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Some Matroid Theory and a Peek into Oriented Matroids

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Abstract

The term matroid was first used by Whitney in a 1935 paper about the abstract properties of linear independence. As the name implies, it has something to do with matrices, and the motivation was to try to generalize the idea of dependence in matrices and graphs. Since then, the structure of matroids has led to a greater understanding of many aspects of combinatorial theory, simplified the proofs of several important theorems, and it has had many applications in combinatorics. This work will introduce several of the axiom systems of matroids and prove their equivalence, and important examples will be brought to light. We will also look at the important duality concept for matroids. Lastly we shall investigate a refinement of matroids, namely oriented matroids, and try to motivate this structure's applicability in mathematics.

Contents

1 Matroids

In this section we will explore some of the axiom systems of matroids and give some, hopefully, illuminating examples of mathematical objects which are matroids. We will also prove the equivalences between some of the axiom systems.

1.1 Independent sets and bases

Definition 1 (Matroid Axioms). A matroid is a finite set E and a collection $\mathcal{I}$ of subsets such that the following axioms are fulfilled:
(I1) The empty set is in $\mathcal{I}$.
(I2) (Hereditary principle) If I_1 is in $\mathcal{I}$ and $I_2 \subseteq I_1$ then I_2 in $\mathcal{I}$.
(I3) If I_1, I_2 are in $\mathcal{I}$ and $|I_1| = |I_2| + 1$ then there exists x in $I_1 - I_2$ such that $I_2 \cup x$ is in $\mathcal{I}$.

We will usually denote the matroid by $(E, \mathcal{I})$. The subsets of $\mathcal{I}$ are called independent and the sets of E which are not in $\mathcal{I}$ are called dependent.
Not surprisingly, for a collection of vectors E in a vector space, the set $\mathcal{I}$ here correspond to all subsets of E with independent vectors. Other concepts in vector space theory such as bases, rank, closure and hyperplanes have their counterpart in matroids, but we will begin exploring with the notion of independent sets above.

Proposition 1. *If E is a finite set of vectors in a vector space V, then the collection of sets of linearly independent vectors in E, call it $\mathcal{I}$, together with E form a matroid $(E, \mathcal{I})$.*

Proof. Clearly $\emptyset \in \mathcal{I}$, since it is difficult to create a linear combination equal to zero with some coefficients not zero without any vectors.
We can also see that (I2) is obviously true.
Suppose the condition in (I3) is true, that $|I_1| = |I_2| + 1$ but not the conclusion. Let $\dim(X)$, where X is a set of vectors, mean the dimension of the subspace spanned by vectors in X. We then know that $\dim(I_1) = \dim(I_2) + 1$. For all $x \in I_1$, $I_2 \cup x$ is a linear dependent set, which means every vector in I_1 is in the linear span of I_2. This however contradicts what we know about the dimension of I_2. So (I3) must be true. $\square$

The collection of independent sets $\mathcal{I}$ of a matroid M will sometimes be denoted $\mathcal{I}(M)$.

Example 1.1. Let $E = \{a, b, c, d, e, f, g\}$. The linearly independent sets of column vectors below, indexed by E, is our set $\mathcal{I}$ in the matroid $(E, \mathcal{I})$.

$$\begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \end{array}.$$

In this example

$$\mathcal{I} = \{\{b\},\{c\},\{d\},\{e\},\{f\},\{b,c\},\{b,d\},\{b,e\},\{b,f\},\{c,d\},\{c,e\},\{c,f\},$$
$$\{d,e\},\{d,f\},\{e,f\},\{b,c,d\},\{b,c,e\},\{b,d,c\},\{b,d,f\},\{b,e,f\},\{b,e,f\},$$
$$\{c,d,f\},\{c,e,f\},\{d,e,f\}\}$$

There is a similar property to (I3) called the augmentation principle, which is sometimes taken as an axiom instead of (I3). We will prove the equivalence in the next proposition.

Proposition 2. $(E,\mathcal{I})$ *is a matroid if and only if* $\mathcal{I}$ *satisfies (I1) and (I2) together with the augmentation principle (I3'), which states that:*
If $X, Y \in \mathcal{I}$ *and* $|X| < |Y|$, *then there is a* $Z \subseteq Y - X$ *such that* $X \cup Z \in \mathcal{I}$ *and* $|X \cup Z| = |Y|$.

Proof. Assume $(E,\mathcal{I})$ is a matroid and $|X| < |Y|$. By the hereditary principle (I2) there exists a $Y^* \subset Y$ in $\mathcal{I}$, such that $|Y^*| = |X| + 1$. From (I3) we know we can add an element from $Y^* - X$ to X. Repeating this procedure we can add elements to X until $X \cup Z \in \mathcal{I}$ and $|X \cup Z| = |Y|$ for $Z \subseteq Y - X$. For the converse, assume (I1),(I2) and (I3') are true. So suppose we have X and Y such that $|Y| = |X| + 1$. From (I3') we can see that there must exist a set $Z \subset Y - X$ such that $|Z| = 1$ and $X \cup Z \in \mathcal{I}$, which is precisely what (I3) says. □

As mentioned above, we can take further inspiration from vector spaces and talk about maximal independent sets, or bases rather. Maximal sets are sets not properly contained in any other set. Given the bases, $\mathcal{I}$ is the collection of all subsets of the bases.

Proposition 3. *All the bases of a matroid are of the same cardinality.*

Proof. If we have the case that $|B_1| < |B_2|$, we can use the augmentation principle to find a Z such that $B_1 \cup Z \in \mathcal{I}$ and $|B_1 \cup Z| = |B_2| > |B_1|$, which contradicts the maximality of B_1. □

Theorem 1 (Base Axioms). *A collection of subsets of* E, *call it* $\mathcal{B}$, *are the bases of a matroid if and only if*
(B1) $\mathcal{B}$ *is non-empty.*

(B2) *If $B_1, B_2 \in \mathcal{B}$ and x is in $B_1 - B_2$, there exists y in $B_2 - B_1$ such that $B_1 \cup y - x$ is in $\mathcal{B}$.*

Proof. So assume $\mathcal{B}$ is the collection of the maximal independent sets, the bases, of a matroid. (B1) is obviously satisfied. Given the conditions in (B2), $|B_1 - x| < |B_2|$, and by (I3) there must exist $y \in B_2 - B_1$ such that $(B_1 \cup y) - x \in \mathcal{B}$.

Before we go further we need the following lemma:

Lemma 1. *All the sets in $\mathcal{B}$ have the same cardinality.*

Wait a minute. Did we not just prove this in Proposition 3? No, we proved that the bases, the maximum independent sets have this property. We have yet to establish that the set of bases is equivalent to $\mathcal{B}$. Let us continue.

Proof. Suppose it is not true and we have $|B_1| < |B_2|$. By (B2) we can exchange elements in B_1 with elements from B_2, call the resulting set after each exchange B_{new}, until $B_{new} \subset B_2$ and $B_{new} \in \mathcal{B}$. Now we get into trouble. Since $|B_2| > |B_{new}|$, $B_2 - B_{new}$ is non-empty, but $B_{new} - B_2$ is empty, which contradicts (B2) since we cannot exchange elements. □

Now let $\mathcal{B}$ be a collection of subsets satisfying (B1-B2), and define $\mathcal{I}$ to be the collection of subsets X of E such that $X \subset B$ for some $B \in \mathcal{B}$. (I1-I2) are clear. So we need to prove (I3).
Assume $X, Y \in \mathcal{I}$ and that $|Y| = |X|+1$. By definition $Y \subset B_1$ and $X \subset B_2$ for some $B_1, B_2 \in \mathcal{B}$. We now have the following situation:

$$X = \{x_1, ..., x_n\} \tag{1}$$
$$B_1 = \{x_1, ...x_n, b_1, ..., b_q\} \tag{2}$$
$$Y = \{y_1, ..., y_{n+1}\} \tag{3}$$
$$B_2 = \{y_1, ..., y_{n+1}, c_1, ..., c_{q-1}\}. \tag{4}$$

In the above we have used that all sets in $\mathcal{B}$ have the same cardinality. We can now utilize (B2) as follows. If we have the case that every single b_k is in B_2, then at least one of them has to be in Y. So let us assume this is not the case. Then we can start exchanging the elements $b_1, ...b_k$ for elements in B_2. One of two things will now happen. Either, there will come a swap where we must swap a b_k for a y_l, for some k, l, in which case we are done, or we will have gone $q-1$ steps and our modified set in $\mathcal{B}$ is $B_{new} = \{x_1, ...x_n, c_1, ..., c_{q-1}, b_q\}$, and at this step either $b_q \in Y$ or we must swap b_q for some $y_m \in Y$. In either case we will have that $X \cup y_l$, for $y_l \in Y$, is a subset of some B in $\mathcal{B}$, which proves (I3).

□

In Example 1.1, the bases are all the sets in $\mathcal{I}$ with three members.

1.2 Circuits

We have so far introduced two sets of axioms which define a matroid, those for independent sets and those for bases. A third is one relating to the minimal dependent subsets of a set E, that is, the dependent sets not properly contained in any other dependent set. These are called the circuits of a matroid. The independent sets are all those subsets which contain no circuit.

Theorem 2 (**Circuit Axioms**). *Let $\mathcal{C}$ be a collection of subsets of a set E. Then $\mathcal{C}$ are the circuits of a matroid if and only if*
(C1) *The empty set is not in $\mathcal{C}$.*
(C2) *If $C_1, C_2 \in \mathcal{C}$ and $C_1 \neq C_2$ then C_1 cannot be a subset of C_2.*
(C3)*(Circuit elimination) If C_1 and C_2 are distinct sets in $\mathcal{C}$, and $z \in C_1 \cap C_2$, there exists a circuits C_3 such that $C_3 \subset (C_1 \cup C_2) - z$.*

Proof. Let $\mathcal{C}$ be the minimal dependent sets of a matroid on E. Then (C1) and (C2) are clear. Not quite so obvious is (C3). Assume we have the situation as in (C3) and that there does not exist such a set C_3. Then $(C_1 \cup C_2) - z$ is an independent set. Also, by (C2) there exists an $x \in C_1$ such that $x \notin C_2$. We also know $C_1 - x$ must be an independent set. Using the augmentation principle there exists $Z \subset (C_1 \cup C_2) - z$ such that $(C_1 - x) \cup Z$ is independent and $|(C_1 - x) \cup Z| = |(C_1 \cup C_2) - z|$. Since C_1 and C_2 are distinct, either $C_1 \subset Z \cup (C_1 - x)$ or $C_2 \subset Z \cup (C_1 - x)$, which is a contradiction.
Now assume $\mathcal{C}$ is a collection of subsets of E such that (C1-C3) are true. Let $\mathcal{I}$ be the collection of subsets of E which contain no member of $\mathcal{C}$. We shall prove these are the independent sets of the matroid.
(I1) is clearly true. Also if I_1 contain no $C \in \mathcal{C}$ then $I_2 \subset I_1$ cannot do so either. This implies (I2) is satisfied.
Suppose that $I_1, I_2 \in \mathcal{I}$ and $|I_1| = |I_2| + 1$.We will prove (I3) by induction over the cardinality of the difference $I_2 - I_1$. That is, we will show that for every $|I_2 - I_1| = n \in \mathbb{N}$, (I3) is true. If $n = 0$ then $I_2 \subset I_1$ and (I3) is true. Moving on to $n = 1$, let $I_2 = \{y_1, ...y_m\}$ and y_1 (the index does not really matter) be the only element in $I_2 - I_1$ and $x_1, x_2 \in I_1 - I_2$. If (I3) was not true, then we would have that sets $\{y_1, .., y_m, x_1\}$ and $\{y_1, ..., y_m, x_2\}$ contained members of $\mathcal{C}$. Either $\{x_1, y_1\}$ or $\{x_2, y_1\}$ have to be subsets of these members of $\mathcal{C}$, else a subset of either I_1 or I_2 would be in $\mathcal{C}$, a contradiction. Using (C3) on these two members of $\mathcal{C}$ with $y_1 = z$ would now imply we have a set $C \in \mathcal{C}$ contained in I_1, a contradiction. So the base case $n = 1$ is true.

Before we go further we need to show that if (I2) is true and (I3) is true for $|I_2 - I_1| \leq k$ then (I3') is also true for $|I_2 - I_1| \leq k$. The proof of this works the same as the one in Proposition 2: Suppose that (I2) is true and (I3) is true for $|I_2 - I_1| \leq k$. Given $|I_1| > |I_2|$ and $|I_2 - I_1| \leq k$, we can then find $Y^* \in \mathcal{I}$ such that $Y^* \subset I_1$ and $|Y^*| = |I_2| + 1$ (by (I2)). We can also see that $|I_2 - Y^*| \leq k$. By (I3) we can then add an element from $Y^* - I_2$ to I_2. We can repeat this procedure until we have augmented I_2 with $Z \subset I_1 - I_2$ such that (I3') is fulfilled.

Now then, assume (I3) is true for $|I_2 - I_1| \leq k$ and we want to show it is true for $|I_2 - I_1| = k + 1$. What we do is we remove one element from I_2 which is in $I_2 - I_1$. Call this new set I_2^*, which by (I2) is in $\mathcal{I}$. Then $|I_2^* - I_1| = k$ and $|I_1| = |I_2^*| + 2$. Our induction assumption means that (I2) is true and (I3) is for $|I_2 - I_1| \leq k$, which means we can use (I3') (shown the previous paragraph). We then augment I_2^* and find $I_3 \in \mathcal{I}$ such that $|I_3| = |I_1|$. We can then see that $|I_2 - I_3| = 1$. So then we have the same situation as in the base case $n = 1$, which means we can find an $x \in I_3 \cap I_1$ such that $I_2 \cup x \in \mathcal{I}$, and thus (I3) is true for every $n = |I_2 - I_1|$. □

We can derive an interesting relation between the circuits and the independent sets of a matroid.

Proposition 4. *If I is an independent set of a matroid $(E, \mathcal{I})$, then for $x \in E$, $I \cup x$ contains at most one circuit.*

Proof. Let A be an independent set and x an element in E. Suppose there are two distinct circuits $C_1, C_2 \subset A \cup x$. Then since x must be in both of these circuits, by (C3) there exists a third circuit such that $C_3 \subset (C_1 \cup C_2) - x) \subset A$, which contradicts the independence of A. □

For a base B and element x, the unique circuit which is a consequence of the lemma above is called the fundamental circuit of x in B.

The circuits in the vector matroid in Example 1.1 are

$$\{a\}, \{c, d, e\}, \{b, c, f\}, \{b, d, e, f\}.$$

For the base $\{d, e, f\}$ and element b, the fundamental circuit is in this case $\{b, d, e, f\}$. It might also be of interest to see the circuit elimination in action on the above circuits. We have the circuits $\{c, d, e\}$ and $\{b, c, f\}$, and in the example we can see that:

$$\begin{aligned}
&c + d - e = 0 \\
&b + c - f = 0 \\
&\implies \\
&b + (e - d) - f = 0.
\end{aligned}$$

In the last equation we have substituted c for $e - d$ which means we have a dependent set of vectors, i.e. a set containing a circuit, with vectors from $\{c, d, e\} \cup \{b, c, f\} - c$.
When talking about circuits, perhaps the structure it most naturally applies to is the edge set E of a graph, where a circuit is a set of edges forming a simple cycle, which we will refer to in the text as just a cycle.

Proposition 5. *Let E be the set of edges of a graph and $\mathcal{C}$ be the collection of subsets of E which are edge sets of cycles. Then $\mathcal{C}$ is the set of circuits of a matroid on E.*

Proof. You cannot create a cycle without any edges, so (C1) is true.
A (simple) cycle cannot properly contain another cycle which means (C2) must be true.
Now assume we have the case that X and Y are edge sets of distinct cycles P_1 and P_2, and $z \in X \cap Y$. That is, the two cycles share an edge z, say between the the two vertices (v_k, v_l). Now we will construct a cycle. Starting from v_k, sooner or later P_1 and P_2 will diverge, which they must since they are distinct. Let w_a be this vertex at which they first diverge. Now let w_b be the first vertex at which they converge again, which again they must, since they will both eventually reach v_l. Now there are edges in X from w_a to w_b which form a path, and then there is another path with edges in Y that go from w_b to w_a. Together these edges form a cycle with edges in $(X \cup Y) - z$, which proves (C3). $\square$

We define a spanning forest of a graph to be a union of spanning trees in the connected components.
Proposition 6. *The edge sets of the spannings forests of a graph are the bases of the the matroid associated with G.*

Proof. Since the sets of cycles are the circuits of the matroid M, and the edges in a spanning forest is a maximal set of edges which does not contain a cycle, it follows. $\square$

Example 1.2. Looking at the graph below, it easy to see that

$$\{a\}, \{c, d, e\}, \{b, c, f\}, \{b, d, e, f\}$$

form the circuits.

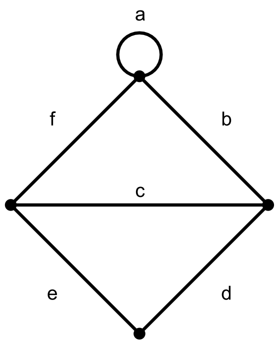


Figure 1: Graphical Matroid.

That the circuits in Example 1.1 and 1.2 are the same means that the matroids *isomorphic* to each other, they have the same structure. Formally we define two matroid M_1 and M_2 over sets E_1 and E_2 to be isomorphic to each other if there exists a bijection ϕ $E_1 \to E_2$ such that C_1 is a circuit in M_1 if and only if $\phi(C_1)$ is a circuit in M_2. We of course have equivalent definitions in regards to independent sets and bases.

We have now introduced two different mathematical objects which share a matroid structure, graphical and matroids over vector spaces, called representable matroids. One might wonder if every matroid is, up to isomorphism, graphical or representatable.
First off, if a matroid has two loops (circuit of one element), it cannot be vectorial, since the only loop among vectors is the zero vector, and there are indeed examples of matroids which are not graphic. We can find an example of such a matroid in another important class of matroids called Uniform matroids.
For a matroid $M(E)$, the rank of a subset X of E is the cardinality of the maximal independent set contained in X. A natural extension of the rank function in vectors spaces. If we have a matroid over a set of n elements and a set I is independent if and only if $|I| \leq k$, then this is a uniform matroid of rank k, denoted $U_{k,n}$. The smallest example of a non-graphic matroid is then $U_{2,4}$, see [3]. Consider an attempt to create a graph out of such a matroid. Take three of the edges and they must form a cycle, so they create a 'triangle', as in Figure 2. Now for the last edge, there are three possibilities. Either the last edge is not attached to any vertex in the triangle, it has one endpoint in the triangle or both endpoints are in the triangle. No matter where we put it, we break the definition of the uniform matroid. In the first two cases we get sets of three which do not contain a cycle, and in the last case we get a 2-cycle.

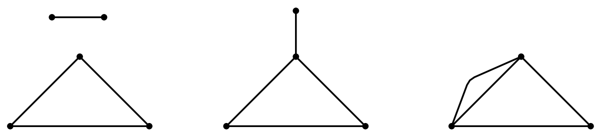
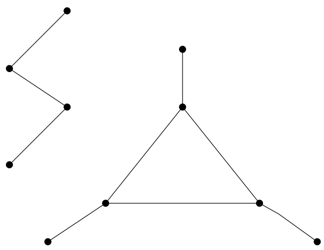
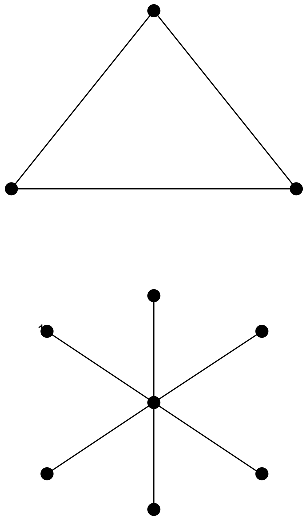


Figure 2: Three possiblities of last edge if there existed graphical $U_{2,4}$.

Important to note about the isomorphism between matroids is that it does not necessarily coincide with isomorphism between graphs, as the example in Figure 3 demonstrates. The matroid structure is the same, there is only the one circuit and the same number of elements (edges), but the graphs are not isomorphic.



(a) Graph A.



(b) Graph B.

Figure 3: Two matroid isomorphic but not graph isomorphic graphs.

1.3 The Greedy Algorithm

Another interesting way to characterize matroids is by their connection to the greedy algorithm. It turns out that not only can we apply the algorithm on matroids, but it is the only structure on which we can do so. This gives an application of matroid theory, where if we want to know if the greedy algorithm is applicable, one way to do so is to verify that the structure is that of a matroid.

Let us first review just what the greedy algorithm does. Let G be a connected graph G and $w(e)$ a function from the edge set $E(G)$ into $\mathbb{R}$. Then

for $X \subset E(G)$, $w(X)$, or if you will, the weight of X, is $\sum_{x \in X} w(x)$. What the algorithm then does is it finds the set of edges X of a spanning tree with $w(X)$ minmized.
So how does it work? We begin by choosing an edge of minimal weight, and then we continue choosing edges not previously chosen of minimal weight and such that we do not form a cycle with our earlier chosen edges. When we can not find any new edges, the algorithm stops and we have a minimal weight spanning tree.
The above algorithm is a special case of a more general problem. Suppose E is a set and $\mathcal{I}$ a collection of subset such that (I1) and (I2) are satisfied. Let w be a weight function similar as before, but from E to $\mathbb{R}$ instead. Finding a maximal member B of $\mathcal{I}$ such that $w(B)$ is maximal is then an optimization problem. In the case of our graph, if try to find such a B for the function $-w$, we will find a minimal weight spanning tree. The greedy algorithm on a pair $(\mathcal{I}, w)$ then proceeds as follows:
We begin picking any e such that $\{e\} \in \mathcal{I}$ and has greatest possible weight. Then as one could guess, we add elements to this set making sure that our new set is still in $\mathcal{I}$ and the new elements has greatest possible weight. When we cannot do this any more we are done.

Proposition 7. *If our set E and collection of subsets $\mathcal{I}$ defines a matroid M, then the above algorithm will produce a maximal member of $\mathcal{I}$ of maximal weight for a given weight function from E to $\mathbb{R}$.*

Proof. Let B_G be the set resulting from the algorithm. It follows that $|B_G| = \text{rank}(M) = n$, since our algorithm will stop when we get to a base $B_G = \{e_1, ..., e_n\}$. Suppose there is a set $B_f = \{f_1, ..., f_n\}$, where the elements are ordered in decreasing weight, such that $w(e_j) < w(f_j)$ for some $e_j \in B_G$ and $f_j \in B_F$. Let k be the least integer such $w(e_k) < w(f_k)$ is satisifed. By (I3), we can complement $\{e_1, ..., e_{k-1}\}$ by an element $f_l \in \{f_1, ..., f_k\}$ such that $w(f_l) \geq w(f_k)$, and this new set is independent. But if this is so, then surely we would have picked this element at the k'th step in the algorithm, a contradiction. □

As stated earlier, we will prove that not only does the greedy algorithm work for matroids, but this is the only case where it works.

Theorem 3 (**Greedy Axioms**)**.** *If $\mathcal{I}$ is a collection of subsets of E, then $(E, \mathcal{I})$ is a matroid if and only if the following conditions are met:*

(I1) *The empty set is in $\mathcal{I}$.*
(I2) *If $I_1, I_2 \in \mathcal{I}$ and $I_1 \subset I_2$, then $I_1 = I_2$.*
(G) *For a given weight function from E to $\mathbb{R}$, the greedy algorithm produces a maximal member of $\mathcal{I}$ of maximal weight.*

Proof. We have indeed shown that (G) is also fulfilled if $(E, \mathcal{I})$ is a matroid. So assume that (I1), (I2) and (G) are true. Well obviously (I1) and (I2) are true so we have to prove I3.
Suppose we have $|I_1| = |I_2| + 1$ but there is no $e \in I_1$ such that $I_2 \cup e \in \mathcal{I}$. Define a weight function w such that $w(x) = 1$ if $x \in I_2$, $w(x) = \epsilon < 1$ if $x \in I_1 - I_2$ and $w(x) = 0$ else. Then our algorithm will pick everything in I_2 first and by assumption cannot pick elements from $I_1 - I_2$. Thus the maximal independent set of maximal weight $I_2' \supset I_2$ has weight $|I_2|$. But the maximal independent set $I_1' \supset I_1$ will have at least weight $(|I_2| + 1)\epsilon > |I_2|$ if $\epsilon > \frac{|I_2|}{|I_2|+1}$, which contradicts the fact that the algorithm should find a set of maximal weight. □

1.4 Other equivalent ways to define a matroid

There are plenty more interesting ways to define a matroid which will be equivalent to the above systems. It is possible to find an axiomatisation using the rank function, and we can also use the closure operator, which is a function that adds all possible elements to a set such that the rank does not increase. Hyperplanes, or rather maximal subsets where the rank is one less than the rank of a base, can also do the job. The fact that so many abstractions of functions and properties lead to the same structure speaks for the matroid abstraction.

2 Matroid duality

Significantly, for every matroid we can find a dual structure that is also a matroid, and these two structures determine each other. This conept of duality coincide with interesting operations on the different mathematical objects which matroids abstract.
As an example of this more general property, let us look at the graphical matroid G in Figure 4. We define a cut to be a set of edges whose removal from the graph increases the number of connected components of the graph. The claim is that the collection of all possible minimum cuts form the circuits of a new matroid. We will postpone the proof of this, but begin with an example. In the graphical matroid in Figure 4, the mimimal cuts are highlighted. This graph is also a planar graph and for a planar graph we can find its geometric dual. Let us do this. As can be seen, any planar graph divides the plane into regions, all but one finite. We call these regions the faces of the planar graph. In each face F we choose a point v_F which is not on an edge or at a vertex. If two regions F and F' share a boundary of edges $\{e_1, .., e_n\}$, we connect v_F and $v_{F'}$ by edges $\{e_1', ...e_n'\}$, where each

e_i' pass through only e_i. The graph in Figure 5 is the geometric dual of the planar graph G. Notice that each cycle in this geometric dual corresponds to a minimal cut in G, if we allow ourselves see an edge e_i' as the same edge as e_i. Very interesting, and we will come back to this soon.

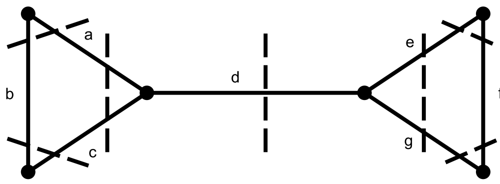


Figure 4: A graph G and its minimal cuts indicated.

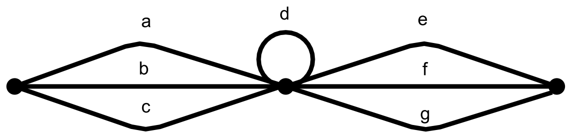


Figure 5: The geometric dual of G in Figure 4.

We can also easily find a dual by working with the bases. Given a matroid M on a set E, it is the case that for every base B in M, every set $E - B$ defines a set of bases on a new matroid on E.

Example 2.1. Let $E = \{a, b, c, d, e, f, g\}$, that is the index set of the column vectors below:

$$\begin{array}{ccccccc} a & b & c & d & e & f & g \end{array}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

So why do the sets $E - B$ for all bases of the matroid M make up a new matroid over E? This might look strange since certainly for instance $\{a, b\}$ is no base in the vector space $\mathbb{R}^4$? The point here is that it is only the abstract relation between the sets that matters: do they fulfill the base axioms? That would in itself take some work to check, so we will move directly to proving the general case.

First a lemma:

Lemma 2. *Let B_1 and B_2 be bases of a matroid M, then for every $x \in B_1 - B_2$ there exists a $y \in B_2 - B_1$ such that $(B_2 - y) \cup x$ is a base of M.*

Proof. Well, the set x is independent by I2, and by the augmentation principle we can augment $\{x\}$ with a set $Z \subset B_2$ such that $x \cup Z$ is a base. Then clearly our y is the only element left in $B_2 - Z$ □

Theorem 4. *Given a matroid M on a set E and its set of bases $\mathcal{B}$, the set $\{E - B | B \in \mathcal{B}\}$ is the set of bases of a matroid M^* on E. M^* is called the dual matroid of M, and we also have that $(M^*)^* = M$.*

Proof. Since $\mathcal{B}$ is non-empty so is $\{E - B | B \in \mathcal{B}\}$. Let us denote the sets in $\{E - B | B \in \mathcal{B}\}$ by $\mathcal{B}^*$. Now let $B_1^* = E - B_1$ and $B_2^* = E - B_2$. Then $B_1^* - B_2^* = B_2 - B_1$ and $B_2^* - B_1^* = B_1 - B_2$. So given $x \in B_1^* - B_2^*$ we want to find $y \in B_2^* - B_1^*$ such that $(B_1^* \cup y) - x$ is in $\mathcal{B}^*$. Well this is equivalent to finding for any $x \in B_2 - B_1$ a $y \in B_1 - B_2$ such that $(B_1 - y) \cup x$ is a base. Well this is precisely what the lemma above says, so M^* is indeed a matroid on E. The symmetry involved then surely implies that $(M^*)^* = M$.

□

This also proves that the dual structure defined in example 2.1 is indeed a matroid. The bases of our dual matroid M^* are called the cobases of the matroid M. Likewise we have similar names for other matroid concepts, such as cocircuits and the corank function.

Proposition 8. *A subset X of E is a base of a matroid M on E if and only if it has a non-null intersection with every cocircuit of M and is minimal in regards to this property.*

Proof. So suppose X is a base. Then $E - X$ is a base in the dual matroid and this cobase of course does not contain any cocircuits, which means at least parts of all possible cocircuits are in X. Suppose X was not minimal with regards to this property, but $X - A$ for some non-empty set $A \subset X$ had this property. Then $E - (X - A)$ would not contain any cocircuits and thus be independent in M^*, but $|E - (X - A)| > |E - X|$ which contradicts the fact that $E - X$ is a base in M^*.
Now assume X has a non-null intersection with every cocircuit of M. Then $E - X$ must be independent, and if X is minimal with this property then $E - X$ must be a maximal independent set, that is a cobase, which implies X is a base of the matroid. □

From this we are led to a more direct connection between the matroid and its cocircuits.

Proposition 9. *Let M be a matroid over a ground set E. Let X^* be a subset of E. Then X^* is a cocircuit of the matroid if and only if it has a non-null* *intersection with every base of the matroid M and is minimal in regards to this property.*

Proof. Assume X^* is cocircuit of M. From Proposition 8, every base has a non-null intersection with every cocircuit and then of course every cocircuit has a non-null intersection with every base. To show it is minimal, assume there is a strict subset Y^* of X^* that has this property. Then Y^* is independent in M^*. This implies $E - Y^*$ contains a base, which is a contradiction, since Y^* contains part of each base. Thus X^* is minimal.
Now assume X^* has non-empty intersection with every base of the matroid M. This is equivalent to X^* not contained in any cobase, which means it is (co)dependent, and a minimal (co)dependent set is cocircuit. □

Now we can also prove that given a graph G, the minimal cuts will always define a dual matroid. To do this we will prove a lemma first.
Lemma 3. *A set of edges X is a cut of a graph if and only if X has a non-empty intersection with the edge set of every spanning forest of G.*

Proof. Suppose X is a cut. If X did not have a non-empty intersection with the edge set of every spanning forest, the removal of X could not increasae the number of components.
Now assume X has a non-empty intersection with the edge set of every spanning forest. Let Y be the edge set of a spanning forest. Removing $X \cap Y$ from Y will of course increase the number of components in a graph. □

Proposition 10. *The minimal cuts of a graph G are the cocircuits of the matroid of G.*

Proof. Let M be the matroid structure of G. By Lemma 3 each minimal cut will have a non-empty intersection with the edge set of every spanning forest of G and be minimal with regards to this property. Since the set edges of the spanning forests are the bases of M, Proposition 8 then implies that the minimal cuts must be the cocircuits of M. □

In graph theory, a simple graph is a graph without loops or multiple edges, and much research focus mainly on these types of graphs. Similarly, in matroid theory we have also simple matroids, which are matroids without loops and parallell elements. A loop is a circuit of only one element, and f and g are called paralell elements if $\{f, g\}$ is a circuit. However, simple matroids do not play as central a role as simple graphs, and this is in large because simple matroids are not closed under duality, which can be easily demonstrated. For instance, in any graph where a single edge separates two components, the dual will have a cocircuit of one element, a loop.
One must also wonder whether the different classes of matroids are closed

under duality. For instance, are graphic matroids closed under duality? The answer to this is no, and the graph K_5 provides an example of this. To show this we first prove the following useful proposition.

Proposition 11. *Given a graphic matroid M, M is isomorphic to some matroid $M(G)$ where G is a connected graph.*

Proof. Let us begin with any graph that has the matroid structure of M. If the graph has one component, we are done. So assume the graph has G has n components, $H_1, ..., H_n$. We now take a vertex from each component and merge them together to one new vertex, with all associated edges going into this new vertex. Call this new graph G'. It is clear that that we still have the old cycles in our new graph. Can there be new cycles with edges from the old components? Impossible, since this would imply 2 paths going to and from at least 2 different components, and there can only be one (passing through the merged vertex). □

Proposition 12. *The dual matroid of $M(K_5)$ is not a graphical matroid.*

Proof. We will prove this by contradiction and assume the dual matroid is graphical. Let $M = M^*(K_5)$ and let G be a graph with the matroid structure of M. We can assume G is connected by Proposition 11. Now $M(K_5)$ has 10 elements and rank 4, since a spanning tree is of 4 edges. This means M has 10 elements and rank 6, since $E(G) - E(T)$, where T is a spanning tree of K_5, will be a base in M. Let T^* be a spanning tree of G, and thus of 6 edges. By the identity $|E(T^*)| = |V(T^*)| - 1$, G must have 10 edges and 7 vertices. This gives an average vertex degree of $2|E(G)/|V(T)| = 20/7$, which is less than three. So G has at least one vertex of degree at most 2. This means there exists a vertex we can cut off with a minimal cut of at most 2 edges. This minimal cut will define a cocircuit of M, which means $M^* = M(K_5)$ must have a circuit of order 1 or 2, which implies a cycle of 1 or 2 edges in K_5. This in turn would mean we have either a loop or two parallell edges in K_5, a contradiction. □

As hinted at in the beginning of this section, there is a connection between geometric duality in graphs and matroid duality. Indeed, we have the following proposition:

Proposition 13. *If G is a planar graph and H^* is a geometric dual of this graph, then the matroid structure of H^* is the matroid dual of the matroid of G.*

We refer the proof of this to Welsh [2].

The class of representable matroids, however, is closed under duality. To prove this we begin with an easy lemma.

Lemma 4. *Let A be any $m \times n$ matrix over a field and $M(A)$ the associated matroid. Changing place of two columns does not change the matroid structure.*

Proof. Clearly the linear dependecies are the same, all we have done is changed the names of two vectors, in principle. $\square$

Theorem 5. *If M is representable over the field F, then so is the dual matroid M^*.*

Proof. Assume M has ground set E, $|E| = n$ and the rank is r. Then let A, an $r \times n$ matrix of rank r be a matrix representation of M. The matrix A can also be viewed as a linear transformation from F^n to F^r. The kernel of this transformation, that is all $x \in F^n$ such that $Ax = 0$ has dimension $n - r$.
Now let B be a $n \times (n - r)$ matrix such that the corresponding linear transformation F^{n-r} to F^n spans the kernel of A. The claim is now that the transponate of B, B^T, is a matrix representation of the dual matroid M^*.
Let us imagine a $1 - 1$ correspondence between the columns $\{a_1, ...a_n\}$ of A and the rows $\{b'_1, ..., b'_n\}$ of B, say a_j to b'_j. So first column of A to first row of B and so on. This also creates a correspondence between the columns of A and those of B^T, namely e_j to b_j.
Now what we want to prove is every set of r columns in A are linearly independent if and only if the complement of the corresponding columns in B^T are linearly independent. For example the r columns $e_1, e_2, ..., e_r$ would be independent if and only if $b_{r+1}, .., b_n$ are independent. If we can prove this, then the bases of the matroid $M(B^T)$ will mirror the base structure in M^*, and thus prove the theorem. Given our correspondence and the fact that reordering columns do not change the underlying matroid, for every such selection we can put the selected r columns first in A, which also puts the complement of $(n - r)$ columns as the last vectors in B^T. Another reformulation of what we then want to prove is that the first r columns of A are linearly dependent if and only if the last $(n - r)$ columns in B^T are linearly dependent.
What we do know is there exists $y = (y_1, \ldots, y_r, 0, \ldots, 0) \in F^n$ with $y \neq 0$ and such that $Ay = 0$ (1)
if and only if
there exists a non-zero $z \in F^{(n-r)}$ such that $Bz = y = (y_1, ...y_r, 0, \ldots, 0) \neq 0$. (2)
(1) is equivalent to that the first r columns in A are dependent.
We can see that

$$B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$$

where B_1 is a $r \times (n-r)$ matrix and B_2 is a $(n-r) \times (n-r)$ matrix. (2) then implies that $B_2 z = 0$, and since $z \neq 0$, B_2 must be singular. This then implies that the last $(n-r)$ rows of B and of course then the last $n-r$ columns of B^T are linearly dependent. Suppose then that for $z \neq 0$, $B_2 z = 0$. Since B has rank $n-r$, the kernel of the linear transformation corresponding to B is only the zero vector, and thus $Bz = (y_1, ..., y_r, 0, \ldots, 0) \neq 0$. This proves that the last columns of B^T being linearly dependent is equivalent to (2) which is equivalent to the first r columns in A being linearly dependent. $\square$

From the proof of this theorem 5 we also learn that the dual of a rank r matroid represented by a rank r matrix is represented by any rank $(n-r)$ matrix D which has a rowspace that spans the nullspace of A (and with same number of columns as A). The rowspace of this matrix D is the orthogonal space of the rowspace of A.

3 Oriented Matroids

In our our examples above of matroids, there is more information that could be of interest and recorded. For instance the sign of the coefficients in our linear dependencies, or looking at directed graphs instead of undirected graphs. This leads to another structure called *oriented matroids*. We will define and give examples of this structure and show how it is related to matroids.

3.1 Oriented Circuits and Covectors

In the linear dependencies which gave us a set of circuits in Example 1.1, if we note the *sign* of the coefficients, we could encode the information in the circuits like this:

$$\mathcal{C}_{\pm} = \left\{ \begin{matrix} + + - \\ c\ d\ e \end{matrix}, \begin{matrix} + + - \\ b\ c\ f \end{matrix}, \begin{matrix} + + - - \\ b\ e\ d\ f \end{matrix}, \begin{matrix} - - + \\ c\ d\ e \end{matrix}, \begin{matrix} - - + \\ b\ c\ f \end{matrix}, \begin{matrix} - - + + \\ b\ e\ d\ f \end{matrix} \right\} \quad (*)$$

These are signed sets and a given signed set X consists of a positive and a negative part $X = (X^+, X^-)$, where X^+ and X^- are two sets. We define the support of X, $\underline{X}$, to be $X^+ \cup X^-$. The signed sets of circuits and ground set E of vectors are a special case of an oriented matroid, which we will define now.

Definition 2 (**Circuit Axioms for Oriented Matroids**)**.** A collection $\mathcal{C}_O$ of signed subsets of a ground set E are the signed circuits of an oriented matroid if and only if the follow properties hold:
(CO1) $(\emptyset, \emptyset)$ is not in $\mathcal{C}_O$
(CO2) If C is in $\mathcal{C}_O$, then so is $-C$.
(CO3) If C_1,C_2 are in $\mathcal{C}_O$ and $\underline{C_1} \subset \underline{C_2}$, then $C_1 = \pm C_2$.

(CO4) If C_1, C_2 are in $\mathcal{C}_O$, $C_1 \neq -C_2$ and $x \in C_1^+ \cap C_2^-$, then there exists $C_3 \in \mathcal{C}_O$ such that

$$C_3^+ \subset (C_1^+ \cup C_2^+) - x$$
$$C_3^- \subset (C_1^- \cup C_2^-) - x$$

Notice that forgetting about signs, (CO1),(CO3) and (CO4) reduce to our circuit axioms for matroid, and thus every oriented matroid has an underlying matroid. Not every matroid can be oriented however [1].
Given a representable matroid on ground set E, do an assignation of signs as in ($*$) give us an oriented matroid? (CO1),(CO2),(CO3) are clear. The conditions in (CO4) would imply we have two linear equations equal to zero with a variable x in common but with different signs of the coefficients infront of that variable. Simply solving for x in one of these equations and then substituting it into the other would give us a new linear dependency among the vectors we want.
There is another useful notation of these signed sets, whereby we simply note each signed set's sign vector in $\{+,-,0\}^E$ where E is the ground set. Thus the signed set $(\{c,d\},\{e\})$ with $E=\{a,b,c,d,e,f\}$ would be given by $(00++-0)$. We will use this latter notation in the text now, and also when we refer to just circuits, we will mean signed circuits of an oriented matroid, unless stated otherwise.
Given a directed graph and an orientation on the cycles, each cycle will have a set of positive edges and a set of negative edges. Unsurprisingly, these signed cycles make up the signed circuits of an oriented matroid, and the proof of this is not much different from the one with ordinary matroids, but we will refer to Bjorner[1].

Example 3.1. The directed graph in Figure 6 is an example of a graphic oriented matroid.

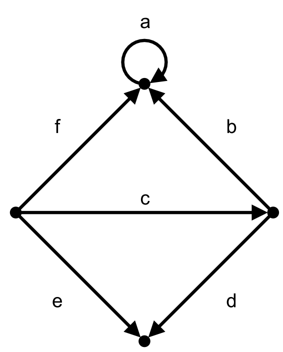


Figure 6: A Directed Graph and Oriented Matroid.

Before we go further we will define a few important operations on these signed sets:

$$X \circ Y = \begin{cases} X_e & X_e \neq 0 \\ Y_e & \text{otherwise} \end{cases}$$

is called the composition of two sign vectors and

$$S(X,Y) = \{e \in E : X_e = -Y_e \neq 0\}$$

is called the separation of two sign vectors.
For example, if our ground set $E = \{1,2,3,4,5,6,7\}$, $X = (+0-++-0)$ and $Y = (-++0+-0)$ then $X \circ Y = (++-++-0)$ and $S(X,Y) = \{1,3\}$. We will say that two sign vectors X and Y are orthogonal if either $\underline{X} \cap \underline{Y} = \emptyset$ or $S(X,Y) \neq \emptyset$.

As in matroid theory, in oriented matroids we have a notion of duality:

Proposition 14. *Let M be an oriented matroid on a ground set E with signed circuits $\mathcal{C}_O$. Then there is a way to assign signs to the cocircuits of underlying matroid $\underline{M}$ such that for this new set of signed subsets, $\mathcal{C}_O^*$, if $X \in \mathcal{C}_O^*$ then $X \perp Y$ for all $Y \in \mathcal{C}_O$. This collection of signed subsets $\mathcal{C}_O^*$ is then the signed circuits of a dual oriented matroid of M, denoted M^*. Also $(M^*)^* = M$.*

For proof of this, see Bjorner[1].

Now we will introduce a set of axioms for oriented matroids for which we have not presented a clear matroid analogue. These are the covector axioms. Yes, you guessed right: Covectors are the *vectors* of the dual oriented matroid, uniquely determined by the oriented matroid.
Theorem 6 (**Covector Axioms**). *A collection of signed subsets of E, call it $\mathcal{L}$, are the covectors of an oriented matroid if and only if the following is true of $\mathcal{L}$.*
(CV1) *The zero sign vector is in $\mathcal{L}$.*
(CV2) *$X \in \mathcal{L}$ implies $-X \in \mathcal{L}$.*
(CV3) *$X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$.*
(CV4) *$X, Y \in \mathcal{L}$ and $e \in S(X,Y)$ implies that there exists a signed set $Z \in \mathcal{L}$ such that $Z_e = 0$ and $Z_f = (X \circ Y)_f$ for $f \in E - S(X,Y)$.*

This equivalence theorem we will not prove here, but also refer to Bjorner[1]. Importantly, one can easily translate between the signed cocircuits and covectors. The signed cocircuits are the covectors of minimal support and the covectors are all possible compositions of the signed cocircuits, see [1].

4 Applications

Oriented matroids have several important applications in mathematics, and in the following section we shall try to give a motivation for why this is. We will look mainly at central and affine arrangements of hyperplanes and what is known as Oriented Matroid Programming, a generalization of linear programming.

4.1 Arrangements of hyperplanes

A central arrangement of hyperplanes in $\mathbb{R}^d$ is a set of hyperplanes $\mathcal{A} = \{H_1, ..., H_n\}$ that go through the origin. Each hyperplane divide $\mathbb{R}^d$ in half and the set of them partition the space. For all vectors x in $\mathbb{R}^d$ we can associate a sign vector in $\{+, -, 0\}^{\mathcal{A}}$ where each sign is dependent on which side of a particular hyperplane x is. A sign equaling 0 would mean vector x is on the plane. Let us call the set of all sign vectors L.
Let A be an $n \times d$ matrix with rows being normals of the hyperplanes. Then the sign of component j of $y = Ax$ will signify on which side of a the hyperplane with normal in the j'th row of A it is. The set of sign vectors of $\{Ax \,|\, x \in \mathbb{R}^d\}$ is then such a set L. Another way to represent this arrangement of hyperplanes is by intersecting the hyperplanes with the unit sphere S^{d-1} and look at the induced sphere system $\mathcal{A} \cap S^{d-1}$, see Figure 7 for an example of this.
Interpreting these sign vectors as signed subsets on a ground set
$\mathcal{A} = \{H_1, ..., H_n\}$ we will now prove that the sign vectors fulfill the covector axioms, that $L = \mathcal{L}$ of an oriented matroid.
Since $A * 0 = 0$, the zero sign vector is in L which proves (CV1). For (CV2), if the sign vector of Ax is X then the sign vector of $A(-x)$ will have sign vector $-X$. Let x and y have sign vectors X and Y respectively, given small enough δ, $A(x + \delta y) = Ax + \delta Ay$ will have sign vector $X \circ Y$. If $H_j \in S(X, Y)$, where X and Y are sign vectors corresponding to sign vectors Ax and Ay, the hyperplane H_j separates the two points x and y. Choose r such that $0 < r < 1$ and $z = x + r(y - x)$ is on this separating hyperplane. $A(x+r(y-x)) = A((1-r)x+ry) = (1-r)A(x)+rA(y)$. Since $0 < r < 1$, the sign vectors of $(1 - r)A(x)$ and $rA(y)$ will not have opposite signs on hyperplanes in $\mathcal{A} - S(X, Y)$, and thus the sign vector of z show that (CV4) is true.

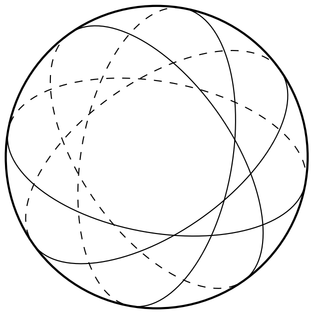


Figure 7: Sphere system in $\mathbb{R}^3$.

This means that every arrangement of hyperplanes gives rise to an oriented matroid, and significantly, the converse is almost true. We will not go into it here, but any simple (analogous to simple matroid) oriented matroid gives rise to a collection not of spheres (which implies an arrangement of hyperplanes), but of pseudospheres. Spheres that are topologically equivalent to a sphere, or rather, "wiggly" spheres, if you will, see Bjorner[1]. This seems to imply that a sphere system is a natural environment to work in, when studying oriented matroids, and is also easier to visualize.
For an arrangement of hyperplanes, and thus spheres, what we are often interested in is the geometric incidence relationship between the faces, faces being the regions into which the hyperplanes divide the space. For this end we can define a facial incidence relation, which for the sign vectors mean a relation defined thus: X is a face Y, denoted $(X \leq Y)$, if $X_i = Y_i$ for all $X_i \neq 0$. The poset, partially ordered set, L is called the combinatorial structure of the arrangement. So for example, in Figure 7, we have the big regions on the sphere. The edges of these big regions are faces of their corresponding big region, and similarly, each edge will have vertices as faces.
This notion of geometric incidence relation is abstracted for oriented matroids. Given our set of covectors, we say define $X \leq Y$ if $X_i \neq 0$ implies $X_i = Y_i$. This partial order coincides with the facial incidence relation in arrangements of hyperplanes [5], which implies the oriented matroid captures a lot of combinatorial information about an arrangement.

4.2 Affine arrangements

Oriented matroids also model general, or affine, arrangements very well. Now the affine structure itself is an oriented matroid, but what is often done if we have an affine arrangement $\mathcal{A} = \{H_1, ..., H_n\}$ in R^d, is to embed this arrangement into R^{d+1}, $(R^d, 1)$ more specifically. We will then get a central arrangement if we let $\mathcal{A}' = \{H_1', ..., H_n', H_g\}$ where each $H_e' = \text{span}(H_e)$ in R^{d+1} for H_e in the original affine arrangement, and $H_g = \{x \in R^{d+1} \mid x_{d+1} = 0\}$.

H_g plays the role of an infinity element. The sign vectors for our affine arrangement correspond to sign vectors in the halfspace with $X_g = +$. Then the partial order of sign vectors with $X_g = +$ will have the same incidence relationship as our affine arangement [1]. The point of this embedding is it is more convenient to work with central arrangements, and we encode the fact some of the faces in the affine arangement are infinite and other finite, with the infinity element H_g.

4.3 Oriented matroid programing

Oriented matroid programming, OMP, has both extended linear programming to a more general abstract setting and it has also led to further development of the simplex method [5]. We will show how the information recorded by oriented matroids can be used to solve linear optimization problems. Given a linear program in the form below, one can translate this problem to an oriented matroid and try to solve it in this setting instead.
Linear Program:

$$\begin{aligned}&\max \boldsymbol{c}^T\boldsymbol{x} - d\\ &A\boldsymbol{x} \leq \boldsymbol{b}\\ &\boldsymbol{x} \geq \boldsymbol{0}.\end{aligned}$$

So for example:

$$\begin{array}{lc} & \max f(x,y) = x + y + 2 \\ 1 & 2x - y \leq 5 \\ 2 & 2y - x \leq 3 \\ & x, y \geq 0. \end{array}$$

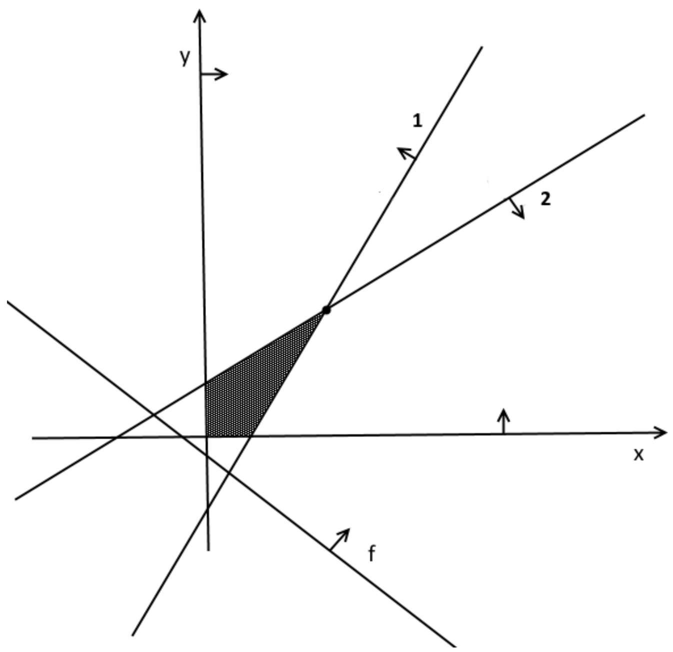


Figure 8: Linear Program.

What we want to do in the linear program above is maximize over a region which is defined by an affine arrangement of hyperplanes. This regions is shaded in Figure 8. As we showed in the previous section we can embed this arrangement into $(\mathbb{R}^2, 1)$, and we get a central arrangement of hyperplanes in $\mathbb{R}^3$ which can be expressed thus, and seen in Figure 9:

$$\begin{array}{cc} f & x + y + 2z \geq 0 \\ 1 & 2x - y - 5z \leq 0 \\ 2 & 2y - x - 3z \leq 0 \\ 3 & x \geq 0 \\ 4 & y \geq 0 \\ g & z \geq 0 \end{array}$$

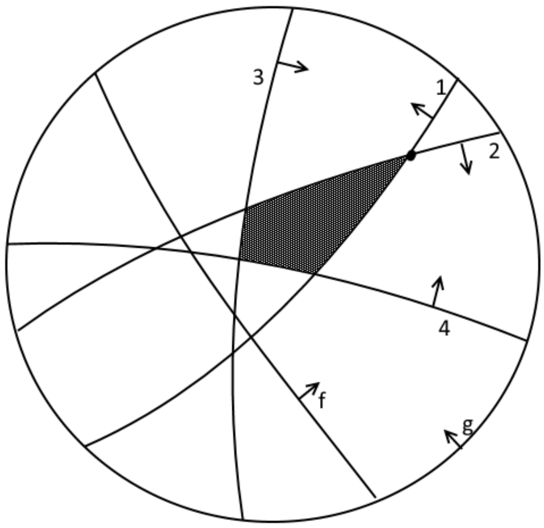


Figure 9: Upper hemisphere of sphere arrangement in $\mathbb{R}^3$.

It is perhaps not so strange that the maximum value will be at an extreme point. It can also be shown that if a vertex is not a maximum point, then there is an edge connected to the point that is an increasing direction. Moving along this edge will then either get us to another extreme point or go on to infinity, in which case the value over the region is unbounded. This fact that you go from vertex to vertex is what the simplex algorithm utilizes.
Now the vertices of this central arrangement correspond to cocircuits in the oriented matroid. Standing on a such a cocircuit Y, we can in the oriented matroid both deduce if it corresponds to a feasible vertex, that is a vertex in the feasible region (each element in the covectors ≥ 0), and also find, if it exists, a feasible increasing direction. A feasible increasing direction will be a circuit Z such that $Z_g = 0$ and $Z_f = +$ and $Y \circ Z$ is in the feasible region. In Figure 9 this corresponds to finding an edge from a vertex, in the feasible region, which ends on the positive side of f. In the oriented matroid we can then find an optimal cocircuit, a cocircuit without an increasing direction, which will correspond to an optimal vertex in our arrangement[1]. If there is such a vertex, that is.
This combinatorial structure which oriented matroids encode then extends the linear programming concept, since not all matroids will correspond to a linear structures, where linear is in the sense that we can find an arrangement of hyperplanes with this oriented matroid structure. This is all well and good, but work in OMP has also led to many new discoveries of algorithms

for linear programming, perhaps most famously Bland's smallest index rule [5].

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