SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Knot theory

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Mattias Selin

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

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Mattias Selin

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Handledare: Alexander Berglund

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Abstract

A mathematical knot can be thought of as a piece of string with the ends tied together. This piece of string might be tangled in such a way that it cannot be untangled into a circular piece of string without cutting it up. This is the type of object studied in knot theory.

The goal of this bachelor's thesis is to investigate some of the famous knot polynomials that exists. In particular, the Jones polynomial and the HOMFLY polynomial. We begin by giving an introduction to the field of knot theory aimed at giving the reader the necessary definitions and concepts to fully grasp the rest of the thesis. Some basic well known facts are also presented and proved. The reader is expected to be familiar with basic euclidean topology and elementary abstract algebra. However, the unacquainted reader should be able to understand the lion's share of this thesis. The thesis contains many visual examples to aid the readers intuition about the concepts presented.

In Section 2 we discuss the concept of link invariants and give a short overview. We take a closer look at the Jones Polynomial.

In Section 3 we give a short introduction to the HOMFLY polynomial followed by an algorithm for calculating it. The algorithm is presented as pseudocode and is based on the written explanation provided in *An Introduction to Knot Theory* by W.B. Raymond Lickorish. We end Section 3 by investigating a possible generalization of the HOMFLY polynomial to elements of a noncommutative ring. We present some new results giving a necessary condition for the noncommutative ring and a partial answer to the sufficiency of the condition.

Acknowledgements

I would like to thank my supervisor Alexander Berglund for his support and insightful advice.

Contents

1 Introduction

In this section we define the essential concepts that we will need in the sections to come. There are many visual examples of knots and links to give the reader an intuition of what the strict definitions are trying to describe. At the end of the section we touch briefly on the concept of prime knots, a knot theoretical analogue of prime numbers.

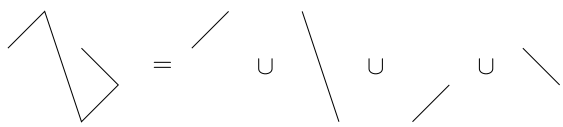
1.1 Links and knots

Before we can indulge in the more interesting parts of knot theory, some basic definitions are required. For example, *what* exactly is a knot? Knots are in fact a special case of the more general concept of a link.

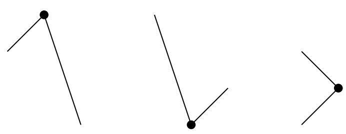
Definition 1.1.1. Let $f : [0, 1] \to \mathbb{R}^3$ be any continuous function. Let $C = \operatorname{Im} f \subseteq \mathbb{R}^3$. Then we will call such a set, C, a **curve**.

Definition 1.1.2. A curve C is **piecewise linear** if there exists a finite partition of the curve $C = L_1 \cup \ldots \cup L_n$ such that L_i is a line segment for $i = 1 \ldots n$ and $L_i \cap L_{i+1} = \{x_i\} \quad i = 1 \ldots n - 1$.

Example 1.1.3.



And the intersection of each consecutive pair of line segments consists of 1 point:

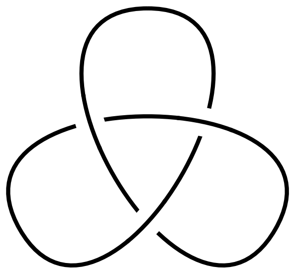


Definition 1.1.4. A **link of n components** is a subset of $\mathbb{R}^3$ that consists of n disjoint simple piecewise linear closed curves[1].

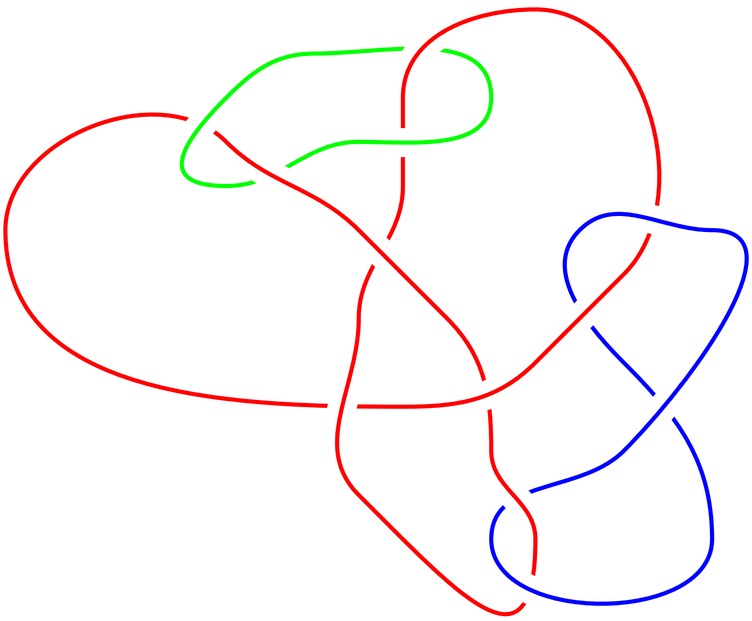
A simple closed curve is a subset of $\mathbb{R}^3$ that is homeomorphic to S^1. The reason links are defined in this way, as piecewise linear curves, is traditionally to simplify some calculations and to exclude so called wild knots. As long as the curves we work with are well-behaved, and can be approximated by piecewise linear curves, smooth curves pose no problems. We will draw links with smooth curves.

Definition 1.1.5. A **knot** is a link of 1 component.

Example 1.1.6. The trefoil knot[1].



Example 1.1.7. A link of 3 components.



An intuitive way to think of a link is as a number of possibly interlinked circular pieces of string. If we do not cut the pieces of string up we can move

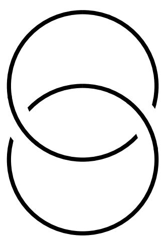
[1]Sometimes called the left-trefoil knot. We shall call it *the* trefoil knot. The right-trefoil knot shall be referred to as *the mirror image of the trefoil knot*

them around in any way as long as they do not pass through each other. Doing this kind of movement does not really change the link. We need a precise mathematical definition to describe this.

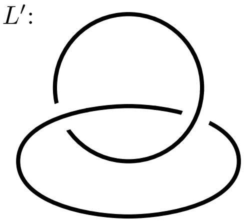
Definition 1.1.8. Two links L and M are **equivalent** if there exists an orientation-preserving homeomorphism $a : \mathbb{R}^3 \to \mathbb{R}^3$ such that $a(L) = M$.[2]

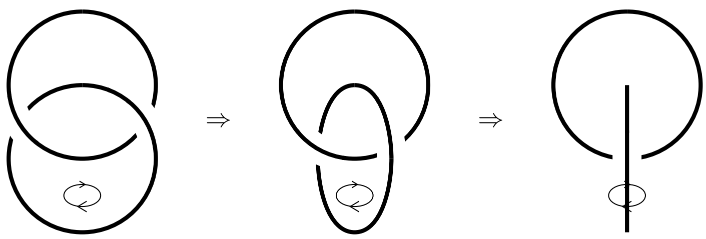
Example 1.1.9. Consider the links L and L' depicted below. Consider rotating the lower component of L as seen below, followed by a slight stretching of it. Result looks like L'. Since we have only slid around the components of L, without letting them pass through each other, there exists an orientation-preserving self-homeomorphism of $\mathbb{R}^3$ that produces the same result. Thus L and L' are equivalent links.

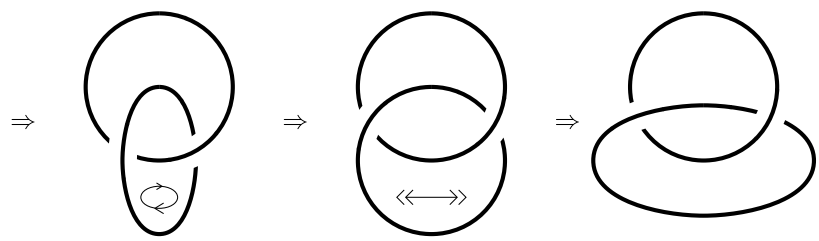
L:



L':







We are only interested in the equivalence classes generated by this equivalence relation[3] when we talk about knots and links. For the rest of this thesis we shall say "a knot" or "a link" when we in fact mean the equivalence class of such. We will write $K = L$ to indicate that links K and L are equivalent.

[2] $a(L)$ denotes the image of L under a.

[3] A proof of that this is an equivalence relation is left as an exercise for the reader.

1.2 Link diagrams

In Examples 1.1.6,1.1.7 and 1.1.9 we represented different links with a kind of diagram. This kind of diagram is called a *link diagram*, or a *knot diagram* if the link represented is in fact a knot.

In the definition below we mention an open neighbourhood. We shall assume that euclidean spaces $\mathbb{R}^n$ are endowed with the standard topology and any subspace has the induced subspace topology.

Definition 1.2.1. Let L be a link. A **link diagram** D **of** L is a pair $(p(L), s)$ where $p(L)$ is the image of a linear map $p : \mathbb{R}^3 \to \mathbb{R}^2$ satisfying the conditions given below and s is a map satisfying the conditions given below.

- $\forall x \in p(L)$ exactly one of the following holds.
  1. $p^{-1}(x) = \{y\}$
  2. $p^{-1}(x) = \{y_1, y_2\}$ and $\exists$ an open neighbourhood[4] N of x such that $\forall y \in N \setminus \{x\}$, $p^{-1}(y) = \{y'\}$. If this is the case we call x a *crossing point*.
- s : {crossing points of $p(L)$} $\to p^{-1}(x)$ assigns to each crossing point x the point in $p^{-1}(x)$ with greatest z-coordinate in the standard basis.

We call a crossing point x together with $s(x)$ a *crossing* of the diagram.

Proposition 1.2.2. *Link diagrams have a finite number of crossings.*

Proof. Let L be a link of n components. Then by definition each component of L is a simple closed piecewise linear curve. Let $D = (p(L), s)$ be a link diagram of L. Let A and B in the claims below be two different line segments in the piecewise linear components of L.

> *Claim.* Let A and B be two disjoint line segments in a partition of the components of L. Then $p(A) \cap p(B)$ contains at most one point.
>
> *Proof.* Assume that there exists two points $x, y \in p(A) \cap p(B)$ such that $x \neq y$. Since $x \in p(A) \cap p(B)$ and the line segments A and B are disjoint, x must be a crossing point. Similarly, y must also be a crossing point. Since x and y are in $p(A)$ there exists $x_A, y_A \in A$ such that
>
> $$p(x_A) = x \text{ and } p(y_A) = y$$
>
> A is a line segment, and thus convex, so $tx_A + (1-t)y_A \in A$ for all $t \in [0, 1]$.
>
> $$p \text{ linear} \implies$$
>
> $$p(tx_A + (1-t)y_A) = tp(x_A) + (1-t)p(y_A) = tx + (1-t)y \in p(A)$$

[4] Open as a subspace of $p(L)$

Similarly, since x and y are in $p(B)$ there exists $x_B, y_B \in B$ such that

$$p(x_B) = x \text{ and } p(y_B) = y$$

$$\implies p(tx_B + (1-t)y_B) = tp(x_B) + (1-t)p(y_B) = tx + (1-t)y \in p(B)$$

Thus, $tx + (1-t)y \in p(A) \cap p(B)$ for all $t \in [0,1]$. This implies that $p^{-1}(tx + (1-t)y) = \{tx_A + (1-t)y_A, tx_B + (1-t)y_B\}$. So any point $tx + (1-t)y,\ t \in [0,1]$ is a crossing point.
y is a crossing point, so by definition there exists an open neighbourhood N of y such that every point in $N \setminus \{y\}$ is not a crossing point. Since N is open there exists an open neighbourhood of $N' \subseteq \mathbb{R}^2$ such that $N = N' \cap p(L)$. And since N' is open and $y \in N'$, there exist an open ball $B_\epsilon(y) \subseteq N'$ with ϵ such that $|x-y| > \epsilon > 0$. Where $|x-y|$ denotes the euclidean distance between the points x and y. Note that $|x-y| > 0$ since $x \neq y$ by assumption.
Let $t_\epsilon = \frac{\epsilon}{2|x-y|}$ and let $z = t_\epsilon x + (1-t_\epsilon)y$. Then $t_\epsilon > 0$ and

$$t_\epsilon = \frac{\epsilon}{2|x-y|} < \frac{|x-y|}{2|x-y|} = \frac{1}{2} < 1$$

Therefore $z \in p(A) \cap p(B)$ and z is a crossing point. However,

$$\begin{aligned}
|z-y| &= |t_\epsilon x + (1-t_\epsilon)y - y| \\
&= |t_\epsilon x - t_\epsilon y| \\
&= |t_\epsilon(x-y)| \\
&= t_\epsilon |x-y| \\
&= \frac{\epsilon}{2|x-y|}|x-y| \\
&= \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

So $z \in B_\epsilon(y) \subseteq N$. Note that $z \neq y$ since $|z-y| = \epsilon/2 > 0$. This implies that $z \in N \setminus \{y\}$ which is a contradiction since z is a crossing point. The claim is proven.

We shall now consider the case where two line segments of L are not disjoint. By the definition of piecewise linear we see that two line segments are not disjoint if and only if their intersection contains exactly one point.

Claim. Let A and B be two line segments of L such that there exists a point $x \in A \cap B$. Then $p(A) \cap p(B) = \{p(x)\}$.

Proof. Assume there exists $y \in p(A) \cap p(B)$ such that $y \neq p(x)$. Then there exists $y_A \in A$ and $y_B \in B$ such that $p(y_A) = p(y_B) = y$. Note that $y_A \neq y_B$ since

$$y_A = y_B \implies y_A, y_B \in A \cap B \iff y_A = y_B = x \implies y = p(y_A) = p(x)$$

which cannot be the case since by assumption $y \neq p(x)$. Therefore $y_A \neq y_B$. This implies that $p^{-1}(y) = \{y_A, y_B\}$ and thus y is a crossing point. A is a line segment, $x, y_A \in A$ and p linear. This implies that

$$\begin{aligned} p(tx + (1-t)y_A) &= tp(x) + (1-t)p(y_A) \\ &= tp(x) + (1-t)y \in p(A), \quad \forall t \in [0,1] \end{aligned}$$

Similarly, B is a line segment, $x, y_B \in B$ and p linear implies that

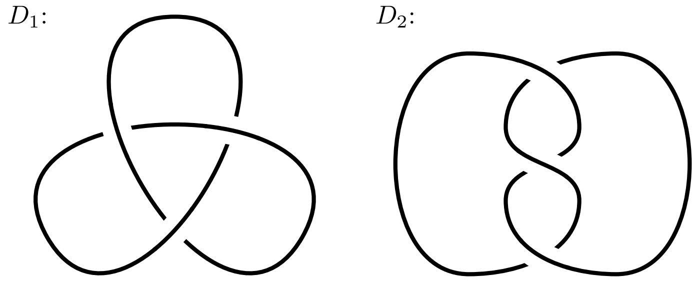
$$\begin{aligned} p(tx + (1-t)y_B) &= tp(x) + (1-t)p(y_B) \\ &= tp(x) + (1-t)y \in p(B), \quad \forall t \in [0,1] \end{aligned}$$

Thus, $tp(x) + (1-t)y \in p(A) \cap p(B), \quad \forall t \in [0,1]$. By the same argument as in the previous claim, every open neighbourhood N of the crossing point y will contain a crossing point $tp(x) + (1-t)y$ not equal to y. This leads to a contradiction. We conclude that $p(A) \cap p(B) = \{p(x)\}$ and our claim is proven.

The claims prove that for any pair of line segments (A, B) of L the intersection of the images $p(A) \cap p(B)$ contains at most one point. By the definition of a link, there exists a finite partition of L as a union of line segments. And since p is linear the image $p(L)$ can be written as a union of finitely many images of line segments. Since there are finitely many pairs of distinct line segments (A, B) the claims imply that there are finitely many crossings. □

One link might have multiple link diagrams. And for two link diagrams it might not be immediately clear that they represent the same link.

Example 1.2.3. Two diagrams, D_1 and D_2, both representing the trefoil knot.



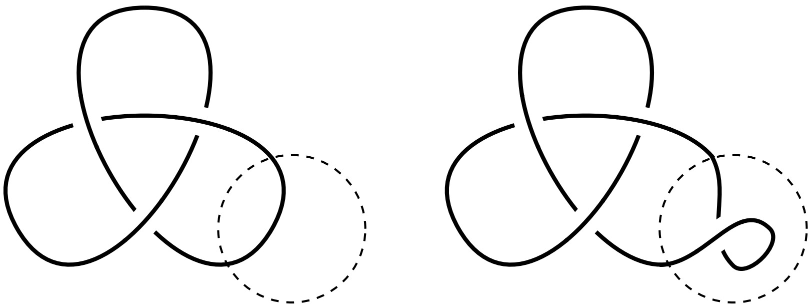
A part looking like $\times$ in our link diagrams represents a crossing x. The solid line represents the image of a neighbourhood around $s(x)$ and is called an *overpass*. The broken line represents the image of a neighbourhood around $p^{-1}(x) \setminus s(x)$ and is called an *underpass*.

Example 1.2.4. Both of the diagrams D_1 and D_2 in Example 1.2.3 have three crossings.

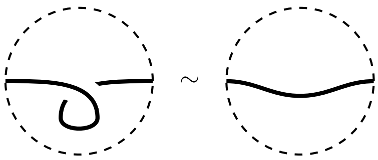
We will sometimes talk about a *strand* of a link. This is usually only to explain an idea and thus a strict definition is unnecessary.

A very common practice in knot theory is to consider *local changes* within a link diagram. That is, we look at two diagrams that differ only within a small simply connected region.

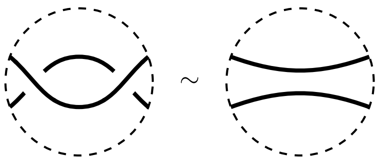
Example 1.2.5. A local change. It should be clear to the reader that these knots are equivalent.



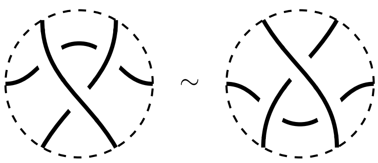
The local change in Example 1.2.5 is a special type of local change. It is what is called a *Reidemeister move*. These moves are traditionally categorized as three types. Below we use the symbol $\sim$ to indicate that two diagrams are equivalent if they only differ by the local change indicated.



Type I Moves of type I adds or removes a simple loop in any direction.



Type II Moves of type II moves a strand over/under another. The reverse of such moves are also considered type II.



Type III Moves of type III moves a strand from one side of a crossing to the other side. It moves the strand either over the crossing or under it.

It is known that the link diagrams of any two equivalent links are related by a finite sequence of Reidemeister moves and orientation-preserving self-homeomorphisms of the plane $\mathbb{R}^2$. For a complete proof see [4, Theorem 4.1.1, p. 50].

We will sometimes talk about "disjoint" components of a link, but by definition the components of a link are already disjoint as sets. We formalize what we mean by disjoint components.

Definition 1.2.6. Let L be a link. Let A, B be a partition of the components of L. A and B are called **disjoint** if there exists an embedding $e : D^3 \to \mathbb{R}^3$ such that $A \subset \operatorname{Im} e$ and $B \subset \mathbb{R}^3 \setminus \operatorname{Im} e$ where $D^3 = \{x \in \mathbb{R}^3 \mid |x| \leqslant 1\}$.

If a link L can be partitioned into two disjoint sets of components A and B we say that L is the *disjoint union* of A and B where A and B are regarded as links in their own right. We denote this by

$$L = A \sqcup B.$$

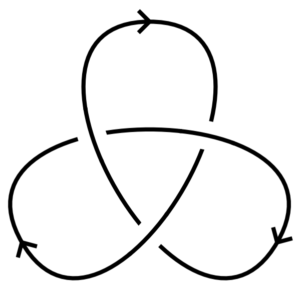
Intuitively, one may think of parts of a link as disjoint if they are not "linked" with each other.

1.3 Oriented links

Up until now we have only talked about links where the strands have no defined direction. Just as we can have undirected graphs and directed graphs, we can define an *oriented link*. This is an important concept so let us make it precise.

Definition 1.3.1. An **oriented link** L is a link where each component has been assigned an orientation.

Example 1.3.2. An orientation of the trefoil knot.



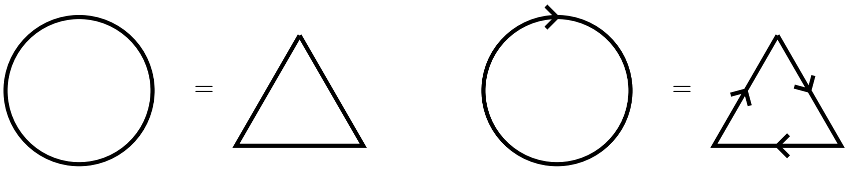
Equivalence between oriented links is defined analogously as for unoriented links[5]. Since we have two choices of orientation for each component of a link, there are a total of 2^n possible choices of orientations for a link of n components. In the case of the trefoil knot both orientations happen to give equivalent oriented knots, but this is not generally the case.

[5] Just replace "links" with "oriented links" Definition 1.1.8

1.4 Knot sum

One important knot is the one that is not knotted.

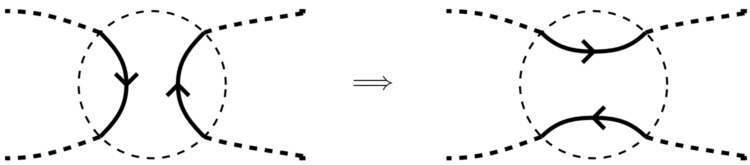
Definition 1.4.1. The knot that is equivalent to a triangle is called **the unknot**.



Unoriented unknot Oriented unknot

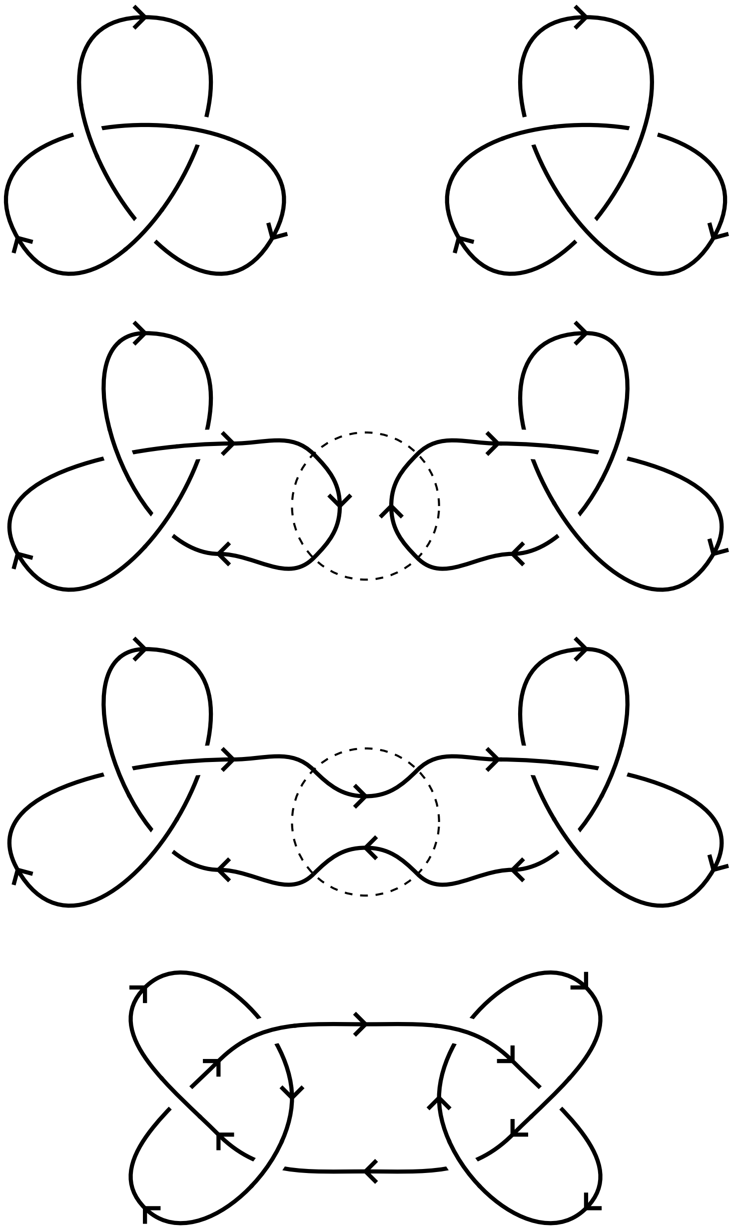
For two oriented knots there is a natural way to join them. The knot formed by joining knots K and L is denoted $K + L$ and called the *knot sum* of K and L.

Definition 1.4.2. Let K and L be oriented knots. Embed them in $\mathbb{R}^3$ in such that a way that there exists a link diagram where their images are disjoint. Extend a strand from each knot into a previously empty ball embedded in $\mathbb{R}^3$ and perform the local change as indicated in the figure below.



The result is the **knot sum** of K and L.

Example 1.4.3. The knot sum of the trefoil knot and its mirror image[6].

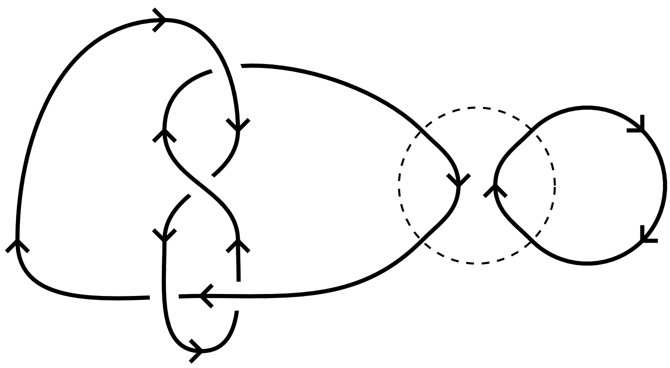


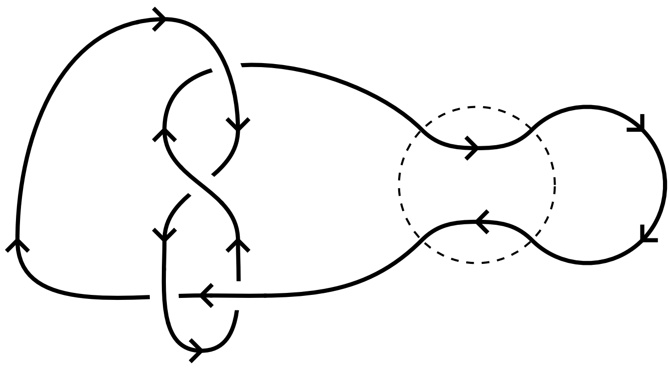
It can be realized with a little bit of careful thought that the knot sum is

[6]Some links are equivalent to their mirror image. The trefoil is not.

well-defined. By well-defined we mean that the knot sum $K + L$ of K and L is equivalent to any other knot sum $K' + L'$ where K' and L' are equivalent to K and L. It is also easily seen that the knot sum is commutative and associative[1, p. 6]. It is clear that the unknot acts as a neutral element.

Example 1.4.4. Let K be an oriented figure-eight knot and let $\circlearrowright$ denote the oriented unknot. Then $K + \circlearrowright = K$.





Some natural questions arise. Which knots can be written as the sum of two other knots? Is there always such a decomposition for a given knot? Given that we have a neutral element $\circlearrowright$, which knots have an additive inverse? To answer these questions we need to consider the concept of *prime knots*.

1.5 Prime knots

Let K be an unoriented knot. There are two possible orientations we can give K. Call the two associated oriented knots K^+ and K^-. If K^+ can be decomposed as $K^+ = K_1^+ + K_2^+$ then clearly we can decompose $K^- = K_1^- + K_2^-$ where K_1^- and K_2^- are opposite orientations of K_1^+ and K_2^+ of the same two underlying unoriented knots K_1 and K_2. So the decomposition of an unoriented knot into two unoriented knots is well-defined. We write $K = K_1 + K_2$ when K,K_1 and K_2 can be given orientations such that the corresponding knot sum expression holds.

A prime knot is *not* the sum of two other knots. But of course, in the same way that the prime number 7 can be written as $7 = 7 \cdot 1 = 7 \cdot 1 \cdot \ldots \cdot 1$, any knot K can be written as $K = K + \bigcirc = K + \bigcirc + \ldots + \bigcirc$ where $\bigcirc$ is the unoriented unknot.

Definition 1.5.1. Let $K \neq \bigcirc$ be a knot. Then K is a **prime knot** if

$$K = K_1 + K_2 \implies K_1 = \bigcirc \text{ or } K_2 = \bigcirc$$

So a knot is prime if it is not the unknot and the only decomposition of the knot is the trivial decomposition into the original knot and the unknot. Now follows a few basic results about prime knots.

Theorem 1.5.2. *Let $K \neq \bigcirc$ be a knot. Then there is a decomposition*

$$K = K_1 + \ldots + K_m$$

that is unique up to the ordering of the summands where K_i is a prime knot $\forall i$.

Proof. See [1, Theorem 2.12, p. 21]. □

It follows directly from this theorem that no knot $K \neq \bigcirc$ has an additive inverse, because if it did, then any decomposition of a knot would not be unique. We formulate this a s a proposition for oriented knots in Proposition 1.5.3 below. We shall use the term "prime knot" for oriented knots as well.

Proposition 1.5.3. *The only oriented knot that has an additive inverse is the neutral element $\circlearrowright$.*

Proof. Let K and L be oriented knots such that $K + L = \circlearrowright$. If $K \neq \circlearrowright$ then by Theorem 1.5.2 K can be decomposed uniquely as

$$K = K_1 + \ldots + K_m$$

where K_i is a prime knot $\forall i = 1, \ldots, m$. If $L = \circlearrowright$ then $\circlearrowright = K + L = K + \circlearrowright = K$. But $K \neq \circlearrowright$. So $K \neq \circlearrowright \implies L \neq \circlearrowright$. Therefore L can be decomposed in the same manner.

$$L = L_1 + \ldots + L_n$$

Let M be any prime knot. Then since the knot sum is associative

$$M = M + \circlearrowright = M + K + L = M + K_1 + \ldots + K_m + L_1 + \ldots + L_n.$$

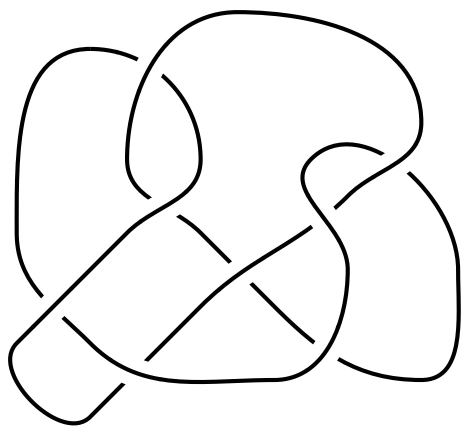
So the knot $M + K + L \neq \circlearrowright$ has a decomposition into the sum of one prime knot, M, and a decomposition into a sum of $m + n + 1$ prime knots. This is a contradiction since Theorem 1.5.2 states that the decomposition of a $M+K+L$ is unique up to ordering of the summands. Hence we conclude that our assumption $K \neq \circlearrowright$ was false and indeed, it must be the case that $K = \circlearrowright$. A symmetric argument for L shows that also $L = \circlearrowright$. □

This means that while the oriented knots do not form a group when equipped with the knot sum operation, they do form what is known as a *monoid*. It can be noted from the example above the this monoid is a zerosumfree monoid. Other names for zerosumfree monoid include conical, centerless or positive monoid.

2 Link invariants

Given two knots, K and L, it is not always easy to see if they are equivalent or not. In Example 1.2.3 we saw two representations of the trefoil knot given as knot diagrams. Early attempts at tabulating all knots represented by knot diagrams with a limited number of crossing famously included two diagrams that actually represented the same knot. It took at least 89 years until this error would be corrected by Perko in 1974[2]. It may not even be obvious that for a knot K, $K \neq \bigcirc$ or $K = \bigcirc$.

Example 2.0.1. A twisted version of the unknot.



Definition 2.0.2. Let T be any set. A function $f : \{Links\} \to T$ is called a **link invariant** if for any two links L_1 and L_2,

$$L_1 \text{ equivalent to } L_2 \implies f(L_1) = f(L_2)$$

If the domain of the function f is instead $\{Knots\}$ we call f a **knot invariant**.

Note that any link invariant induces a knot invariant by a restriction of the domain. An invariant for oriented links is called an oriented link invariant.

Link invariants tell us something "essential" about a knot. The use of the word essential here is motivated by the fact that if we change a link, without letting strands pass through each other, the value of the invariant remains. Only when we let strands pass through each other can we get a different value. Notice that Definition 2.0.2 does not require f to be injective. Hence, if K and

L are links for which $f(K) = f(L)$, we can not conclude that they equal. Link invariants generally only helps us *distinguish* links by identifying an essential property that a pair of links do not share.

A description of a link can be given in many ways, but it is often possible to translate this description into a link diagram of the link. It is therefore sensible to try to find link invariants that can be calculated from any link diagram of a link.

One family of link invariants are the "minimal"-type invariants. These are invariants whose value of a knot is the minimal value over all representations. We provide a few examples.

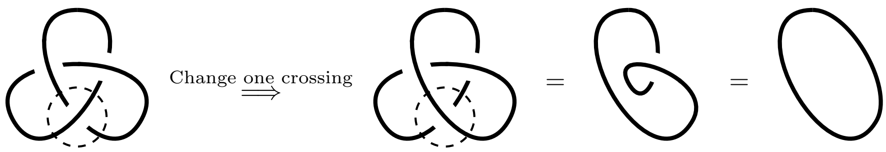
Example 2.0.3. The crossing number of a link is the minimal number of crossings a diagram of the link can have. Let *Crossing* denote this invariant.

$$Crossing(\bigcirc) = 0$$

$$Crossing(\text{trefoil knot}) = 3$$

This shows that the trefoil knot is actually knotted, i.e not equivalent to the unknot.

Example 2.0.4. The unknotting number is a knot invariant. The unknotting number of a knot K is the minimal number of times one has to change a crossing of a knot diagram of K from $\times$ to $\times$ so that the knot of the resulting knot diagram is equivalent to the unknot. The unknotting number of the trefoil knot is 1.



The interested reader is encouraged to read about the bridge number and the stick number in the literature.

All these "minimal"-type invariants share a common drawback. Since they are the minimum over an oftentimes infinite set, it can be difficult to calculate their value. For example, in Example 2.0.3 we stated that the crossing number of the trefoil knot is 3, but since we cannot possibly check every knot diagram of the trefoil knot we cannot in a straightforward manner verify that this value is the minimum. We need to come up with some innovative proof.

It would be preferable if we could calculate the value of a knot invariant at a knot K from *any* knot diagram of K, and always get the same value. This type of knot invariant is what will be considered in the following subsections.

2.1 The Jones Polynomial

The Jones polynomial is an oriented link invariant that assigns a Laurent polynomial with coefficients in $\mathbb{Z}$ to each oriented link. By the restriction of the domain to knots, the Jones polynomial is also a knot invariant. Such invariants are often called *knot polynomials*.

Definition 2.1.1.
Let $\mathbf{D}$ denote the set of all unoriented link diagrams.

Let $\vec{\mathbf{D}}$ denote the set of all oriented link diagrams.

Many link invariants are presented as functions from $\mathbf{D}$ or $\vec{\mathbf{D}}$ to some set. As long as the function gives the same value for any link diagram of equivalent knots, such a function can be extended to a function that is an actual link invariant.

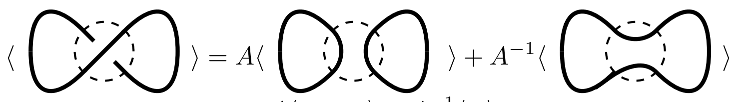
To define the Jones polynomial we need to present the *Kauffman bracket*. Note however that the Kauffman bracket itself is *not* a link invariant.

Definition 2.1.2. The **Kauffman bracket** is the function $\langle \cdot \rangle : \mathbf{D} \to \mathbb{Z}[A^{-1}, A]$ defined recursively by

- $\langle \bigcirc \rangle = 1$
- $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$
- $\langle \times \rangle = A\langle)\,(\rangle + A^{-1}\langle \asymp \rangle$

The Kauffman bracket does not necessarily assign the same value to different link diagrams of equivalent knots. The $\bigcirc$ in the definition needs to be the diagram of a knot with zero crossings. The equation $\langle \times \rangle = A\langle)\,(\rangle + A^{-1}\langle \asymp \rangle$ should be read as $\langle D_1 \rangle = A\langle D_2 \rangle + A^{-1}\langle D_3 \rangle$ where D_1, D_2 and D_3 are link diagrams differing only by a local change where D_1 has a part similar $\times$, D_2 has a part similar to $)\,($ and D_3 has a part similar to $\asymp$.

Example 2.1.3.



$$
\begin{aligned}
&= A\langle \bigcirc \sqcup \bigcirc \rangle + A^{-1}\langle \bigcirc \rangle \\
&= A((-A^{-2} - A^{2})\langle \bigcirc \rangle) + A^{-1}\langle \bigcirc \rangle \\
&= (-A^{-1} - A^{3} + A^{-1})\langle \bigcirc \rangle \\
&= (-A^{3})\langle \bigcirc \rangle = -A^{3}
\end{aligned}
$$

Since the Kauffman bracket is defined recursively it is not certain that it is well-defined or even defined for all unoriented link diagrams. From the third equation we see that we can calculate the value of the Kauffman bracket for a link with n crossings by calculating the value for two links with $n-1$ crossings.

This means that if we can show that we can calculate a base case we can by induction calculate the Kauffman bracket of any link diagram with a finite number of crossings. From Proposition 1.2.2 we know that any link diagram has a finite number of crossings. Now consider the base case where we have a link diagram with zero crossings. This must clearly be the disjoint union of a number of unknots. If we in this case have $m > 1$ unknots we use the second equation to get a link diagram of $m-1$ disjoint unknots and recursively calculate the value of the Kauffman brackets. We see that we will always reach the case where all we have left is the unknot, $\bigcirc$, from where we by the first equation have an explicit value. Thus the Kauffman bracket is always defined for a link diagram.

Proposition 2.1.4. *The Kauffman bracket is well-defined.*

Before we prove the proposition we introduce a convenient piece of notation. Just like $\langle ? \rangle$ denotes the value of the Kauffman polynomial for a link diagram that differs only locally by ? from the other link diagrams in the same equation, we will use the notation $\langle ?, ? \rangle$ to denote a the value of a link diagram that differs locally at two places from the other link diagrams in the same equation. To make the equations more readable and to distinguish `)(` from parentheses we will circle the crossing symbols in the following proof.

Proof. Let $D \in \mathbf{D}$ be a link diagram with 0 crossings. We then have, by applying the second equation of Definition 2.1.2,

$$\langle D \rangle = \begin{cases} (-A^{-2} - A^2)^{m-1}, & \text{if } m > 1 \\ 1, & \text{if } m = 1 \end{cases}$$

where m is the number of components of D. Thus, the Kauffman bracket is well-defined for all link diagrams with 0 crossings.

Let $D \in \mathbf{D}$ be a link diagram with 1 crossing. By the third equation of Definition 2.1.2 the value of $\langle D \rangle$ is uniquely defined by the values of two link diagrams with 0 crossings. This implies that the Kauffman bracket is well-defined for all link diagrams with 1 crossing.

Let $D \in \mathbf{D}$ be a link diagram with $n \geqslant 2$ crossings. To show that the value of $\langle D \rangle$ is well-defined we need to show that the order in which we "remove" the crossings of D does not affect the calculated value. It is enough to check that the same value is acquired when switching two adjacent "crossing-removals". Since D has at least 2 crossings we may write $\langle D \rangle = \langle \otimes, \otimes \rangle$ where we shall refer to these indicated crossings as *the first* and *the second* crossing. Applying the third equation of Definition 2.1.2 to the first crossing yields

$$\langle \otimes, \otimes \rangle = A \langle \textcircled{)(}, \otimes \rangle + A^{-1} \langle \textcircled{$\asymp$}, \otimes \rangle \tag{1}$$

We now apply the equation to the second crossing of the new diagrams.

$$\langle \textcircled{)(}, \otimes \rangle = A \langle \textcircled{)(}, \textcircled{)(} \rangle + A^{-1} \langle \textcircled{)(}, \textcircled{$\asymp$} \rangle$$

$$\langle \asymp, \otimes \rangle = A\langle \asymp,)(\rangle + A^{-1}\langle \asymp, \asymp \rangle$$

Inserting this into equation 1 we get

$$\begin{aligned}
\langle \otimes, \otimes \rangle &= A\langle)(, \otimes \rangle + A^{-1}\langle \asymp, \otimes \rangle \\
&= A(A\langle)(,)(\rangle + A^{-1}\langle)(, \asymp \rangle) + A^{-1}(A\langle \asymp,)(\rangle + A^{-1}\langle \asymp, \asymp \rangle) \\
&= A^2\langle)(,)(\rangle + \langle)(, \asymp \rangle + \langle \asymp,)(\rangle + A^{-2}\langle \asymp, \asymp \rangle \\
&= /\text{Rearrange the terms}/ \\
&= A^2\langle)(,)(\rangle + A^{-2}\langle \asymp, \asymp \rangle + \langle)(, \asymp \rangle + \langle \asymp,)(\rangle \qquad (2)
\end{aligned}$$

If we instead begin by applying the equation to the second crossing we get,

$$\langle \otimes, \otimes \rangle = A\langle \otimes,)(\rangle + A^{-1}\langle \otimes, \asymp \rangle \qquad (3)$$

Now conversely apply the equation to the first crossing of the new diagrams.

$$\langle \otimes,)(\rangle = A\langle)(,)(\rangle + A^{-1}\langle \asymp,)(\rangle$$

$$\langle \otimes, \asymp \rangle = A\langle)(, \asymp \rangle + A^{-1}\langle \asymp, \asymp \rangle$$

Inserting this into equation 3 we get

$$\begin{aligned}
\langle \otimes, \otimes \rangle &= A\langle \otimes,)(\rangle + A^{-1}\langle \otimes, \asymp \rangle \\
&= A(A\langle)(,)(\rangle + A^{-1}\langle \asymp,)(\rangle) + A^{-1}(A\langle)(, \asymp \rangle + A^{-1}\langle \asymp, \asymp \rangle) \\
&= A^2\langle)(,)(\rangle + \langle \asymp,)(\rangle + \langle)(, \asymp \rangle + A^{-2}\langle \asymp, \asymp \rangle \\
&= /\text{Rearrange the terms}/ \\
&= A^2\langle)(,)(\rangle + A^{-2}\langle \asymp, \asymp \rangle + \langle)(, \asymp \rangle + \langle \asymp,)(\rangle \qquad (4)
\end{aligned}$$

Equation 2 shows the value of $\langle D \rangle$ calculated by removing the first crossing followed by the second. Equation 4 shows the value of $\langle D \rangle$ calculated by removing the second crossing followed by the first. Since they are equal we can conclude that the Kauffman bracket of a link diagram with $n \geqslant 2$ crossings is uniquely defined by an expression of 4 associated link diagrams with $n-2$ crossings each. By induction over the number of crossings we get that the Kauffman bracket is well-defined for all link diagrams. $\square$

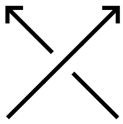
For a function from $\mathbf{D}$ or $\overrightarrow{\mathbf{D}}$ to be a link invariant, it is clear that it has to be invariant under orientation-preserving self-homeomorphisms of the plane as well as invariant under the Reidemeister moves. From Example 2.1.3 we see that the Kauffman bracket is not invariant under a Reidemeister Type I move. It is however invariant under orientation-preserving self-homeomorphisms of the plane and under Reidemeister Type II and Type III moves[1]. The following is a lemma from [1].

Lemma 2.1.5. *If a link diagram is changed by a Reidemeister Type I move, the value of the Kauffman bracket changes in the following way*

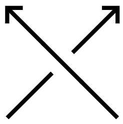
$$\langle \text{-}\!\circ\!\text{-} \rangle = -A^3\langle \text{—} \rangle, \quad \langle \text{-}\!\circ\!\text{-} \rangle = -A^{-3}\langle \text{—} \rangle$$

Proof. See [1, Lemma 3.2, p. 24]. $\square$

Definition 2.1.6. Let x be a crossing of an oriented link diagram. By an orientation-preserving self-homeomorphism of $\mathbb{R}^2$ a neighbourhood of x either looks like



or



$sgn(x) = 1$ $\quad\quad$ $sgn(x) = -1$

We define the **sign** of a crossing as the value of the function $sgn : \{Crossings\} \to \{-1, 1\}$ where

$$sgn(\times) = 1$$

$$sgn(\times) = -1$$

A crossing with a sign of $+1$ is called a *positive* crossing and similarly a crossing with a sign of -1 is called a *negative* crossing. By letting the strands pass through each others we can make a crossing switch between being positive and negative.

Definition 2.1.7. Let $D \in \overrightarrow{\mathbf{D}}$ be an oriented link diagram. Let x be a crossing of D. By performing a **crossing switch** on x we refer to the operation of constructing a new oriented link diagram D' where the only difference is the sign of the crossing corresponding to x. If the crossing in D' is labelled x' then

$$sgn(x) = -sgn(x')$$

Definition 2.1.8. Let $D \in \overrightarrow{\mathbf{D}}$ be an oriented link diagram. Then

$$w(D) = \sum_{x \in Cr(D)} sgn(x)$$

where $Cr(D)$ is the set of crossings of D. The value $w(D)$ is called the **writhe** of the diagram D.

It can be easily verified that the writhe of an oriented link diagram does not change when we change the link diagram by a Reidemeister Type II or Type III move. However, by a Type I move we either add a new negative crossing or add a new positive crossing. We can use this fact and combine the Kauffman bracket with the writhe in such a way that the effects of a Type I move cancel out.

Theorem 2.1.9. *Let* $X : \overrightarrow{\mathbf{D}} \to \mathbb{Z}[A^{-1}, A]$ *be*

$$X(D) = (-A)^{-3w(D)} \langle D \rangle$$

where $w(\cdot)$ *is the writhe function and* $\langle \cdot \rangle$ *is the Kauffman bracket. Then* X *is a link invariant for oriented links.*

Proof. See [1, Theorem 3.5, p. 26]. □

The link invariant X defined above is called *the normalized bracket polynomial*.

Definition 2.1.10. Let D be a link diagram for the oriented link L. Let $X : \overrightarrow{\mathbf{D}} \to \mathbb{Z}[A^{-1}, A]$ be the normalized bracket polynomial. The **Jones polynomial** $V(L)$ is defined as

$$V(L) = (X(D))_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

where $(\cdot)_{t^{1/2}=A^{-2}}$ denotes substitution of A^{-2} in $X(D) \in \mathbb{Z}[A^{-1}, A]$, by $t^{1/2}$ into an expression in $\mathbb{Z}[t^{-1/2}, t^{1/2}]$.

The Jones polynomial distinguishes between many different links[7]. It also has the nice property that for two oriented knots K_1 and K_2, $V(K_1 + K_2) = V(K_1)V(K_2)$. The Jones polynomial is thus a monoid homomorphism between the monoid of oriented knots and the monoid of formal Laurent polynomials in $t^{-1/2}$ and $t^{1/2}$ with multiplication as the monoid operation.

3 The HOMFLY polynomial

The HOMFLY polynomial is a link invariant that assigns a homogeneous Laurent polynomial of degree 0 in three variables to each link[3]. A homogenous polynomial is a polynomial where every non-zero term have the same degree.

Definition 3.0.1. The **HOMFLY polynomial** is the link invariant $P[\,\cdot\,] : \overrightarrow{\mathbf{D}} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ defined recursively by

- $P[\circlearrowright](x, y, z) = 1$
- $xP[\times](x, y, z) + yP[\times](x, y, z) + zP[)(](x, y, z) = 0$

Theorem 3.0.2. *There is a unique oriented link invariant*

$$P : \overrightarrow{\mathbf{D}} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$$

such that

$$P[\circlearrowright](x, y, z) = 1$$

$$xP[\times](x, y, z) + yP[\times](x, y, z) + zP[)(](x, y, z) = 0$$

This P is the HOMFLY polynomial.

Proof. See [1, Theorem 15.2, p. 168]. □

[7]Not all thought.

The HOMFLY polynomial encompasses both the Jones Polynomial and the famous Alexander-Conway polynomial in the following sense.

Proposition 3.0.3. *Let L be an oriented link. Let $D \in \overrightarrow{\mathbf{D}}$ be a diagram of L. Then*

$$V(D) = P[D](t, -t^{-1}, t^{1/2} - t^{-1/2})$$

and

$$\Delta(D) = P[D](1, -1, t^{1/2} - t^{-1/2})$$

where $V(D)$ is the Jones polynomial and $\Delta(D)$ is the Alexander-Conway polynomial.

Proof. See [3, Remark 2]. $\square$

This means that HOMFLY polynomial is strictly better at distinguishing links than both the Jones polynomial and the Alexander-Conway polynomial. Let L_1 and L_2 be link diagrams of two non-equivalent oriented links. It follow directly from Proposition 3.0.3 that

$$P[L_1] = P[L_2] \implies V(L_1) = V(L_2)$$

which is equivalent to

$$V(L_1) \neq V(L_2) \implies P[L_1] \neq P[L_2]$$

$$\iff$$

$$V(\cdot) \text{ can distinguish links } L_1 \text{ and } L_2 \implies P[\,\cdot\,] \text{ can distinguish } L_1 \text{ and } L_2.$$

The same is of course true for the Alexander-Conway polynomial.

3.1 Calculation

We will now describe how to calculate the HOMFLY polynomial from an oriented link diagram using a recursive algorithm. This is the method described in [1, Theorem 15.2, p. 168]. Before we present the algorithm, some definitions and a proposition are required.

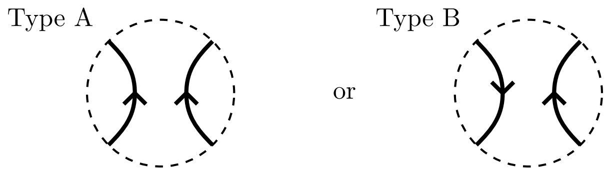
Proposition 3.1.1. *Let D be an oriented link diagram with zero crossings and n disjoint components. Then*

$$P[D](x, y, z) = (-z^{-1}(x + y))^{n-1}$$

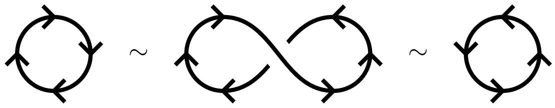
Proof. Since the only oriented knot with zero crossings is the oriented unknot it follows that D is a disjoint union of n oriented unknots.

We will prove the proposition by induction on the number of components n. If $n = 1$, $P[D] = P[\circlearrowright] = 1$ and we are done.

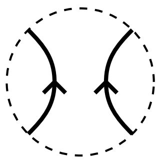
Now, assume the formula holds for all diagrams of N disjoint oriented unknots where $N \geqslant 1$. Let D be an oriented link diagram of $N+1$ disjoint oriented unknots. $N + 1 \geqslant 2$ so D has at least two components. Then there exists a region of D that is similar to either



where the strands are from different components. If D has a region similar to type B, we can always find an equivalent diagram with a region similar to type A by reversing the orientation of one of the components. A reversal of one of the components can be achieved by performing two Reidemeister Type I moves on a component of D as follows.



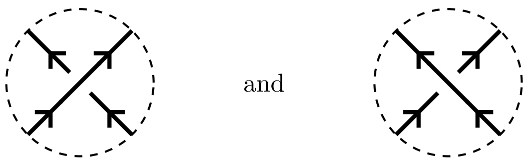
This shows that if D has no region similar to type A there is always an equivalent diagram D' that does. Note that D' also has zero crossings and $N+1$ components. And since P is an oriented link invariant $P[D] = P[D']$. We can therefore assume that D has a region similar to



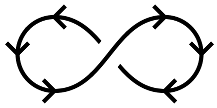
Consider the HOMFLY equation

$$xP[D_+](x,y,z) + yP[D_-](x,y,z) + zP[D](x,y,z) = 0$$

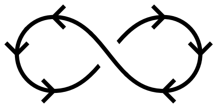
where D_+ and D_- is are diagrams that differ from D only at the region similar to type A such that the equation holds. The corresponding regions in D_+ and D_- look like



respectively. Since we assumed that the strands in the type A region were from different components of D we have that D_+ and D_- are diagrams with N disjoint components and one crossing. The two components with strands in the region in D have been "joined" in D_+ and D_-. This new joined component look like



and



in D_+ and D_- respectively. In either diagram the component is equivalent to an oriented unknot. Therefore there exists diagrams $D'_+ \sim D_+$ and $D'_- \sim D_-$ that are disjoint unions of N oriented unknots. By the inductive hypothesis $P[D'_+] = P[D'_-] = (-z^{-1}(x+y))^{N-1}$ and since $D'_+ \sim D_+, D'_- \sim D_-$ it follows that

$$P[D_+] = P[D_-] = (-z^{-1}(x+y))^{N-1}$$

We combine this with the HOMFLY equation and get

$$x(-z^{-1}(x+y))^{N-1} + y(-z^{-1}(x+y))^{N-1} + zP[D] = 0$$

$$\Longleftrightarrow$$

$$(x+y)(-z^{-1}(x+y))^{N-1} + zP[D] = 0$$

$$\Longleftrightarrow$$

$$zP[D] = -(x+y)(-z^{-1}(x+y))^{N-1}$$

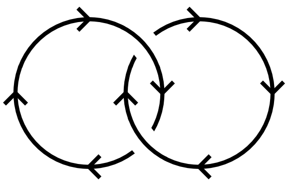
$$\Longleftrightarrow$$

$$\begin{aligned} P[D] &= -z^{-1}(x+y)(-z^{-1}(x+y))^{N-1} \\ &= (-z^{-1}(x+y))^N \end{aligned}$$

By induction the formula holds for any number of components. □

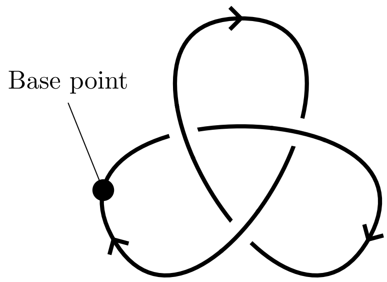
Definition 3.1.2. An **ordered** oriented link is an oriented link L together with a bijective function $r_L : \{\text{Components of } L\} \to \{1, \ldots, n\}$ where n is the number of components of L.

Example 3.1.3. The Hopf link with the two components numbered giving them an order.



Definition 3.1.4. Let D be an oriented link diagram. A **base point** of a component C of D is a point $b \in C$ that does not lie on a crossing of D.

Example 3.1.5. The trefoil with a base point chosen for its only component.



Definition 3.1.6. A **based** oriented link diagram D is an oriented link diagram together with a base point for each component of D.

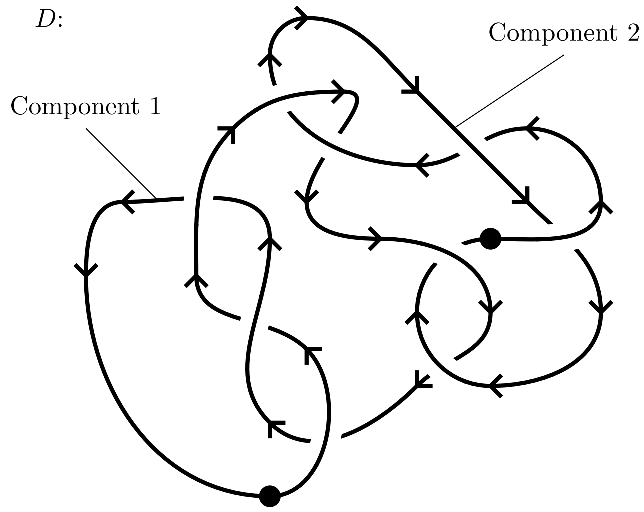
Note that an oriented, based and ordered link diagram induces an ordering of its crossings in a natural way. Consider writing down a list of the crossings as follows. Start at the base point of component 1 and follow along the orientation. When we encounter a crossing we add it to the end of our list if it is not already in the list. Anytime we arrive back at the base point of a component n we jump to the base point in component $n+1$ and continue there. If there are no more components we are done and our list contains each crossing exactly once. This describes an ordering of the crossings. Note that in total every crossing will be encountered twice, once as an underpass and once as an overpass.

Definition 3.1.7. Let D be an oriented, based and ordered link diagram. The associated **ascending diagram**, αD, is an oriented link diagram derived from D by performing crossing switches on D so that the first time any crossing is encountered, following the method described above, it is encountered as an underpass.

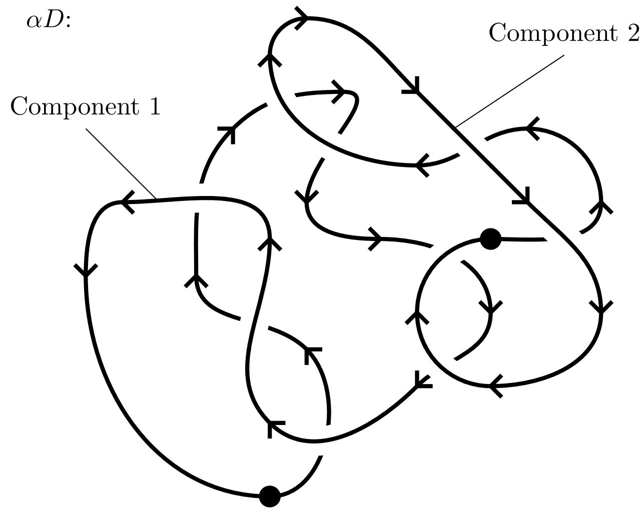
Note that each component of an ascending diagram is equivalent to the oriented unknot and the diagram as a whole is equivalent to a disjoint union of oriented unknots.

Example 3.1.8. Let D be an oriented, based and ordered link diagram. αD is the associated ascending diagram. Notice that if we start at the basepoint of component 1 of αD and follow along the orientation of the link, the first time we encounter any crossing it is as an underpass. After following the component a full lap, we go to the base point of the next component, component 2. Note that as we follow the orientation of the component each crossing that we have not encountered before is first encountered as an underpass.

D:



αD:



We are now ready to present the algorithm as pseudocode followed by some explanations and important observations.

Algorithm 1 HOMFLY polynomial calculation

procedure CALCULATEHOMFLY(Diagram D)
    if $D.numberOfCrossings = 0$ **then**
        Polynomial p := calculateHomflyNoCrossings(D) ▷ Easy
        return p
    end if
    Diagram aD := calculateAscendingDiagram(D)
    SetOfCrossings $differingCrossings$:= getChangedCrossings(D,aD)
    Diagram $curD$:= aD ▷ Current diagram
    Polynomial P_{curD} := calculateHomflyAscending(aD) ▷ Easy
    for each Crossing γ in $differingCrossings$ **do**
        Diagram E := copy($curD$)
        E.removeCrossing(γ) ▷ Replaces the crossing with a
        Polynomial P_E := calculateHOMFLY(E) ▷ Recursive step
        if γ is positive crossing in $curD$ **then**
            Polynomial P_{next} := solve for P_{next} in

$$x \cdot P_{curD}(x, y, z) + y \cdot P_{next}(x, y, z) + z \cdot P_E(x, y, z) = 0$$

            $curD$.doCrossingSwitch(γ)
            $P_{curD} := P_{next}$
        else
            Polynomial P_{next} := solve for P_{next} in

$$x \cdot P_{next}(x, y, z) + y \cdot P_{curD}(x, y, z) + z \cdot P_E(x, y, z) = 0$$

            $curD$.doCrossingSwitch(γ)
            $P_{curD} := P_{next}$
        end if
    end for
    return P_{curD}
end procedure

Code comments

Line 1 Let Diagram represent the data structure of an oriented, based and ordered link diagram.

Line 3 If D has no crossings we can simply count the number of components[8] and apply Proposition 3.1.1.

Line 9 Since any ascending diagram is the diagram of a link equivalent to a disjoint union of unkots we can again apply Proposition 3.1.1.

Line 7 We can label crossings and interpreted them as being shared by multiple diagrams.

Line 10 We start a loop where we successively perform crossing switches on $curD$ until we arrive at D.

Line 12 E now has one less crossing than D.

Line 13 Since E has one less crossing than D we will eventually reach the base case of the recursion handled on line 2.

Line 15 By solving for P_{next} we get the polynomial for a diagram that differs from $curD$ only at the crossing γ.

Line 16 And here we transform $curD$ into this new diagram by performing a crossing switch on γ.

Line 19 If the crossing γ is negative in $curD$ the polynomials P_{curD} and P_{next} switch places in the equation.

Line 24 When we arrive here all crossings of $curD$ differing from D have been changed. As we have updated P_{curD} after each crossing switch, the polynomial P_{curD} is the HOMFLY polynomial of D.

This gives us a way to calculate the HOMFLY polynomial under the assumption that it is well-defined for all oriented link diagrams and is an oriented link invariant. Proving that it is well-defined involves showing that the value calculated by the algorithm is independent of which base points we choose and what ordering of the components we choose. One also needs to show that the value produced by the algorithm does not depend on the order in which we handle the crossings of $differingCrossings$. Finally one needs to show that oriented link diagrams related by Reidemeister moves and orientation-preserving self-homeomorphisms of $\mathbb{R}^2$ give the same value. All of this is show in the proof of Theorem 3.0.2 which can be found in [1, p. 168].

3.2 Generalizing the HOMFLY polynomial to $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$

$\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ forms what is known as a *commutative ring*. That this ring is commutative is easily realized since we consider elements like x^2y^{-1} and $y^2xy^{-3}x$ to be equal. If we do not recognise such elements to be equal we get

[8]How we do this depends heavily on the data structure chosen to represent an oriented link diagram.

a noncommutative ring denoted $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$. $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ consists of elements of the form $\sum_{i=1}^{n} a_i b_i$ where $a_i \in \mathbb{Z}$ and b_i is on the form $b_i = \prod_{j=1}^{m_i} g_{ij}^{e_{ij}}$ where $g_{ij} \in \{x, y, z\}$ and $e_{ij} \in \mathbb{Z}$. We do not impose commutativity for the multiplication in this ring. We do however impose the relations

$$g^0 = 1, \quad \forall g \in \{x, y, z\} \text{ and } g^n g^m = g^{n+m}, \quad \forall g \in \{x, y, z\}, \forall n, m \in \mathbb{Z}.$$

Example 3.2.1. In $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$,

$$3x^3y^2z^{-1} \neq 3y^2z^{-1}x^3 \neq 3x^3y^3z^3y^{-1}z^{-4}$$

but

$$(xy^2) \cdot (y^{-5}z^2) = xy^2y^{-5}z^2 = xy^{-3}z^2$$

We aim to generalize the HOMFLY polynomial to a link invariant that for each oriented link diagram assigns an element of $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$ where $\sim$ is some congruence relation on $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$. We will place some restrictions on $\sim$.

Note that there is a natural way to map elements of $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ to $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ by simplifying them.

Definition 3.2.2. Let $ab : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ be the ring homomorphism defined by

$$ab(g^n) = g^n, \quad \forall g \in \{x, y, z\}, \forall n \in \mathbb{Z}$$

That ab is a ring homomorphism means that

$$ab(1) = 1$$

$$ab(p + q) = ab(p) + ab(q), \quad \forall p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$$
$$ab(pq) = ab(p)ab(q), \quad \forall p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$$

Example 3.2.3. Let $p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$.

$$p = xyx^{-1}$$

$$q = zx - xz + y$$

Neither p nor q can be further simplified. However, $ab(p), ab(q) \in \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. And $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ is commutative.

$$ab(p) = xyx^{-1} = xx^{-1}y = y$$

$$ab(q) = zx - xz + y = zx - zx + y = y$$

So even though $p \neq q$, $ab(p) = ab(q)$.

We shall for the rest of this subsection only consider congruence relations $\sim$ that are compatible with ab. That is, we require that

$$p \sim q \implies ab(p) = ab(q), \quad \forall p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}.$$

One way to construct $\sim$ so that the above holds is the following.

Example 3.2.4. Define $\sim$ to be the congruence relation on $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ generated by

$$p \sim q \iff ab(p) = ab(q)$$

It is easily checked that this is a congruence relation. It is also clear that in this case we get that $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim \,\cong\, \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ where $\cong$ denotes that the rings are isomorphic. However, this is not a very interesting case as we already know that the HOMFLY polynomial is well-defined.

We shall investigate the minimal constraints placed on $\sim$ for the HOMFLY equations

$$\begin{cases} P[\bigcirc] = 1 \\ xP[\positivecrossing] + yP[\negativecrossing] + zP[\smoothing] = 0 \end{cases}$$

to uniquely define a link invariant with $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$ as range. But before we do this we introduce some concepts to aid our reasoning.

Definition 3.2.5. An ordered triplet of oriented link diagrams (D_+, D_-, D_0) is called a **skein-triplet** if they only differ in a local region where D_+ looks like $\positivecrossing$, D_- looks like $\negativecrossing$ and D_0 looks like $\smoothing$.

Definition 3.2.6. A **template diagram with n free crossings** is a function $D : \{\positivecrossing, \negativecrossing, \smoothing\}^n \to \overrightarrow{\mathbf{D}}$ such that

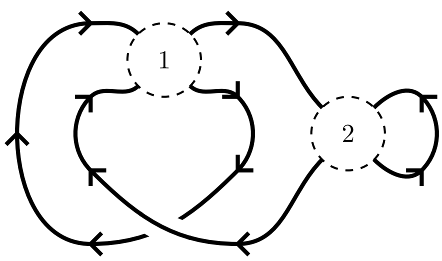
$$(D(x_1, \ldots, \underbrace{\positivecrossing}_{i:\text{th}}, \ldots, x_n), D(x_1, \ldots, \underbrace{\negativecrossing}_{i:\text{th}}, \ldots, x_n), D(x_1, \ldots, \underbrace{\smoothing}_{i:\text{th}}, \ldots, x_n))$$

is a skein-triplet for all $i = 1, \ldots, n$.

We can interpret a template diagram with n free crossings as an incomplete diagram with n regions where we may fill in any of $\{\positivecrossing, \negativecrossing, \smoothing\}$. If we fix $m \leqslant n$ of the arguments of a template diagram D with n free crossings we can interpret it as a template diagram with $n - m$ free crossings. By a template diagram of 0 free crossings we simply mean an oriented link diagram. We shall also refer to *the total number of crossings* of a template diagram D as the number of crossings in an oriented link diagram $D(x_1, \ldots, x_n)$ where $x_i \in \{\positivecrossing, \negativecrossing, \smoothing\}$.

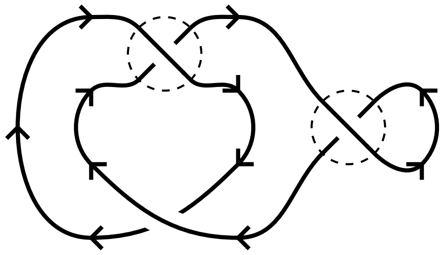
Example 3.2.7. Let D be a template diagram with two free crossings. We can represent D as an incomplete diagram with two regions where one of $\{\positivecrossing, \negativecrossing, \smoothing\}$ can be fitted. We mark the regions with 1 and 2 representing which region corresponds to which argument of $D(\cdot, \cdot)$.

D:



Consider filling in region 1 with a positive crossing, $\times$, and region 2 with a negative crossing, $\times$. There is only one way to do this while respecting the orientation of the diagram. In region 1 we have to rotate $\times$ 90 degrees clockwise to fit it in and in region 2 we have to rotate $\times$ 180 degrees. This gives us the oriented link diagram $D(\times, \times)$.

$D(\times, \times)$:



Definition 3.2.8. Let $Q[\,\cdot\,] : \overrightarrow{\mathbf{D}} \to \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$ denote a possibly well-defined, oriented link invariant that satisfies

- $Q[\circlearrowleft](x, y, z) = 1$
- $xQ[D(\times)](x, y, z) + yQ[D(\times)](x, y, z) + zQ[D()()](x, y, z) = 0$ for every template diagram D with one free crossing.

where $\sim$ is some congruence relation on $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ satisfying

$$p \sim q \implies ab(p) = ab(q), \quad \forall p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$$

A priori it is not clear that the definition uniquely defines a link invariant $Q[\,\cdot\,]$ for every compatible $\sim$. However, if it indeed is well-defined, we can infer some constraints placed on $\sim$.

The condition that $p \sim q \implies ab(p) = ab(q)$ allows us to "extend" $ab : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ to a ring homomorphism $ab_\sim : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$.

Proposition 3.2.9. *Given a congruence relation $\sim$ on $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ such that*

$$p \sim q \implies ab(p) = ab(q), \quad \forall p, q \in \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$$

there exists a unique ring homomorphism $ab_\sim : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim \to \mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ such that

$$ab_\sim \circ \pi = ab$$

where π is the quotient map of $\sim$.

Proof. Define $ab_\sim([p]_\sim) = p$ where $[p]_\sim \in \mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}/\sim$ is the congruence class of p. We first show that $ab_\sim$ is well-defined.

Let $p, q \in \mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}$, $p \sim q$. Then $[p]_\sim = [q]_\sim$.

$$ab_\sim([p]_\sim) = p \in \mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$$

$$ab_\sim([q]_\sim) = q \in \mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$$

But $p \sim q$ implies that $ab(p) = ab(q)$ so

$$ab_\sim([p]_\sim) = p = ab(p) = ab(q) = q = ab_\sim([q]_\sim)$$

So $ab_\sim$ is well-defined.

$$ab_\sim \circ \pi(p) = ab_\sim([p]_\sim) = p = ab(p)$$

So $ab_\sim \circ \pi = ab$.

Finally, assume there exists a function $u : \mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}/\sim \to \mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ such that $u \circ \pi = ab$. Then

$$u \circ \pi(p) = u([p]_\sim) = ab(p) = p \implies u([p]_\sim) = p \implies u = ab_\sim$$

Thus $ab_\sim$ is unique. $\square$

We shall denote $ab_\sim$ by ab when it is clear that we mean $ab_\sim$ by examining the domain.

Proposition 3.2.10. *If $Q[\,\cdot\,]$ is well-defined then $ab \circ Q = P$ where P is the HOMFLY polynomial.*

Proof.
Q satisfies

- $Q[\circlearrowright] = 1$
- $xQ[D(\times_+)] + yQ[D(\times_-)] + zQ[D()()] = 0$ for every template diagram D with one free crossing.

This implies that for every template diagram D with one free crossing,

$$ab(xQ[D(\times_+)] + yQ[D(\times_-)] + zQ[D()()]) = ab(0)$$

$$\iff$$

$$x\ ab(Q[D(\times_+)]) + y\ ab(Q[D(\times_-)]) + z\ ab(Q[D()()]) = 0$$

$$\iff$$

$$x(ab \circ Q[D(\times_+)]) + y(ab \circ Q[D(\times_-)]) + z(ab \circ Q[D()()]) = 0$$

Similarly,

$$ab(Q[\circlearrowright]) = ab(1) = ab(x^0) = x^0 = 1$$

So $ab \circ Q : \overrightarrow{\mathbf{D}} \to \mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ is an oriented link invariant that satisfies

- $ab \circ Q[\circlearrowright] = 1$
- $x\ ab \circ Q[⤯] + y\ ab \circ Q[⤰] + z\ ab \circ Q[)(] = 0$

Theorem 3.0.2 tells us that there is only one such invariant and that is the HOMFLY polynomial. Hence,

$$ab \circ Q = P$$

$\square$

The symbols ⤯ and ⤰ can be a bit hard to distinguish and make long computations hard to follow. We shall for the rest of this thesis use the following symbols instead.

$\oplus$ = ⤯	(A positive crossing)
$\ominus$ = ⤰	(A negative crossing)
$⓪$ =)(	(No crossing)

Lemma 3.2.11. *If $Q[\cdot]$ is well-defined, there exists a template diagram D with 2 crossings such that $Q[D(\ominus, ⓪)] \neq Q[D(⓪, \ominus)]$.*

Proof. Assume that no such template diagram exists. This implies that for every template diagram D

$$Q[D(\ominus, ⓪)] - Q[D(⓪, \ominus)] = 0$$

Apply $ab : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim \to \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ to both sides of the equation. This gives us that for any template diagram D,

$$ab(Q[D(\ominus, ⓪)] - Q[D(⓪, \ominus)]) = ab(0) = 0$$

$$\Longleftrightarrow$$

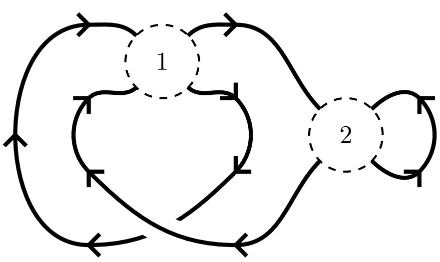
$$ab(Q[D(\ominus, ⓪)]) - ab(Q[D(⓪, \ominus)]) = 0$$

$$\Longleftrightarrow \text{/Proposition 3.2.10/} \Longleftrightarrow$$

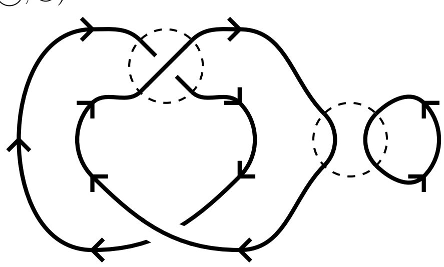
$$P[D(\ominus, ⓪)] = P[D(⓪, \ominus)].$$

Consider the following template diagram D.

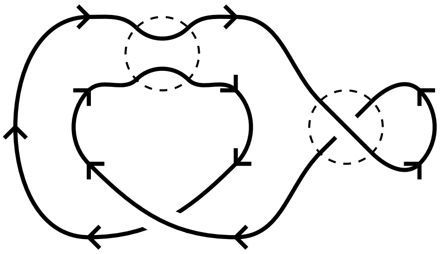
D:



$D(\ominus, ⓪)$:



$D(\mathbb{0}, \ominus)$:



$$\begin{aligned} P[D(\ominus, \mathbb{0})] =& P[\circlearrowright \sqcup \circlearrowright \sqcup \circlearrowright] \\ &= /\text{Proposition 3.1.1}/ \\ &= (-z^{-1}(x+y))^2 \\ &= z^{-2}(x^2+2xy+y^2) \\ &= z^{-2}x^2 + 2z^{-2}xy + z^{-2}y^2 \end{aligned}$$

but

$$P[D(\mathbb{0}, \ominus)] = P[\circlearrowright] = 1 \neq z^{-2}x^2 + 2z^{-2}xy + z^{-2}y^2$$

This is a contradiction. Thus, there must exist a template diagram D such that $Q[D(\ominus, \mathbb{0})] \neq Q[D(\mathbb{0}, \ominus)]$. $\square$

An important property of the HOMFLY equation

$$xP[\oplus](x,y,z) + yP[\ominus](x,y,z) + zP[\mathbb{0}](x,y,z) = 0$$

is that given any two of the three involved polynomials, $P[\oplus](x,y,z)$,$P[\ominus](x,y,z)$ or $P[\mathbb{0}](x,y,z)$, the remaining one can be explicitly found.

Example 3.2.12.

$$xP[\oplus](x,y,z) + yP[\ominus](x,y,z) + zP[\mathbb{0}](x,y,z) = 0$$

$$\Longleftrightarrow$$

$$P[\oplus](x,y,z) = -x^{-1}yP[\ominus](x,y,z) - x^{-1}zP[\mathbb{0}](x,y,z)$$

$$\Longleftrightarrow$$

$$P[\ominus](x,y,z) = -y^{-1}xP[\oplus](x,y,z) - y^{-1}zP[\mathbb{0}](x,y,z)$$

$$\Longleftrightarrow$$

$$P[\mathbb{0}](x,y,z) = -z^{-1}xP[\oplus](x,y,z) - z^{-1}yP[\ominus](x,y,z)$$

Note that we have not used the commutativity of $\mathbb{Z}[x^{\pm1}, y^{\pm1}, z^{\pm1}]$ to derive these equations. In fact the corresponding equations for Q are true in the context of $\mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}/\sim$. We will use this in the proof of the following proposition which is our first new result.

Proposition 3.2.13. *If Q is well-defined then*

$$yx^{-1}z \sim zx^{-1}y$$

Proof. Similar to the proof of Proposition 2.1.4 we shall consider how the value of Q depend on the order in which two crossing switches are performed. We shall refer to the equation

$$xQ[\oplus](x,y,z) + yQ[\ominus](x,y,z) + zQ[\mathbb{0}](x,y,z) = 0$$

as *the recursive equation.*

Let $Q[A,B]$ be a shorthand for $Q[D(A,B)]$ where D is some template diagram with 2 free crossings and $n \geqslant 2$ total number of crossings.

The proof is quite technical and we shall use the notation $\bar{x} = x^{-1}$ in favour of space.

The value of $Q[D]$ can be recursively calculated using an algorithm similar to Algorithm 1 where the datatype Polynomial is replaced by a datatype corresponding to an element in $\mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}/\sim$. Since Q is well-defined the value produced by the algorithm should always be the same. The order of the calculations may however differ depending on the order of crossings in $differingCrossings$ on line 7 of the algorithm. Each time a crossing γ is handled in the loop of the algorithm a crossing switch is performed on γ and $curD$ moves one step closer towards the ascending diagram. Like in the proof of Proposition 2.1.4 we proceed by considering the value of Q we get by performing two crossing switches in different order. Because of the recursive step, we can express the value $Q[D]$ of an oriented link diagram D as an expression of

1. Q-values of oriented diagrams of fewer crossings than D.
2. The Q-value of an oriented link diagram two steps "closer" to an ascending diagram.

Given a template diagram D there are only five diagrams with lower number of crossings that show up in the calculation of $D(A,B)$.

- $D(\oplus, \mathbb{0})$
- $D(\mathbb{0}, \oplus)$
- $D(\ominus, \mathbb{0})$
- $D(\mathbb{0}, \ominus)$
- $D(\mathbb{0}, \mathbb{0})$

However, the Q-value of these five are not independent.

$$Q[\oplus, \mathbb{0}] = -\bar{x}yQ[\ominus, \mathbb{0}] - \bar{x}zQ[\mathbb{0}, \mathbb{0}] \tag{5}$$

$$Q[\mathbb{0}, \oplus] = -\bar{x}yQ[\mathbb{0}, \ominus] - \bar{x}zQ[\mathbb{0}, \mathbb{0}] \tag{6}$$

This shows that any diagram of $n-1$ crossings that show up in our calculations can be expressed in terms of $Q[\mathbb{0}, \ominus]$,$Q[\ominus, \mathbb{0}]$ and $Q[\mathbb{0}, \mathbb{0}]$. We shall use this.

Under the assumption that Q is well-defined it follows that any recursive calculation must be independent on the order of crossing switches. Consider the case where we have a diagram $D(\oplus, \ominus)$ and we need to switch both crossings to get to $D(\ominus, \oplus)$, which is a diagram closer to an ascending diagram.

Our goal is to express $Q[\oplus, \ominus]$ in terms of $Q[\ominus, \oplus]$ and polynomials of diagrams with lower number of crossings, such as $Q[\mathbb{0}, \ominus]$,$Q[\ominus, \mathbb{0}]$ and $Q[\mathbb{0}, \mathbb{0}]$. We apply the recursion equation to the first crossing followed by an application to the second.

$$Q[\oplus, \ominus] = -\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\mathbb{0}, \ominus]$$

Now apply the recursive equation to the second crossing of $D(\ominus, \ominus)$.

$$Q[\ominus, \ominus] = -\bar{y}xQ[\ominus, \oplus] - \bar{y}zQ[\ominus, \mathbb{0}]$$

Combining these we get

$$\begin{aligned}
Q[\oplus, \ominus] &= -\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\mathbb{0}, \ominus] \\
&= -\bar{x}y(-\bar{y}xQ[\ominus, \oplus] - \bar{y}zQ[\ominus, \mathbb{0}]) - \bar{x}zQ[\mathbb{0}, \ominus] \\
&= Q[\ominus, \oplus] + \bar{x}zQ[\ominus, \mathbb{0}] - \bar{x}zQ[\mathbb{0}, \ominus] \\
&= Q[\ominus, \oplus] + \bar{x}z(Q[\ominus, \mathbb{0}] - Q[\mathbb{0}, \ominus]) \qquad (7)
\end{aligned}$$

We apply the recursion equation to the second crossing followed by an application to the first.

$$Q[\oplus, \ominus] = -\bar{y}xQ[\oplus, \oplus] - \bar{y}zQ[\oplus, \mathbb{0}]$$

Now apply the recursive equation to the first crossing of $D(\oplus, \oplus)$.

$$Q[\oplus, \oplus] = -\bar{x}yQ[\ominus, \oplus] - \bar{x}zQ[\mathbb{0}, \oplus]$$

Combining these we get

$$\begin{aligned}
Q[\oplus, \ominus] &= -\bar{y}xQ[\oplus, \oplus] - \bar{y}zQ[\oplus, \mathbb{0}] \\
&= -\bar{y}x(-\bar{x}yQ[\ominus, \oplus] - \bar{x}zQ[\mathbb{0}, \oplus]) - \bar{y}zQ[\oplus, \mathbb{0}] \\
&= Q[\ominus, \oplus] + \bar{y}zQ[\mathbb{0}, \oplus] - \bar{y}zQ[\oplus, \mathbb{0}] \\
&= Q[\ominus, \oplus] + \bar{y}z(Q[\mathbb{0}, \oplus] - Q[\oplus, \mathbb{0}])
\end{aligned}$$

We note that

$$
\begin{aligned}
&Q[\mathbb{0},\oplus] - Q[\oplus,\mathbb{0}]\\
&= /\text{Apply equation 5 and equation 6}/\\
&= -\bar{x}yQ[\mathbb{0},\ominus] - \bar{x}zQ[\mathbb{0},\mathbb{0}] - (-\bar{x}yQ[\ominus,\mathbb{0}] - \bar{x}zQ[\mathbb{0},\mathbb{0}])\\
&= -\bar{x}yQ[\mathbb{0},\ominus] - \bar{x}zQ[\mathbb{0},\mathbb{0}] + \bar{x}yQ[\ominus,\mathbb{0}] + \bar{x}zQ[\mathbb{0},\mathbb{0}]\\
&= -\bar{x}yQ[\mathbb{0},\ominus] + \bar{x}yQ[\ominus,\mathbb{0}]\\
&= \bar{x}y(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus])
\end{aligned}
$$

Combining these two equations we get

$$
\begin{aligned}
Q[\oplus,\ominus] &= Q[\ominus,\oplus] + \bar{y}z(Q[\mathbb{0},\oplus] - Q[\oplus,\mathbb{0}])\\
&= Q[\ominus,\oplus] + \bar{y}z(\bar{x}y(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus]))\\
&= Q[\ominus,\oplus] + \bar{y}z\bar{x}y(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus]) \qquad (8)
\end{aligned}
$$

That the transition from $D(\oplus,\ominus)$ to $D(\ominus,\oplus)$ is independent of the order of the crossing switches implies that the right hand sides of equations 7 and 8 must coincide. It follows that

$$Q[\ominus,\oplus] + \bar{x}z(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus]) = Q[\ominus,\oplus] + \bar{y}z\bar{x}y(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus])$$

$$\Longleftrightarrow$$

$$\bar{x}z(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus]) = \bar{y}z\bar{x}y(Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus])$$

If Q is well-defined then this equality holds for any template diagram D. Lemma 3.2.11 tells us that there exists at least one template diagram such that $Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus] \neq 0$. We can therefore divide by $Q[\ominus,\mathbb{0}] - Q[\mathbb{0},\ominus]$ on both sides.

$$\Longrightarrow$$

$$\bar{x}z = \bar{y}z\bar{x}y$$

$$\Longleftrightarrow$$

$$y\bar{x}z = z\bar{x}y$$

Since both sides of the equation are elements of $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$ it is equivalent to

$$y\bar{x}z \sim z\bar{x}y$$

Which is what we wanted to show. □

Note that $yx^{-1}z \sim zx^{-1}y$ is a necessary condition on the congruence relation for Q to even have a chance to be well-defined. A natural question is whether it is a sufficient condition. We give a partial result to this in Proposition 3.2.15 below.

Also note that $yx^{-1}z \sim zx^{-1}y$ is a weaker condition than commutativity. Commutativity of $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$ clearly implies $yx^{-1}z \sim zx^{-1}y$ but the converse is not true.

Proposition 3.2.14. *There exists a congruence relation $\sim$ on $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$ such that*

$$yx^{-1}z \sim zx^{-1}y$$

but

$$yz \nsim zy$$

Proof.
The quaternions, denoted $\mathbb{H}$, is an associative noncommutative division algebra over $\mathbb{R}$. Every element of $\mathbb{H}$ is a linear sum of the base elements 1,i,j and k which have the following multiplication table.

$\cdot$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Let $\sim$ be generated by the relation $yx^{-1}z \sim zx^{-1}y$. Define a ring homomorphism $f : \mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim \to \mathbb{H}$ by

$$f(x^n) = (-i)^n, \quad \forall n \in \mathbb{Z}$$

$$f(y^n) = (i+j)^n, \quad \forall n \in \mathbb{Z}$$

$$f(z^n) = (i-j)^n, \quad \forall n \in \mathbb{Z}$$

Since the quaternions is an associative division algebra, q^{-1} is well-defined for all $q \in \mathbb{H}$. We first show that the ring homomorphism is well-defined. $yx^{-1}z \sim zx^{-1}y$ implies that $yx^{-1}z = zx^{-1}y$ in $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$.

$$\begin{aligned}
f(yx^{-1}z) &= f(y)f(x^{-1})f(z) \\
&= (i+j)(-i)^{-1}(i-j) \\
&= (i+j)i(i-j) \\
&= (i^2+ji)(i-j) \\
&= (-1-k)(i-j) \\
&= -i+j-ki+kj \\
&= -i+j-j-i \\
&= -2i
\end{aligned} \tag{9}$$

$$\begin{aligned}
f(zx^{-1}y) &= f(z)f(x^{-1})f(y) \\
&= (i-j)(-i)^{-1}(i+j) \\
&= (i-j)i(i+j) \\
&= (i^2-ji)(i+j) \\
&= (-1+k)(i+j) \\
&= -i-j+ki+kj \\
&= -i-j+j-i \\
&= -2i \qquad (10)
\end{aligned}$$

Since the right hand sides of equations 9 and 10 are equal, the ring homomorphism f is well-defined.

Now, suppose that $yx^{-1}z \sim zx^{-1}y$ implies $yz \sim zy$. This implies that $yz = zy$ in $\mathbb{Z}\{x^{\pm1}, y^{\pm1}, z^{\pm1}\}/\sim$ and that

$$f(yz) = f(zy).$$

But

$$\begin{aligned}
f(yz) &= f(y)f(z) \\
&= (i+j)(i-j) \\
&= i^2 - ij + ji - j^2 \\
&= -1-k-k-(-1) \\
&= -2k
\end{aligned}$$

and

$$\begin{aligned}
f(zy) &= f(z)f(y) \\
&= (i-j)(i+j) \\
&= i^2 + ij - ji - j^2 \\
&= -1+k+k-(-1) \\
&= 2k \neq -2k
\end{aligned}$$

We have a contradiction and we conclude that with the congruence relation generated by $yx^{-1}z \sim zx^{-1}y$,

$$yz \nsim zy.$$

$\square$

Proposition 3.2.15. *If $yx^{-1}z \sim zx^{-1}y$ then the recursive computation of Q by an algorithm similar to Algorithm 1 is independent of the order of crossing switches.*

Proof. We need to show that for any diagram D the Q-value that the algorithm produces does not depend on the order in which the crossing switches are handled. It is enough to show that a transposition of two crossing switches gives the same result. We will use the same shorthands as in the proof of Proposition 3.2.13. That is, $\bar{x} = x^{-1}$ and $Q[A, B] = Q[D(A, B)]$.

Assume we have a template diagram D with two free crossings. We get four cases of possible crossing switches when changing two crossings:

$$D(\oplus, \oplus) \to D(\ominus, \ominus)$$

$$D(\ominus, \ominus) \to D(\oplus, \oplus)$$

$$D(\oplus, \ominus) \to D(\ominus, \oplus)$$

$$D(\ominus, \oplus) \to D(\oplus, \ominus)$$

Since the forth case can be seen as the third with the labels of the free crossings of D swapped, they are symmetrical cases and we need only consider one of them. This gives us just three cases to consider.

We restate the equations from the proof of Proposition 3.2.13.

$$Q[\oplus, \mathbb{0}] = -\bar{x}yQ[\ominus, \mathbb{0}] - \bar{x}zQ[\mathbb{0}, \mathbb{0}] \tag{5 revisited}$$

$$Q[\mathbb{0}, \oplus] = -\bar{x}yQ[\mathbb{0}, \ominus] - \bar{x}zQ[\mathbb{0}, \mathbb{0}] \tag{6 revisited}$$

Case $D(\oplus, \ominus) \to D(\ominus, \oplus)$**:**

This case was already considered in the proof of Proposition 3.2.13. Recall that the independency of the crossing switches for this case is equivalent to the expressions

$$Q[\ominus, \oplus] + \bar{x}z(Q[\ominus, \mathbb{0}] - Q[\mathbb{0}, \ominus])$$

and

$$Q[\ominus, \oplus] + \bar{y}z\bar{x}y(Q[\ominus, \mathbb{0}] - Q[\mathbb{0}, \ominus])$$

being equal. From the assumption of this proposition we have that in $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$

$$y\bar{x}z = z\bar{x}y \iff \bar{x}z = \bar{y}z\bar{x}y.$$

$$\implies Q[\ominus, \oplus] + \bar{x}z(Q[\ominus, \mathbb{0}] - Q[\mathbb{0}, \ominus]) = Q[\ominus, \oplus] + \bar{y}z\bar{x}y(Q[\ominus, \mathbb{0}] - Q[\mathbb{0}, \ominus])$$

We have proved that $yx^{-1}z \sim zx^{-1}y$ implies that the case $D(\oplus, \ominus) \to D(\ominus, \oplus)$ is independent on the order of crossing switches.

Case $D(\oplus, \oplus) \to D(\ominus, \ominus)$**:**

Assume $yx^{-1}z \sim zx^{-1}y$.

Our goal is to express $Q[\oplus, \oplus]$ in terms of $Q[\ominus, \ominus]$ and Q-values of diagrams with lower number of crossings, such as $Q[\mathbb{0}, \ominus]$,$Q[\ominus, \mathbb{0}]$ and $Q[\mathbb{0}, \mathbb{0}]$.

We apply the recursion equation to the first crossing followed by an application to the second.

$$Q[\oplus, \oplus] = -\bar{x}yQ[\ominus, \oplus] - \bar{x}zQ[\mathbb{0}, \oplus]$$

Now apply the recursive equation to the second crossing of $D(\ominus, \oplus)$.

$$Q[\ominus, \oplus] = -\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\ominus, \mathbb{0}]$$

Combining these we get

$$\begin{aligned}
Q[\oplus, \oplus] &= -\bar{x}yQ[\ominus, \oplus] - \bar{x}zQ[\mathbb{0}, \oplus] \\
&= -\bar{x}y(-\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\ominus, \mathbb{0}]) - \bar{x}zQ[\mathbb{0}, \oplus] \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\ominus, \mathbb{0}] - \bar{x}zQ[\mathbb{0}, \oplus] \\
&= /\text{Apply equation } 6/ \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\ominus, \mathbb{0}] - \bar{x}z(-\bar{x}yQ[\mathbb{0}, \ominus] - \bar{x}zQ[\mathbb{0}, \mathbb{0}]) \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\ominus, \mathbb{0}] + \bar{x}z\bar{x}yQ[\mathbb{0}, \ominus] + \bar{x}z\bar{x}zQ[\mathbb{0}, \mathbb{0}] \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}(y\bar{x}zQ[\ominus, \mathbb{0}] + z\bar{x}yQ[\mathbb{0}, \ominus]) + \bar{x}z\bar{x}zQ[\mathbb{0}, \mathbb{0}]
\end{aligned} \tag{11}$$

We apply the recursion equation to the second crossing followed by an application to the first.

$$Q[\oplus, \oplus] = -\bar{x}yQ[\oplus, \ominus] - \bar{x}zQ[\oplus, \mathbb{0}]$$

Now apply the recursive equation to the first crossing of $D(\oplus, \ominus)$.

$$Q[\oplus, \ominus] = -\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\mathbb{0}, \ominus]$$

Combining these we get

$$\begin{aligned}
Q[\oplus, \oplus] &= -\bar{x}yQ[\oplus, \ominus] - \bar{x}zQ[\oplus, \mathbb{0}] \\
&= -\bar{x}y(-\bar{x}yQ[\ominus, \ominus] - \bar{x}zQ[\mathbb{0}, \ominus]) - \bar{x}zQ[\oplus, \mathbb{0}] \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\mathbb{0}, \ominus] - \bar{x}zQ[\oplus, \mathbb{0}] \\
&= /\text{Apply equation } 5/ \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\mathbb{0}, \ominus] - \bar{x}z(-\bar{x}yQ[\ominus, \mathbb{0}] - \bar{x}zQ[\mathbb{0}, \mathbb{0}]) \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}y\bar{x}zQ[\mathbb{0}, \ominus] + \bar{x}z\bar{x}yQ[\ominus, \mathbb{0}] + \bar{x}z\bar{x}zQ[\mathbb{0}, \mathbb{0}] \\
&= \bar{x}y\bar{x}yQ[\ominus, \ominus] + \bar{x}(y\bar{x}zQ[\mathbb{0}, \ominus] + z\bar{x}yQ[\ominus, \mathbb{0}]) + \bar{x}z\bar{x}zQ[\mathbb{0}, \mathbb{0}]
\end{aligned} \tag{12}$$

Since the terms $+\bar{x}y\bar{x}yQ[\ominus, \ominus]$ and $+\bar{x}z\bar{x}zQ[\mathbb{0}, \mathbb{0}]$ appear in the right hand sides of both equations 11 and 12, the right hand sides are equal if and only if the expressions

$$\bar{x}(y\bar{x}zQ[\ominus, \mathbb{0}] + z\bar{x}yQ[\mathbb{0}, \ominus])$$

and

$$\bar{x}(y\bar{x}zQ[\mathbb{0}, \ominus] + z\bar{x}yQ[\ominus, \mathbb{0}])$$

are equal. $y\bar{x}z \sim z\bar{x}y$ implies that in $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$

$$y\bar{x}z = z\bar{x}y$$

$$\Longrightarrow$$

$$\begin{aligned}
&\bar{x}(y\bar{x}zQ[\ominus,\mathbb{0}] + z\bar{x}yQ[\mathbb{0},\ominus])\\
=&\bar{x}(z\bar{x}yQ[\ominus,\mathbb{0}] + z\bar{x}yQ[\mathbb{0},\ominus])\\
=&\bar{x}(z\bar{x}yQ[\ominus,\mathbb{0}] + y\bar{x}zQ[\mathbb{0},\ominus])\\
=&\bar{x}(y\bar{x}zQ[\mathbb{0},\ominus] + z\bar{x}yQ[\ominus,\mathbb{0}])
\end{aligned}$$

We have proved that $yx^{-1}z \sim zx^{-1}y$ implies that the case $D(\oplus,\oplus) \to D(\ominus,\ominus)$ is independent of the order of crossing switches.

Case $D(\ominus,\ominus) \to D(\oplus,\oplus)$**:**

Assume $yx^{-1}z \sim zx^{-1}y$.

Our goal is to express $Q[\ominus,\ominus]$ in terms of $Q[\oplus,\oplus]$ and Q-values of diagrams with lower number of crossings, such as $Q[\mathbb{0},\ominus]$,$Q[\ominus,\mathbb{0}]$ and $Q[\mathbb{0},\mathbb{0}]$.

We apply the recursion equation to the first crossing followed by an application to the second.

$$Q[\ominus,\ominus] = -\bar{y}xQ[\oplus,\ominus] - \bar{y}zQ[\mathbb{0},\ominus]$$

Now apply the recursive equation to the second crossing of $D(\oplus,\ominus)$.

$$Q[\oplus,\ominus] = -\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\oplus,\mathbb{0}]$$

Combining these we get

$$\begin{aligned}
Q[\ominus,\ominus] &= -\bar{y}xQ[\oplus,\ominus] - \bar{y}zQ[\mathbb{0},\ominus]\\
&= -\bar{y}x(-\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\oplus,\mathbb{0}]) - \bar{y}zQ[\mathbb{0},\ominus]\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] + \bar{y}x\bar{y}zQ[\oplus,\mathbb{0}] - \bar{y}zQ[\mathbb{0},\ominus]\\
&= /\text{Apply equation 5}/\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] + \bar{y}x\bar{y}z(-\bar{x}yQ[\ominus,\mathbb{0}] - \bar{x}zQ[\mathbb{0},\mathbb{0}]) - \bar{y}zQ[\mathbb{0},\ominus]\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}x\bar{y}z\bar{x}yQ[\ominus,\mathbb{0}] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\mathbb{0},\ominus]\\
&= /z\bar{x}y = y\bar{x}z/\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}x\bar{y}y\bar{x}zQ[\ominus,\mathbb{0}] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\mathbb{0},\ominus]\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\ominus,\mathbb{0}] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\mathbb{0},\ominus]\\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}z(Q[\ominus,\mathbb{0}] + Q[\mathbb{0},\ominus]) - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] \quad (13)
\end{aligned}$$

We apply the recursion equation to the second crossing followed by an application to the first.

$$Q[\ominus,\ominus] = -\bar{y}xQ[\ominus,\oplus] - \bar{y}zQ[\ominus,\mathbb{0}]$$

Now apply the recursive equation to the first crossing of $D(\ominus,\oplus)$.

$$Q[\ominus,\oplus] = -\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\mathbb{0},\oplus]$$

Combining these we get

$$\begin{aligned}
Q[\ominus,\ominus] &= -\bar{y}xQ[\ominus,\oplus] - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= -\bar{y}x(-\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\mathbb{0},\oplus]) - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] + \bar{y}x\bar{y}zQ[\mathbb{0},\oplus] - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= /\text{Apply equation 6}/ \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] + \bar{y}x\bar{y}z(-\bar{x}yQ[\mathbb{0},\ominus] - \bar{x}zQ[\mathbb{0},\mathbb{0}]) - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}x\bar{y}z\bar{x}yQ[\mathbb{0},\ominus] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= /z\bar{x}y = y\bar{x}z/ \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}x\bar{y}y\bar{x}zQ[\mathbb{0},\ominus] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}zQ[\mathbb{0},\ominus] - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] - \bar{y}zQ[\ominus,\mathbb{0}] \\
&= \bar{y}x\bar{y}xQ[\oplus,\oplus] - \bar{y}z(Q[\mathbb{0},\ominus] + Q[\ominus,\mathbb{0}]) - \bar{y}x\bar{y}z\bar{x}zQ[\mathbb{0},\mathbb{0}] \qquad (14)
\end{aligned}$$

The right hand sides are equal since $Q[\mathbb{0},\ominus]+Q[\ominus,\mathbb{0}] = Q[\ominus,\mathbb{0}]+Q[\mathbb{0},\ominus]$. We have proved that $yx^{-1}z \sim zx^{-1}y$ implies that the case $D(\ominus,\ominus) \to D(\oplus,\oplus)$ is independent of the order of crossing switches.

This concludes the proof. □

Note that Proposition 3.2.15 is just a partial result and that further constraints may be placed on $\sim$ as necessary conditions for the computation to be independent of which base points we choose, what ordering of components we choose and that the computation is actually gives the same results for diagrams related by Reidemeister moves and orientation-preserving self-homeomorphisms of the plane.

In closing we would like to shortly discuss the relation $yx^{-1}z \sim zx^{-1}y$. The relation is derived from the equation

$$xQ[D(\oplus)](x,y,z) + yQ[D(\ominus)](x,y,z) + zQ[D(\mathbb{0})](x,y,z) = 0$$

which has some symmetry for x and y. It stands to reason that $xy^{-1}z \sim zy^{-1}x$ should also be true. By a simple computation this is actually an equivalent condition. In fact, the relation is true for any permutation of the symbols x,y and z.

Proposition 3.2.16. *As elements of* $\mathbb{Z}\{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}/\sim$,

$$yx^{-1}z = zx^{-1}y \iff xy^{-1}z = zy^{-1}x \iff xz^{-1}y = yz^{-1}x$$

Proof.

$$\boxed{yx^{-1}z = zx^{-1}y}$$

$$\iff$$

$$x^{-1}z = y^{-1}zx^{-1}y$$

$$\iff$$

$$z = xy^{-1}zx^{-1}y$$

$$\iff$$

$$zy^{-1} = xy^{-1}zx^{-1}$$

$$\iff$$

$$\boxed{zy^{-1}x = xy^{-1}z}$$

$$\iff$$

$$y^{-1}x = z^{-1}xy^{-1}z$$

$$\iff$$

$$x = yz^{-1}xy^{-1}z$$

$$\iff$$

$$xz^{-1} = yz^{-1}xy^{-1}$$

$$\iff$$

$$\boxed{xz^{-1}y = yz^{-1}x}$$

□

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