.5. *Consider the RHP (Γ, J) where J is a 2×2 matrix. If there exists a solution, then this solution is unique.*

Proof. This is a proof that can be found in e.g. Theorem 7.18 in [3] but adjusted to our settings.

Let m be a solution to the RHP (Γ, J). First of all, we want to prove that m^{-1} exists. since m is a solution to the RHP, $\det(m(z))$ is an analytic function away from Γ. Since $\det(J(z)) = 1$,

$$\det(m_+(z)) = \det(m_-(z)) \det(J(z)) = \det(m_-(z)).$$

We want to use this equality to see that we can extend $\det(m(z))$ to an entire function. Let $z' \in \Gamma$, and let $\gamma : [t_1, t_2] \to \Gamma$ be the parametrization in the definition of locally $L^2(\Gamma)$ convergence. Since m is a 2×2 matrix we get from the convergence of m to m_+ that

$$\int_{t_0}^{t_1} \left|\det\left(m\left(\gamma(t) + i\epsilon\gamma'(t)\right)\right) - \det\left(m_+(\gamma(t))\right)\right| dt \to 0 \tag{7}$$

as $\epsilon \to 0$. Let $t_0 \leq s_0 < s_1 \leq t_1$ and let $\mathcal{C}_\epsilon$ be the contour on the $+$-side of Γ that consists of the contour $\gamma_\epsilon = \gamma([s_0, s_1]) + i\epsilon\gamma'([s_0, s_1])$ and with part of a half circle connecting $\gamma(s_0) + i\epsilon\gamma'(s_0)$ and $\gamma(s_1) + i\epsilon\gamma'(s_1)$, oriented counterclockwise. Let $\epsilon > 0$ be so small that the intersection of the interior of all $\mathcal{C}_{\epsilon'}$, $0 \leq \epsilon' < \epsilon$ contains an accumulation point. Let z be in that intersection. Then

$$\det(m(z)) = \int_{\mathcal{C}_\epsilon} \frac{\det(m(w))}{w - z} \frac{dw}{2\pi i}. \tag{8}$$

Let $s = \sup_{t\in[s_0,s_1]} |\gamma''(t)|$ and $s_z = \inf_{t\in[s_0,s_1]} |\gamma(t) - z|$, from (7),

$$\begin{aligned}
&\left|\int_{\gamma_\epsilon} \frac{\det(m(w))}{w - z} \frac{dw}{2\pi i} - \int_{\gamma_0} \frac{\det(m_+(w))}{w - z} \frac{dw}{2\pi i}\right| \\
&\leq \frac{1 + \epsilon s}{s_z - \epsilon} \int_{s_0}^{s_1} \left|\det\left(m\left(\gamma(t) + i\epsilon\gamma'(t)\right)\right) - \det\left(m_+(\gamma(t))\right)\right| \frac{dt}{2\pi} \\
&+ \frac{|z|\epsilon}{(s_z - \epsilon)s_z} \int_{s_0}^{s_1} |\det(m_+(\gamma(t)))| \frac{dt}{2\pi} \\
&\to 0
\end{aligned}$$

as $\epsilon \to 0$ since $\det(m_+) \in L^1([s_0.s_1])$ which can be seen from (7). From the above calculations and from Fubini's Theorem,

$$\int_0^\epsilon \int_{\gamma_{\epsilon'}} \frac{\det(m(w))}{w - z} \frac{dw}{2\pi i} d\epsilon' \to 0$$

and from Chebyshev's inequality we can conclude that

$$\int_0^\epsilon \frac{\det(m(\gamma(t) + i\epsilon'\gamma'(t)))}{\gamma(t) + i\epsilon'\gamma'(t) - z} (\gamma'(t) + i\epsilon\gamma''(t)) \frac{d\epsilon'}{2\pi} \to 0 \tag{9}$$

for almost every $t \in (t_0, t_1)$ as $\epsilon \to 0$. Hence, by letting $\epsilon \to 0$ in (8),

$$\det(m(z)) = \int_{\mathcal{C}_0} \frac{\det(m(w))}{w - z} \frac{dw}{2\pi i}$$

with $\det(m(w)) = \det(m_+(w))$ for $w \in \Gamma$. By the same argument,

$$0 = \int_{\mathcal{C}_{0,-}} \frac{\det(m(w))}{w - z} \frac{dw}{2\pi i}$$

where $\mathcal{C}_{\epsilon,-}$ is the correspondent to $\mathcal{C}_\epsilon$ on the $-$-side of Γ. Hence, if $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_{0,-} \backslash \gamma([s_0.s_1])$, where s_0 and s_1 are such that (9) is true, then

$$\det(m(z)) = \int_{\mathcal{C}} \frac{\det(m(w))}{w-z} \frac{dw}{2\pi i}.$$

Of course the same argument is true if z is on the $-$-side of Γ. Hence

$$\det(m(z)) = \int_{\mathcal{C}} \frac{\det(m(w))}{w-z} \frac{dw}{2\pi i}$$

for all z in the interior of $\mathcal{C}$ away from Γ by the Uniqueness Theorem of analytic functions. Since the right hand side defines an analytic function in the interior of $\mathcal{C}$, we can extend $\det(m(z))$ over Γ close to z'. This is possible to do for all $z' \in \Gamma$, hence we can extend $\det(m(z))$ to an entire function. But since $\det(m(z))$ is bounded and $\det(m(z)) \to 1$ as $|z| \to \infty$, for some z, we can conclude that $\det(m(z))$ is constant and equal to one. Hence $m(z)^{-1}$ exists for all $z \in \mathbb{C}$.

Assume now that m_1 and m_2 are solutions to (Γ, J) and let $m = m_1 m_2^{-1}$, which is well defined since m_2^{-1} exists. Then

$$m_+ = (m_{1-}J)(m_{2-}J)^{-1} = m_{1-}m_{2-} = m_-.$$

With the same argument as used above, we can extend m to an entire function. Moreover $m(z) \to I$ as $|z| \to \infty$. Hence $m(z) = I$ for all $z \in \mathbb{C}$ which implies that

$$m_1(z) = m_2(z)$$

for all $z \in \mathbb{C}\backslash\Gamma$. □

For a solution to the RHP (Γ, J) we can, in some circumstances, use an operator on $L^2(\Gamma)^{2\times 2}$ defined as

$$C_w h = C_-(hw_+) + C_+(hw_-) \tag{10}$$

where $w = w_- + w_+$ and w_+ and w_- are defined as

$$J = (I - w_-)^{-1}(I + w_+).$$

for some factorization. In this factorization we have a lot of freedom, but the factorization needs to be done in such a way that w_+ and w_- are bounded and in $L^2(\Gamma)$. The following theorem is true under certain condition on Γ and J.

Theorem 2.6. *Consider the operator defined in* (10). *Assume that* $(I - C_w)^{-1}$ *exists as a bounded operator. Then with*

$$\mu = (I - C_w)^{-1} I$$

and

$$m = I + (C(\mu w))(z),$$

m solves the RHP (Γ, J).

Remark. We need to be a bit careful what we mean with $(I - C_w)^{-1}I$ if Γ is unbounded. What we mean is that $\mu - I \in L^2(\Gamma)$ such that $(I - C_w)(\mu - I) = C_w I$. Not that m still is well defined due to the condition on w_+ and w_-.

Proof. We will only prove this for the circle and for the real line, with extra assumptions on the real line. These are the cases that will be used later in this thesis. For the more general proof and the exact assumptions on Γ and J see Theorem 7.103 in [3].

We start with the proof for the circle. First of all, since w is bounded, $\mu w \in L^2(\mathbb{T})$, so m is a well defined analytic function on $\mathbb{C}\backslash\mathbb{T}$ from Lemma 2.1. From the same lemma we know that C converges to C_+ and C_- in the proper way. Hence

$$\begin{aligned} m_+ &= I + C_+(\mu w) \\ &= I + C_+(\mu w_-) + C_-(\mu w_+) + C_+(\mu w_+) - C_-(\mu w_+) \\ &= I + C_w(\mu) + \mu w_+ \\ &= \mu(I + w_+). \end{aligned} \tag{11}$$

In the same way

$$m_- = \mu(I - w_-).$$

Hence

$$m_+ = \mu(I - w_-)(I - w_-)^{-1}(I + w_+) = m_- J.$$

From (3) it is direct that

$$m = I + \mathcal{O}(z^{-1})$$

as $|z| \to \infty$.

For the case of the real line, the same calculations of the first part is still valid. What we need to prove, is that $m \to I$. The extra assumptions that we will add are that μ and w are analytic in some strip containing the real line, $\mu w \in L^1(\mathbb{R})$ and that there exists an $\epsilon > 0$ such that μw is bounded for $|\mathrm{Im}(z)| \leq \epsilon$. Then if $\mathrm{Im}(z) > 0$, we can deform the integration contour to Γ_ϵ such that the distance from z to Γ_ϵ is greater then ϵ. Then

$$\begin{aligned} |m(z) - I| &= \left| \int_{\Gamma_\epsilon} \frac{\mu(z')w(z')}{z' - z} dz' \right| \\ &\leq \left(\frac{1}{\epsilon} \int_{\mathbb{R}} |\mu(z')w(z')| dz' + \pi \sup_{|\mathrm{Im}(z')| \leq \epsilon} |\mu(z')w(z')| \right). \end{aligned}$$

If $\mathrm{Im}(z) < 0$, we can deform the contour in the other direction. Hence $m(z)$ is bounded as $|z| \to \infty$ and it is no problem to see that $m(z) \to I$ is $z \to \infty$ along the imaginary axes. $\square$

Corollary 2.7. *If*

$$\|J - I\|_\infty < \frac{1}{\|C_-\|_{L^2}}$$

then the RHP (Γ, J) *has a solution.*

Proof. To find an explicit solution we can use Theorem 2.6 with the factorization

$$J = I^{-1} J.$$

Then $w_- = 0$ and $w_+ = J - I$, hence

$$C_w h = C_-(h w_+).$$

Since C_- is a bounded operator

$$\begin{aligned} \|C_w h\|_{L^2} &= \|C_-(h w_+)\|_{L^2} \\ &\leq \|C_-\|_{L^2} \|h(J - I)\|_{L^2} \\ &\leq \|C_-\|_{L^2} \|J - I\|_\infty \|h\|_{L^2} \\ &< \|h\|_{L^2}, \end{aligned}$$

so

$$\|C_w\| < 1. \tag{12}$$

And since this is less than one we can use the Neumann series to see that

$$(I - C_w)^{-1}$$

exists as a bounded operator. □

To prevent this thesis from becoming to long, we will not go any deeper in the theory of RHP.

2.3 Integrable operators

An important object in the proof of the main results is integrable operators. We will give the definition and some properties of integrable operators and a connection to RHP. The material here are based on [2], except Lemma 2.11 which is included to give a tool to see if we can use the theory.

Recall that an integral operator K is an operator of the form

$$Kf(z) = \int K(z, z') f(z') dz'$$

for some kernel $K(z, z')$.

Definition 2.6. Let $\Gamma \subset \mathbb{C}$ be an oriented contour. An integral operator $K : L^2(\Gamma) \to L^2(\Gamma)$ is *integrable* if it has a kernel of the form

$$K(z, z') = \sum_{k=1}^{n} \frac{f_k^{(1)}(z) f_k^{(2)}(z')}{z - z'}$$

for $z, z' \in \Gamma$ for some functions $f_k^{(1)}, f_k^{(2)} \in L^2(\Gamma)$, $k = 1, \ldots, n$.

A kernel of this type has possible singularities at $z = z'$. One way to interpret this in quite general settings is with the Principle value integral (see [2]). In this thesis the functions and Γ will be especially nice, which makes it to an removable singularity. Moreover, it is possible to do the analysis with the Cauchy operators already introduced and therefore we do not need any knowledge about the Principle value integral. But the reader with understanding about it, can make these calculations more general.

First of all we will establish some general facts. Consider the Hilbert space $\mathcal{H} = L^2(\Gamma)$ and the operator A on $\mathcal{H}$ which is the multiplication with z, that is,

$$Ah(z) = zh(z).$$

Let $\mathcal{E}$ be the space of operators whose commutator with A is of finite rank, that is, all operators K on $\mathcal{H}$ such that

$$[A, K] = AK - KA$$

is of finite rank.

Remark. For us Γ will be the unit circle or the real line.

Lemma 2.8. *Assume that K is an integrable operator and that $(I - K)^{-1}$ exists as a bounded operator. Assume further that $R = (I - K)^{-1}K$ is an integral operator. Then R is an integrable operator.*

Proof. For the proof, we will first show that $R \in \mathcal{E}$ if $K \in \mathcal{E}$ and then that the integral operators in $\mathcal{E}$ are precisely the integrable operators on $\mathcal{H}$.

For the first assertion we consider the equality

$$\begin{aligned} [A, R] &= A(I - K)^{-1} - (I - K)^{-1}A \\ &= (I - K)^{-1}[A, K](I - K)^{-1} \end{aligned} \tag{13}$$

where the right hand side is of finite rank as long as $[A, K]$ is of finite rank. For the second assertion, assume that K is an integrable operator with the kernel

$$K(z, z') = \sum_{k=1}^{n} \frac{f_k^{(1)}(z) f_k^{(2)}(z')}{z - z'}.$$

Then

$$[A, K]h(z) = \int_\Gamma (z - z')K(z, z')h(z')dz'$$
$$= \sum_{k=1}^{n} (h, f_k^{(2)}) f_k^{(1)}(z),$$

hence $K \in \mathcal{E}$. Assume now that R is an integral operator in $\mathcal{E}$, that is

$$Rh(z) = \int_\Gamma R(z, z')h(z')dz'.$$

Then

$$[A, R]h(z) = \int_\Gamma (z - z')R(z, z')h(z')dz',$$

but since $[A, R]$ is of finite rank, there exists functions $F_k^{(1)}, F_k^{(2)} \in L^2(\Gamma)$ such that

$$[A, R]h(z) = \sum_{k=1}^{n} (h, F_k^{(2)}) F_k^{(1)}(z).$$

This implies that

$$\int_\Gamma \left((z - z')R(z, z') - \sum_{k=1}^{n} F_k^{(1)}(z)F_k^{(2)}(z') \right) h(z')dz' = 0$$

for all $h \in L^2(\Gamma)$. Hence

$$R(z, z') = \sum_{k=1}^{n} \frac{F_k^{(1)}(z)F_k^{(2)}(z')}{z - z'}.$$

□

For an operator B let $\overline{B}$ be the operator defined as

$$\overline{B}h(z) = \overline{B\overline{h}(z)}$$

and let B^T be the operator $\overline{B}^*$. Then

$$\int_\Gamma Bg(z)h(z)dz = \int_\Gamma g(z)B^T h(z)dz.$$

Theorem 2.9. *Assume that K is an integrable operator with kernel*

$$K(z, z') = \sum_{k=1}^{n} \frac{f_k^{(1)}(z)f_k^{(2)}(z')}{z - z'}.$$

Assume further that $(I-K)^{-1}$ exists and that $R = (I-K)^{-1}K$ is an integral operator. Then R is an integrable operator with kernel

$$R(z, z') = \sum_{k=1}^{n} \frac{F_k^{(1)}(z)F_k^{(2)}(z')}{z - z'}$$

where $F_k^{(1)} = (I-K)^{-1}f_k^{(1)}$ and $F_k^{(2)} = (I-K^T)^{-1}f_k^{(2)}$.

Proof. By Lemma 2.8 we know that R is an integrable operator with

$$R(z, z') = \sum_{k=1}^{n} \frac{F_k^{(1)}(z)F_k^{(2)}(z')}{z - z'}$$

for some functions $F_k^{(1)}$, $F_k^{(2)}$. We want to find a relation between the kernel of K and the kernel of R. Let $h \in L^2(\Gamma)$, then

$$[A, R]h(z) = \int_\Gamma (z - z')R(z, z')h(z')dz'$$

and

$$\begin{aligned}
&(I-K)^{-1}[A, K](I-K)^{-1}h(z) \\
&= (I-K)^{-1} \int_\Gamma (z - z')K(z, z')(I-K)^{-1}h(z')dz' \\
&= \sum_{k=1}^{n}(I-K)^{-1}f_k^{(1)}(z) \int_\Gamma f_k^{(2)}(z')(I-K)^{-1}h(z')dz' \\
&= \int_\Gamma \sum_{k=1}^{n}(I-K)^{-1}f_k^{(1)}(z)((I-K)^{-1})^T f_k^{(2)}(z')h(z')dz'
\end{aligned}$$

Since this is true for all $h \in L^2(\Gamma)$ we get from (13) that

$$F_k^{(1)} = (I-K)^{-1}f_k^{(1)}$$

and

$$F_k^{(2)} = ((I-K)^{-1})^T f_k^{(2)} = (I-K^T)^{-1}f_k^{(2)}.$$

The last equality can be seen e.g. since the equalities

$$\int_\Gamma (((I-K)^{-1})^T(I-K^T)h_1(z)h_2(z)dz = \int_\Gamma h_1(z)h_2(z)dz$$

and

$$\int_\Gamma (I-K^T)(((I-K)^{-1})^T h_1(z)h_2(z)dz = \int_\Gamma h_1(z)h_2(z)dz$$

hold for all $h_1, h_2 \in L^2(\Gamma)$. □

Given an integrable operator together with the functions in Definition 2.6, we will denote the vector $f^{(1)}$ and $f^{(2)}$ as the vectors formed by these functions,

$$f^{(1)} = (f_1^{(1)}, \dots, f_n^{(1)})^T$$

and

$$f^{(2)} = (f_1^{(2)}, \dots, f_n^{(2)})^T.$$

Theorem 2.10. *Let Γ be an oriented contour in $\mathbb{C}$ such that C_+ and C_- are bounded operators on $L^2(\Gamma)$. Let K and R be as in Theorem 2.9 and let m be the unique solution to the RHP (Γ, J) where*

$$J = I - 2\pi i f^{(1)}(f^{(2)})^T.$$

Assume further that $f_k^{(1)}, f_k^{(2)} \in L^\infty(\Gamma)$ and that they are analytic in some neighborhood of Γ. We will also assume that $f^{(1)}(z)^T f^{(2)}(z) = 0$ for all $z \in \Gamma$ and that J^{-1} exists. Then

$$F^{(1)} = m_+ f^{(1)}$$

and

$$F^{(2)} = (m_+)^{-T} f^{(2)}$$

where $-T$ denotes the inverse of the transpose.

Proof. Fist of all we want to express K in terms of the Cauchy operator. Note that since $f^{(1)}(z)^T f^{(2)}(z) = 0$ can we see that

$$\sum_{k=1}^{n} f_k^{(1)}(z) C_-(f_k^{(2)} h)(z) = \sum_{k=1}^{n} f_k^{(1)}(z) C_+(f_k^{(2)} h)(z).$$

We can therefore extend

$$\sum_{k=1}^{n} f_k^{(1)}(z) C(f_k^{(2)} h)(z)$$

to an analytic function in some neighborhood of Γ. We can write

$$Kh(z) = \int_{\Gamma \setminus B} \sum_{k=1}^{n} \frac{f_k^{(1)}(z) f_k^{(2)}(z')}{z - z'} h(z') dz' + \int_B \sum_{k=1}^{n} K(z, z') h(z') dz' \quad (14)$$

for some ball around z. But it is clear from the definition of the Cauchy transform that the right hand side is equal to

$$-2\pi i \sum_{k=1}^{n} f_k^{(1)}(z) C(f_k^{(2)} h)(z)$$

for $z \in B$, where this is the extended function over Γ. Let the size of B tend to zero, then, since $z' = z$ is a removable singularity, the last term tends to zero and therefore, the first term tends to

$$-2\pi i \sum_{k=1}^{n} f_k^{(1)}(z) C(f_k^{(2)} h)(z).$$

Hence

$$Kh = -2\pi i \sum_{k=1}^{n} f_k^{(1)} C_-(f_k^{(2)} h).$$

Let $D : L^2(\Gamma)^{n\times n} \to L^2(\Gamma)^n$ be defined as

$$Dh(z) = h(z) f^{(1)}(z)$$

and let $E : L^2(\Gamma)^n \to L^2(\Gamma)^{n\times n}$ be defined as

$$Eh(z) = -2\pi i C_-(h(f^{(2)})^T)(z).$$

If we let K act componentwise on a column vector $h \in L^2(\Gamma)^n$ then

$$Kh(z) = DEh(z).$$

Now consider the operator ED,

$$\begin{aligned} EDh(z) &= C_-(h(-2\pi i f^{(1)}(f^{(2)})^T)(z) \\ &= C_w(h)(z) \end{aligned}$$

where C_w is defined by (10) with $w_- = 0$ and $w_+ = J - I$, that is

$$C_w h = C_-(h w_+).$$

From the commutation formula,

$$(I - DE)^{-1} = I + D(I - ED)^{-1} E,$$

and from Theorem 2.9

$$\begin{aligned} F^{(1)} &= (I - K)^{-1} f^{(1)} \\ &= f^{(1)} + D(I - C_w)^{-1} C_w I \\ &= D(I - C_w)^{-1} I \\ &= \mu f^{(1)} \\ &= m_+(I - 2\pi i f^{(1)}(f^{(2)})^T)^{-1} f^{(1)} \\ &= m_+ f^{(1)}. \end{aligned}$$

In the fourth equality we have used the definition of μ in Theorem 2.9 and in the fifth equality we have used (11). To see the last equation note that

$$(I - 2\pi i f^{(1)}(f^{(2)})^T)f^{(1)} = f^{(1)}.$$

In similar way, we can see that

$$F^{(2)} = \tilde{m}_+ f^{(2)}$$

where $\tilde{m}$ is a solution to

$$\tilde{J} = I + 2\pi i f^{(2)}(f^{(1)})^T.$$

Note that $\tilde{J} = J^{-T}$ and we can therefore observe that m^{-T} solves the RHP $(\Gamma, \tilde{J})$. By the assumption on uniqueness for the RHP (Γ, J) we can conclude that $\tilde{m} = m^{-T}$. □

We will end this section with a lemma that is not related to Integrable operators, but that gives a family of operators that fulfills the assumption in Theorem 2.9. Our operators will be in this family.

Lemma 2.11. *Let $K\phi$ be a bounded operator on $L^2(\Gamma)$ where Γ is a contour in $\mathbb{C}$ such that $L^2(\Gamma)$ is a Hilbert space. Assume that K is an integral operator on $L^2(\Gamma)$ with kernel in $L^2(\Gamma \times \Gamma)$ and $\phi \in L^\infty(\Gamma)$. If*

$$\|\phi\|_\infty \|K\| < 1$$

then

$$(I - K\phi)^{-1}$$

exists as a bounded operator and

$$(I - K\phi)^{-1}K\phi$$

is an integral operator.

Proof. A stronger result follows from general theory (see e.g. [11] Theorem VI.23). For completeness we will include another proof.

For the first assertion we can use the Neumann series, since

$$\|K\phi\| \le \|\phi\|_\infty \|K\| < 1.$$

For the second assertion we need to work a bit more. We will prove that $(I - K\phi)^{-1}K\phi$ is equal to the operator defined by the kernel

$$\sum_{k=1}^{\infty} (K\phi)^k(z, z')$$

for all $h \in L^2(\Gamma)$. The first step is to show that this series makes sense.

By Fubini's Theorem and (62) we can see that $(K\phi)^k$ is an integral operator for all $k \in \mathbb{N}$ with kernel

$$(K\phi)^k(z,z') = \int_\Gamma (K\phi)^{k-1}(z,w)K(w,z')\phi(z')dw$$
$$= \phi(z')K^T((K\phi)^{k-1}(z,\cdot))(z').$$

Note that by Fubini's Theorem and recursively, the last term makes sense with

$$K^T h(z) = \int_\Gamma K(z',z)h(z')dz'.$$

This implies that for almost every $z \in \Gamma$,

$$\begin{aligned}\|(K\phi)^k(z,\cdot)\|_{L^2(\Gamma)} &\leq \|\phi\|_\infty \|K^T\| \|(K\phi)^{k-1}(z,\cdot)\|_{L^2(\Gamma)} \\ &\leq (\|\phi\|_\infty \|K^T\|)^{k-1} \|(K\phi)(z,\cdot)\|_{L^2(\Gamma)}. \qquad (15)\end{aligned}$$

From the definition of K^T we can see that $\|K^T\| = \|K\|$. Now, let $h \in L^2(\Gamma)$ and let

$$\begin{aligned}H_N(z) &= \sum_{k=1}^N (K\phi)^k h(z) \\ &= \sum_{k=1}^N \int_\Gamma (K\phi)^k(z,z')h(z')dz'.\end{aligned}$$

Then from (15)

$$\sum_{k=1}^N \int_\Gamma |(K\phi)^k(z,z')h(z')||dz'|$$
$$\leq \|(K\phi)(z,\cdot)\|_{L^2(\Gamma)} \|h\|_{L^2(\Gamma)} \sum_{k=1}^N (\|\phi\|_\infty \|K\|)^{k-1}_{L^2(\Gamma)}$$

where the right hand side converges by the assumption $\|\phi\|_\infty \|K\| < 1$. Hence, by the Monotone Convergence Theorem and Chebyshev's Inequality,

$$\sum_{k=1}^\infty (K\phi)^k(z,z')h(z')$$

converges absolutely for almost every $z, z' \in \Gamma$. Moreover, since

$$\| \|(K\phi)(z,\cdot)\|_{L^2(\Gamma)} \|_{L^2(\Gamma)} = \|(K\phi)\|_{L^2(\Gamma\times\Gamma)}$$

and by Lebesgue Dominated Convergence Theorem,

$$\begin{aligned}&\left\| H_N - \int_\Gamma \sum_{k=1}^{\infty} (K\phi)^k(\cdot, z')h(z')dz' \right\|_{L^2(\Gamma)} \\ &\leq \|(K\phi)\|_{L^2(\Gamma\times\Gamma)} \|h\|_{L^2(\Gamma)} \sum_{k=N+1}^{\infty} (\|\phi\|_\infty \|K\|_{L^2(\Gamma)})^{k-1} \\ &\to 0\end{aligned}$$

as $N \to \infty$. But from the Neumann series

$$H_N \to \sum_{k=1}^{\infty} (K\phi)^k h = (I - K\phi)^{-1} K\phi$$

in $L^2(\Gamma)$. Since this is true for all $h \in L^2(\Gamma)$ we get that $(I - K\phi)^{-1}K\phi$ is an integral operator. □

3 Proof of Theorem 1.4

In this section we will prove Theorem 1.4. First we give a relation between the moment generating function and a Fredholm determinant. We will later relate the Fredholm determinant to an integral that we can understand for large n.

To prove Theorem 1.4 we will consider the moment generating function of $X_n(G_n) - \mathbb{E}[X_n(G_n)]$,

$$\mathbb{E}\left[e^{\lambda(X_n(G_n) - \mathbb{E}[X_n(G_n)])} \right],$$

and prove that it converges to a Gaussian for all λ in some disc around zero (see Section 30 in [1]). To choose the disc, let $0 < \delta < 1$, let $\epsilon' > 0$ be such that $|1 - e^z| < \delta$ if $|z| < \epsilon'$ and let $c' = \sup_{x \in \mathbb{R}} |G(x)|$. Let $\epsilon = \frac{\epsilon'}{c'+1}$ and assume for what follows, that $|\lambda| < \epsilon$.

Lemma 3.1 (Vandermonde determinant)**.**

$$\det(x_k^{\ell-1})_{k,\ell=1}^n = \prod_{1 \leq k < \ell \leq n} (x_k - x_\ell).$$

Proof. Denote

$$D_n(x_1, \ldots, x_n) = \det(x_k^{\ell-1})_{k,\ell=1}^n.$$

Note that $D_n(x_1, \ldots, x_n)$ is a polynomial of degree $n-1$ in the variable x_n

with zeros $x_1, \dots, x_{n-1}$. By comparing the leading term, we can see that

$$\begin{aligned} D_n(x_1, \dots, x_n) &= D_{n-1}(x_1, \dots, x_{n-1}) \prod_{1 \le \ell \le n-1} (x_n - x_\ell) \\ &= \prod_{1 \le k < \ell \le n-1} (x_k - x_\ell) \prod_{1 \le \ell \le n-1} (x_n - x_\ell) \\ &= \prod_{1 \le k < \ell \le n} (x_k - x_\ell) \end{aligned}$$

where the second equality is by induction. For $n = 2$, the relation is obvious. $\square$

An important relation is Andreief's identity, which can be found e.g. in [7]

Lemma 3.2. *Let ϕ_k, ψ_k be measurable functions such that $\phi_k \psi_\ell \in L^1(\mathbb{T})$ for all $k, \ell = 1, \dots, n$, then*

$$\begin{aligned} &\int_{\mathbb{T}^n} \det(\phi_k(z_\ell))_{k,\ell}^n \det(\phi_k(z_\ell))_{k,\ell}^n dz_1 \dots dz_n \\ &= n! \det \left(\int_{\mathbb{T}} \phi_k(z) \psi_\ell(x) dz \right)_{k,\ell=1}^n . \end{aligned}$$

Proof. This is an exercise of using the definition of determinants,

$$\begin{aligned} &\int_{\mathbb{T}^n} \det(\phi_k(z_\ell))_{k,\ell}^n \det(\phi_k(z_\ell))_{k,\ell}^n dz_1 \dots dz_n \\ &= \sum_{\sigma \in S_n} \int_{\mathbb{T}^n} \mathrm{Sign}(\sigma) \prod_{\ell=1}^n \phi_\ell(z_{\sigma(\ell)}) \det(\phi_k(z_\ell))_{k,\ell}^n dz_1 \dots dz_n \\ &= n! \int_{\mathbb{T}^n} \prod_{\ell=1}^n \phi_\ell(z_\ell) \det(\phi_k(z_\ell))_{k,\ell}^n dz_1 \dots dz_n \\ &= n! \int_{\mathbb{T}^n} \sum_{\sigma \in S_n} \mathrm{Sign}(\sigma) \prod_{\ell=1}^n \phi_\ell(z_\ell) \phi_{\sigma(\ell)}(z_\ell) dz_1 \dots dz_n \\ &= n! \det \left(\int_{\mathbb{T}} \phi_\ell(z) \phi_k(z) \right)_{k,\ell=1}^n dz. \end{aligned}$$

In the third equality we have swapped the columns according to σ and then changed the order of integration. The other equalities are by the definition of determinants. $\square$

Given a function $\varphi \in L^2(\mathbb{T})$ let $T_n(\varphi)$ denote the $n \times n$ Toeplitz matrix given by

$$T_n(\varphi) = \begin{pmatrix} \hat{\varphi}(0) & \hat{\varphi}(-1) & \cdots & \hat{\varphi}(1-n) \\ \hat{\varphi}(1) & \hat{\varphi}(0) & \cdots & \hat{\varphi}(2-n)) \\ \vdots & \vdots & & \vdots \\ \hat{\varphi}(n-1) & \hat{\varphi}(n-2) & \cdots & \hat{\varphi}(0) \end{pmatrix}$$

where

$$\hat{\varphi}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(e^{i\theta})e^{-ik\theta}d\theta$$

is the k:th Fourier coefficient. The Toeplitz matrix has an interesting relation to the characteristic function. Next lemma shows this relation.

Lemma 3.3. *Let $\varphi_n(e^{i\theta}) = e^{\lambda G_n(\theta)}$. Then*

$$\mathbb{E}\left[e^{\lambda X_n(G_n)}\right] = \det(T_n(\varphi_n)).$$

Proof. Let $\phi_k(\theta) = e^{i(k-1)\theta}$ and $\psi_k(\theta) = e^{-i(k-1)\theta}$. Use the probability measure (1) to see that

$$\begin{aligned}
&\mathbb{E}\left[e^{\lambda X_n(G_n)}\right] = \mathbb{E}\left[\prod_{k=1}^{n} e^{\lambda G_n(\theta_k)}\right] \\
&= \frac{1}{n!(2\pi)^n}\int_{[-\pi,\pi]^n} \prod_{k=1}^{n} e^{\lambda G_n(\theta_k)} \det(\phi_k(\theta_\ell))_{k,\ell}^n \det(\psi_k(\theta_\ell))_{k,\ell}^n d\theta_1 \dots d\theta_n \\
&= \frac{1}{n!(2\pi)^n}\int_{[-\pi,\pi]^n} \det(e^{\lambda G_n(\theta_\ell)}\phi_k(\theta_\ell))_{k,\ell}^n \det(\psi_k(\theta_\ell))_{k,\ell}^n d\theta_1 \dots d\theta_n \\
&= \det\left(\int_{-\pi}^{\pi} e^{\lambda G_n(\theta)}\phi_k(\theta)\psi_\ell(\theta)\frac{d\theta}{2\pi}\right)_{k,\ell}^n \\
&= \det(T_n(\varphi_n)).
\end{aligned}$$

The second equality is the definition of expectation value together with Lemma 3.1, the forth equality is Andreief's identity and in the last equality we have used that $\phi_k(\theta) = e^{i(k-1)\theta}$ and $\psi_k(\theta) = e^{-i(k-1)\theta}$. $\square$

We will now relate the moment generating function with the determinant of an integral operator. Let $K_n : \mathbb{C} \times \mathbb{C}\backslash\{(0,0)\} \to \mathbb{C}$ be defined as

$$K_n(z, z') = \frac{1}{2\pi i}\frac{z^n(z')^{-n} - 1}{z - z'} \tag{16}$$

if $z' \neq z$ and

$$K_n(z, z) = \frac{1}{2\pi i}\frac{n}{z}. \tag{17}$$

Lemma 3.4. *Let, φ_n and G_n be as before, then*

$$\det(T_n(\varphi_n)) = \det\left(I + K_n\left(e^{\lambda G_n} - 1\right)\right).$$

Remark. The right hand side is the determinant of an operator on $L^2(\mathbb{T})$,

$$h \mapsto h + \int_{\mathbb{T}} K_n(\cdot, z')\left(\varphi_n(z') - 1\right)h(z')dz'.$$

For this we need to consider the Fredholm determinant, see Appendix.

Proof. To prove this, we will compare the eigenvalues of the matrix on the left hand side with the eigenvalues of the operator on the right hand side.

First of all, we can use the fact that K_n is a projection to see that

$$(I + K_n(\varphi_n - 1))z^k = \sum_{\ell=0}^{n-1} \hat{\varphi}_n(\ell - k)z^\ell$$

if $0 \leq k < n$ and

$$(I + K_n(\varphi_n - 1))z^k = z^k + \sum_{\ell=0}^{n-1} \hat{\varphi}_n(\ell - k)z^\ell$$

if $k < 0$ or $k \geq n$. Assume that $h \in L^2(\mathbb{T})$,

$$h(z) = \sum_{k=-\infty}^{\infty} \hat{h}(k)z^k,$$

is an eigenvector to the operator $I+K_n(\varphi_n-1)$ with eigenvalue μ. If $\hat{h}(k) \neq 0$ for some $k < 0$ or $k \geq n$, then, since $\{z^k\}$ is a basis for $L^2(\mathbb{T})$, the above shows that $\mu = 1$. Assume $h \in L^2(\mathbb{T})$ with

$$h(z) = \sum_{k=0}^{n-1} \hat{h}(k)z^k.$$

Then a straight forward calculation shows that h is an eigenvector to $I + K_n(\varphi_n - 1)$ with eigenvalue μ if and only if $(\hat{h}(0), \ldots, \hat{h}(n-1))^T$ is an eigenvector to $T_n(\varphi_n)$ with eigenvalue μ. Hence

$$\det(T_n(\varphi_n)) = \det\left(1 + K_n\left(e^{\lambda G_n} - 1\right)\right).$$

□

What we have done so far is to translate the problem from a probability problem to an analysis problem. What we will do now is to understand the right hand side of previous lemma. Compare with Theorem 1.5. In that theorem this is the starting point.

We will do the proof with some simplifications and then extend the result. For what follows assume that $G \in C^\infty$ and that $\theta_0 = 0$. Let

$$\tilde{G}_n(z) = \sum_{k=-N_n}^{N_n} \hat{G}_n(k)z^k,$$

where N_n will be chosen later. This is an analytic function in $\mathbb{C}\backslash\{0\}$. If n is big then G_n has support in $[-\pi, \pi]$ and $\tilde{G}_n(e^{i\theta})$ approximates $G_n(\theta)$. Let $\tilde{\varphi}_n(z) = e^{\lambda \tilde{G}_n(z)}$, then both $\tilde{\varphi}_n(z)$ and $\tilde{\varphi}_n(z)^{-1}$ are analytic in $\mathbb{C}\backslash\{0\}$.

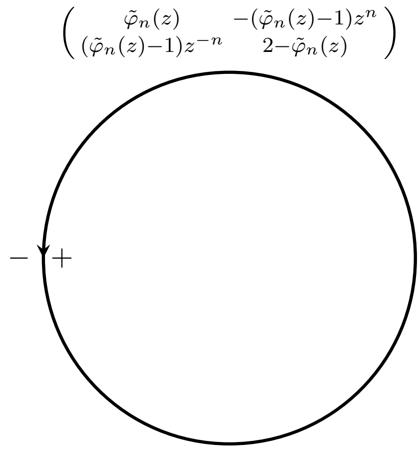


Figure 2: The contour and jump matrix for the RHP in Theorem 3.5.

The following theorem is where we will use the theory about integrable operators from the preliminaries. It expresses the Fredholm determinant in terms of integrals. Let

$$f_n^{(1)}(z) = (z^n, 1)^T$$

and

$$f_n^{(2)}(z) = \frac{1 - e^{\lambda \tilde{G}_n(z)}}{2\pi i}(z^{-n}, -1)^T.$$

Then

$$K_n\left(1 - e^{\lambda \tilde{G}_n}\right)(z, z') = \frac{f_n^{(1)}(z)^T f_n^{(2)}(z')}{z - z'}$$

that is $K_n\left(1 - e^{\lambda \tilde{G}_n}\right)$ is an integrable operator.

Theorem 3.5. *If m is a solution to the RHP $(\mathbb{T}, J_m)$ where $J_m : \mathbb{T} \to \mathbb{C}^{2\times 2}$ is the jump matrix*

$$J_m(z) = \begin{pmatrix} \tilde{\varphi}_n(z) & -(\tilde{\varphi}_n(z) - 1)z^n \\ (\tilde{\varphi}_n(z) - 1)z^{-n} & 2 - \tilde{\varphi}_n(z) \end{pmatrix},$$

and

$$F_n^{(1)} = m_+ f_n^{(1)} \tag{18}$$

and

$$F_n^{(2)} = (m_+)^{-T} f_n^{(2)}. \tag{19}$$

Then

$$
\begin{aligned}
&\log\det\left(I+K_n\left(e^{\lambda'\tilde{G}_n}-1\right)\right)\\
&=-\int_\gamma\int_{\mathbb{T}}\int_{\mathbb{T}}\tilde{G}_n(z')e^{\lambda\tilde{G}_n(z')}\frac{z^n(z')^{-n}-1}{(z-z')^2}F_n^{(1)}(z')^TF_n^{(2)}(z)dz'\frac{dz}{2\pi i}d\lambda\\
&+\int_\gamma\operatorname{Tr}\left(\tilde{G}_ne^{\lambda\tilde{G}_n}K_n\right)d\lambda \qquad (20)
\end{aligned}
$$

where γ is a straight line from 0 to λ'.

Proof. In this proof we will use theory about determinants and traces for operators (see Appendix) and the theory about integrable operators.

Since K_n is of finite rank and since the space of trace class operators is an ideal in the space of bounded operators, we can see that $K_n\left(e^{\lambda\tilde{G}_n}-1\right)$ and $K_n\tilde{G}_ne^{\lambda\tilde{G}_n}$ are trace class operators, moreover, since $\|K_n\|=1$ and $|e^{\lambda\tilde{G}_n(z)}-1|<1$ we can use the Neumann series to see that $\left(I+K_n\left(e^{\lambda\tilde{G}_n}-1\right)\right)^{-1}$ exists, for all $\lambda\in\gamma$. Moreover

$$
\sup_{z\in\mathbb{T}}\left|\frac{e^{h\tilde{G}_n(z)-1}}{h}-\tilde{G}_n\right|\to 0
$$

as $h\to 0$ and by (50)

$$
\left\|\frac{K_n\left(e^{(\lambda+h)\tilde{G}_n}-1\right)-K_n\left(e^{\lambda\tilde{G}_n}-1\right)}{h}-K_n\tilde{G}_ne^{\lambda\tilde{G}_n}\right\|_1\to 0
$$

as $h\to 0$. Hence, we can use Lemma A.10 to differentiate the left hand side with respect to λ for $\lambda\in\gamma$. For the following calculations, we will also use (56) and (57),

$$
\begin{aligned}
&\frac{\partial}{\partial\lambda}\log\det\left(I+K_n\left(e^{\lambda\tilde{G}_n}-1\right)\right)\\
&=\operatorname{Tr}\left(\left(I+K_n\left(e^{\lambda\tilde{G}_n}-1\right)\right)^{-1}\left(K_n\tilde{G}_ne^{\lambda\tilde{G}_n}\right)\right)\\
&=\operatorname{Tr}\left(\left(\left(I+K_n\left(e^{\lambda\tilde{G}_n}-1\right)\right)^{-1}-I\right)\left(K_n\tilde{G}_ne^{\lambda\tilde{G}_n}\right)\right)\\
&+\operatorname{Tr}\left(K_n\tilde{G}_ne^{\lambda\tilde{G}_n}\right)\\
&=-\operatorname{Tr}\left(K_n\tilde{G}_ne^{\lambda\tilde{G}_n}R_n\right)\\
&+\operatorname{Tr}\left(K_n\tilde{G}_ne^{\lambda\tilde{G}_n}\right)
\end{aligned}
$$

where

$$
R_n=(I+K_n(e^{\lambda\tilde{G}_n}-1))^{-1}K_n(e^{\lambda\tilde{G}_n}-1).
$$

From Theorem 2.9, Theorem 2.10 and Lemma 2.11,

$$R_n(z,z') = -\frac{F_n^{(1)}(z)^T F_n^{(2)}(z')}{z-z'}.$$

Hence, by (62) and Theorem A.6,

$$\begin{aligned}
&\operatorname{Tr}\left(K_n \tilde{G}_n e^{\lambda \tilde{G}_n} R_n\right) \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \left(K_n(z,z') \tilde{G}_n e^{\lambda \tilde{G}_n} R_n(z',z)\right) dz' dz \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z-z')^2} F_n^{(1)}(z')^T F_n^{(2)}(z) dz' \frac{dz}{2\pi i}.
\end{aligned}$$

Use (18) and (19) and that $f_n^{(1)}(z)^T f_n^{(2)}(z) = 0$ to see that $F_n^{(1)}(z)^T F_n^{(2)}(z) = 0$. From (18) we can see that $F_n^{(1)}$ is a restriction of an analytic function, in particular it tells us that the integral is well defined in the usual sense. Put these calculations together and integrate over γ to get the result. □

We will now introduce some explicit functions that will help us to be able to evaluate the integral in the previous theorem for large n. Let

$$\begin{aligned}
H_n^{(1)}(z) &= \left(e^{\lambda (C_+ \tilde{G}_n)(z)} \tilde{\varphi}_n(z)^{-1} z^n, e^{-\lambda (C_+ \tilde{G}_n)(z)}\right)^T \\
&= \left(e^{\lambda (C_- \tilde{G}_n)(z)} z^n, e^{-\lambda (C_+ \tilde{G}_n)(z)}\right)^T
\end{aligned}$$

and

$$\begin{aligned}
H_n^{(2)}(z) &= \frac{1-\tilde{\varphi}_n(z)}{2\pi i} \left(e^{-\lambda (C_+ \tilde{G}_n)(z)} z^{-n}, e^{\lambda (C_+ \tilde{G})(z)} \tilde{\varphi}_n(z)^{-1}\right) \\
&= \frac{1-\tilde{\varphi}_n(z)}{2\pi i} \left(e^{-\lambda (C_+ \tilde{G}_n)(z)} z^{-n}, e^{\lambda (C_- \tilde{G})(z)}\right) \qquad (21)
\end{aligned}$$

We can extend $H_n^{(1)}$ to an analytic function away from zero by

$$H_n^{(1)}(z) = \left(e^{\lambda (C \tilde{G}_n)(z)} \tilde{\varphi}_n(z)^{-1} z^n, e^{-\lambda (C \tilde{G}_n)(z)}\right)^T \qquad (22)$$

for $\rho_n < |z| < 1$, where ρ_n is chosen in the following theorem, and

$$H_n^{(1)}(z) = \left(e^{\lambda (C \tilde{G}_n)(z)} z^n, e^{-\lambda (C \tilde{G}_n)(z)} \tilde{\varphi}_n(z)^{-1}\right)^T$$

for $1 < |z| < \rho_n^{-1}$. From (18) we can see that we can do a similar extension of $F_n^{(1)}$. As mentioned above, the idea to introduce $H_n^{(1)}$ and $H_n^{(2)}$ is that they are close to $F_n^{(1)}$ and $F_n^{(2)}$ respectively as n tends to infinity. The exact statement is done in the following theorem.

Theorem 3.6. *Let β and γ be such that $\alpha < \beta < \gamma < 1$ and*

$$\beta - \alpha < 1 - \gamma. \tag{23}$$

Recall that α is the rate of scaling. Let

$$\rho_n = (1 - n^{-\gamma}) \tag{24}$$

and $N_n \in \mathbb{N}$ be the biggest integer smaller than n^β. Let $r_n = 1 - \frac{n^{-\gamma}}{8}$ and let $\mathcal{C}_n$ be the circle with radius r_n. Then

$$\begin{aligned}
&\Big| \int_{\mathbb{T}} \int_{\mathbb{T}} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z - z')^2} F_n^{(1)}(z')^T F_n^{(2)}(z) dz' \frac{dz}{2\pi i} \\
&- \int_{\mathbb{T}} \int_{\mathcal{C}_n} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z - z')^2} H_n^{(1)}(z')^T H_n^{(2)}(z) dz' \frac{dz}{2\pi i} \Big| \\
&\leq d_n e^{-\frac{n^{1-\gamma}}{8\pi}}.
\end{aligned}$$

where $d_n = 3 \cdot 2^{16} \pi^3 e \int_{\mathbb{R}} |G(s)| ds \epsilon n^{\beta - \alpha + 3\gamma} e^{\frac{9ec}{\pi} \epsilon n^{\beta - \alpha}}$.

Remark. The constants introduced in Theorem 3.6 are chosen in a very specific way to make the proof work. When necessary we will point back to the specific choice.

The proof is rather long and technical, therefore it is postponed to section 5.1. By the choice of the constants we can see that $e^{-\frac{n^{1-\gamma}}{8\pi}}$ is the dominated factor, so the right hand side tends to zero when n tends to infinity. Hence, we can consider the integral in Theorem 3.5 with $H_n^{(1)}$ and $H_n^{(2)}$ instead of $F_n^{(1)}$ and $F_n^{(2)}$. To summarize, Theorem 3.5, that uses the theory of integrable operators, gives us a way to understand the Fredholm determinant in terms of an integral, and Theorem 3.6, that uses the theory of RHP, gives us a way to evaluate these integrals as n tends to infinity.

Lemma 3.7. *With $\tilde{G}_n$, $H_n^{(1)}$ and $H_n^{(2)}$ defined earlier, the equality*

$$\begin{aligned}
&\int_{\mathbb{T}} \int_{\mathcal{C}_n} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z - z')^2} H_n^{(1)}(z') H_n^{(2)}(z) dz' \frac{dz}{2\pi i} \\
&= -2\lambda \sum_{k=1}^{N_n} k \hat{G}_n(k) \hat{G}_n(-k) - n \hat{G}_n(0) + \operatorname{Tr}\left(\tilde{G}_n e^{\lambda \tilde{G}_n} K_n \right)
\end{aligned}$$

holds.

Proof. The following is a lot of tedious calculations. The first equality comes from (22), (21) and a change of order of integration, which can be done since everything is analytic away from zero.

$$
\begin{aligned}
&\int_{\mathbb{T}}\int_{\mathcal{C}_n} \tilde{G}_n(z')e^{\lambda\tilde{G}_n(z')}\frac{z^n(z')^{-n}-1}{(z-z')^2}H_n^{(1)}(z')H_n^{(2)}(z)dz'\frac{dz}{2\pi i}\\
&=\frac{1}{2\pi i}\int_{\mathcal{C}_n}\int_{\mathbb{T}}\tilde{G}_n(z')e^{\lambda\tilde{G}_n(z')}\left(1-e^{\lambda\tilde{G}_n(z)}\right)\frac{z^n(z')^{-n}-1}{(z-z')^2}\\
&\times\left(e^{\lambda(C\tilde{G}_n(z')-\tilde{G}_n(z')-C_+\tilde{G}_n(z))}(z')^n z^{-n}-e^{\lambda(C_-\tilde{G}_n(z)-C\tilde{G}_n(z'))}\right)\frac{dz}{2\pi i}dz' \quad (25)
\end{aligned}
$$

Expand $(z-z')^{-2}$ into a power series to see that (25) is equal to

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathcal{C}_n}\tilde{G}_n(z')e^{\lambda C\tilde{G}_n(z')}\sum_{k=1}^{\infty}k\int_{\mathbb{T}}\left(e^{-\lambda C_+\tilde{G}_n(z)}-e^{-\lambda C_-\tilde{G}_n(z)}\right)z^{-k}\frac{dz}{2\pi i z}(z')^{k-1}dz'\\
&-\frac{1}{2\pi i}\int_{\mathcal{C}_n}\tilde{G}_n(z')e^{\lambda C\tilde{G}_n(z')}\sum_{k=1}^{\infty}k\int_{\mathbb{T}}\left(e^{-\lambda C_+\tilde{G}_n(z)}-e^{-\lambda C_-\tilde{G}_n(z)}\right)z^{-k-n}\frac{dz}{2\pi i z}(z')^{k+n-1}dz'\\
&-\frac{1}{2\pi i}\int_{\mathcal{C}_n}\tilde{G}_n(z')e^{-\lambda(C\tilde{G}_n(z')-\tilde{G}_n(z'))}\sum_{k=1}^{\infty}k\int_{\mathbb{T}}\left(e^{\lambda C_-\tilde{G}_n(z)}-e^{\lambda C_+\tilde{G}_n(z)}\right)z^{-k+n}\frac{dz}{2\pi i z}(z')^{k-n-1}dz'\\
&+\frac{1}{2\pi i}\int_{\mathcal{C}_n}\tilde{G}_n(z')e^{-\lambda(C\tilde{G}_n(z')-\tilde{G}_n(z'))}\sum_{k=1}^{\infty}k\int_{\mathbb{T}}\left(e^{\lambda C_-\tilde{G}_n(z)}-e^{\lambda C_+\tilde{G}_n(z)}\right)z^{-k}\frac{dz}{2\pi i z}(z')^{k-1}dz'.
\end{aligned}
\quad (26)
$$

For the following step we will use that $\widehat{e^{\pm\lambda C_+\tilde{G}_n}}(k)=0$ if $k<0$ and $\widehat{e^{\pm\lambda C_-\tilde{G}_n}}(k)=0$ if $k>0$. We will also deform the curve of integration, which is possible to do since all the power series converges. So (26) is equal to

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathbb{T}}\tilde{G}_n(z')e^{\lambda C_+\tilde{G}_n(z')}\sum_{k=1}^{\infty}k\widehat{e^{-\lambda C_+\tilde{G}_n}}(k)(z')^{k-1}dz'\\
&-\frac{1}{2\pi i}\int_{\mathbb{T}}\tilde{G}_n(z')e^{\lambda C_+\tilde{G}_n(z')}\sum_{\ell=n+1}^{\infty}(\ell-n)\widehat{e^{-\lambda C_+\tilde{G}_n}}(\ell)(z')^{\ell-1}dz'\\
&-\frac{1}{2\pi i}\int_{\mathbb{T}}\tilde{G}_n(z')e^{-\lambda C_-\tilde{G}_n(z')}\sum_{\ell=1-n}^{0}\ell\widehat{e^{\lambda C_-\tilde{G}_n}}(\ell)(z')^{\ell-1}dz'\\
&-\frac{1}{2\pi i}\int_{\mathbb{T}}\tilde{G}_n(z')e^{-\lambda C_-\tilde{G}_n(z')}\frac{n}{z}\sum_{\ell=1-n}^{\infty}\left(\widehat{e^{\lambda C_-\tilde{G}_n}}-\widehat{e^{\lambda C_+\tilde{G}_n}}\right)(z')^{\ell}dz' \quad (27)
\end{aligned}
$$

In the next step we will use that

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\mathcal{C}_n} \tilde{G}_n(z')e^{\lambda C_+\tilde{G}_n(z')}\sum_{\ell=n+1}^{\infty}(\ell-n)\widehat{e^{-\lambda C_+\tilde{G}_n}}(\ell)(z')^{\ell-1}dz' &= 0,\\
\frac{1}{2\pi i}\int_{\mathcal{C}_n} \tilde{G}_n(z')e^{-\lambda C_-\tilde{G}_n(z')}\sum_{\ell=-\infty}^{-n}\ell\widehat{e^{\lambda C_-\tilde{G}_n}}(\ell)(z')^{\ell-1}dz' &= 0,\\
\frac{1}{2\pi i}\int_{\mathcal{C}_n} \tilde{G}_n(z')e^{-\lambda C_-\tilde{G}_n(z')}\sum_{\ell=-\infty}^{-n}\left(\widehat{e^{\lambda C_-\tilde{G}_n}}-\widehat{e^{\lambda C_+\tilde{G}_n}}\right)(\ell)(z')^{\ell}dz' &= 0.
\end{aligned}$$

This is true since $N_n < n$ and since $\widehat{e^{\pm\lambda C_+\tilde{G}_n}}(k) = 0$ if $k < 0$ and $\widehat{e^{\pm\lambda C_-\tilde{G}_n}}(k) = 0$ if $k > 0$, which implies that there is no power of z equal to -1. Then (27) is equal to

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{\mathbb{T}} \tilde{G}_n(z')(-\lambda C_+\tilde{G}'_n(z')dz' - \frac{1}{2\pi i}\int_{\mathbb{T}} \tilde{G}_n(z')\lambda C_-\tilde{G}'_n(z')dz'\\
&-\frac{1}{2\pi i}\int_{\mathcal{C}_n} \tilde{G}_n(z')\frac{n}{z}dz' + \frac{1}{2\pi i}\int_{\mathbb{T}} \tilde{G}_n(z')e^{\lambda\tilde{G}_n(z')}\frac{n}{z'}dz'
\end{aligned}\tag{28}$$

For the last steps (4) and (5) will be used, also the relation between the Fourier coefficients of G_n and the Fourier coefficients of $\tilde{G}_n$. Hence (28) is equal to

$$\begin{aligned}
&-\lambda\int_{\mathbb{T}} \tilde{G}_n(z')\sum_{k=1}^{\infty}k\left(\hat{\tilde{G}}_n(k)(z')^{k-1}+\hat{\tilde{G}}_n(-k)(z')^{-k-1}\right)\frac{dz'}{2\pi i}\\
&-n\hat{\tilde{G}}_n(0)+\operatorname{Tr}\left(\tilde{G}_n e^{\lambda\tilde{G}_n}K_n\right)\\
&=-2\lambda\sum_{k=1}^{\infty}k\hat{\tilde{G}}_n(k)\hat{\tilde{G}}_n(-k)-n\hat{\tilde{G}}_n(0)+\operatorname{Tr}\left(\tilde{G}_n e^{\lambda\tilde{G}_n}K_n\right)\\
&=-2\lambda\sum_{k=1}^{N_n}k\hat{G}_n(k)\hat{G}_n(-k)-n\hat{G}_n(0)+\operatorname{Tr}\left(\tilde{G}_n e^{\lambda\tilde{G}_n}K_n\right).
\end{aligned}$$

□

Lemma 3.8. *If we consider $\tilde{G}_n$ as a function on $[-\pi,\pi)$, that is,*

$$\theta \mapsto \tilde{G}_n(e^{i\theta}),$$

then

$$\mathbb{E}\left[e^{\lambda X_n(\tilde{G}_n)}\right] = e^{n^{1-\alpha}\frac{\lambda}{2\pi}\mathcal{F}(G)(0)}e^{\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2d\xi}\left(1+\mathcal{O}\left(n^{-\alpha}\right)\right)$$

as $n\to\infty$.

Proof. By (20), Theorem 3.6 and Lemma 3.7 we can see that

$$
\begin{aligned}
&\log\det\left(I + K_n\left(e^{\lambda'\tilde{G}_n} - 1\right)\right) - n\lambda'\hat{G}_n(0) \\
&= (\lambda')^2 \sum_{k=1}^{N_n} k\hat{G}_n(k)\hat{G}_n(-k) + \mathcal{O}\left(d_n e^{-\frac{n^{1-\gamma}}{8\pi}}\right) \\
&= \frac{(\lambda')^2}{4\pi^2}\frac{1}{n^\alpha}\sum_{k=0}^{N_n}\frac{k}{n^\alpha}\mathcal{F}(G)\left(\frac{k}{n^\alpha}\right)\mathcal{F}(G)\left(-\frac{k}{n^\alpha}\right) + \mathcal{O}\left(d_n e^{-\frac{n^{1-\gamma}}{8\pi}}\right) \\
&\to \frac{(\lambda')^2}{4\pi^2}\int_0^\infty \xi\mathcal{F}(G)(\xi)\mathcal{F}(G)(-\xi)d\xi
\end{aligned}
\tag{29}
$$

as $n \to \infty$. In the second equality we have used the calculations

$$
\begin{aligned}
\hat{G}_n(k) &= \int_{-\pi}^{\pi} G\left(n^\alpha\theta\right) e^{-ik\theta}\frac{d\theta}{2\pi} \\
&= \frac{1}{2\pi}\int_{-n^\alpha\pi}^{n^\alpha\pi} G(s)e^{-i\frac{k}{n^\alpha}s}\frac{ds}{n^\alpha} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(s)e^{-i\frac{k}{n^\alpha}s}\frac{ds}{n^\alpha} \\
&= \frac{1}{2\pi n^\alpha}\mathcal{F}(G)\left(\frac{k}{n^\alpha}\right).
\end{aligned}
\tag{30}
$$

To see the limit, remember the remark after Theorem 3.6 that $d_n e^{-\frac{n^{1-\gamma}}{8\pi}} \to 0$ as $n \to \infty$, because of the specific choice of β and γ. Note also that the sum is a Riemann sum. To find the rate of convergence we need to find the rate of convergence of the Riemann sum.

To find the convergence of the Riemann sum, let

$$
h(\xi) = \xi\mathcal{F}(G)(\xi)\mathcal{F}(G)(-\xi)
$$

then h is continuously differentiable. Consider

$$
\begin{aligned}
\left|\int_0^{\frac{N_n}{n^\alpha}} h(\xi)d\xi - \frac{1}{n^\alpha}\sum_{k=0}^{N_n} h\left(\frac{k}{n^\alpha}\right)\right| &\le \sum_{k=0}^{\infty}\int_{k/n^\alpha}^{(k+1)/n^\alpha}\left|h(\xi) - h\left(\frac{k}{n^\alpha}\right)\right| d\xi \\
&\le \sum_{k=0}^{\infty}\int_{k/n^\alpha}^{(k+1)/n^\alpha}\frac{1}{n^\alpha}|h'(x(\xi))|d\xi \\
&= \frac{1}{n^\alpha}\int_0^\infty |h'(x(\xi))|d\xi
\end{aligned}
$$

where we have used the Mean Value Theorem, so $x(\xi) \in [k/n^\alpha, \xi]$. Note that $h'(x(\xi)) \to h'(\xi)$ as $n \to \infty$ and both $\mathcal{F}(G)$ and $\mathcal{F}(G)'$ is bounded by some C and $\frac{C}{\xi^2}$ away from zero. This implies that $|h'|$ is bounded by some

D and $\frac{D}{\xi^3}$ away from zero. Hence $|h'(x(\xi))|$ is dominated by D for $0 \leq \xi \leq 2$ and $\frac{D}{(\xi-1)^3}$ if $\xi > 2$. We can now use the Lebesgue's Dominated Convergence Theorem to conclude

$$n^{\alpha} \left| \int_0^{\frac{N_n}{n^{\alpha}}} h(\xi) d\xi - \frac{1}{n^{\alpha}} \sum_{k=0}^{N_n} h\left(\frac{k}{n^{\alpha}}\right) \right| \to c \leq \int_0^{\infty} |h'(\xi)| d\xi < \infty.$$

Since G is smooth and h is defined by the Fourier transform of G, we can find an ℓ such that $(\beta - \alpha)\ell \geq 1$, remember (23), and h is of order $\xi^{-\ell}$, hence

$$\int_{\frac{N_n}{n^{\alpha}}}^{\infty} |h(\xi)| d\xi \leq c \left(\frac{1}{n^{\beta} n^{-\alpha}}\right)^{\ell+1} \leq c\frac{1}{n}$$

for some constant c. Hence the limit in (29) is of order $n^{-\alpha}$.

By using the fact that

$$e^{n^{-\alpha}} = 1 + \frac{e^{n^{-\alpha}} - 1}{n^{-\alpha}} n^{-\alpha}$$

and that $\mathcal{F}(G)(-\xi) = \overline{\mathcal{F}(G)(\xi)}$, since G is real valued, and by Lemma 3.3 and Lemma 3.4 we can conclude that

$$\mathbb{E}\left[e^{\lambda X_n(\tilde{G}_n)}\right] = e^{n^{1-\alpha} \frac{\lambda}{2\pi} \mathcal{F}(G)(0)} e^{\lambda^2 \frac{1}{4\pi^2} \int_0^{\infty} \xi |\mathcal{F}(G)(\xi)|^2 d\xi} (1 + \mathcal{O}\left(n^{-\alpha}\right))$$

as $n \to \infty$. □

Before proceeding, note that the right hand side barely depends on α. It is only the first term in the exponent, which will turn out to be the expectation value of $X_n(G_n)$, and the rate of convergence that depends on α. We can also note that the condition $G \in C^{\infty}$ is unnecessarily strong, it would suffice that $G \in C^{\ell}$ and we would still have the same rate of convergence. But ℓ depends on α, so α gives some kind of contribution when it comes to rate of convergence and how smooth a function needs to be in order not to lose the convergence rate.

Lemma 3.8 tells us that we understand the moment generating function for the function we approximated G with. We will now show that we understand the limit for all functions in Theorem 1.4. This will be done by using some probabilistic results that makes it possible to extend this result to a wider class of functions. We will now state the facts we need. These are standard and can be found in e.g. [7].

Lemma 3.9. *Let h be a bounded function on $[-\pi, \pi)$, then*

$$\mathbb{E}[X_n(h)] = \mathrm{Tr}(K_n h)$$

and

$$\mathrm{Var}(X_n(h)) = \mathrm{Tr}(K_n h^2) - \mathrm{Tr}\left((K_n h)^2\right).$$

Proof. Consider the logarithm of the moment generating function

$$\log \mathbb{E}\left[e^{\lambda X_n(h)}\right].$$

Expend this into a power series in λ, that is

$$\log \mathbb{E}\left[e^{\lambda X_n(h)}\right] = \lambda \mathbb{E}[X_n(h)] + \frac{\lambda^2}{2}(\mathbb{E}[X_n(h)^2] - \mathbb{E}[X_n(h)]^2) + \mathcal{O}(\lambda^3),$$

where

$$\mathbb{E}[X_n(h)^2] - \mathbb{E}[X_n(h)]^2 = \mathrm{Var}(X_n(h)).$$

Now use Lemma 3.3, Lemma 3.4, (61) and expand the right hand side,

$$\begin{aligned}\log \mathbb{E}\left[e^{\lambda X_n(h)}\right] &= \log\det\left(I + K_n\left(e^{\lambda h} - 1\right)\right)\\ &= \mathrm{Tr}\log\left(I + K_n\left(e^{\lambda h} - 1\right)\right)\\ &= \lambda\,\mathrm{Tr}(K_n h) + \frac{\lambda^2}{2}\left(\mathrm{Tr}(K_n h^2) - \mathrm{Tr}\left((K_n h)^2\right)\right) + \mathcal{O}(\lambda^3)\end{aligned}$$

where the logarithm is defined by a power series for small λ. Compare the two series to get the relation. $\square$

Lemma 3.10. *Let h_1 and h_2 be bounded real valued functions on $[-\pi, \pi)$ such that*

$$\sum_{k=0}^{\infty} k|\hat{h}(k)|^2 < \infty$$

and such that the expectation values of $X_n(h_1)$ and $X_n(h_2)$ are zero. Then

$$\left|\mathbb{E}\left[e^{i\lambda X_n(h_1)} - e^{i\lambda X_n(h_2)}\right]\right| \le \sqrt{2}|\lambda| \left(\sum_{k=0}^{\infty} k|\widehat{(h_1 - h_2)}(k)|^2\right)^{\frac{1}{2}}.$$

Proof. First we bound the left hand side with the square root of the variance,

$$\begin{aligned}\left|\mathbb{E}\left[e^{i\lambda X_n(h_1)} - e^{i\lambda X_n(h_2)}\right]\right| &\le \mathbb{E}\left|e^{i\lambda(X_n(h_1) - X_n(h_2))} - 1\right|\\ &\le |\lambda|\mathbb{E}|X_n(h_1) - X_n(h_2)|\\ &\le |\lambda|\,\mathrm{Var}(X_n(h_1 - h_2))^{\frac{1}{2}}.\end{aligned}$$

In the first inequality, we have used that h_1 and h_2 are real, in the second, that $|e^{it} - 1| \le |t|$ for $t \in \mathbb{R}$ and in the last we have used Cauchy-Swartz inequality and that the $\mathbb{E}[X_n(h_1)] = \mathbb{E}[X_n(h_2)] = 0$. Now, let $h\left(e^{i\theta}\right) = h_1(\theta) - h_2(\theta)$, that is we are viewing the difference as a function on the unit circle. Then, from Lemma 3.9 and Theorem A.6 together with (62),

$$\mathrm{Var}(X_n(h)) = n\int_{\mathbb{T}} h(z)^2 \frac{dz}{2\pi i z} - \int_{\mathbb{T}}\int_{\mathbb{T}} h(z')h(z)K_n(z, z')K_n(z', z)dz'dz.$$

Note that

$$K_n(z,z')K_n(z',z) = \frac{1}{2\pi iz}\frac{1}{2\pi iz'}\sum_{k=-(n-1)}^{n-1}(n-|k|)\left(\frac{z}{z'}\right)^k,$$

with this it is easy to see that the right most term is equal to

$$-2\sum_{k=1}^{n-1}k\hat{h}(k)\hat{h}(-k) + 2n\sum_{k=1}^{n-1}\hat{h}(k)\hat{h}(-k) + n\hat{h}(0)^2.$$

Write

$$h(z) = \sum_{k=-\infty}^{\infty}\hat{h}(k)z^k$$

to see that

$$n\int_{\mathbb{T}}h(z)^2\frac{dz}{2\pi iz} = 2n\sum_{k=1}^{\infty}\hat{h}(k)\hat{h}(-k) + n\hat{h}(0)^2.$$

Hence, since h is real valued,

$$\begin{aligned}\mathrm{Var}(X_n h) &= 2\sum_{k=1}^{\infty}\min(n,k)|\hat{h}(k)|^2\\ &\leq 2\sum_{k=0}^{\infty}k|\hat{h}(k)|^2.\end{aligned}$$

□

We are now ready to extend our result. The next lemma states that the statement is true for the function we started with, that is a smooth functions with compact support.

Lemma 3.11. *Let G be as before. Then*

$$\mathbb{E}\left[e^{i\lambda(X_n(G_n)-\mathbb{E}[X_n(G_n)])}\right] \to e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi}$$

as $n\to\infty$.

Proof. From Lemma 3.9 and Theorem A.6,

$$\mathbb{E}[\lambda X_n(G_n)] = \mathrm{Tr}(\lambda K_n G_n) = n^{1-\alpha}\frac{\lambda}{2\pi}\mathcal{F}(G)(0) \tag{31}$$

and

$$\mathbb{E}[\lambda X_n(\tilde{G}_n)] = n^{1-\alpha}\frac{\lambda}{2\pi}\mathcal{F}(G)(0).$$

Note that this is exactly the first term in the exponent of the moment generating function. From Lemma 3.10 and since $\hat{G}_n(k) = \hat{\tilde{G}}_n(k)$ for $|k| \leq N_n$,

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{i\lambda(X_n(G_n)-\mathbb{E}[X_n(G_n)])} - e^{i\lambda(X_n(\tilde{G}_n)-\mathbb{E}[X_n(\tilde{G}_n)])}\right]\right| \\
&\leq \sqrt{2}|\lambda| \left(\sum_{k=N_n+1}^{\infty} k|\hat{G}_n(k)|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}|\lambda| \left(\frac{1}{4\pi^2 n^\alpha}\sum_{k=N_n+1}^{\infty} \frac{k}{n^\alpha}\left|\mathcal{F}(G)\left(\frac{k}{n^\alpha}\right)\right|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{2}|\lambda| \left(\frac{1}{4\pi^2}\int_{\frac{N_n}{n^\alpha}}^{\infty} \xi\,|\mathcal{F}(G)(\xi)|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Where the last term converges to zero as n tends to infinity. Hence

$$\begin{aligned}
\mathbb{E}\left[e^{i\lambda(X_n(G_n)-\mathbb{E}[X_n(G_n)]}\right] &= \mathbb{E}\left[e^{i\lambda(X_n(\tilde{G}_n)-\mathbb{E}[X_n(\tilde{G}_n)]}\right] \\
&\quad + \mathbb{E}\left[e^{i\lambda(X_n(G_n)-\mathbb{E}[X_n(G_n)])} - e^{i\lambda(X_n(\tilde{G}_n)-\mathbb{E}[X_n(\tilde{G}_n)])}\right] \\
&\quad \to e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi}
\end{aligned}$$

as $n \to \infty$. □

Unfortunately we lose the rate of convergence here. It is because we have to consider the characteristic function instead of the moment generating function, which we have worked with before, to be able to use the estimation in Lemma 3.10.

So far we have proved Theorem 1.4 for smooth functions with compact support and with $\theta_0 = 0$. We will now extend it to all functions in Theorem 1.4 and for arbitrary θ_0, which will complete the proof.

Proof of Theorem 1.4. Let $G \in L^2(\mathbb{R})$ be a continuous real valued function, such that

$$\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi < \infty.$$

Remember that this defines a seminorm. Let H be a smooth real valued function with compact support. Without loss of generality we can assume that both $X_n(G_n)$ and $X_n(H_n)$ has expectation value zero. Consider the estimation

$$\left|\mathbb{E}\left[e^{i\lambda X_n(G_n)}\right] - e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(G))(\xi)|^2 d\xi}\right|$$
$$\leq \left|\mathbb{E}\left[e^{i\lambda X_n(H_n)}\right] - e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(H))(\xi)|^2 d\xi}\right| \tag{32}$$
$$+ \left|\mathbb{E}\left[e^{i\lambda X_n(G_n)} - e^{i\lambda X_n(H_n)}\right]\right| \tag{33}$$
$$+ \left|e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(H)(\xi)|^2 d\xi} - e^{-\lambda^2\frac{1}{4\pi^2}\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi}\right|. \tag{34}$$

We will choose H and n such that (32), (33) and (34) are small.

Let H such that

$$\int_0^\infty \xi|\mathcal{F}(G-H)(\xi)|^2 d\xi$$

is so small such that (33) is small by a similar argument as in Lemma 3.11. We can assume that

$$\int_0^\infty \xi|\mathcal{F}(H)(\xi)|^2 d\xi \leq \int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi$$

and hence,

$$0 \leq \int_0^\infty \xi|\mathcal{F}(H)(\xi)|^2 d\xi - \int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi$$
$$\leq \left(\int_0^\infty \xi|\mathcal{F}(H-G)(\xi)|^2 d\xi\right)^{\frac{1}{2}} \left(\left(\int_0^\infty \xi|\mathcal{F}(H)(\xi)|^2 d\xi\right)^{\frac{1}{2}} + \left(\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi\right)^{\frac{1}{2}}\right).$$

This implies that (34) is less than

$$\left|1 - e^{-\lambda^2\frac{1}{4\pi^2}\left(\int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi - \int_0^\infty \xi|\mathcal{F}(G)(\xi)|^2 d\xi\right)}\right|$$

which is small by the choice of H. From Lemma 3.11 we can choose n such that (32) is small. Hence the result follows for $\theta_0 = 0$.

For the case if $\theta_0 \neq 0$ we use a little trick. Let φ_n be as before and let $\varphi_{n,0} = e^{\lambda G(n^\alpha(\theta-\theta_0))}$. Let n be so big that $-n^\alpha\pi, n^\alpha\pi, -n^\alpha(\pi-\theta_0), n^\alpha(\pi-\theta_0)$ are not in the support of G. Then

$$\begin{aligned}
\hat{\varphi}_{n,0}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda G(n^\alpha(\theta-\theta_0))} e^{-ik\theta} d\theta \\
&= e^{ik\theta_0}\frac{1}{2\pi}\int_{-\pi-\theta_0}^{\pi-\theta_0} e^{\lambda G(n^\alpha(\theta'))} e^{-ik\theta'} d\theta' \\
&= e^{ik\theta_0}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\lambda G(n^\alpha(\theta'))} e^{-ik\theta'} d\theta' \\
&= e^{ik\theta_0}\hat{\varphi}_n(k).
\end{aligned}$$

Especially $\hat{\varphi}_n(0) = \hat{\varphi}_{n,0}(0)$. Consider the determinant,

$$\det(T(\varphi_{n,0})) =$$

$$\begin{vmatrix} \hat{\varphi}_n(0) & e^{-i\theta_0}\hat{\varphi}_n(-1) & \cdots & e^{i(1-n)\theta_0}\hat{\varphi}_n(1-n) \\ e^{i\theta_0}\hat{\varphi}_n(1) & \hat{\varphi}_n(0) & \cdots & e^{i(2-n)\theta_0}\hat{\varphi}_n(2-n) \\ \vdots & \vdots & & \vdots \\ e^{i(n-1)\theta_0}\hat{\varphi}_n(n-1) & e^{i(n-2)\theta_0}\hat{\varphi}_n(n-2) & \cdots & \hat{\varphi}_n(0) \end{vmatrix}$$

Multiply the k:th row with $e^{-ik\theta_0}$ and the k:th column with $e^{ik\theta}$. Doing this in the above equality the left hand side does not change and the right hand side becomes $\det(T(\varphi_n))$. Hence from Lemma 3.3 the result follows for all θ_0. □

4 Proof of Theorem 1.5

In this section we will prove Theorem 1.5. The main ideas in this proof are the same as in the proof of Theorem 1.4. But, as mentioned before, in this proof we will not go into the relation with moment generating functions.

The assumptions in Theorem 1.5 are made in such a way that a specific choice of properties will be fulfilled. These properties are stated below, although most of them are not needed before we solve the RHP.

Lemma 4.1. *Assume the same assumptions as in Theorem 1.5. Then there exists a strip S containing the real line such that f can be extended to an analytic function in this strip such that*

$$f(z) = \mathcal{O}(z^{-2})$$

as $|z| \to \infty$. Moreover, there exists an $\epsilon > 0$ and a δ, $0 < \delta < 1$, such that if $|\lambda| < \epsilon$ then $|1 - \varphi_{n,t}(z)| < \delta$ where

$$\varphi_{n,t}(x) = 1 - t(1 - e^{\lambda f_n(x)}),$$

in

$$S_n = \{nz; z \in S\}$$

and hence $\varphi_{n,t}(z)^{-1}$ exists as an analytic function.

Proof. Without loss of generality we can assume that the first and second derivative of the Fourier transform has the same bound as the Fourier transform. Let $0 < b < a$ and $S = \{z \in \mathbb{C}; |\text{Im}(z)| < b\}$. To see that f can be extended, define

$$f(z) = \int_{\mathbb{R}} \mathcal{F}(f)(\xi) e^{i\xi z} \frac{d\xi}{2\pi} \tag{35}$$

for $z \in S$. This is well defined since

$$\int_{\mathbb{R}} |\mathcal{F}(f)(\xi)e^{i\xi z}|\frac{d\xi}{2\pi} \leq A\int_{\mathbb{R}} e^{-|\xi|a}e^{|\xi|b}\frac{d\xi}{2\pi} < \infty. \tag{36}$$

We want to show that this defines an analytic function. For that, define

$$f_N(z) = \int_{\mathbb{R}} 1_{[-N,N]}\mathcal{F}(f)(\xi)e^{i\xi z}\frac{d\xi}{2\pi},$$

this defines an analytic function such that $f_N \to f$ as $N \to \infty$ uniformly on any compact subset of S. Hence f is analytic. This is basically a part of a weaker version of the Paley-Wiener Theorem.

By integration by part twice, which is possible by the assumption on the first and second derivative, (35) shows that

$$f(z) = \mathcal{O}(z^{-2}).$$

Let $0 < \delta < 1$ and let $\epsilon' > 0$ be such that $|1 - e^z| < \delta$ if $|z| < \epsilon'$. Let $\epsilon = \frac{\epsilon'}{c}$ where $|f(z)| \leq c$ in S, which exists by (36). If $\lambda \in \mathbb{C}, |\lambda| < \epsilon$ then by the triangle inequality

$$|1 - \varphi_{n,t}(z)| < \delta$$

for $z \in S_n$. Hence $\varphi_{n,t}(z)$ is non-zero and $\varphi_{n,t}(z)^{-1}$ is therefore analytic in S_n. $\square$

Note that this lemma implies that $1-\varphi_{n,t}$ and $1-\varphi_{n,t}^{-1}$ belongs to $L^2(\mathbb{R}+inb')$ and $L^1(\mathbb{R}+inb')$ for any $|b'| < b$.

Let log denote the principle branch of logarithm and define $g_{n,t} : \mathbb{R} \to \mathbb{C}$ as

$$g_{n,t}(x) = \log(\varphi_{n,t}(x)).$$

Lemma 4.2. *With $g_{n,t}$ defined above it is a well defined Schwartz function. If $z \in \mathbb{C}\backslash\mathbb{R}$ then*

$$|Cg_{n,t}(z)| \leq d\frac{n}{|Im(z)|}$$

where d is a constant not depending on z, t or n.

Proof. First of all, in the proof of Lemma 4.1, we saw that $\varphi_{n,t}$ attains values close to one. Therefore we can write

$$g_{n,t}(x) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(t(e^{\lambda f_n(x)} - 1))^k}{k}$$

and it is a well defined smooth function. Since the function $\frac{e^z-1}{z}$ is continuous close to zero and since $|\lambda f(x)| < \epsilon$, we can assume that ϵ is small enough to make sure that

$$\left|\frac{e^{\lambda f(x)} - 1}{\lambda f(x)}\right| \leq 2$$

for all real x. Then, since $|e^{\lambda f(x)} - 1| < \delta < 1$, we have the estimate

$$
\begin{aligned}
|g_{n,t}(x)| &\leq \sum_{k=1}^{\infty} \frac{|e^{\lambda f_n(x)} - 1|^k}{k} \\
&\leq |e^{\lambda f_n(x)} - 1| \frac{1}{1-\delta} \\
&\leq |\lambda f_n(x)| \frac{2}{1-\delta}.
\end{aligned}
$$

From this and the fact that

$$
g'_{n,t}(x) = \frac{\varphi'_{n,t}(x)}{\varphi_{n,t}}
$$

we can conclude that $g_{n,t}$ is a Schwartz function. Moreover

$$
\begin{aligned}
|\mathcal{F}(g_{n,t})(\xi)| &\leq \int_{\mathbb{R}} |g_{n,t}(x)| dx \\
&\leq n \frac{2|\lambda|}{1-\delta} \int_{\mathbb{R}} |f(x)| dx \\
&< \infty.
\end{aligned}
$$

If $\mathrm{Im}(z) > 0$ we can see from (6) that

$$
|Cg_{n,t}(z)| \leq \frac{|\lambda|}{\pi(1-\delta)} \int_{\mathbb{R}} |f(x)| dx \frac{n}{\mathrm{Im}(z)}.
$$

$\square$

We have now a sufficient understanding of the functions we will work with. Before proceeding to integrable operators, we need to show that $K(1-\varphi_{n,t})$ actually is a trace class operator, that is, that the left hand side in Theorem 1.5 makes sense. For theory about trace class operators see Appendix.

Lemma 4.3. *Let $\phi \in L^2(\mathbb{R})$ be such that*

$$
\int_{\mathbb{R}} (1+y^2)|\phi(y)|^2 dy < \infty.
$$

Then the operator

$$
(K\phi)h(x) = \int_{\mathbb{R}} \frac{\sin(\pi(x-y))}{\pi(x-y)} \phi(y) h(y) dy
$$

defines a trace class operator.

Proof. The first step is to show that it actually is a bounded linear operator. Let

$$g(x) = \frac{\sin(\pi x)}{\pi x}.$$

Then the operator K can be expressed as a convolution,

$$Kh = g * h.$$

A straight forward calculation shows that

$$\frac{\sin(\pi x)}{\pi x} = \mathcal{F}^{-1}(1_{[-1,1]})(x).$$

Hence, by using the Fourier transform,

$$\begin{aligned} Kh &= \mathcal{F}^{-1}(\mathcal{F}(g)\mathcal{F}(h)) \\ &= \mathcal{F}^{-1}(1_{[-1,1]}\mathcal{F}(h)). \end{aligned}$$

This tells us that it is a linear bounded operator. Moreover we can see that it is a projection, that is

$$K^2 = K. \tag{37}$$

We will now prove that it is a trace class operator. Let $\psi(x) = (i+x)^{-1}$. Consider the kernel of the commutator $[K, \psi]$,

$$\begin{aligned} [K,\psi](x,y) &= K(x,y)(\psi(y) - \psi(x)) \\ &= K(x,y)\frac{1}{i+y}\frac{1}{i+x}(x-y) \end{aligned}$$

which defines a finite rank operator. Let K act from the left and $\psi^{-1}\phi$ from the right on the left hand side, and use (37), to see that

$$K\phi - K\psi K\psi^{-1}\phi$$

is of finite rank. We want to show that the second term is a trace class operator to see the $K\phi$ is a trace class operator. By the calculations

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)\psi(y)|^2 dxdy = \int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 dx \int_{\mathbb{R}} \frac{1}{1+y^2} dy < \infty$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |K(x,y)\phi(y)\psi(y)|^2 dxdy = \int_{\mathbb{R}} \left(\frac{\sin(\pi x)}{\pi x}\right)^2 dx \int_{\mathbb{R}} (1+y^2)|\phi(y)|^2 dy < \infty,$$

we can use the estimate (52) and Theorem A.7, to see that

$$\|K\psi K\psi^{-1}\phi\|_1 \le \|K\psi\|_2 \|K\psi^{-1}\phi\|_2 < \infty.$$

Hence $K\phi$ is a trace class operator. $\square$

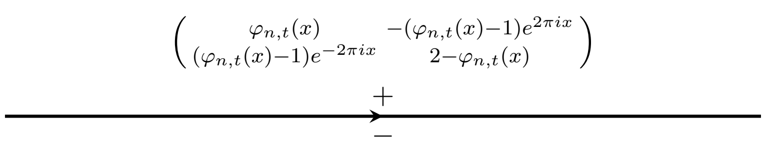


Figure 3: The contour and jump matrix for the RHP in Theorem 4.4.

We will now proceed to use the theory about integrable operators, as in the first problem. Let

$$f_{n,t}^{(1)}(x) = (e^{i\pi x}, e^{-i\pi x})^T$$

and

$$f_{n,t}^{(2)}(y) = \frac{1-\varphi_{n,t}(x)}{2\pi i}(e^{-i\pi y}, -e^{i\pi y})^T.$$

Then the kernel of the operator $K(1-\varphi_{n,t})$ is given by

$$\frac{f_{n,t}^{(1)}(x)^T f_{n,t}^{(2)}(y)}{x-y}.$$

The following theorem gives a relation between the Fredholm determinant and an integral, which we can evaluate asymptotically.

Theorem 4.4. *Let the contour Γ coincide with the real line oriented from $-\infty$ to $+\infty$ and let $J_m : \Gamma \to \mathbb{C}^{2\times 2}$ be the jump matrix*

$$J_m(x) = \begin{pmatrix} \varphi_{n,t}(x) & -(\varphi_{n,t}(x)-1)e^{2\pi ix} \\ (\varphi_{n,t}(x)-1)e^{-2\pi ix} & 2-\varphi_{n,t}(x) \end{pmatrix}.$$

If m is the solution to the RHP (Γ, J_m),

$$F_{n,t}^{(1)} = m_+ f_{n,t}^{(1)} \tag{38}$$

and

$$F_{n,t}^{(2)} = (m_+)^{-T} f_{n,t}^{(2)}, \tag{39}$$

then

$$\log\det\left(I + K\left(e^{\lambda f_n} - 1\right)\right) = -\int_0^1 \frac{1}{t}\int_{\mathbb{R}} F_{n,t}^{(1)\prime}(x)^T F_{n,t}^{(2)}(x)\,dx\,dt. \tag{40}$$

Proof. From Lemma 4.3 we get that $tK\left(e^{\lambda f_n}-1\right)$ and $K(e^{\lambda f_n}-1)$ are trace class for all $t \in [0,1]$. Since $\|K\| = 1$ and $tK\left(e^{\lambda f_n}-1\right) < 1$, if we choose δ wisely in Lemma 4.1, we can use the Neumann series to see that $\left(1+tK\left(e^{\lambda f_n}-1\right)\right)^{-1}$ exists for all t. Hence, we can use Lemma A.10 to differentiate the left hand side of the function

$$h(t) = \log\det(I + K(\varphi_{n,t}-1))$$

with respect to t, the limit we need to check is trivial. For the following calculations, we will also use (56),

$$\begin{aligned} h'(t) &= \operatorname{Tr}\left(\left(1+tK\left(e^{\lambda f_n}-1\right)\right)^{-1} K(e^{\lambda f_n}-1)\right) \\ &= \frac{1}{t}\operatorname{Tr}(R_{n,t}) \end{aligned} \tag{41}$$

where

$$R_{n,t} = \left(1+tK\left(e^{\lambda f_n}-1\right)\right)^{-1} tK(e^{\lambda f_n}-1).$$

From Theorem 2.9, Theorem 2.10 and Lemma 2.11,

$$R_{n,t}(x,y) = -\frac{F_n^{(1)}(x)^T F_n^{(2)}(y)}{x-y}$$

and since $F_{n,t}^{(1)}$ is differentiable we can let $x \to y$ to see that

$$R_{n,t}(x,x) = -F_n^{(1)\prime}(x)^T F_n^{(2)}(x).$$

$F_{n,t}^{(1)}$ can actually be extended to an analytic function in a neighborhood of the real line, see the beginning of the proof of Theorem 4.5. By the decay of f_n, Lemma 4.3 implies that $K(e^{\lambda f_n}-1)(x^2+1)$ is trace class. By using (50) we can see that

$$\begin{aligned} &\|K(e^{\lambda f_n}-1) - 1_{[-N,N]}K(e^{\lambda f_n}-1)1_{[-N,N]}\|_1 \\ &\leq \|1_{[-N,N^c]}K(e^{\lambda f_n}-1)1_{[-N,N]}\|_1 + \|K(e^{\lambda f_n}-1)1_{[-N,N]^c}\|_1 \\ &\leq 2\|K(e^{\lambda f_n}-1)(x^2+1)\|_1 \|(x^2+1)^{-1}1_{[-N,N]^c}\|_\infty \\ &\to 0 \end{aligned}$$

as $N \to \infty$. Since $B_1(L^2(\mathbb{R}))$ forms an ideal, we can use Theorem A.6,

$$\operatorname{Tr}(R_{n,t}) = -\int_{\mathbb{R}} F_n^{(1)\prime}(x)^T F_n^{(2)}(x)dx.$$

Use this in (41) and integrate from 0 to 1 to get the result. □

The next step is to understand the integral in the previous theorem for large n. This is done by approximate $F_n^{(1)}$ and $F_n^{(2)}$ with explicit functions.

Theorem 4.5. *Let*

$$G_{n,t}^{(1)}(x) = \left(e^{(C-g_{n,t})(x)}e^{i\pi x}, e^{-(C+g_{n,t})(x)}e^{-i\pi x}\right)^T$$

and

$$G_{n,t}^{(2)}(y) = \frac{1-\varphi_{n,t}(x)}{2\pi i}\left(e^{-(C+g_{n,t})(y)}e^{-i\pi y}, -e^{(C-g_{n,t})(y)}e^{i\pi y}\right).$$

Then

$$\left| \int_{\mathbb{R}} \left(F_{n,t}^{(1)'}(x) F_{n,t}^{(2)}(x) - G_{n,t}^{(1)'}(x) G_{n,t}^{(2)}(x) \right) dx \right| \leq d' n t e^{-2\pi n \rho}$$

where d' does not depend on n or t.

The proof of this theorem is rather long and technical, as in the other problem, it is postponed to section 5.2. By the previous lemma, we can calculate (40) with $G_{n,t}^{(1)}$ and $G_{n,t}^{(2)}$ instead of $F_{n,t}^{(1)}$ and $F_{n,t}^{(2)}$. The calculation is done in the following lemma.

Lemma 4.6. *With $G_n^{(1)}$ and $G_n^{(2)}$ defined in Theorem 4.5,*

$$\begin{aligned} & -\int_0^1 \frac{1}{t} \int_{\mathbb{R}} G_{n,t}^{(1)\prime}(x)^T G_{n,t}^{(2)}(x) dx dt \\ &= \frac{1}{4\pi^2} \int_0^\infty \xi \mathcal{F}(\lambda f_n)(\xi) \mathcal{F}(\lambda f_n)(-\xi) d\xi + \int_{\mathbb{R}} \lambda f_n(\xi) d\xi. \end{aligned} \tag{42}$$

Proof. A straightforward differentiation of $G_{n,t}^{(1)}(x)$, which can be done due to Lemma 2.4 and a simplification yields that (42) can be written as

$$\frac{1}{2\pi i} \int_0^1 \int_{\mathbb{R}} \frac{\varphi_n(x) - 1}{\varphi_{n,t}(x)} \left((C_+ g_{n,t})'(x) + (C_- g_{n,t})'(x) + 2\pi i \right) dx dt. \tag{43}$$

By Fubini's Theorem and some calculations

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\varphi_n(x) - 1}{\varphi_{n,t}(x)} \left((C_+ g_{n,t})'(x) + (C_- g_{n,t})'(x) \right) dx \\ &= \frac{1}{4\pi^2} \int_{\mathbb{R}} \frac{\partial}{\partial t} g_{n,t}(x) \int_0^\infty \xi \left(\mathcal{F}(g_{n,t})(\xi) e^{i\xi x} + \mathcal{F}(g_{n,t})(-\xi) e^{-i\xi x} \right) d\xi dx \\ &= \frac{1}{4\pi^2} \int_0^\infty \xi \left(\mathcal{F}(g_{n,t})(\xi) \frac{\partial}{\partial t} \int_{\mathbb{R}} g_{n,t}(x) e^{i\xi x} dx + \mathcal{F}(g_{n,t})(-\xi) \frac{\partial}{\partial t} \int_{\mathbb{R}} g_{n,t}(x) e^{-i\xi x} \right) dx d\xi \\ &= \frac{1}{4\pi^2} \frac{\partial}{\partial t} \int_0^\infty \xi \mathcal{F}(g_{n,t})(\xi) \mathcal{F}(g_{n,t})(-\xi) d\xi. \end{aligned}$$

Since

$$g_{n,0}(x) = 0$$

and

$$g_{n,1}(x) = \lambda f_n(x),$$

(43) can be written as

$$\frac{1}{4\pi^2} \int_0^\infty \xi \mathcal{F}(\lambda f_n)(\xi) \mathcal{F}(\lambda f_n)(-\xi) d\xi + \int_{\mathbb{R}} \lambda f_n(\xi) d\xi.$$

□

At this point, we can express the determinant with an integral which we can approximately evaluate for big n. Hence, we have all pieces to prove Theorem 1.5.

Proof of Theorem 1.5. Note first that

$$\mathcal{F}(f_n)(\xi) = n\mathcal{F}(f)(n\xi)$$

and

$$\mathcal{F}(f)(-\xi) = \overline{\mathcal{F}(f)(\xi)},$$

since f is real valued. By Theorem 4.4, Theorem 4.5 and Lemma 4.6,

$$\begin{aligned}
&\log\det\left(I + K\left(e^{\lambda f_n} - 1\right)\right) \\
&= \lambda\int_{\mathbb{R}} f_n(\xi)d\xi + \frac{1}{4\pi^2}\int_0^\infty n\xi\mathcal{F}(\lambda f)(n\xi)\mathcal{F}(\lambda f)(-n\xi)nd\xi + \mathcal{O}(ne^{-2\pi in\rho}) \\
&= \lambda n\int_{\mathbb{R}} f(\xi)d\xi + \frac{1}{4\pi^2}\lambda^2\int_0^\infty \xi|\mathcal{F}(f)(\xi)|^2 d\xi + \mathcal{O}(ne^{-2\pi in\rho}).
\end{aligned}$$

Hence

$$\det\left(I + K\left(e^{\lambda f_n} - 1\right)\right) = e^{\lambda n\int_{\mathbb{R}} f(\xi)d\xi + \lambda^2\int_0^\infty \xi|\mathcal{F}(f)(\xi)|^2\frac{d\xi}{4\pi^2}}(1 + \mathcal{O}(e^{-2\pi n\rho})).$$

□

5 Solution to Riemann-Hilbert problem and asymptotics

In this section we will solve the Riemann-Hilbert problems which we postponed. To solve the RHP, in both problems, we will do transformations from one Riemann-Hilbert problem to another, where every transformation is not to difficult and such that it is possible to go back. But every step will make us come closer to a problem we can solve. This is called Deift - Zhou steepest descent technique (see [3]).

In both these cases there will appear a lot of constants. The actually value of these constants is not of importance but we will still keep track of them to be able to have control of the dependence on different parameters.

5.1 Proof of Theorem 3.6

The Deift / Zhou steepest descent technique will be done by the transformations

$$m \to S \to R$$

and R will be a solution to a problem we can solve explicitly for big n.

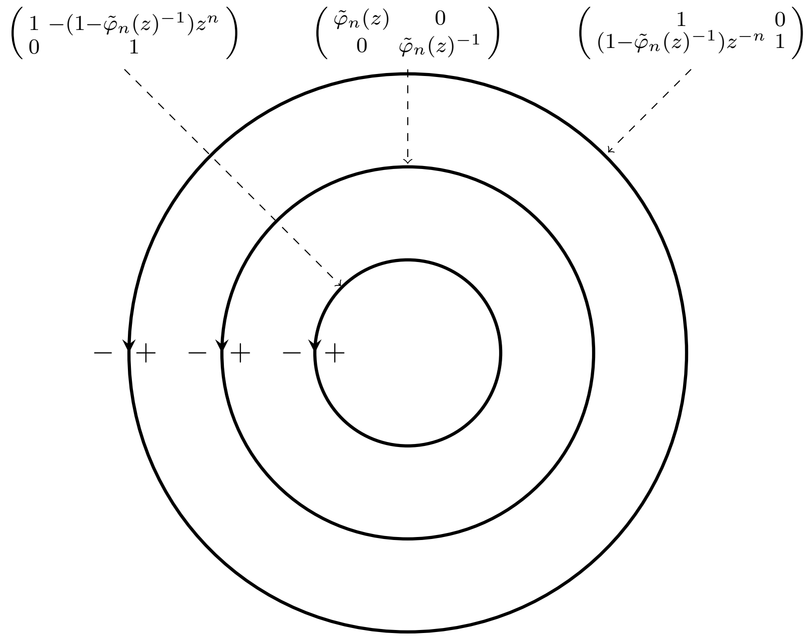


Figure 4: The contour and jump matrix for the RHP in Lemma 5.1.

The first step is called the opening of the lens. This is a technique where one pushes some part of the jump matrix away from the previous contour in such a way that the part of the jump matrix that one has pushed converges to the identity matrix.

Lemma 5.1. *If m solves the RHP $(\mathbb{T}, J_m)$, define $S : \mathbb{C}\backslash\Gamma \to \mathbb{C}$ as*

$$\begin{aligned}
S =& m, && |z| < \rho_n \\
S =& m \begin{pmatrix} 1 & -(1-\tilde{\varphi}_n(z)^{-1})z^n \\ 0 & 1 \end{pmatrix}^{-1} && \rho_n < |z| < 1 \\
S =& m \begin{pmatrix} 1 & 0 \\ (1-\tilde{\varphi}_n(z)^{-1})z^{-n} & 1 \end{pmatrix} && 1 < |z| < \rho_n^{-1} \\
S =& m && 1 < |z|.
\end{aligned}$$

Let $\Gamma = \{z \in \mathbb{C}; |z| \in \{1, \rho_n, \rho_n^{-1}\}\}$. Then S solves the RHP (Γ, J_S) where

$$\begin{aligned}
J_S =& \begin{pmatrix} 1 & -(1-\tilde{\varphi}_n(z)^{-1})z^n \\ 0 & 1 \end{pmatrix} && |z| = \rho_n \\
J_S =& \begin{pmatrix} \tilde{\varphi}_n(z) & 0 \\ 0 & \tilde{\varphi}_n(z)^{-1} \end{pmatrix} && |z| = 1 \\
J_S =& \begin{pmatrix} 1 & 0 \\ (1-\tilde{\varphi}_n(z)^{-1})z^{-n} & 1 \end{pmatrix}^{-1} && |z| = \rho_n^{-1}.
\end{aligned}$$

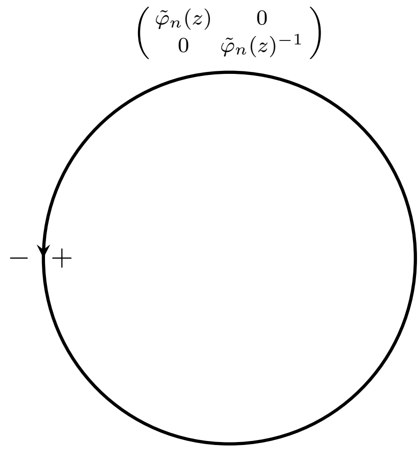


Figure 5: The contour and jump matrix for the RHP in Lemma 5.2.

Proof. This is a check of Definition 2.4. (i) follows since $\tilde\varphi_n^{-1}$ is analytic for $\rho_n < |z| < \rho^{-1}$. To see (ii), factorize J_m to the form

$$\begin{pmatrix} 1 & 0 \\ (1-\tilde\varphi_n(z)^{-1})z^{-n} & 1 \end{pmatrix} \begin{pmatrix} \tilde\varphi_n(z) & 0 \\ 0 & \tilde\varphi_n(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -(1-\tilde\varphi_n(z)^{-1})z^{n} \\ 0 & 1 \end{pmatrix}.$$

With this factorization the jumps for $|z| = \rho_n$ and $|z| = \rho_n^{-1}$ follows by definition. For the jump on the unit circle, use the jump of m,

$$\begin{aligned} S_+ &= m_+ \begin{pmatrix} 1 & -(1-\tilde\varphi_n(z)^{-1})z^{n} \\ 0 & 1 \end{pmatrix}^{-1} \\ &= m_- \begin{pmatrix} 1 & 0 \\ (1-\tilde\varphi_n(z)^{-1})z^{-n} & 1 \end{pmatrix} \begin{pmatrix} \tilde\varphi_n(z) & 0 \\ 0 & \tilde\varphi_n(z)^{-1} \end{pmatrix} \\ &= S_- \begin{pmatrix} \tilde\varphi_n(z) & 0 \\ 0 & \tilde\varphi_n(z)^{-1} \end{pmatrix}. \end{aligned}$$

Finally (iii) is true since it is true for m. □

The idea here is that this RHP is close to a RHP that we can solve. That is, the jumps on $\Gamma\backslash\mathbb{T}$ tends to the identity matrix as n tends to infinity. If $\tilde\varphi_n^{-1}$ did not depend on n we would pretty much be done by general theory. But since it do depend on n we have to be careful. But by the specific choice of β, γ and ρ_n, remember (23) and (24), we can make sure that the jumps tends to the identity matrix anyway.

Consider the part of the problem that is on the unit circle, this is a problem we can solve.

Lemma 5.2. *Let J_P be the jump matrix*

$$J_P = \begin{pmatrix} \tilde{\varphi}_n(z) & 0 \\ 0 & \tilde{\varphi}_n(z)^{-1} \end{pmatrix},$$

and $P : \mathbb{C}\backslash\mathbb{T} \to \mathbb{C}^{2\times 2}$ be

$$P(z) = \begin{pmatrix} e^{\lambda(C\tilde{G}_n)(z)} & 0 \\ 0 & e^{-\lambda(C\tilde{G}_n)(z)} \end{pmatrix}.$$

Then P solves the RHP $(\mathbb{T}, J_P)$.

Proof. This is also a check of Definition 2.4. From Lemma 2.1 we get (i) and that $C_+ - C_- = I$. Hence

$$\begin{aligned} P_+(z) &= \begin{pmatrix} e^{\lambda(C_+\tilde{G}_n)(z)} & 0 \\ 0 & e^{-\lambda(C_+\tilde{G}_n)(z)} \end{pmatrix} \\ &= \begin{pmatrix} e^{\lambda(C_-\tilde{G}_n)(z)} & 0 \\ 0 & e^{-\lambda(C_-\tilde{G}_n)(z)} \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_n(z) & 0 \\ 0 & \tilde{\varphi}_n(z)^{-1} \end{pmatrix} \\ &= P_-(z)J_P(z). \end{aligned}$$

For (iii) we can use (3). □

The following lemma makes it precise what we mean by saying that S and P are close.

Lemma 5.3. *Define $R : \Gamma \to C^{2\times 2}$ as*

$$R = SP^{-1}$$

and let $J_R = P_- J_S P_+^{-1}$ then R solves the RHP (Γ, J_R). Moreover

$$|(R - I)(z)| \leq \frac{8\pi\|\mu\|_{L^2(\Gamma)}}{\min\{\rho_n - |z|, \rho_n^{-1} - |z|\}} e^{-\frac{n^{1-\gamma}}{4\pi}}$$

as $n \to \infty$.

Remark. The absolute value on a matrix is meant as the absolute value of each entry.

Proof. That R solves the RHP (Γ, J_R) is again just a check of Definition 2.4. We can directly see that (i) and (iii) are true, since they are true for S and P^{-1}. For (ii) note that $P_- = P_+ = P$ on $\Gamma\backslash\mathbb{T}$. Hence

$$\begin{aligned} R_+ &= S_+ P_+^{-1} \\ &= S_- P_-^{-1} P_- J_S P_+^{-1} \\ &= R_- J_R \end{aligned}$$

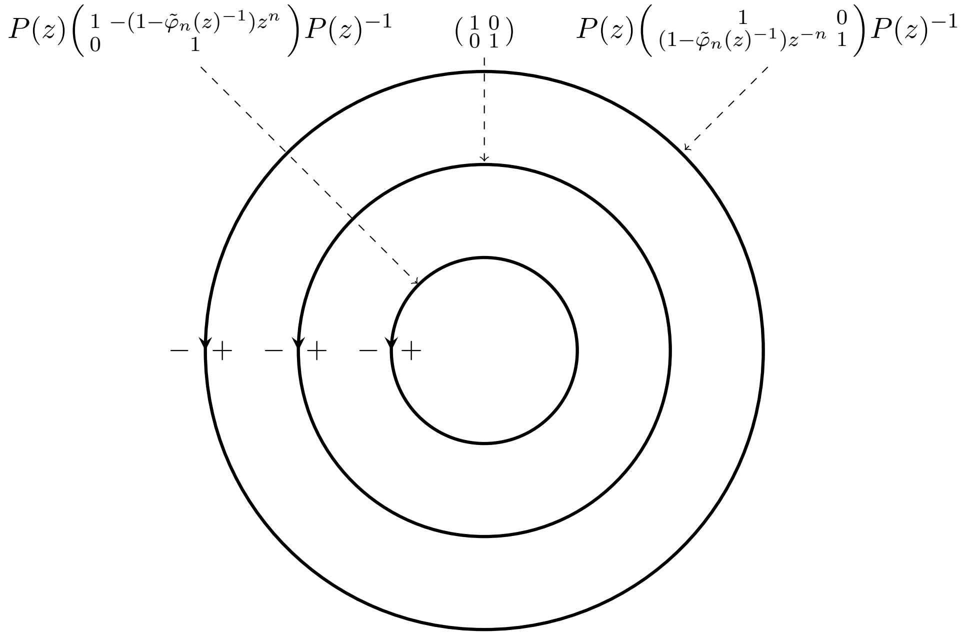


Figure 6: The contour and jump matrix for the RHP in Lemma 5.3.

on Γ. For the second assertion remember that $\rho_n = (1 - n^{-\gamma})$ and that $\beta < \gamma$. By (30) we can get a bound on $\tilde{G}_n$. If $|z| = \rho_n$ or $|z| = \rho_n^{-1}$ then, by the specific choice of β and γ,

$$\begin{aligned} |\tilde{G}_n(z)| &\leq \frac{c}{2\pi n^{\alpha}}(2N_n+1)\rho_n^{-N_n} \\ &\leq \frac{3c}{2\pi} n^{\beta-\alpha}(1-n^{-\gamma})^{-n^{\beta}} \\ &\leq \frac{3ec}{2\pi} n^{\beta-\alpha} \end{aligned} \tag{44}$$

for big enough n where $c = \int_{\mathbb{R}} |G(s)| ds$. From the calculations of the Cauchy operator on the circle we can get the same bound for $|C\tilde{G}_n(z)|$. Let n be so big such that $\frac{1}{2\pi} \leq \frac{\ln(1-n^{-\gamma})}{-n^{-\gamma}}$, then

$$\begin{aligned} \rho_n^n &= (1-n^{-\gamma})^n \\ &\leq e^{\frac{-n^{1-\gamma}}{2\pi}}. \end{aligned} \tag{45}$$

Let $|z| = 1$, then

$$J_R = P_- J_S P_+^{-1} = P_- J_P P_+^{-1} = I.$$

If $|z| = \rho_n$, let

$$E(z) = \begin{pmatrix} 0 & -(1-\tilde{\varphi}_n(z)^{-1})z^n \\ 0 & 0 \end{pmatrix}$$

and if $|z| = \rho_n^{-1}$, let

$$E(z) = \begin{pmatrix} 0 & 0 \\ -(1 - \tilde{\varphi}_n(z)^{-1})z^{-n} & 0 \end{pmatrix},$$

then

$$J_R = PJ_SP^{-1} = P(I + E)P^{-1} = I + PEP^{-1}.$$

From the above and (44) we can conclude the estimation

$$\begin{aligned} \|J_R - I\|_{\infty(\Gamma)} &\leq e^{2|\lambda|\|C\tilde{G}_n\|_{\infty(\Gamma\setminus\mathbb{T})}}(1 + e^{|\lambda|\|\tilde{G}_n\|_{\infty(\Gamma\setminus\mathbb{T})}})\rho_n^n \\ &\leq 2e^{\frac{1}{2\pi}(\epsilon 9ecn^{\beta-\alpha} - n^{1-\gamma})} \\ &\leq 2e^{-\frac{n^{1-\gamma}}{4\pi}} \end{aligned}$$

as $n \to \infty$. Theorem 2.6 and Corollary 2.7 implies that

$$R = I + (C(\mu w))(z)$$

solves (Γ, J_R) for big enough n. Before proceeding, we should note that this is not the Cauchy operator on the unit circle, but the Cauchy operator on Γ. But since Γ is a disjoint union of three circles, and since $h_1 \in L^2(\mathbb{T})$ if and only if the function $h_2(z) = h_1\left(\frac{z}{r}\right)$ belongs to $L^2(\{|z| = r\})$ for some positive r, it is easy to use the Cauchy operator on the circle to understand the Cauchy operator on Γ.

Since $J_R - I = 0$ on the real line and $\mu \in L^2(\Gamma)$, we can see that for any $z \in \mathbb{C}$ with $\rho_n < |z| < \rho_n^{-1}$ that

$$\begin{aligned} |(R - I)(z)| &\leq \int_{\Gamma\setminus\mathbb{T}} \frac{|\mu(w)||(J_R - I)(w)|}{|w - z|}|dw| \\ &\leq \frac{8\pi\|\mu\|_{L^2(\Gamma)}}{\min\{\rho_n - |z|, \rho_n^{-1} - |z|\}}e^{-\frac{n^{1-\gamma}}{4\pi}} \end{aligned}$$

In the second inequality we have used that μ is diagonal, which can bee seen from the definition of μ. □

We have now found a solution of R and we have a relation between R and m. By tracing back from R to m and and from (18) we get the relation

$$F_n^{(1)}(z) = H_n^{(1)}(z) + (R(z) - I)H_n^{(1)}(z). \tag{46}$$

Proof of Theorem 3.6. From the estimate (44) and from the Maximum Modulus Principle, we can see that if $\rho_n < |z| < \rho_n^{-1}$ then $|\tilde{G}_n(z)| \leq \frac{3ec}{2\pi}n^{\beta-\alpha}$ and $|e^{\lambda C\tilde{G}_n(z)}| \leq e^{\frac{3ec\epsilon}{2\pi}n^{\beta-\alpha}}$. By the choice of r_n and similar calculations as (45),

$$r_n^{-n} \leq e^{\frac{n^{1-\gamma}}{16\pi}}.$$

Hence for $r_n \le |z| \le 1$,

$$|H_n^{(1)}(z)| \le \left(e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} e^{\frac{n^{1-\gamma}}{16\pi}}, e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} \right)$$

and from (46) and from the previous lemma,

$$|F_n^{(1)}(z) - H_n^{(1)}(z)| \le \frac{8\pi\|\mu\|_{L^2(\Gamma)}}{|\rho_n - r_n|} e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} \left(e^{-\frac{3n^{1-\gamma}}{16\pi}}, e^{-\frac{n^{1-\gamma}}{4\pi}} \right).$$

For $z \in \mathbb{T}$,

$$|H_n^{(2)}(z)| \le \frac{1}{2\pi} \left(e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}}, e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} \right).$$

Since $\det(J_R) = 1$ we get from the the proof of Theorem 2.5 that $\det(R) = 1$ and then from the inverse formula for 2×2 matrices, that is

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}^{-1} = \frac{1}{\det(R)} \begin{pmatrix} R_{22} & -R_{12} \\ -R_{21} & R_{11} \end{pmatrix},$$

it is clear that R^{-T} has the same kind of asymptotic as R as $n \to \infty$. Hence in a similar way,

$$|F_n^{(2)} - H_n^{(2)}| \le \frac{8\|\mu\|_{L^2(\Gamma)}}{|\rho_n - 1|} e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} \left(e^{-\frac{n^{1-\gamma}}{4\pi}}, e^{-\frac{n^{1-\gamma}}{4\pi}} \right).$$

By the definition of μ and with the Neumann series, it is not difficult to see that $\mu \le 8\pi$ for big enough n. This implies that

$$|F_n^{(2)}(z)| \le 2|H_n^{(2)}(z)|.$$

Now consider

$$\begin{aligned}
&\int_{\mathbb{T}} \int_{\mathcal{C}_n} \left| (F_n^{(1)}(z')F_n^{(2)}(z) - H_n^{(1)}(z')H_n^{(2)}(z)) \right| |dz'||dz| \\
&\le \int_{\mathcal{C}_n} |F_n^{(1)}(z') - H_n^{(1)}(z')||dz'| \int_{\mathbb{T}} |F_n^{(2)}(z)||dz| \\
&+ \int_{\mathcal{C}_n} |H_n^{(1)}(z')||dz'| \int_{\mathbb{T}} |F_n^{(2)}(z) - H_n^{(2)}(z)||dz| \\
&\le 2^9\pi^4 n^\gamma e^{\frac{6ec}{\pi}\epsilon n^{\beta-\alpha}} e^{-\frac{3n^{1-\gamma}}{16\pi}}.
\end{aligned}$$

Since the integrand in the first term in the estimate in Lemma 3.6 is analytic in some open annulus containing $\mathbb{T}$ and $\mathcal{C}_n$ we can see that

$$\begin{aligned}
&\left| \int_{\mathbb{T}} \int_{\mathbb{T}} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z - z')^2} F_n^{(1)}(z')^T F_n^{(2)}(z) dz' \frac{dz}{2\pi i} \right. \\
&\left. - \int_{\mathbb{T}} \int_{\mathcal{C}_n} \tilde{G}_n(z') e^{\lambda \tilde{G}_n(z')} \frac{z^n (z')^{-n} - 1}{(z - z')^2} H_n^{(1)}(z')^T H_n^{(2)}(z) dz' \frac{dz}{2\pi i} \right| \\
&\le \frac{3 \cdot 2^7 ec}{\pi} \epsilon n^{\beta-\alpha+2\gamma} e^{\frac{3ec}{\pi}\epsilon n^{\beta-\alpha}} e^{\frac{n^{1-\gamma}}{16\pi}} \int_{\mathbb{T}} \int_{\mathcal{C}_n} \left| (F_n^{(1)}(z')F_n^{(2)}(z) - H_n^{(1)}(z')H_n^{(2)}(z)) \right| |dz'||dz| \\
&\le 3 \cdot 2^{16} \pi^3 ec\epsilon n^{\beta-\alpha+3\gamma} e^{\frac{9ec}{\pi}\epsilon n^{\beta-\alpha}} e^{\frac{-n^{1-\gamma}}{8\pi}}.
\end{aligned}$$

$\square$

This concludes the proof of Theorem 1.4.

5.2 Proof of Theorem 4.5

The Deift - Zhou steepest descent technique will be done by the transformations

$$m \to T \to S \to R$$

and R will be a problem we can solve explicitly for big n. Since we have seen similar transformations between different RHP in the proof of Theorem 3.6, we will not go into all details. Denote $\varphi_t = \varphi_{1,t}$ and $g_t = g_{1,t}$.

The first step is to do a transformation so that the function will not vary with n.

Lemma 5.4. *Define* $T : \mathbb{C}\backslash\Gamma \to \mathbb{C}^{2\times 2}$ *as*

$$T(z) = m(nz).$$

Then T *solves the Riemann-Hilbert Problem* (Γ, J_T) *where*

$$J_T(x) = J_m(nx) = \begin{pmatrix} \varphi_t(x) & -(\varphi_t(x)-1)e^{2\pi inx} \\ (\varphi_t(x)-1)e^{-2\pi inx} & 2-\varphi_t(x) \end{pmatrix}.$$

Proof. This is just a direct check of Definition 2.4. $\square$

It is now possible to open the lens as we did in the first problem.

Lemma 5.5. *Let* $0 < \rho < b$ *and let* $\Gamma_S = \{z \in \mathbb{C}; |z| \in \{-\rho, 0, \rho\}\}$ *oriented from left to right. Define* $S : \mathbb{C}\backslash\Gamma_S \to \mathbb{C}^{2\times 2}$ *as*

$$\begin{aligned}
S(z) =& T(z) && \rho < Im(z) \\
S(z) =& T(z)\begin{pmatrix} 1 & -(1-\varphi_t(z)^{-1})e^{2\pi inz} \\ 0 & 1 \end{pmatrix}^{-1} && 0 < Im(z) < \rho \\
S(z) =& T(z)\begin{pmatrix} 1 & 0 \\ (1-\varphi_t(z)^{-1})e^{-2\pi inz} & 1 \end{pmatrix} && -\rho < Im(z) < 0 \\
S(z) =& T(z) && Im(z) < -\rho
\end{aligned}$$

Then S *solves the RHP* (Γ_S, J_S) *where*

$$\begin{aligned}
J_S(z) =& \begin{pmatrix} 1 & -(1-\varphi_t(z)^{-1})e^{2\pi inz} \\ 0 & 1 \end{pmatrix} && Im(z) = \rho \\
J_S(z) =& \begin{pmatrix} \varphi_t(z) & 0 \\ 0 & \varphi_t(z)^{-1} \end{pmatrix} && Im(z) = 0 \\
J_S(z) =& \begin{pmatrix} 1 & 0 \\ (1-\varphi_t(z)^{-1})e^{-2\pi inz} & 1 \end{pmatrix} && Im(z) = -\rho.
\end{aligned}$$

Proof. This is also e check of Definition 2.4. By Lemma 4.1 we get (i). To see (ii) is straight forward with the factorization

$$J_T(z) = \begin{pmatrix} 1 & 0 \\ (1-\varphi_t(z)^{-1})e^{-2\pi inz} & 1 \end{pmatrix} \begin{pmatrix} \varphi_t(z) & 0 \\ 0 & \varphi_t(z)^{-1} \end{pmatrix} \begin{pmatrix} 1 & -(1-\varphi_t(z)^{-1})e^{2\pi inz} \\ 0 & 1 \end{pmatrix}.$$

For (iii) we note that $1-\varphi_t(z)^{-1}$ has the same asymptotic as $f(z)$ hence from Lemma 4.1

$$\left(1-\varphi_t(z)^{-1}\right)e^{\pm 2\pi nz} = \mathcal{O}(z^{-2})$$

as $|z| \to \infty$, $-\rho < \mathrm{Im}(z) < \rho$. □

The idea of this transformation is that the parts on ρ and $-\rho$ converges to the identity matrix. Therefore this solution is close to the solution to the problem with the jump only on the real line. Since φ_t does not depend on n this is clear from general theory, but of course we will do this properly.

Lemma 5.6. *Let g_t be the function defined in Lemma 4.2 and $J_P : \mathbb{C}\backslash\Gamma \to \mathbb{C}^{2\times 2}$ be defined as*

$$J_P(z) = \begin{pmatrix} \varphi_t(z) & 0 \\ 0 & \varphi_t(z)^{-1} \end{pmatrix}.$$

Then

$$P(z) = \begin{pmatrix} e^{Cg_t(z)} & 0 \\ 0 & e^{-Cg_t(z)} \end{pmatrix}.$$

solves the RHP (Γ, J_P).

Proof. This is a consequence of Lemma 4.2, Lemma 2.3 together with the observation

$$(Cg_t)(z) = \mathcal{O}(z^{-1}) \tag{47}$$

as $|z| \to \infty$ which can be seen by integration by part. □

In this last transformation we will make it precise in what sense S and P are close.

Lemma 5.7. *Define the function $R : \mathbb{C}\backslash\Gamma_R \to \mathbb{C}^{2\times 2}$ as*

$$R = SP^{-1}.$$

Then R solves the RHP (Γ_R, J_R) where $\Gamma_R = \Gamma_S$, $J_R = I$ on Γ and $J_R = PJ_SP^{-1}$ on $\Gamma_R\backslash\Gamma$. Moreover

$$|R(z) - I| \le \frac{c}{|\rho - Im(z)|}e^{2\pi n\rho}$$

for some constant c.

Proof. From Lemma 4.1 and Lemma 4.2 we can see

$$\begin{aligned}\|J_R - I\|_{\infty(\Gamma_R)} &\leq \|e^{\pm 2Cg_t}(1-\varphi_t^{-1})\|_{\infty(\Gamma_R\setminus\Gamma)}e^{-2\pi\rho n} \\ &\leq \frac{e^{\frac{2d}{\rho}}}{1-\delta}\|e^{\lambda f}-1\|_{\infty(\Gamma_R\setminus\Gamma)}e^{-2\pi\rho n}\end{aligned}$$

on Γ_R,

$$\|J_R - I\|_{L^1(\Gamma_R)} \leq \frac{e^{\frac{2d}{\rho}}}{1-\delta}\|e^{\lambda f}-1\|_{L^1(\Gamma_R)}e^{-2\pi\rho n}.$$

and

$$\|J_R - I\|_{L^2(\Gamma_R)} \leq \frac{e^{\frac{2d}{\rho}}}{1-\delta}\|e^{\lambda f}-1\|_{L^2(\Gamma_R)}e^{-2\pi\rho n}.$$

From the remark under Lemma 4.1 all the norms above are bounded. As in the first problem, we want to use Theorem 2.6 and Corollary 2.7 to see that

$$R = I + (C(\mu w))(z)$$

solves (Γ_R, J_R) for big enough n. Note that this is the Cauchy operator on Γ_R but as in the circle case, this does not give us any problem. We will therefore show that μ is analytic in the strip S, defined in the proof of Lemma 4.1. For that, note first, that if h is analytic in S then $C_w h$ is the restriction of an analytic function in S. This can be seen by deforming the contour which is possible since $w = J_R - I$ is analytic in S. That is

$$\begin{aligned}
&C_w^{k+1}I(z) = C(C_w^k(I)w)(z) - C_w^k(I)w(z), && \rho < \mathrm{Im}(z),\\
&C_w^{k+1}I(z) = C(C_w^k(I)(z), && -\rho < \mathrm{Im}(z) < \rho,\\
&C_w^{k+1}I(z) = C(C_w^k(I)w)(z) + C_w^k(I)w(z), && \mathrm{Im}(z) < -\rho.
\end{aligned}$$

Therefore

$$\sum_{k=1}^{N} C_w^k I$$

is analytic for all N. For any $z \in S$ we can deform the contour in the integral slightly, if needed, so by similar estimations as above, we can see that

$$|C_w^k I(z)| \leq \left(ce^{-2\pi(\rho-\epsilon)n}\right)^k$$

for some constant c and some $0 < \epsilon < \rho$ that compensates for the deformations. Hence for big enough n the sequence $\sum_{k=1}^{N} C_w^k I(z)$ converges uniformly on any compact subset of S. Hence

$$\mu = I + (I - C_w)^{-1}C_w I$$

is analytic in S. From the above calculations and the remark under Lemma 4.1, there is no problem to see that $\mu w \in L^1(\Gamma_R)$ and are bounded close to Γ_R. Hence we can use Theorem 2.6 and Corollary 2.7.

Since $J_R - I = 0$ on the real line and $\mu - I \in L^2(\Gamma_R)$, for any $z \in \mathbb{C}$ with $|\mathrm{Im}(z)| < \rho$ we can see that

$$\begin{aligned}|(R-I)(z)| &\leq \int_{\Gamma_R \setminus \Gamma} \frac{|\mu(w)||(J_R - I)(w)|}{|w-z|}|dw| \\ &\leq \frac{1}{\rho - |\mathrm{Im}(z)|}\left(\|J_R - I\|_{L^1(\Gamma_R)} + 2\|\mu - I\|_{L^2(\Gamma_R)}\|J_R - I\|_{L^2(\Gamma_R)}\right) \\ &\leq \frac{e^{\frac{2d}{\rho}}\left(\|e^{\lambda f} - 1\|_{L^1(\Gamma_R)} + 2\|\mu - I\|_{L^2(\Gamma_R)}\|e^{\lambda f} - 1\|_{L^2(\Gamma_R)}\right)}{(1-\delta)(\rho - |\mathrm{Im}(z)|)} e^{-2\pi\rho n}.\end{aligned}$$

Note that with the absolute value on a matrix we mean the absolute value on each entry. □

Proof of Theorem 4.5. Remember the definition of $F_{n,t}^{(1)}$, (38). We can extend that to an analytic function in the strip S_n by

$$\begin{aligned} F_{n,t}^{(1)}(z) &= m(z) f_{n,t}^{(1)}(z) & 0 < \mathrm{Im}(z) < n\rho \\ F_{n,t}^{(1)}(z) &= m(z) J_m(z) f_{n,t}^{(1)}(z) & -n\rho < \mathrm{Im}(z) < 0. \end{aligned}$$

This is possible since everything is analytic in this strip. We can do the same for $G_{n,t}^{(1)}$ with

$$G_{n,t}^{(1)}(z) = \left(e^{(Cg_{n,t})(z)}\varphi_{n,t}(z)^{-1}e^{i\pi z}, e^{-(Cg_{n,t})(z)}e^{-i\pi z}\right)^T \tag{48}$$

for $0 < \mathrm{Im}(z) < n\rho$ and

$$G_{n,t}^{(1)}(z) = \left(e^{(Cg_{n,t})(z)}e^{i\pi z}, e^{-(Cg_{n,t})(z)}\varphi_{n,t}(z)^{-1}e^{-i\pi z}\right)^T$$

for $-n\rho < \mathrm{Im}(z) < 0$. Then, by tracing back from R to m, we get that

$$F_{n,t}^{(1)}(z) = G_{n,t}^{(1)}(z) + \left(R\left(\frac{z}{n}\right) - I\right) G_{n,t}^{(1)}(z) \tag{49}$$

for $-n\rho < \mathrm{Im}(z) < n\rho$.

From Lemma 4.2, Lemma 4.1 and (48) we can found the bound

$$|G_{n,t}^{(1)}(z)| \leq \left(\frac{1}{1-\delta}e^{\frac{dn}{|\mathrm{Im}(z)|} - \pi\mathrm{Im}(z)}, e^{\frac{dn}{|\mathrm{Im}(z)|} + \pi\mathrm{Im}(z)}\right).$$

Let $\Gamma_n = \{z \in \mathbb{C}; \mathrm{Im}(z) = \frac{n\rho}{2}\}$ and let $0 < r < \frac{\rho}{2}$ and $\gamma_z = \{w \in \mathbb{C}; |w - z| < r\}$. Let $z \in \Gamma_n$, by Cauchy integral formula and (49),

$$\begin{aligned}|F_{n,t}^{(1)\prime}(z) - G_{n,t}^{(1)\prime}(z)| &= \left|\int_{\gamma_z} \frac{F_{n,t}^{(1)}(w) - G_{n,t}^{(1)}(w)}{(w-z)^2} dw\right| \\ &\leq \frac{\left|R\left(\frac{z+ir}{n}\right) - I\right)\right|}{r^2}\int_{\gamma_z} |G_{n,t}^{(1)}(w)|^2|dw| \\ &\leq ce^{-2\pi n\rho}\left(\frac{e^{\frac{2dn}{n\rho - 2r} - \frac{\pi}{2}(\rho n - 2r)}}{1-\delta}, e^{\frac{2dn}{n\rho - 2r} + \frac{\pi}{2}(\rho n + 2r)}\right).\end{aligned}$$

where

$$c = \frac{2\pi e^{\frac{2d}{\rho}} \left(\|e^{\lambda f} - 1\|_{L^1(\Gamma_R)} + 2\|\mu - I\|_{L^2(\Gamma_R)} \|e^{\lambda f} - 1\|_{L^2(\Gamma_R)} \right)}{r(1-\delta)(\frac{\rho}{2} - \frac{r}{n})}.$$

In the same way

$$\int_{\Gamma_n} |G^{(2)}_{n,t}(z)||dz| \le \frac{nt}{2\pi} \int_{\Gamma_1} |1 - e^{\lambda f(z)}||dz| \left(e^{\frac{2d}{\rho} + \frac{\pi}{2} n\rho}, \frac{e^{\frac{2d}{\rho} - \frac{\pi}{2} n\rho}}{1-\delta} \right)$$

and

$$\int_{\Gamma_n} |F^{(2)}_{n,t}(z) - G^{(2)}_{n,t}(z)||dz| \le c' nte^{-2\pi n\rho} \left(e^{\frac{2d}{\rho} + \frac{\pi}{2} n\rho}, \frac{e^{\frac{2d}{\rho} - \frac{\pi}{2} n\rho}}{1-\delta} \right)$$

where

$$c' = \frac{e^{\frac{2d}{\rho}}}{\pi\rho(1-\delta)} \left(\|e^{\lambda f} - 1\|_{L^1(\Gamma_R)} + 2\|\mu - I\|_{L^2(\Gamma_R)} \|e^{\lambda f} - 1\|_{L^2(\Gamma_R)} \right) \int_{\Gamma_1} |1 - e^{\lambda f(z)}||dz|$$

and the integral converges by the decay of $1 - e^{\lambda f}$. With the same technique as in Lemma 4.1 and Lemma 4.2 we can verify that

$$|Cg'_{n,t}(z)| < \frac{|\lambda|}{\pi(1-\delta)} \int_{\mathbb{R}} |f(x)| dx \frac{n}{\mathrm{Im}(z)^2}$$

and

$$|\varphi'_{n,t}(z)| < \frac{|\lambda| e^{\epsilon}}{n} \left| f'\left(\frac{z}{n}\right) \right|$$

Hence for $z \in \Gamma_n$,

$$|G^{(1)'}_{n,t}(z)| < \left(c''' e^{\frac{2d}{\rho} - \frac{\pi}{2} n\rho}, c'' e^{\frac{2d}{\rho} + \frac{\pi}{2} n\rho} \right)$$

where

$$c'' = \frac{|\lambda|}{n\rho\pi(1-\delta)} \int_{\mathbb{R}} |f(x)| dx + \pi$$

and

$$c''' = c'' + \frac{|\lambda| e^{\epsilon}}{n(1-\delta)^2} \sup_{x \in \mathbb{R}} \left| f'\left(\frac{x}{n} + i\frac{\rho}{2} \right) \right|.$$

Finally

$$\begin{aligned}
&\left| \int_{\mathbb{R}} \left(F^{(1)'}_{n,t}(x) F^{(2)}_{n,t}(x) - G^{(1)'}_{n,t}(x) G^{(2)}_{n,t}(x) \right) dx \right| \\
&= \left| \int_{\Gamma_n} \left(F^{(1)'}_{n,t}(z) F^{(2)}_{n,t}(z) - G^{(1)'}_{n,t}(z) G^{(2)}_{n,t}(z) \right) dz \right| \\
&\le \|F^{(1)'}_{n,t} - G^{(1)'}_{n,t}\|_{\infty(\Gamma_n)} \int_{\Gamma_n} |F^{(2)}_{n,t}(z)||dz| \\
&+ \|G^{(1)'}_{n,t}\|_{\infty(\Gamma_n)} \int_{\Gamma_n} |F^{(2)}_{n,t}(z) - G^{(2)}_{n,t}(z)||dz| \\
&\le d' tne^{-2\pi n\rho}.
\end{aligned}$$

where we can choose d' such that it does not depend on t, n or z. This is not difficult to see by just looking at all the constants and note, from the definition, that $\|\mu - I\|_{L^2(\Gamma_R)}$ is uniformly bounded. $\square$

This concludes the proof of Theorem 1.5.

A Appendix

In this appendix we will give a brief introduction to trace class operators and the Fredholm determinant. We will not prove all results, especially not all deep results, but we will prove some, which gives some insight how to work within this space. For a rigorous treatment, see [6].

On what follows let $\mathcal{H}$ be a separable Hilbert space.

Definition A.1. Let A be a compact operator on $\mathcal{H}$ and let $\sigma_k^2(A)$ be the eigenvalues of A^*A with the ordering

$$\sigma_1^2(A) \geq \sigma_2^2(A) \cdots > 0.$$

Then $\sigma_k(A) > 0$ is called the *singular values* of A.

Recall that the eigenvalues of a self adjoint compact operator converges to zero and that the eigenvalues are real. Moreover if A is compact then A^*A is a self adjoint compact operator. If x is an eigenvector to the eigenvalue $\sigma_k^2(A)$ then

$$(Ax, Ax) = \sigma_k^2(A)(x, x)$$

which implies that $\sigma_k^2(A) > 0$.

Definition A.2. Let A be a compact operator on $\mathcal{H}$. Then A is *trace class* if

$$\sum_{k=1}^{\infty} \sigma_k(A) < \infty.$$

The space of trace class operators form a Banach space, with the norm

$$\|A\|_1 = \sum_{k=1}^{\infty} \sigma_k(A),$$

denoted by $B_1(\mathcal{H})$ (see [6], Theorem IV.5.1). Before proceeding we will state some properties of this space. Let $B_\infty(\mathcal{H})$ denote the space of operators with

$$\|A\|_\infty = \sigma_1(A) < \infty.$$

This is the space of bounded operators on $\mathcal{H}$ (see [6] (IV.2.2)). We can can also define a closely related space, $B_2(\mathcal{H})$, it consists of all compact operators A with

$$\sum_{k=1}^{\infty} \sigma_k(A)^2 < \infty$$

with the norm

$$\|A\|_2 = \left(\sum_{k=1}^{\infty} \sigma_k(A)^2 \right)^{\frac{1}{2}}.$$

An operator in this space is called Hilbert-Schmidt operator.

Lemma A.1. *Let $A \in B_1(\mathcal{H})$, $B \in B_\infty(\mathcal{H})$ and $C, D \in B_2(\mathcal{H})$. Then*

$$\|A\|_\infty \le \|A\|_1$$

$$\|AB\|_1 \le \|A\|_1 \|B\|_\infty \tag{50}$$

$$\|BA\|_1 \le \|A\|_1 \|B\|_\infty \tag{51}$$

$$\|CD\|_1 \le \|C\|_2 \|D\|_2. \tag{52}$$

For the proof, see Proposition IV.5.4 and Lemma IV.7.2 in [6]. In particular this lemma tells us that if $A, B \in B_1(\mathcal{H})$ then

$$\|AB\|_1 \le \|A\|_1 \|B\|_1 \tag{53}$$

and hence $B_1(\mathcal{H})$ is a Banach algebra. Moreover, since the space of compact operators form an ideal in $B_\infty(\mathcal{H})$, so do the space of trace class operators.

Lemma A.2. *If A and B are operators of finite rank, then*

$$|\det(I + A) - \det(I + B)| \le \|A - B\|_1 e^{\|A\|_1 + \|B\|_1 + 1} \tag{54}$$

and

$$|\operatorname{Tr}(A)| \le \|A\|_1. \tag{55}$$

For the proof, see Theorem IV.5.2 and Corollary IV.3.4 in [6]

Lemma A.3. *For a bounded operator A, it is trace class if and only if there exists a sequence of finite rank operators $\{A_n\}$ such that*

$$\|A - A_n\|_1 \to 0$$

as $n \to \infty$.

For the proof, see Theorem IV.5.1 in [6].

With these two lemmas in hand it is possible to extend the notion of determinant and traces to the space of trace class operators.

Corollary A.4. *If A is a finite rank operator, then the functions*

$$A \to \det(I + A)$$

and

$$A \to \operatorname{Tr}(A)$$

can be continuous extended from the finite rank operators to the trace class operators. That is $\det(I + A)$ and $\operatorname{Tr}(A)$ are well defined for all trace class operators.

Proof. This is an ordinary continuation procedure. Let $\{A_n\}$ be a sequence as in Lemma A.3. Then $\{A_n\}$ is a Cauchy sequence in $B_1(\mathcal{H})$. From (54) we can see that $\{\det(I + A_n)\}$ is a Cauchy sequence in $\mathbb{C}$. Therefore, the definition

$$\det(I + A) = \lim_{n\to\infty} \det(I + A_n)$$

makes sense as long as it does not matter which sequence $\{A_n\}$ is chosen. The independence of sequence follows from (54). Assume $\{A'_n\}$ also converges to A, let $d = \lim \det(I + A_n)$ and $d' = \lim \det(I + A'_n)$. Then

$$|d-d'| \leq |d-\det(I+A_n)| + \|A_n - A'_n\|_1 e^{\|A_n\|_1+\|A'_n\|_1+1} + |\det(I+A'_n) - d'|.$$

For the trace, we can use the same procedure with (55) and the additivity of traces for finite rank operators. □

We will now state some properties, that we needed, that is known for matrices, that is, finite rank operators, and can be extended to all trace class operators.

Lemma A.5. *If A and B are trace class operators, C is a bounded operator and α and β are complex numbers, then*

$$\operatorname{Tr}(\alpha A + \beta B) = \alpha \operatorname{Tr}(A) + \beta \operatorname{Tr}(B) \tag{56}$$
$$\operatorname{Tr}(AC) = \operatorname{Tr}(CA) \tag{57}$$
$$\det(I + A)\det(I + B) = \det(I + A + B + AB) \tag{58}$$
$$\det(I + AC) = \det(I + CA) \tag{59}$$
$$\frac{1}{\det(I + A)} = \det(I - A(I + A)^{-1}) \tag{60}$$
$$\det(I + (e^A - I)) = e^{\operatorname{Tr}(A)} \tag{61}$$

where (60) *is true if $(I + A)^{-1}$ exists in $B_\infty(\mathcal{H})$.*

Proof. We will only prove the last equality and leave the others as an exercise. The idea is the same for all equalities but the last requires most effort.

Consider the bounded operator

$$e^A - I = \sum_{k=1}^{\infty} \frac{1}{k!} A^k,$$

by (53) the series defines a trace class operator. Let $\{A_n\}$ be a sequence that converges to A as in Lemma A.3. We want to show that $\{e^{A_n} - I\}$ converges in the same way to $e^A - I$. If that is true, then

$$\begin{aligned}\det\left(I + \left(e^A - I\right)\right) &= \lim_{n\to\infty} \det\left(I + \left(e^{A_n} - I\right)\right)\\ &= \lim_{n\to\infty} e^{\operatorname{Tr}(A_n)}\\ &= e^{\operatorname{Tr}(A)},\end{aligned}$$

by the definition of determinants and traces for trace class operators and since the equality holds for finite rank operators.

From (50) and (51) we can see that

$$\|A_n^k - A^k\|_1 \leq \|A_n - A\|_1 \|A_n^{k-1}\|_\infty + \|A_n^{k-1} - A^{k-1}\|_1 \|A\|_\infty$$

and, by induction, the right hand side converges to zero as n tends to infinity. Now given an $\epsilon > 0$ find an $N \in \mathbb{N}$ such that

$$\sum_{k=N+1}^{\infty} \frac{1}{k!} \|A_n\|_1^k < \frac{\epsilon}{3},$$

for all n, which is possible since $\|A_n\|_1 \to \|A\|_1$. Let n be so big such that

$$\sum_{k=1}^{N} \|A_n^k - A^k\|_1 < \frac{\epsilon}{3}.$$

Then

$$\|(e^{A_n} - I) - (e^A - I)\|_1 \leq \sum_{k=1}^{\infty} \frac{1}{k!} \|A_n^k - A^k\|_1 < \epsilon.$$

□

For some operators on $L^2([a,b])$, it is possible to express the trace as an integral.

Theorem A.6. *Let A be an integral trace class operator on $L^2([a,b])$ with continuous kernel $A(x,y)$ on $[a,b] \times [a,b]$. Then*

$$\mathrm{Tr}(A) = \int_a^b A(x,x)dx.$$

This is Theorem IV.8.1. in [6].

If R and K are integral operators with kernels $R(z,z')$ and $K(z,z')$ then if we can change order of integration, we can see that the kernel of RK is given by

$$RK(z,z') = \int R(z,w)K(w,z')dw. \tag{62}$$

With this observation we can find the trace for products of operators.

A motivation why we introduced $B_2(\mathcal{H})$, is because it can be easier to work with then $B_1(\mathcal{H})$. One reason why it is easier is the following lemma (see [6] Theorem IV.7.7).

Theorem A.7. *Let $K(x,y)$ be a measurable function on $\mathbb{R} \times \mathbb{R}$. Then the integral operator defined by K is a Hilbert-Schmidt operator in $L^2(\mathbb{R})$ if and only if*

$$\int_{\mathbb{R}} |K(x,y)|^2 dxdy < \infty.$$

Next theorem will not be used but has a nice result, so we will include it for completeness of this introduction. It tells us that the relation between eigenvalues and traces and determinants are still valid. This can seems as a obvious result, but require a lot of work to prove. See [6] Theorem IV.6.1.

Theorem A.8 (Lidskii's Theorem)**.** *Let A be a trace class operator and let $\lambda_k(A)$ be the eigenvalues to A. Then*

$$\mathrm{Tr}(A) = \sum_{k=1}^{\infty} \lambda_k(A)$$

and

$$\det(I + A) = \prod_{k=1}^{\infty} (1 + \lambda_k(A)).$$

Lemma A.9. *Assume that $\|A\|_1 < 1$, then*

$$\det(I + A) = e^{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \mathrm{Tr}(A^k)}.$$

Proof. Since $\|A\|_\infty \leq \|A\|_1 < 1$ we can define $\log(I + A)$ as

$$\log(I + A) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} A^k$$

and from (53) this series also converges in $B_1(\mathcal{H})$. Moreover

$$(I + A) = e^{\log(I+A)}.$$

Hence

$$\begin{aligned} \det(I + A) &= \det e^{\log(I+A)} \\ &= e^{\mathrm{Tr} \log(I+A)} \\ &= e^{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \mathrm{Tr}(A^k)} \end{aligned}$$

where the second equality is (61) and the last equality is by continuity of traces. $\square$

Next lemma will be of great importance for us.

Lemma A.10. *Assume that the function $t \mapsto A(t)$ defines a function from some open set or line segment γ in $\mathbb{C}$ to $B_1(\mathcal{H})$. Assume further that there exists an operator $A'(t) \in B_1(\mathcal{H})$ such that*

$$\left\| \frac{A(t+h) - A(t)}{h} - A'(t) \right\|_1 \to 0$$

as $h \to 0$ if $t + h \in \gamma$ and that $(I - A(t))^{-1}$ exists in $B_\infty(\mathcal{H})$. Then

$$\frac{d}{dt} \log \det(I + A(t)) = \mathrm{Tr}\left(A'(t)(I + A(t))^{-1}\right).$$

Proof. Consider the quotient

$$\begin{aligned}\frac{\det(I + A(t+h))}{\det(I + A(t))} &= \det(I + A(t+h))\det(I - A(t)(I + A(t))^{-1}) \\ &= \det(I + (A(t+h) - A(t))(I + A(t))^{-1})\end{aligned}$$

where the first equality is (60) and the second equality is (58) and some algebraic manipulations. From the assumption, we can see that for small h the difference $A(t+h) - A(t)$ is small. Hence, with (50) we can make

$$\|(A(t+h) - A(t))(I + A(t))^{-1}\|_1 < 1.$$

From Lemma A.9 we can conclude that

$$\frac{\det(I + A(t+h))}{\det(I + A(t))} = e^{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \operatorname{Tr}\left(((A(t+h)-A(t))(I+A(t))^{-1})^k\right)}.$$

Now

$$\begin{aligned}&\frac{d}{dt}\log\det(I + A(t)) \\ &= \lim_{h\to 0}\frac{1}{h}\log\left(\frac{\det(I + A(t+h))}{\det(I + A(t))}\right) \\ &= \lim_{h\to 0}\frac{1}{h}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\operatorname{Tr}\left(((A(t+h) - A(t))(I + A(t))^{-1})^k\right) \\ &= \operatorname{Tr}\left(A'(t)(I + A(t))^{-1}\right).\end{aligned}$$

The last equality is due to linearity and continuity of traces and since $\operatorname{Tr}\left(((A(t+h) - A(t))(I + A(t))^{-1})^k\right)$ is of order h^k. □

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