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The Pick Interpolation Theorem and Some Related Topics

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Abstract

In this report the classic Nevanlinna-Pick-Schur interpolation problem is dealt with. The focus is on understanding the theory and Pick's original work and how it can be used in applications.

Denna rapport behandlar det klassiska ämnet Nevalinna-Pick-Schur-interpolation. Dessutom diskuteras ett antal tillämpningar och lösningar presenteras.

Författaren vill rikta ett stort tack till min handledare Yishao Zhou för inspiration och tålamod. Tack även till Jan-Erik Björk av samma anledningar.

1 Introduction

In his paper [7], Pick stated and proved the following statement:
Theorem. (The Pick Interpolation Theorem) *Given pairs of complex numbers* (z_α, w_α) *(*$\alpha = 1, 2, ...n$*) with* z_α *in the open unit disc* $\mathbb{D}$ *and* w_α *in the closed unit disc* $\overline{\mathbb{D}}$*, the necessary and sufficient conditions for the existence of an analytic function* $f : \mathbb{D} \to \overline{\mathbb{D}}$ *such that* $f(z_\alpha) = w_\alpha$ *for* $\alpha = 1, 2, ..., n$*, are that the so-called* Pick matrix P_n *is positive semi-definite, and the Pick matrix* P_n *is of the form*

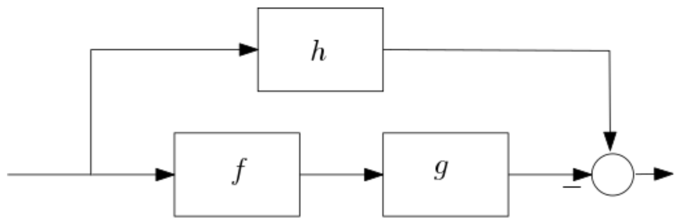
$$\left(\frac{1 - w_\alpha \overline{w}_\beta}{1 - z_\alpha \overline{z}_\beta} \right)_{\alpha,\beta=1,2,...,n}.$$

When P_n *is positive semi-definite there is a finite Blaschke product of degree at most* n *which solves* $f(z_\alpha) = w_\alpha$ *for* $\alpha = 1, 2, ..., n$.

This is a remarkable result which makes the theorem algebraically checkable, with a digital computer for example. Thus it is very useful as far as the applications are concerned. The aim of this report is to understand Pick's original proof given in [7], and solutions provided by Nevanlinna and Schur and how the theorem can be applied in a diversity of applications, in particular, in circuit theory and modern robust control theory.

We start with some simple examples to show how the theorem can be applied.

Example 1. (Optimizing the command response, [6]) In $\mathcal{H}_\infty$ controller synthesis problems we are asked to analyze the design of a reference signal preflight's in command tracking applications. Such a command response optimization can be depicted by the flow diagram above



where the plant model g is supposed to be a given stable rational transfer function (i.e. g has no poles in $\overline{\mathbb{C}^+}$) and h is a given stable rational transfer function with desired command response properties. The design task is to find a stable rational prefilter with a transfer function f such that $\|h - gf\|_\infty$ is minimized. Here the ∞-norm is defined as

$$\|h\|_\infty = \sup_{\omega \in \mathbb{R}} |h(i\omega)|, \quad i^2 = 1.$$

An unstable prefilter is unacceptable in practical applications because it results in unbounded control signals and actuator saturation.

Now if g has no zeros in $\overline{\mathbb{C}^+}$, then we may simply set $f = g^{-1}h$. In case g has zeros in $\mathbb{C}^+$, the plant inverse leads to an unstable prefilter unless $\mathbb{C}^+$-poles of g^{-1} happen to be cancelled by zeros of h. Thus, when g has right-half-plane zeros, the requirement that the prefilter be stable forces us to accept some error between gf and h:

$$e = h - gf \quad \Longleftrightarrow \quad f = g^{-1}(h - e).$$

Assume now that the right-half-plane zeros of g are $z_1, z_2, ..., z_m$ of multiplicity one, the prefilter will be stable if and only if

$$e(z_\alpha) = h(z_\alpha), \quad \alpha = 1, 2, ..., m,$$

since the unstable poles of g^{-1} will be cancelled by the zeros of $h - e$. The previous conditions are called *interpolation conditions/constraints.* This is an example of the Pick interpolation problem. This is because the half-plane can be one-to-one mapped to the unit disc by the Möbius transform (see any text book in complex analysis or the Appendix). The Pick matrix then has its components of the form

$$\frac{w_\alpha + \overline{w}_\beta}{z_\alpha + \overline{z}_\beta}.$$

More concrete we consider the functions $g = \frac{s-1}{s+2}$, $h = \frac{s+1}{s+3}$. Obviously g has a single zero at $s = 1$ so there is a single interpolation condition

$$e(1) = h(1) = \frac{s+1}{s+3}\Big|_{s=1} = \frac{1}{2}.$$

The Pick matrix should be (in the half-place case) a scalar $\frac{w_1+\overline{w}_1}{z_1+\overline{z}_1} = \frac{1}{2} > 0$. By the Pick Theorem the solution for e exists.

Next we consider two interpolation conditions. Let $f = \frac{(s-1)(s-2)}{(s+3)^2}$ $h = \frac{2}{3(s+3)}$. The interpolation conditions are

$$e(1) = h(1) = \frac{1}{6}, \quad e(2) = h(2) = \frac{2}{15}.$$

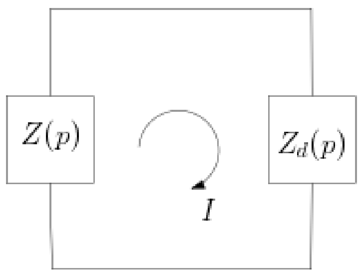
In this case the Pick matrix is

$$\begin{pmatrix} \frac{1/3}{1+1} & \frac{1/6+2/15}{1+2} \\ \frac{1/6+2/15}{1+2} & \frac{4/15}{2+2} \end{pmatrix} = \begin{pmatrix} 1/6 & 1/10 \\ 1/10 & 1/15 \end{pmatrix}.$$

It is easy to check that it is positive definite. Again by the Pick Theorem there is a function e interpolating the conditions above.

Example 2. (Design of a small signal oscillator of an active impedance [11])

Assume there is an active impedance (i.e. a tunnel diode) $Z_d(p)$ available. The design of a small signal oscillator involves, at least in the preliminary conceptual stage, the problem of embedding $Z_d(p)$ in an appropriate passive environment $Z(p)$ in order to achieve a prescribed set of modes, see the following Figure of the embedding of an active 2-terminal device in a passive environment.



The complex frequencies p_α which correspond to a non-zero circulating current I satisfy the equation

$$Z(p_\alpha) + Z_d(p_\alpha) = 0 \Leftrightarrow Z(p_\alpha) = -Z_d(p_\alpha).$$

Hence, the problem of achieving a given set of frequencies $p_i, ..., p_n$ in $\mathbb{C}^+$ is equivalent to generating a positive real function $Z(p)$ which assumes the values $-Z_d(p_\alpha)$ at the given points $p_\alpha, \alpha = 1, ..., n$. By positive real function we mean $Z(s)$ is $\mathcal{H}_\infty$ and satisfies

$$Z(i\omega) + \overline{Z(i\omega)} \geq 0.$$

For example, let us consider the three pairs $(1, 2)$, $(2, \frac{5}{2})$, $(3, \frac{10}{3})$. Is it possible to find a positive real function $Z(s)$ such that $Z(1) = 2$; $Z(2) = \frac{5}{2}$, $Z(3) = \frac{10}{3}$?

Youla and Saito [11] raised the following question: *Given n pairs $(p_1, z_1), (p_2, z_2), ..., (p_n, z_n)$ with* $\operatorname{Re} p_\alpha > 0$*,* $\alpha = 1, ..., n$ *what are the necessary and sufficient conditions for the existence of a positive real function $Z(p)$ satisfying the interpolation conditions*

$$Z(p_\alpha) = z_\alpha, \alpha = 1, ..., n\,?$$

They showed that the answer is that *the Pick-matrix of $n \times n$*

$$P = \left(\frac{\bar{z}_\alpha + z_\beta}{\bar{p}_\alpha + p_\beta} \right)_{\alpha,\beta=1,\ldots,n}$$

must be non-negative definite.

For the three given pairs we form the Pick-matrix

$$P = \begin{pmatrix} 2 & 3/2 & 4/3 \\ 3/2 & 5/4 & 7/6 \\ 4/3 & 7/6 & 10/9 \end{pmatrix}$$

A straightforward computation shows that the principal minors of P

$$2 > 0, \quad \begin{vmatrix} 2 & 3/2 \\ 3/2 & 5/4 \end{vmatrix} = 1/4 > 0, \text{ and } \begin{vmatrix} 2 & 3/2 & 4/3 \\ 3/2 & 5/4 & 7/6 \\ 4/3 & 7/6 & 10/9 \end{vmatrix} = 0.$$

Hence the Pick matrix is positive semi-definite. Thus the function $Z(p)$ exists.

Note that there are many approaches to the Pick interpolation problems, for example, in operator theory. We will follow Pick, Nevanlinna and Schur, [7, 8, 9], basically from point of view of classical analysis. Thus we sometimes say Nevanlinna-Pick-Schur interpolation problem.

And of course we cant omit the work of the Swedish mathematician Arne Beurling. His formulation and solution of the interpolation problem clarified and developed the problem in a most elegant way. But this leads us to far away from the scope of this work.[2]

This report is organized as follows. In Section 2 we collect some definitions and basic facts necessary for our problems. In particular, we discuss the Schwarz Lemma, the Schwarz-Pick Lemma, finite Blaschke products and their consequences and solve interpolation problems for $n = 1, 2$. They provide some insights and techniques for solving more general Nevanlinna-Pick-Schur interpolation problems. In Section 3 we give a complete proof of the Pick Interpolation Theorem on existence. We shall, in Section 4, prove in detail many of the statements in [7] on the solution of the interpolation problem where the Pick matrix is singular. Section 5 is a short review of Nevanlinnas approach to the problem and Schur's algorithm is presented Section 6. Finally in Section 7 we determine solutions to the examples above together with solutions for feedback stabilization of linear dynamical plants with uncertainty in the gain factor, which is the foundation work for modern robust control theory. We hope to share with our readers some geometric pictures of the topics presented in this report. The last mentioned point was the original motivation for the author to study this subject due to the fact that it brings linear algebra into the picture. We hope that linear algebra and classical analysis deserve a larger space in our mathematical education.

2 The Schwarz Lemma and the Schwarz-Pick Lemma

Now we turn to a short discussion on the Pick interpolation theorem. The theorem concerns finding necessary and sufficient conditions for the existence of an analytic function $f(z_\alpha) = w_\alpha$ and its successive derivatives when z_α is in the open unit disc $\mathbb{D}$ and w_α is in $\overline{\mathbb{D}}$. In this report we focus on the case where z_α's are distinct and simple.

Pick and Nevalinna gave solutions independently in [7], [8]. Picks proof concerns finite sequences whereas Nevanlinna gave a solution for denumerable sets recursively and Schur parameterized the solution set. [9]. Pick showed that the necessary and sufficient conditions are that the Pick matrix

$$P_n = \left(\frac{1 - w_\alpha \overline{w}_\beta}{1 - z_\alpha \overline{z}_\beta} \right)_{\alpha,\beta=1,\dots,n}$$

is positive semi-definite. The matrix above is given for the *Schur class*, $\mathcal{S}$, of functions which is the set of analytic functions from $\mathbb{D}$ to $\overline{\mathbb{D}}$. The necessity of the theorem comes from a derivation of the Pick matrix from the Cauchy and Poisson formulas for analytic functions where the former gives the function value in terms of a line integral around a closed path and the latter in terms of its real part on the border of its definition. The result is the Pick matrix for the *Carathéodory class* of analytic functions with funtions from $\mathbb{D}$ to $\overline{\mathbb{C}^+}$

$$\left(\frac{w_\alpha + \bar{w}_\beta}{1 - \bar{z}_\alpha z_\beta} \right)_{\alpha,\beta=1,\dots,n}.$$

As we have already seen in the previous section this is a third alternative of the Pick Matrix.

For the sufficiency of the proof we will use the maximum modulus theorem, Schwarz lemma, the Schwarz-Pick lemma, and the fact that the determinant of a Hermitian matrix is a quadratic form. For the sake of exposition we recall some basic definitions and theorems in this section.

The following notations are standard in the literature. The open unit disc $\mathbb{D}$ is defined by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and the closed unit disc is $\overline{\mathbb{D}}$. The unit circle is denoted as $\mathbb{T}$.

The term analytic refers to the property of having a derivative at every point where the function is defined. Therefore, an analytic function can be both real valued and complex valued. The function can then be represented with its Taylor series and thus it is infinitely differentiable and integrable. And so they can also be called holomorphic functions, from the greek words *holos* (whole) and *morphe* (form). A property of holomorphic functions is the maximum modulus theorem. The Pick interpolation theorem can be thought of as an extension of the Schwarz Pick Lemma. From this point of view, we review these theorems in this section. Proofs which have impact on interpolation will be given. We discuss these basic results in the light of the Nevanlinna-Pick-Schur interpolation problem.

2.1 The Maximum Modulus Theorem and the Schwarz Lemma

The Maximum Modulus Theorem says essentially that holomorphic functions have their maximum absolute value on the boundary of their domain of definition.

Theorem. [The Maximum Modulus Theorem] *If a holomorphic function attains its maximum in absolute value on an interior point it is a constant.*

Let Ω be a region in the complex plane. If $f(z)$ is analytic and non constant in Ω then its absolute value $|f(z)|$ has no maximum in Ω.

So if $w_0 = f(z)$ is any value in Ω there exits a neighbourhood $|w - w_0| < \epsilon$ contained in the image of Ω. In this neighbourhood there are points of modulus greater than $|w_0|$ and so $|f(z_0)|$ is not the maximum of $|f(z)|$.

The property of the non-vanishing derivative together with the maximum modulus theorem gives us,

Theorem. [The Schwarz Lemma] *If $f \in \mathcal{S}$ and $f(0) = 0$, then*

$$|f(z)| \leq |z|, \quad z \neq 0$$

$$|f'(0)| \leq 1$$

Furthermore, if $|f(z)| = |z|$ for some $0 \neq z \in \mathbb{D}$, or $|f'(0)| = 1$ then $f(z) = ze^{i\theta}$ (for some $\theta \in \mathbb{R}$).

In other words, f rotates z on the unit circle.

Remark. Geometrically this is a holomorphic mapping of $\mathbb{D}$ into $\mathbb{D}$ with the origin fixed. The Schwarz Lemma gives a connection between function theory and geometry in that it relates the modulus of an analytic function to its analytic properties of having a derivative at every point of its definition. □

We recall the proof here since the idea of the proof will be used later.

Proof. When f is analytic it has a Taylor series representation as $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since $f(0) = 0$ we can define

$$g(z) = \begin{cases} f(z)/z & 0 < |z| < 1 \\ a_1 = f'(0) & z = 0 \end{cases}.$$

This is a holomorphic function from $\mathbb{D}$ to $\mathbb{D}$. For $z \in \mathbb{D}$ there exists $r < 1$ such that $|z| < r < 1$. Then by the Maximum Modulus Theorem,

$$|g(z)| \leq \sup_{|\zeta|=r} |g(\zeta)| = \sup_{|\zeta|=r} \frac{|f(\zeta)|}{r} \leq \frac{1}{r}$$

Letting $r \to 1$ yields $|g(z)| \leq 1$, i.e., $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$ as desired.

If $|f(c)| = |c|$ for some non-zero $c \in \mathbb{D}$ of $|f'(0)| = 1$, that is, $|g(c)| = 1$ for some non-zero $|c| < 1$ then g is constant with modulus 1 according to the Maximum Modulus Theorem and hence $f(z) = ze^{i\theta}$ (for some $\theta \in \mathbb{R}$). $\square$

2.2 The Schwarz-Pick Lemma

The Schwarz Lemma is given with the origin as a fixed point. If we choose to have another point fixed we get the first extension of this lemma.

Theorem. [The Schwarz-Pick Lemma] *Let $f \in \mathcal{S}$. Then*

$$\left| \frac{f(z) - f(a)}{1 - f(z)\overline{f(a)}} \right| \leq \left| \frac{z - a}{1 - \bar{a}z} \right|, \quad \forall z, a \in \mathbb{D}, \tag{SP1}$$

and

$$\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}, \qquad \forall z \in \mathbb{D}. \tag{SP2}$$

Furthermore, equality in (SP1) *for some $z, a \in \mathbb{D}$ or in* (SP2) *for some $z \in \mathbb{D}$ occurs if and only if f is automorphism of $\mathbb{D}$, consequently f is a Möbius transform.*

The inequality says that for all $z \in \mathbb{D}$ and $|a| < 1$ the function f contracts the circles with "origin" a.

Recall that an *automorphism* of a region is a bijective conformal mapping of the region to itself. To prove the The Schwarz-Pick Lemma, we first show

Proposition. *Every automorphism $\varphi : \mathbb{D} \to \mathbb{D}$ is a Möbius transform*

$$\varphi(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}$$

for some $\theta \in \mathbb{R}$ (Indeed $\theta = \arg \varphi'(0)$), where $a = \varphi^{-1}(0) \in \mathbb{D}$. In particular, every $\varphi \in \mathrm{Aut}(\mathbb{D})$ extends continuously to a homeomorphism of $\overline{\mathbb{D}}$ onto itself.

Proof. Using the identity

$$\left| \frac{z - a}{1 - \bar{a}z} \right|^2 = 1 - \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}, \ \forall z \in \mathbb{D} \tag{Ic}$$

which is

$$1 - |\varphi(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2}, \ \forall z \in \mathbb{D}$$

and hence $\varphi(\mathbb{D}) \subset \mathbb{D}$; Moreover, a straightforward calculation gives the inverse of φ

$$\varphi^{-1}(z) = e^{-i\theta} \frac{z + ae^{i\theta}}{1 + \bar{a}e^{-i\theta} z}$$

(which can be verified by a routine computation). Now let Φ be the set of automorphisms of the Möbius transforms. It is easy to show that Φ is a group (with composition as the binary operation) and it acts transitively on $\mathbb{D}$. Hence if φ is a automorphism of $\mathbb{D}$ there exists $\psi \in \Phi$ such that $\psi \circ \varphi(0) = 0$; thus it suffices to show that every automorphism φ of $\mathbb{D}$ leaving 0 fixed is of the form $\varphi(z) = e^{i\theta} z$ for some real number θ and hence belongs to Φ. But if we apply the Schwarz Lemma to φ and φ^{-1}, we see that $|\varphi'(0)| = 1$ and the statement follows from the Schwarz Lemma. □

Proof of Schwarz-Pick Lemma. Pick an $a \in \mathbb{D}$. Let

$$\varphi_1(z) = \frac{z - a}{1 - \bar{a}z}, \quad \varphi_2(z) = \frac{z - f(a)}{1 - \overline{f(a)}z}.$$

By the Proposition above they belong to $\mathrm{Aut}(\mathbb{D})$. Now we apply the Schwarz Lemma to $\varphi_2 \circ f \circ \varphi_1$. That is

$$|(\varphi_2 \circ f \circ \varphi_1)(z)| \leq |z|$$

or

$$|(\varphi_2 \circ f)(z)| \leq |\varphi_1(z)|$$

which is

$$\left| \frac{f(z) - f(a)}{1 - f(z)\overline{f(a)}} \right| \leq \left| \frac{z - a}{1 - \bar{a}z} \right|,$$

as desired. It is the same as

$$\left| \frac{f(z) - f(a)}{(1 - f(z)\overline{f(a)})(z - a)} \right| \leq \left| \frac{1}{1 - \bar{a}z} \right|$$

Thus the second inequality can be obtained by letting $a \to z$. □

Remark 1. As we have studied in the first complex analysis course if w is a point in the upper half-plane (i.e. $\mathrm{Im}\, w > 0$), then

$$\phi_w(z) = \frac{z - w}{z - \bar{w}}$$

is an invertible holomorphic function of z mapping from the upper half-plane onto the unit disc and $\phi_w(w) = 0$. The identity (Ic) for the circle becomes

$$\left| \frac{z - w}{z - \bar{w}} \right|^2 = 1 - \frac{2(\mathrm{Im}\, z)(\mathrm{Im}\, w)}{|z - \bar{w}|^2}.$$

So ϕ_w maps the upper half-plane into the unit disc. Indeed the map is bijective. This is because

$$\phi_w^{-1}(\zeta) = \frac{w - \bar{w}\zeta}{1-\zeta},$$

and

$$\operatorname{Im} \frac{w - \bar{w}\zeta}{1-\zeta} = \frac{(\operatorname{Im} w)(1 - [\zeta|^2)}{|1-\zeta|^2}.$$

If f is a holomorphic mapping of the upper half-plane into itself, and w is an arbitrary point in the upper half-plane, then the $\phi_{f(w)} \circ f \circ \phi_w^{-1}$ maps the unit disc to itself, making the point 0 fixed. The derivative

$$(\phi_{f(w)} \circ f \circ \phi_w^{-1})'(0) = \frac{\phi'_{f(w)}(f(w))f'(w)}{\phi'_w(w)}$$

and

$$\phi'_w(w) = 1/(2i \operatorname{Im} w)$$

Thus the Schwarz Lemma yields

$$|f'(w)| \leq \frac{\operatorname{Im} f(w)}{\operatorname{Im} w}, \quad \operatorname{Im} w > 0.$$

This is the Schwarz Lemma for the upper half-plane. In a similar way the Schwarz-Pick Lemma reads

$$|\phi_{f(w)} \circ f(z)| \leq |\phi_w(z)| \quad \text{equivalently} \quad \left| \frac{f(z) - f(w)}{f(z - \overline{f(w)}} \right| \leq \left| \frac{z-w}{z-\bar{w}} \right|.$$

In fact we can transform the results on the unit circle to apply to a disk with an arbitrary center and an arbitrary radius to any half-plane. In control theory and design the right half-planes and unit circles are common objects. It is useful to know that we can switch back and forth between the two settings so in the sequel we will do so without further discussion. □

Remark 2. It is worth pointing out that every holomorphic function f on the unit disc different from the identity map has at most one fixed point. It is because if $f(z_1) = z_1$ and $f(z_2) = z_2$ for two distinct points z_1 and z_2 in $\mathbb{D}$ (assuming $z_1 = 0$, without loss of generality) then $f(z) = e^{i\theta}z$ for some real number θ. However $e^{i\theta}z_2 = z_2$ leads to $e^{i\theta} = 1$. So f is an identity map on the unit circle. □

Remark 3. The inequality in the Schwarz-Pick Lemma is called an *invariant form.* Below we explain why. Led by the inequality (SP1) we introduce

$$\delta(z_1, z_2) = \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$$

This is the *pseudohyperbolic metric* on $\mathbb{D}$. The Schwarz-Pick Lemma says: *A holomorphic map from $\mathbb{D}$ to $\mathbb{D}$ is Lipschitz continuous in this metric:*

$$\delta(f(z), f(w)) \leq \delta(z, w)$$

The lemma also indicates that the metric is invariant under Möbius transform

$$\delta(\varphi(z), \varphi(w)) = \delta(z, w).$$

In other words we can reformulate the Schwarz-Pick lemma: *A holomorphic self map of the unit disc is either an isometry (preserves distances and norms) or a contraction with respect to the hyperbolic metric.* □

Remark 3. The Schwarz-Pick lemma is an extension of the Schwarz lemma with a new origin a. In the same way the Pick Interpolation Theorem can be seen as a generalization of the lemma but with an arbitrary number of points. In fact, when $n = 2$ the Pick Interpolation Theorem is equivalent to the Schwarz-Pick lemma. This can be proven directly. Note that P_n is positive semidefinite, denoted by $P_n \geq 0$, if and only if $1 - |w_1| \geq 0$ and the $\det(P_2) \geq 0$, which is

$$\frac{(1 - |w_1|^2)(1 - |w_2|^2)}{|1 - \bar{w}_1 w_2|^2} \geq \frac{(1 - |z_1|^2)(1 - |z_2|^2)}{|1 - \bar{z}_1 z_2|^2}$$

By the useful identity (Ic) the last inequality can be rewritten as

$$\left| \frac{w_1 - w_2}{1 - \bar{w}_1 w_2} \right| \leq \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right|$$

which is (SP1). □

2.3 Finite Blaschke products

Finite Blaschke products play an important role in solutions of the Nevanlinna-Pick-Schur interpolation problems. It is due to the following theorem by Fatou [5]:

Theorem. *If f is analytic on $\mathbb{D}$ and $|f(z)| \to 1$ as $|z| \to 1$, then f is a finite Blaschke product.*

Proof. Note that $|f(z)| \to 1$ uniformly as $|z| \to 1$. Then there is an $r < 1$ such that f is non-vanishing on the annulus $\{z : r \leq |z| < 1\}$. Consequently, f has at most a finite number of zeros in $\mathbb{D}$. Let B be the finite Blaschke product formed from the zeros of f, counted with multiplicity. Then f/B and B/f are analytic in $\mathbb{D}$ and their moduli tend uniformly to 1 as we approach $\mathbb{T}$. By the Maximum Modulus Principle $|f/B| \leq 1$ and $|B/f| \leq 1$ on $\mathbb{D}$, so f/B is constant on $\mathbb{D}$. Since this constant must be unimodular, proving that f is a unimodular scalar multiple of B. □

The theorem says when the function f is uniquely determined and of modulus 1 it can be described by the finite Blaschke product:

$$B(z) = |c| \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a}_i z}.$$

Here the a_i's are the finite list of the zeros of $B(z)$ and c is a constant with modulus 1, thus can be written as $e^{i\theta}$ for some real number θ.
The uniqueness follows from the fact that if there were two functions f and g, the difference $f - g$ would, thanks to the maximum principle, be zero. Therefore, if $f = B(z)$, then f is unique.

Each finite Blaschke product belongs the disk algebra $\mathcal{A}(\mathbb{D})$, the set of analytic functions on $\mathbb{D}$ that extend continuously on $\overline{\mathbb{D}}$. In fact, the finite Blaschke products are the only elements of $\mathcal{A}(\mathbb{D})$, that map $\mathbb{T}$ into $\mathbb{T}$.
Corollary. *If $f \in \mathcal{A}(\mathbb{D})$ is unimodular on $\mathbb{T}$, then f is a finite Blaschke product.*

Rephrase

$$B(z) = e^{i\theta} \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a}_i z}.$$

It has the properties:

- B is continuous across $\partial\mathbb{D} = \mathbb{T}$;
- $|B| = 1$ on the boundary $\mathbb{T}$;
- $B(z) = 1/\overline{B(1/\bar{z})}$, if $z \in \mathbb{C} \cup \{\infty\}$; and
- B has finitely many zeros in $\mathbb{D}$.

In particular, every finite Blaschke product belongs to $H^\infty = \mathcal{H}_\infty(\mathbb{D})$, the set of bounded analytic functions on $\mathbb{D}$.

Clearly, a finite Blaschke product is a rational function. Recall that if P and Q are two coprime polynomials then the *degree* (or *order*) of the rational function $f = P/Q$ is defined $\deg f = \max\{\deg P, \deg Q\}$. So the finite Blaschke product B described above has degree n. Hence for each $w \in \mathbb{C} \cup \{\infty\}$ the equation $B(z) = w$ has exactly n solutions counted by multiplicity. If $w \in \mathbb{D}$ then these solutions lie in $\mathbb{D}$; if $w \in \mathbb{T}$ then these solutions lie on $\mathbb{T}$.

It is apparent that the set of all finite Blaschke products is closed under pointwise multiplication. Next we show that it is also closed under composition like Möbius transforms, as anticipated since each factor in $B(z)$, called *Blaschke product*, is a Möbius transform

$$M_a(z) = \frac{z - a}{1 - \bar{a} z}.$$

Proposition. *Assume B_1 and B_2 are finite Blaschke products. Then $B_1 \circ B_2$ is a finite Blaschke product. Moreover, if n_1 and n_2 are the degree of B_1 and B_2, respectively, then the degree of $B_1 \circ B_2$ is $n_1 n_2$.*

Proof. Call the zeros of B_1 $a_1, ..., a_{n_1}$. So

$$B_1 = e^{i\theta_1} M_{a_1} M_{a_2} \cdots M_{a_{n_1}}.$$

Then

$$B_1 \circ B_2 = e^{i\theta}(M_{a_1} \circ B_2) \cdot (M_{a_2} \circ B_2) \cdots (M_{a_{n_1}} \circ B_2).$$

If we can show that each $M_{a_k} \circ B_2$ ($k \in \{1, ..., n_1\}$) is a finite Blaschke product of order n_2 then we proved the theorem.

We know that the function $M_{a_k} \circ B_2$ is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$, and unimodular on the boundary $\mathbb{T}$ the Corollary above tells us that $M_{a_k} \circ B_2$ is a finite Blaschke product. Moreover $M_{a_k} \circ B_2(z) = 0$ if and only if $B_2(z) = a_k$. As discussed before the equation $B_2(z) = a_k$ has exactly n_2 solutions in $\mathbb{D}$. Hence, $M_{a_k} \circ B_2$ is a finite Blaschke product of degree n_2. □

Notice that we can also prove that $B_2 \circ M_{a_k}$ is a finite Blaschke product of degree n_2.

Now we discuss the relation between the interpolation problem and finite Blaschke products. The Möbius transformation of the form

$$M(z) = \frac{pz + q}{\bar{q}z + \bar{p}}$$

can be written as

$$M_a(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

It follows from the following algebraic manipulation. Write $q = -pa$ and $a = -q/p$ so $|a| = |q|/|p| < 1$.Then we get

$$M(z) = \frac{pz - pa}{-\bar{p}\bar{a}z + \bar{p}} = \left(\frac{p}{\bar{p}}\right) \frac{z - a}{-\bar{a}z + 1} = \left(\frac{p}{\bar{p}}\right) \frac{z - a}{1 - \bar{a}z}$$

Since

$$\left(\frac{p}{\bar{p}}\right) = e^{i\theta}$$

we get

$$M_a(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z}.$$

The function M is indeed an interpolating function of the conditions $M(0) = q/\bar{p}$, and $M(a) = 0$. In this case the 2×2 Pick matrix is

$$\begin{pmatrix} 1 - \left|\frac{q}{\bar{p}}\right|^2 & 1 \\ 1 & 1/\left(1 - \left|\frac{q}{p}\right|^2\right) \end{pmatrix}$$

Obviously it is semi-definite and the determinant is equal to 0. By the Schwarz-Pick Lemma (or the Pick Interpolation Theorem for $n = 2$), there is an analytic function on $\mathbb{D}$ satisfying the interpolation conditions. The function is a finite Blaschke product of degree 1.

When the matrix is a singular of $n \times n$ we get a Blaschke product of at most $n-1$ degree as stated by Pick.

Finally in this section, we illustrate the usefulness of the identity (Ic) by proving the following equality which is of interest in the study of finite Blaschke products: Let

$$B(z) = \prod_{j=1}^{n} \frac{z - a_j}{1 - \bar{a}_j z},$$

$$B_k(z) = \prod_{j=1}^{k-1} \frac{z - a_j}{1 - \bar{a}_j z}, \quad 2 \leq k \leq n, \quad B_1(z) = 1$$

For every $z \in \mathbb{C} \setminus \mathbb{T}$,

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \sum_{k=1}^{n} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \bar{a}_j z|^2}.$$

We prove this identity by induction on n. When $n = 1$ it is the useful identity (Ic). Assume the equality holds for $n - 1$, i.e.

$$\frac{1 - |B_n(z)|^2}{1 - |z|^2} = \sum_{k=1}^{n-1} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \bar{a}_j z|^2}.$$

Now

$$B(z) = B_n(z) \frac{z - a_n}{1 - \bar{a}_n z}$$

we have, using the identity (Ic),

$$\begin{aligned} 1 - |B(z)|^2 &= 1 - |B_n(z)|^2 \left| \frac{z - a_n}{1 - \bar{a}_n z} \right|^2 \\ =& 1 - |B_n(z)|^2 + |B_n(z)|^2 \left(1 - \left| \frac{z - a_n}{1 - \bar{a}_n z} \right|^2 \right) \\ =& 1 - |B_n(z)|^2 + |B_n(z)|^2 \frac{(1 - |z|^2)(1 - |a_n|^2)}{|1 - \bar{a}_n z|^2}. \end{aligned}$$

By the induction assumption we obtain

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \frac{1 - |B_n(z)|^2}{1 - |z|^2} + |B_n(z)|^2 \frac{1 - |a_n|^2}{|1 - \bar{a}_n z|^2} = \sum_{k=1}^{n} |B_k(z)|^2 \frac{1 - |a_k|^2}{|1 - \bar{a}_k z|^2}.$$

2.4 Three special cases

Now we demonstrate the use of the Schwarz Lemma and the Schwarz-Pick Lemma to solve some problems which will be used later, serving as a preparation for the general interpolation problem.

Problem 1. Show that
(a) if $f \in \mathcal{S}$, $|f(0)| < 1$, then

$$f_1(z) = \begin{cases} \dfrac{f(z) - f(0)}{z(1 - \overline{f(0)}f(z))}, & z \in \mathbb{D} \setminus \{0\} \\ \dfrac{f'(0)}{1 - |f(0)|^2}, & z = 0 \end{cases} \in \mathcal{S}.$$

(b) Let $\rho_0 \in \mathbb{D}$. Then

$$f(z) = \frac{\rho_0 + zg(z)}{1 + z\bar{\rho}_0 g(z)}$$

describes all functions in $\mathcal{S}$ such that $f(0) = \rho_0$, $\forall g \in \mathcal{S}$.
Proof. (a) Since $|f(0)| < 1$, $f(z)$ is not a constant with modulus 1, but it could be a constant less than 1 in modulus. So $|f(z)| < 1$ for all $z \in \mathbb{D}$. We know that the Möbius transform act transitively on $\mathbb{D}$ (by the Proposition) , so $|z| < 1$ and $|w| < 1$ imply

$$\left| \frac{z - w}{1 - z\bar{w}} \right| < 1 \tag{†}$$

implying

$$h(z) := \frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)} \in \mathcal{S}, \quad \forall z \in \mathbb{D},$$

and $h(0) = 0$. By the Schwarz Lemma

$$f_1(z) = \begin{cases} \dfrac{h(z)}{z}, & z \in \mathbb{D} \setminus \{0\} \\ h'(0), & z = 0 \end{cases} \in \mathcal{S}.$$

(b) Assume that $g \in \mathcal{S}$, we have

$$|zg(z)| < 1 \quad \forall z \in \mathbb{D}.$$

Taking z as $zg(z)$ and w as ρ_0 in (†) we can conclude that

$$f(z) = \frac{\rho_0 + zg(z)}{1 + \bar{\rho}_0 zg(z)} \in \mathcal{S}.$$

Obviously, $f(0) = \rho_0$, so for all $g \in \mathcal{S}$, $f(z) \in \mathcal{S}$. It remains to show that the other direction is also true. Assume now f is a solution, the function f_1 defined in (a) with $f(0) = \rho_0$ lies in $\mathcal{S}$. Solving f in terms of f_1 yields

$$f(z) = \frac{\rho_0 + zf_1(z)}{1 + \bar{\rho}_0 z f_1(z)} \in \mathcal{S}.$$

Hence f is in the desired form in (b). $\square$

Furthermore we can compute the derivative $f'(0)$ in terms if $g(0)$ and ρ_0 in (b):

$$\begin{aligned} f'(z) &= \frac{(g(z) + zg'(z))(1 + \bar{\rho}_0 z g(z)) - (\rho_0 + zg(z))(\bar{\rho}_0 g(z) + z\bar{\rho}_0 g'(z))}{(1 + \bar{\rho}_0 z g(z))^2} \\ &= (zg'(z) + g(z))\frac{1 - |\rho_0|^2}{(1 + \bar{\rho}_0 z g(z))^2} \end{aligned}$$

Then

$$f'(0) = g(0)(1 - |\rho_0|^2).$$

From this computation we can conclude the following: To find $f \in \mathcal{S}$ with presigned values of $f(0)$ and $f'(0)$ it is sufficient to first solve the problem of finding all $f \in \mathcal{S}$ such that $f(0)$ is given. Then finding all f (if any) such that $f'(0)$ is given is to determine all $g \in \mathcal{S}$ such that

$$g(0) = \frac{f'(0)}{1 - |\rho_0|^2}.$$

This problem (i) has no solution if

$$\rho_1 := \frac{f'(0)}{1 - |\rho_0|^2}$$

has modulus greater than 1; (ii) has a unique solution if $|\rho_1| = 1$ and (iii) has infinitely many solutions if $\rho_1 \in \mathbb{D}$.

Problem 2. Given two pairs of numbers (z_1, w_1) and (z_2, w_2) in $\mathbb{D}^2$, find a necessary and sufficient condition for a function $f \in \mathcal{S}$ to exist such that

$$f(z_1) = w_1, \; f(z_2) = w_2$$

and describe the set of of all solutions.
Solution. There are two cases.
(i) If $|w_1| = 1$ the only function in **S** for which $f(z_1) = w_1$ is the constant function $f(z_1) \equiv w_1$. Thus if $w_1 \neq w_2$ the interpolation problem at hand has no solution, and it has a unique solution $f(z) \equiv w_1$ if $w_1 = w_2$.
(ii) $|w_1| < 1$. By Problem 1, a function $f \in \mathcal{S}$ satisfies $f(z_1) = w_1$ if and only if it is of the form

$$f(z) = \frac{w_1 + \frac{z - z_1}{1 - \bar{z}_1 z} g(z)}{1 + \bar{w}_1 \frac{z - z_1}{1 - \bar{z}_1 z} g(z)}, \quad g \in \mathcal{S}$$

Then use the interpolation condition at z_2, $f(z_2) = w_2$ to determine the necessary and sufficient condition for f to exist. This constraint is equivalent to

$$w_2 = \frac{w_1 + \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} g(z_2)}{1 + \bar{w}_1 \frac{z_2 - z_1}{1 - \bar{z}_1 z_2} g(z_2)} \quad \Leftrightarrow \quad g(z_2) = \frac{w_2 - w_1}{1 - \bar{w}_1 w_2} \frac{1 - \bar{z}_1 z_2}{z_2 - z_1} := \rho.$$

Now we have three possibilities depending on the value of ρ:

(i) If $|\rho| > 1$ there is no solution.

(ii) If $|\rho| = 1$ there is a unique solution:

$$f(z) = \frac{w_1 + \frac{z - z_1}{1 - \bar{z}_1 z}\rho}{1 + \bar{w}_1 \frac{z - z_1}{1 - \bar{z}_1 z}\rho} = \frac{(\rho - w_1 \bar{z}_1)(z - \dfrac{-w_1 + \rho z_1}{\rho - w_1 \bar{z}_1})}{(1 - \bar{w}_1 \rho z_1)(1 - \dfrac{\bar{z}_1 - \bar{w}_1 \rho}{1 - \bar{w}_1 \rho z_1} z)}$$

Set $\zeta_1 := \dfrac{-w_1 + \rho z_1}{\rho - w_1 \bar{z}_1}$. By (†) $\zeta_1 \in \mathbb{D}$. Remembering that $|\rho| = 1$, we get

$$\bar{\zeta}_1 = \overline{\left(\frac{-w_1 + \rho z_1}{\rho - w_1 \bar{z}_1} \right)} = \frac{-\bar{w}_1 + \bar{\rho} \bar{z}_1}{\bar{\rho} - \bar{w}_1 z_1} = \frac{\bar{\rho}(\bar{z}_1 - \rho \bar{w}_1)}{\bar{\rho}(1 - \bar{w}_1 \rho z_1)} = \frac{\bar{z}_1 - \rho \bar{w}_1}{1 - \bar{w}_1 \rho z_1}$$

as in the denominator, and

$$\left| \frac{\rho - w_1 \bar{z}_1}{1 - \bar{w}_1 \rho z_1} \right| = \left| \frac{\rho - w_1 \bar{z}_1}{\rho(\bar{\rho} - \bar{w}_1 \rho z_1)} \right| = \frac{1}{|\rho|} = 1.$$

so we can define

$$e^{i\theta} := \frac{\rho - w_1 \bar{z}_1}{\rho(\bar{\rho} - \bar{w}_1 z_1)}, \ \theta \in \mathbb{R}$$

Hence

$$f(z) = \frac{\rho - w_1 \bar{z}_1}{\rho(\bar{\rho} - \bar{w}_1 z_1)} \cdot \frac{z - \zeta_1}{1 - \bar{\zeta}_1 z} = e^{i\theta} \frac{z - \zeta_1}{1 - \bar{\zeta}_1 z}$$

Thus the unique solution is a Blaschke product of degree 1. □

(iii) If $|\rho| < 1$ then there are infinitely many solutions of the form, using the result from Problem 1(b):

$$f(z) = \frac{\rho + \frac{z - z_2}{1 - \bar{z}_2 z} g(z)}{1 + \bar{\rho} \frac{z_2 - z}{1 - \bar{z}_2 z} g(z)}, \quad g \in \mathcal{S}.$$

In other words, the solution is parametrizied by any $g \in \mathcal{S}$.

To summarize, a necessary and sufficient for $f \in \mathcal{S}$ interpolating $f(z_1) = w_1$ and $f(z_2) = w_2$ is $|\rho| \leq 1$. Note that it is equivalent with that the Pick matrix

$$\left(\frac{1 - w_j \bar{w}_k}{1 - z_j \bar{z}_k}\right)_{j,k=1,2}$$

is positive semi-definite by the same reason as indicated in Remark 3 above. $\square$

Problem 3. (Carathéodory) *If $f(z) \in \mathcal{S}$ then there is a sequence $\{B_k\}$ of finite Blaschke products that converges to $f(z)$ pointwise on $\mathbb{D}$.*

Proof. Let $f(z) = c_0 + c_1 z + \ldots$ be the Taylor expansion of f . We shall find the Blaschke product of degree at most m whose first n coefficients match those of f:

$$B_n = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} + d_n z^n + \cdots$$

Now $|c_0| \leq 1$, and from the derivation of the Schur algorithm, we can take

$$B_0 = \frac{z + c_0}{1 + \bar{c}_0 z}.$$

If $|c_0| = 1$, then $B_0 = c_0$ is a Blaschke product of degree 0. Assume that for each $g \in \mathcal{S}$ we have constructed $B_{n-1}(z)$. Let

$$g(z) = \frac{1}{z} \frac{f - f(0)}{1 - \overline{f(0)} f}$$

and let B_{n-1} be a Blaschke product of degree at most $n-1$ such that $g - B_{n-1}$ has $n-1$ zeros at 0. Then $zg - zB_{n-1}$ has n zeros at 0. Define

$$B_n(z) = \frac{zB_{n-1}(z) + f(0)}{1 + \overline{f(0)} z B_{n-1}(z)}$$

Then B_n is a finite Blaschke product of degree $zB_{n-1} \leq n$, and

$$\begin{aligned} f(z) - B_n(z) = & \frac{zg(z) + f(0)}{1 + \overline{f(0)} z g(z)} - \frac{zB_{n-1}(z + f(0)}{1 + \overline{f(0)} z B_{n-1}(z)} \\ = & \frac{(1 - |f(0)|^2) z (g(z) - B_{n-1}(z))}{(1 + \overline{f(0)} z g(z))(1 + \overline{f(0)} z B_{n-1}(z))} \end{aligned}$$

so that $f - B_n$ has a zero of order n at $z = 0$. The proof is complete. $\square$

Remark. Without further discussion we point out that the Carathéodory Theorem is linked to the partial realization problem, see e.g. [3] .

3 The Pick interpolation Theorem

We now present Pick's derivation of the Pick matrix, thus the necessity, and then a complete proof. We start with a quote from a well known author: "The proof of the sufficiency is somewhat complicated". [1]

3.1 Derivation of the Pick Matrix

The necessity of the theorem comes from the Cauchy representation formula for analytic functions:

$$f(z) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(\xi)}{\xi - z}\, d\xi.$$

The formula lets us compute the value of $f(z)$ as soon as we know the values of $f(\xi)$ on the circle K centered at 0 with radius ρ. If we set $\xi = e^{i\theta}$ we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})\frac{e^{i\theta}}{e^{i\theta} - z}\, d\theta.$$

And with the point ζ on the circle we get

$$2\pi w = \int_0^{2\pi} w_K \frac{\zeta}{\zeta - z}\, d\theta \tag{1}$$

where $w = f(z)$ and w_K stands for $f(\zeta)$.

If we consider a point outside the circle K we get an integral of an analytic function

$$0 = \int_0^{2\pi} w_K \frac{\zeta}{\zeta - \frac{\rho^2}{\overline{z}}}\, d\theta$$

Since ζ is on the circle K, $\overline{\zeta}=\frac{\rho^2}{\zeta}$. Taking conjugate of this equality yields

$$0 = \int_0^{2\pi} \overline{w}_K \frac{\overline{\zeta}}{\overline{\zeta} - \frac{\rho^2}{z}}\, d\theta$$

Simplifying we get

$$0 = \int_0^{2\pi} \overline{w}_K \frac{z}{\zeta - z}\, d\theta. \tag{2}$$

Let $w_K = u_K + iv_K$. Add (1) to (2), after simplification,

$$2\pi w = \int_0^{2\pi} u_K \frac{\zeta + z}{\zeta - z}\, d\theta + i\int_0^{2\pi} v_K\, d\theta,$$

or

$$2\pi \overline{w} = \int_0^{2\pi} u_K \frac{\overline{\zeta} + \overline{z}}{\overline{\zeta} - \overline{z}}\, d\theta - i\int_0^{\pi} v_K\, d\theta.$$

We sum the previous two equations with the number pairs (z_α, w_α) in the first equation and (z_β, w_β) in the second and get

$$2\pi(w_\alpha + \overline{w}_\beta) = \int_0^{2\pi} u_K \left(\frac{\zeta + z_\alpha}{\zeta - z_\alpha} + \frac{\overline{\zeta} + \overline{z}_\beta}{\overline{\zeta} - \overline{z}_\beta} \right) d\theta = \int_0^{2\pi} u_K \frac{2(\zeta\overline{\zeta} - z_\alpha \overline{z}_\beta)}{(\zeta - z_\alpha)(\overline{\zeta} - \overline{z}_\beta)} d\theta$$

i.e.

$$\frac{w_\alpha + \overline{w_\beta}}{\rho^2 - z_\alpha \overline{z_\beta}} = \frac{1}{\pi} \int_0^{2\pi} u_K \frac{d\theta}{(\zeta - z_\alpha)(\overline{\zeta} - \overline{z_\beta})}$$

Now let $\rho^2 = 1$ and take the sum from $\alpha, \beta = 1, ..., n$. Multiply with $s_\alpha \overline{s}_\beta$ and sum over α and β we get

$$q_n(s_1, ..., s_n) = \sum_{\alpha,\beta=1}^{n} \frac{w_\alpha + \overline{w_\beta}}{1 - z_\alpha \overline{z_\beta}} s_\alpha \overline{s}_\beta = \frac{1}{\pi} \int_0^{2\pi} u_k \left| \sum_{\alpha}^{n} \frac{s_\alpha}{\zeta - z_\alpha} \right|^2 d\theta \geqq 0.$$

Here q_n is the hermitian form of the variables $s_1, ..., s_n$. Denote $\mathbf{s} = \begin{pmatrix} s_1 \\ \vdots \\ s_n \end{pmatrix}$. In matrix form

$$q_n = \mathbf{s}^H P_n \mathbf{s}$$

where

$$P_n = \left(\frac{w_\alpha + \overline{w}_\beta}{1 - z_\alpha \overline{z}_\beta} \right)_{\alpha,\beta=1}^{n}$$

is the Pick matrix. So we have proved that if there is a Schur function satisfying the interpolation constraints the Pick matrix is positive semi-definite.
Remark. This version is in the setting of the Carathoédory class of functions. In order to get the Pick matrix for Schur functions let us consider the analytic function f on the closed disc. Then the function $F = (1 + f)/(1 - f)$ has a positive real part. Let $F = U + iV$ and so we can represent F by

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} U(e^{i\theta}) d\theta + iV(0).$$

This yields

$$F(z_\alpha) + \overline{F(z_\beta)} = \frac{1}{\pi} \int_0^{2\pi} \frac{1 - z_\alpha \overline{z}_\beta}{(e^{i\theta} - z_\alpha)(e^{-i\theta} - \overline{z}_\beta)} U \, d\theta.$$

Therefore

$$\sum_{\alpha,\beta=1}^{n} \frac{F(z_\alpha) + \overline{F(z_\beta)}}{1 - z_\alpha \overline{z}_\beta} t_\alpha \bar{t}_\beta = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_{\alpha=1}^{n} \frac{t_\alpha}{e^{i\theta} - z_\alpha} \right|^2 U d\theta.$$

Note that by the transformation made in the beginning

$$F(z_\alpha) + \overline{F(z_\beta)} = \frac{2(1 - w_\alpha \overline{w}_\beta)}{(1 - w_\alpha)(1 - \overline{w}_\beta)}.$$

So we get a Hermitian matrix where the elements of the Pick matrix has the form

$$P_n = \left(\frac{1 - w_\alpha \overline{w}_\beta}{1 - z_\alpha \overline{z}_\beta} \right)_{\alpha,\beta=1}^{n}.$$

3.2 Proof of the Pick Interpolation Theorem

Theorem (The Pick Interpolation Theorem) *There is an $f \in \mathcal{S}$ satisfying the interpolation conditions $f(z_\alpha) = w_\alpha$ ($\alpha = 1, ..., n$) if and only if $P_n \geq 0$. When $P_n \geq 0$ there is a Blaschke product of degree at most n which satisfies the interpolation condition.*

Proof. By defintion, the Pick matrix P_n is positive semi-definite if and only if the corresponding Hermitian form is positive semi-definite, that is

$$P_n = \left(\frac{1 - w_\alpha \overline{w}_\beta}{1 - z_\alpha \overline{z}_\beta} \right)_{\alpha,\beta=1}^{n} \geq 0 \iff h_n(s_1, ..., s_n) = \sum_{\alpha,\beta=1}^{n} \frac{1 - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} s_\alpha \bar{s}_\beta \geq 0,\ \forall s \in \mathbb{C}$$

We use induction on n. Note that we have already proved the statement for $n = 1, 2$ as shown in Problems 1 and 2 in the previous section. Now assume that the theorem holds for $n - 1$, $n > 1$. We wish to show that it holds for n, that is $h_n \geq 0$ is equivalent to the existence of $f \in \mathcal{S}$ such that $f(z_\alpha) = w_\alpha$, for $\alpha = 1, ..., n$ with $(z_\alpha, w_\alpha) \in \mathbb{D}^2$.

Suppose $f(z_\alpha) = w_\alpha$. Then it is clear that $|w_n| \leq 1$. If $|w_n| = 1$ then the interpolating function is the constant w_n and $w_\alpha = w_n$, $1 \leq \alpha \leq n - 1$. Suppose that $h_n \geq 0$. Choose now $s_n = 1, s_\alpha = 0$ for $\alpha = 1, ..., n - 1$. This is $|w_n| \leq 1$. When $|w_n| = 1$ we choose $s_\alpha = 0$ for $\alpha \neq \beta (< n), n$. So

$$h_n(0, ..., 1, 0..., 0, 1) \geq 0 \iff \begin{pmatrix} \frac{1-|w_\beta|^2}{1-|z_\beta|^2} & \frac{1-w_\beta \bar{w}_n}{1-z_\beta \bar{z}_n} \\ \frac{1-w_n \bar{w}_\beta}{1-z_n \bar{z}_\beta} & \frac{1-|w_n|^2}{1-|z_n|^2} \end{pmatrix} \geq 0.$$

By (SP1) in The Schwarz-Pick Lemma this implies $w_\beta = w_n$, as shown before. Hence we can choose $B_n = w_n$ if $|w_n| = 1$.

Now we consider $|w_n| < 1$. To make calculations easier we change variables so that z_n and w_n are zero:

$$z'_\alpha = \frac{z_\alpha - z_n}{1 - \bar{z}_n z_\alpha}, \qquad w'_\alpha = \frac{w_\alpha - w_n}{1 - \bar{w}_n w_\alpha} \qquad \alpha = 1, ..., n$$

Apparently $z'_n = 0$, $w'_n = 0$ as desired.

Then there is $f \in \mathcal{S}$ satisfying the interpolation conditions if and only if

$$g(z) = \frac{f\left(\dfrac{z+z_n}{1+\bar{z}_n z}\right) - w_n}{1 - \bar{w}_n f\left(\dfrac{z+z_n}{1+\bar{z}_n z}\right)} \in \mathcal{S}$$

and

$$g(z'_\alpha) = w'_\alpha, \quad \alpha = 1, ..., n.$$

Moreover, if f is a Blaschke product of degree at most n then the same is true for g, and vice versa.

To be able to use the induction assumption we form the Hermitian form h'_n corresponding to the points $\{z'_1, ..., z'_{n-1}, 0\}$ and $\{w'_1, ..., w'_{n-1}, 0\}$ and try to find the relation to h_n.

First we compute:

$$\frac{1 - w'_\alpha \bar{w}'_\beta}{1 - w_\alpha \bar{w}_\beta} = \frac{1 - \frac{w_\alpha - w_n}{1 - \bar{w}_n w_\alpha} \frac{\bar{w}_\beta - \bar{w}_n}{1 - w_n \bar{w}_\beta}}{1 - w_\alpha \bar{w}_\beta} = \frac{1 - |w_n|^2}{(1 - \bar{w}_n w_\alpha)(1 - w_n \bar{w}_\beta)} = u_\alpha \bar{u}_\beta$$

where

$$u_\alpha = \frac{\sqrt{1 - |w_n|^2}}{1 - \bar{w}_n w_\alpha}.$$

In the same way for

$$\frac{1 - z'_\alpha \bar{z}'_\beta}{1 - z_\alpha \bar{z}_\beta} = \frac{1 - |z_n|^2}{(1 - \bar{z}_n z_\alpha)(1 - z_n \bar{z}_\beta)} = v_\alpha \bar{v}_\beta$$

and

$$v_\alpha = \frac{\sqrt{1 - |z_n|^2}}{1 - \bar{z}_n z_\alpha}.$$

We have

$$\frac{1 - w'_\alpha \bar{w}'_\beta}{1 - z'_\alpha \bar{z}'_\beta} s_\alpha \bar{s}_\beta = \frac{1 - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} \left(\frac{u_\alpha}{v_\alpha} s_\alpha\right) \overline{\left(\frac{u_\beta}{v_\beta} s_\beta\right)}$$

Hence

$$h'_n(s_1, ..., s_n) = h_n\left(\frac{u_1}{v_1} s_1, ..., \frac{u_n}{v_n} s_n\right).$$

Therefore

$$h'_n \geq 0 \iff h_n \geqq 0.$$

Consequently, the problem is reduced to the case $z_n = 0, w_n = 0$.

For simplicity we assume that $z_n = 0, w_n = 0$, that is there is $f \in \mathcal{S}$ solving $f(0) = 0$, $f(z_\alpha) = w_\alpha$, $\alpha = 1, ..., n-1$ if and only if there is $g(z)/z \in \mathcal{S}$ solving $g(z_\alpha) = w_\alpha / z_\alpha$,

$\alpha = 1, ..., n-1$. So now we have to know whether there is an $f \in S$ such that $f(0) = 0$ and $f(z_j) = w_j$ for $1 \leqq j \leqq n-1$. Moreover, f is a Blaschke product of degree at most d if and only if g a Blaschke product of degree at most $d-1$. By the induction assumption, $g(z_\alpha) = w_\alpha / z_\alpha$, $\alpha = 1, ..., n-1$ has solution if and only if the Hermitian form

$$\widetilde{h}_{n-1}(s'_1, ..., s'_{n-1}) = \sum_{\alpha,\beta=1}^{n-1} \frac{1 - \frac{w_\alpha}{z_\alpha} \overline{\frac{w_\beta}{z_\beta}}}{1 - z_\alpha \bar{z}_\beta} s'_\alpha \bar{s'_\beta} \geq 0$$

This means that the theorem reduces to show that

$$h_n \geqq 0 \iff \widetilde{h}_{n-1} \geqq 0$$

provided that $z_n = 0, w_n = 0$.

Using, z_n and $w_n = 0$ we get,

$$h_n(s_1, ..., s_n) = |s_n|^2 + 2\mathrm{Re} \sum_{\alpha=1}^{n-1} \bar{s}_\alpha s_n + \sum_{\alpha,\beta=1}^{n-1} \frac{1 - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} s_\alpha \bar{s}_\beta.$$

When completing the square relative to s_α we get two $s_\alpha \bar{s}_\beta$'s so we have to subtract one $g(z_\alpha) = w_\alpha / z_\alpha$, $\alpha = 1, ..., n-1$. It gives

$$h_n(s_1, ..., s_n) = \left| s_n + \sum_{\alpha=1}^{n-1} s_\alpha \right|^2 + \sum_{\alpha,\beta=1}^{n-1} \left(\frac{1 - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} - 1 \right) s_\alpha \bar{s}_\beta$$

But

$$\frac{1 - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} - 1 = \frac{z_\alpha \bar{z}_\beta - w_\alpha \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} = \frac{1 - \left(\frac{w_\alpha}{z_\alpha}\right) \overline{\left(\frac{w_\beta}{z_\beta}\right)}}{1 - z_\alpha \bar{z}_\beta} s_\alpha \bar{s}_\beta$$

yielding

$$\begin{aligned} h_n(s_1, ..., s_n) &= \left| \sum_{\alpha=1}^{n} s_\alpha \right|^2 + \sum_{\alpha,\beta=1}^{n-1} \frac{1 - \left(\frac{w_\alpha}{z_\alpha}\right) \overline{\left(\frac{w_\beta}{z_\beta}\right)}}{1 - z_\alpha \bar{z}_\beta} (s'_\alpha) \overline{(s'_\beta)} \\ &= \left| \sum_{\alpha=1}^{n} s_\alpha \right|^2 + \tilde{h}_{n-1}(z_1 s_1, ..., z_{n-1} s_{n-1}) \end{aligned}$$

It follows that $\tilde{h}_{n-1} \geq 0$ implies $h_n \geq 0$. Letting $s_n = -\sum_{\alpha=1}^{n-1} s_\alpha$ we get the implication $h_n \geq 0 \implies \tilde{h}_{n-1} \geq 0$. This completes the proof. □

Corollary. *Assume $P \geqq 0$. Then*

(i) *$f(z_\alpha) = w_\alpha$, ($\alpha = 1, ..., n$) has a unique solution $f(z) \in \mathcal{S}$ if and only* $\det(P_n) = 0$.

(ii) *If* $\det(P_n) = 0$ *and $m < n$ is the rank of P_n then the interpolating function is a Blaschke product of degree m. Conversely, if a Blaschke product of degree $m < n$ satisfying $f(z_\alpha) = w_\alpha$, $\alpha = 1, ..., n$, then P_n has rank m.*

(iii) *Let* $\det(P_n) > 0$ *and $z \in \mathbb{D}$, $z \neq z_\alpha$, $\alpha = 1, ..., n$. The set of values*

$$W = \{f(z) : f \in \mathcal{S}, f(z_\alpha) = w_\alpha, \alpha = 1, ..., n\}$$

is a nondegenerate closed disc contained in $\mathbb{D}$. If $f \in \mathcal{S}$ such that $f(z_\alpha) = w_\alpha, \alpha = 1, ..., n$, then $f(z) \in \partial W$ if and only if f is a Blaschke product of degree n. Moreover, if $w \in \partial W$ there is a unique solution to the interpolation problem in $\mathcal{S}$ which also solves $f(z) = w$.

Before processing the proof we recall that we have proved

(S1) $h'_n(s_1, ..., s_n) = h_n\left(\frac{u_1}{v_1}s_1, ..., \frac{u_n}{v_n}s_n\right)$.

(S2) $h_n(s_1, ..., s_n) = \left|\sum_{\alpha=1}^{n} s_\alpha\right|^2 + \tilde{h}_{n-1}(z_1 s_1, ..., z_{n-1}s_{n-1})$

Proof. (i) and (ii) Note that the problem is trivial if $|w_n| = 1$ because then $h_n = 0$, $m = 0$ and $B_n = w_n$. So we assume that $|w_n| < 1$.

We can also without loss of generality assume that $z_n = 0, w_n = 0$. By (S1), h_n and h'_n have same rank. By g described in the proof of the Pick Theorem, the original problem has a unique solution if and only if the problem with $g(z'_\alpha) = w'_\alpha$, $\alpha = 1, ..., n$ has a unique solution. Moreover it can be solved by a Blaschke product of degree m if and only if the original one can be solved.

Now that $z_n = w_n = 0$. The original problem has a unique solution if and only if $g(z_\alpha) = w_\alpha/z_\alpha$, $\alpha = 1, .., n-1$ has a unique solution; and $f(z_\alpha) = w_\alpha, \alpha = 1, ..., n$, can be solved with a Blaschke product of degree $m - 1$. Consequently by induction (i) and (ii) will be proved if we can show that

$$\operatorname{rank}(h_n) = 1 + \operatorname{rank}(\tilde{h}_{n-1}) \tag{S3}$$

Denote the Hermitian matrix related to $\tilde{h}_{n-1}$, $\tilde{P}_{n-1} = (p_{\alpha\beta})_{\alpha,\beta=1,...n-1}$. By inspection of (S2) the Hermitian matrix H_n relative to h_n is

$$H_n = \left(\begin{array}{ccc|c} & & & 1 \\ & 1 + z_\alpha \bar{z}_\beta p_{\alpha\beta} & & \vdots \\ & & & 1 \\ \hline 1 & \cdots & 1 & 1 \end{array}\right)$$

By elementary row operations,

$$\left(\begin{array}{c|c} & 1 \\ 1+z_\alpha\bar{z}_\beta p_{\alpha\beta} & \vdots \\ & 1 \\ \hline 1 \ \cdots \ 1 & 1 \end{array}\right) \sim \left(\begin{array}{c|c} & 0 \\ z_\alpha\bar{z}_\beta p_{\alpha\beta} & \vdots \\ & 0 \\ \hline 1 \ \cdots \ 1 & 1 \end{array}\right)$$

These two matrices should have same rank, but the latter has rank $1+\mathrm{rank}(\tilde{P}_n)$.
(iii) Again we assume $z_n = w_n$. Then, using (S3) $\det(\tilde{P}_{n-1}) > 0$. By induction,

$$\widetilde{W} = \{g(z) : g \in \mathcal{S}, g(z_\alpha) = w_\alpha/z_\alpha, \alpha = 1, ..., n-1\}$$

is a closed disc in $\mathbb{D}$. But then $W = \{z\zeta : \zeta \in \widetilde{W}\}$ is a close disc. Since $w \in \partial W$ if and only if $w/z \in \partial\widetilde{W}$, the other claims follow by induction. □

4 Pick's solutions

In his paper [7], Pick proved that if the determinant of the Pick matrix, being positive definite, is zero then the solution to the Nevanlinna-Pick-Schur interpolation problem is a unique real rational function of $n-1$ degree. Also, he constructed the solution. He then argued that the case where the determinant is positive can be reduced to the previous case by a parametrization, thus we get infinitely many solutions. The key in this proof is the following fact, whose proof was a single line in [7]:

Claim 1. *Given the n pairs of numbers $p_\alpha = (z_\alpha, w_\alpha)$, let the principal minors of the Pick matrix P_n be*

$$D(p_1, ..., p_k) = \det\left((p_{\alpha\beta})_{\alpha,\beta=1,...,k}\right) = \det\left(\left(\frac{w_\alpha - \bar{w}_\beta}{z_\alpha - \bar{z}_\beta}\right)_{\alpha,\beta=1,...,k}\right)$$

or explicitly

$$D(p_1, ..., p_k) = \begin{vmatrix} \frac{w_1-\bar{w}_1}{z_1-\bar{z}_1} & \frac{w_1-\bar{w}_2}{z_1-\bar{z}_2} & \cdots & \frac{w_1-\bar{w}_k}{z_1-\bar{z}_k} \\ \frac{w_2-\bar{w}_1}{z_2-\bar{z}_1} & \frac{w_2-\bar{w}_2}{z_2-\bar{z}_2} & \cdots & \frac{w_2-\bar{w}_k}{z_2-\bar{z}_k} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{w_k-\bar{w}_1}{z_k-\bar{z}_1} & \frac{w_k-\bar{w}_2}{z_k-\bar{z}_2} & \cdots & \frac{w_k-\bar{w}_k}{z_k-\bar{z}_k} \end{vmatrix}, \quad k = 1, ..., n.$$

Assume that $D(p_1, ..., p_n) = 0$, then $D(p_1, ..., p_n, p) \geq 0$ with $p = (z, w)$, i.e.

$$D(p_1, ..., p_n, p) = \begin{vmatrix} \frac{w_1 - \bar{w}_1}{z_1 - \bar{z}_1} & \frac{w_1 - \bar{w}_2}{z_1 - \bar{z}_2} & \cdots & \frac{w_1 - \bar{w}_n}{z_1 - \bar{z}_n} & \frac{w_1 - \bar{w}}{z_1 - \bar{z}} \\ \frac{w_2 - \bar{w}_1}{z_2 - \bar{z}_1} & \frac{w_2 - \bar{w}_2}{z_2 - \bar{z}_2} & \cdots & \frac{w_2 - \bar{w}_n}{z_2 - \bar{z}_n} & \frac{w_2 - \bar{w}}{z_2 - \bar{z}} \\ \cdots & \cdots & \cdots & \cdots & \\ \frac{w_n - \bar{w}_1}{z_n - \bar{z}_1} & \frac{w_n - \bar{w}_2}{z_n - \bar{z}_2} & \cdots & \frac{w_n - \bar{w}_n}{z_n - \bar{z}_n} & \frac{w_n - \bar{w}}{z_n - \bar{z}} \\ \frac{w - \bar{w}_1}{z - \bar{z}_1} & \frac{w - \bar{w}_2}{z - \bar{z}_2} & \cdots & \frac{w - \bar{w}_n}{z - \bar{z}_n} & \frac{w - \bar{w}}{z - \bar{z}} \end{vmatrix} \geq 0$$

implies that there are n constants $\lambda_1, ..., \lambda_n$ independent of p (or z, w), such that

$$\lambda_1 \frac{w - \bar{w}_1}{z - \bar{z}_1} + \lambda_2 \frac{w - \bar{w}_w}{z - \bar{z}_w} + \cdots + \lambda_n \frac{w - \bar{w}_n}{z - \bar{z}_n} = 0.$$

Before we prove this proposition, we discuss some of its consequences. Solving this equation for w (call it f_{ext}) we see that f_{ext} is a unique rational function of z of degree $n - 1$:

$$f_{\text{ext}}(z) = \frac{\lambda_1 \frac{\bar{w}_1}{z - \bar{z}_1} + \lambda_2 \frac{\bar{w}_2}{z - \bar{z}_2} + \cdots + \lambda_n \frac{\bar{w}_n}{z - \bar{z}_n}}{\frac{\lambda_1}{z - \bar{z}_1} + \frac{\lambda_2}{z - \bar{z}_2} + \cdots + \frac{\lambda_n}{z - \bar{z}_n}} = \frac{\sum_{k=1}^{n} \lambda_k \bar{w}_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - \bar{z}_j)}{\sum_{k=1}^{n} \lambda_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - \bar{z}_j)}$$

From this fact Pick constructed the solution by setting

$$w = \frac{\psi(z)}{\varphi(z)},$$

where φ and ψ are coprime polynomial of degree $n - 1$, then he claimed, without further argument, that

$$w_\alpha = \frac{\psi(z_\alpha)}{\varphi(z_\alpha)}, \quad \bar{w}_\alpha = \frac{\psi(\bar{z}_\alpha)}{\varphi(\bar{z}_\alpha)}.$$

In order for this to be true we have to show that

Claim 2. *The rational function f_{ext} obtained above is a real rational function, that is, the coefficients of both $\varphi(z)$ and $\psi(z)$ are real.*

Proof. We show that, for each $z \notin \{z, ..., z_n, \bar{z}_1, ..., \bar{z}_n\} \cup \{z : \sum_{k=1}^{n} \lambda_k \prod_{\substack{j=1 \\ j \neq k}}^{n} (z - \bar{z}_j)\}$, $f_{\text{ext}}(\bar{z}) =$

$\overline{f_{\text{ext}}(z)}$ holds. So we compute

$$f_{\text{ext}}(\bar{z}) - \overline{f_{\text{ext}}(z)} = \frac{\displaystyle\sum_{k=1}^{n} \frac{\lambda_k \bar{w}_k}{\bar{z} - \bar{z}_k} \sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\bar{z} - z_j} - \sum_{j=1}^{n} \frac{\bar{\lambda}_j w_j}{\bar{z} - z_j} \sum_{k=1}^{n} \frac{\lambda_k}{\bar{z} - \bar{z}_k}}{\displaystyle\sum_{k=1}^{n} \frac{\lambda_k}{\bar{z} - \bar{z}_k} \sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\bar{z} - z_j}}.$$

It remains to show that the denominator is equal to zero. It equals

$$\begin{aligned}
&\sum_{k=1}^{n} \frac{\lambda_k \bar{w}_k}{\bar{z} - \bar{z}_k} \sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\bar{z} - z_j} - \sum_{j=1}^{n} \frac{\bar{\lambda}_j w_j}{\bar{z} - z_j} \sum_{k=1}^{n} \frac{\lambda_k}{\bar{z} - \bar{z}_k} = \sum_{j,k=1}^{n} \frac{\lambda_k \bar{\lambda}_j \bar{w}_k - \bar{\lambda}_j \lambda_k w_j}{(\bar{z} - \bar{z}_k)(\bar{z} - z_j)} \\
&= \sum_{j,k=1}^{n} \frac{\lambda_k \bar{\lambda}_j (\bar{w}_k - w_j)}{(\bar{z} - \bar{z}_k)(\bar{z} - z_j)} = \sum_{j,k=1}^{n} \frac{\lambda_k \bar{\lambda}_j (\bar{z}_k - z_j)}{(\bar{z} - \bar{z}_k)(\bar{z} - z_j)} \frac{\bar{w}_k - w_j}{\bar{z}_k - z_j} \\
&= \sum_{j,k=1}^{n} \lambda_k \bar{\lambda}_j \frac{\bar{w}_k - w_j}{\bar{z}_k - z_j} \left(\frac{1}{\bar{z} - \bar{z}_k} - \frac{1}{\bar{z} - z_j} \right) \\
&= \sum_{k=1}^{n} \frac{\lambda_k}{\bar{z} - \bar{z}_k} \sum_{j=1}^{n} \bar{\lambda}_j \frac{w_j - \bar{w}_k}{z_j - \bar{z}_k} - \sum_{j=1}^{n} \frac{\bar{\lambda}_j}{\bar{z} - z_j} \sum_{k=1}^{n} \lambda_k \frac{w_j - \bar{w}_k}{z_j - \bar{z}_k} = 0
\end{aligned}$$

thanks to Claim 1

$$\sum_{k=1}^{n} \lambda_k \frac{w_j - \bar{w}_k}{z_j - \bar{z}_k} = 0, \quad j = 1, ..., n$$

and

$$\sum_{j=1}^{n} \bar{\lambda}_j \frac{w_j - \bar{w}_k}{z_j - \bar{z}_k} = 0, \quad k = 1, ..., n,$$

respectively, $\square$

Claim 3. *The real rational function $f_{ext}(z)$ satisfies the interpolation conditions.*

Proof. Since $\lambda_\alpha \neq 0$, $\alpha = 1, ..., n$,

$$\begin{aligned}
f_{\text{ext}}(z_\alpha) &= \frac{\lambda_1 \dfrac{\bar{w}_1}{z_\alpha - \bar{z}_1} + \lambda_2 \dfrac{\bar{w}_2}{z_\alpha - \bar{z}_2} + \cdots + \lambda_n \dfrac{\bar{w}_n}{z_\alpha - \bar{z}_n}}{\dfrac{\lambda_1}{z_\alpha - \bar{z}_1} + \dfrac{\lambda_2}{z_\alpha - \bar{z}_2} + \cdots + \dfrac{\lambda_n}{z_\alpha - \bar{z}_n}} \\
&= \frac{\lambda_1 \dfrac{(\bar{w}_1 - w_\alpha)}{z_\alpha - \bar{z}_1} + \lambda_2 \dfrac{(\bar{w}_2 - w_\alpha)}{z_\alpha - \bar{z}_2} + \cdots + \lambda_n \dfrac{(\bar{w}_n - w_\alpha)}{z_\alpha - \bar{z}_n}}{\dfrac{\lambda_1}{z_\alpha - \bar{z}_1} + \dfrac{\lambda_2}{z_\alpha - \bar{z}_2} + \cdots + \dfrac{\lambda_n}{z_\alpha - \bar{z}_n}} + w_\alpha = w_\alpha
\end{aligned}$$

The last equality holds by Claim 1. $\square$

Now

$$p_{\alpha\beta} = \frac{w_\alpha - \bar{w}_\beta}{z_\alpha - \bar{z}_\beta} = \frac{\dfrac{\psi(z_\alpha)}{\varphi(z_\alpha)} - \dfrac{\psi(\bar{z}_\beta)}{\varphi(\bar{z}_\beta)}}{z_\alpha - \bar{z}_\beta} = \frac{1}{\varphi(z_\alpha)\varphi(\bar{z}_\beta)} \cdot \frac{\varphi(\bar{z}_\beta)\psi(z_\alpha) - \psi(\bar{z}_\beta)\varphi(z_\alpha)}{z_\alpha - \bar{z}_\beta}$$

Let

$$\Omega(z, \bar{z}) := \frac{\varphi(\bar{z})\psi(z) - \psi(\bar{z})\varphi(z)}{z - \bar{z}}.$$

Then

$$p_{\alpha\beta} = \frac{\Omega(z_\alpha, \bar{z}_\beta)}{\varphi(z_\alpha)\varphi(\bar{z}_\beta)}$$

So the principal minors

$$D(p_1, ..., p_m) = \begin{vmatrix} \dfrac{\Omega(z_1, \bar{z}_1)}{\varphi(z_1)\varphi(\bar{z}_1)} & \cdots & \dfrac{\Omega(z_1, \bar{z}_m)}{\varphi(z_1)\varphi(\bar{z}_m)} \\ \dfrac{\Omega(z_2, \bar{z}_1)}{\varphi(z_2)\varphi(\bar{z}_1)} & \cdots & \dfrac{\Omega(z_2, \bar{z}_m)}{\varphi(z_2)\varphi(\bar{z}_m)} \\ \cdots & \cdots & \cdots \\ \dfrac{\Omega(z_m, \bar{z}_1)}{\varphi(z_m)\varphi(\bar{z}_1)} & \cdots & \dfrac{\Omega(z_m, \bar{z}_m)}{\varphi(z_m)\varphi(\bar{z}_m)} \end{vmatrix} = \frac{1}{|\varphi(z_1)|^2 \cdots |\varphi(z_m)|^2} \begin{vmatrix} \Omega(z_1, \bar{z}_1) & \cdots & \Omega(z_1, \bar{z}_m) \\ \Omega(z_2, \bar{z}_1) & \cdots & \Omega(z_2, \bar{z}_m) \\ \cdots & \cdots & \cdots \\ \Omega(z_m, \bar{z}_1) & \cdots & \Omega(z_m, \bar{z}_m) \end{vmatrix}$$

It is positive definite if and only if

$$\Delta_m = \begin{vmatrix} \Omega(z_1, \bar{z}_1) & \cdots & \Omega(z_1, \bar{z}_m) \\ \Omega(z_2, \bar{z}_1) & \cdots & \Omega(z_2, \bar{z}_m) \\ \cdots & \cdots & \cdots \\ \Omega(z_m, \bar{z}_1) & \cdots & \Omega(z_m, \bar{z}_m) \end{vmatrix} > 0, \quad m = 1, ..., n-1$$

Equivalently, the Hermitian form

$$H(s) = \sum_{\alpha,\beta=1}^{n} \Omega(z_\alpha, \bar{z}_\beta) s_\alpha \bar{s}_\beta, \quad s = (s_1, ..., s_n) \in \mathbb{C}^n$$

is positive definite. Write now

$$\varphi(z) = \sum_{k=0}^{n-1} \varphi_k z^k, \;\; \psi(z) = \sum_{k=0}^{n-1} \psi_k z^k,$$

This leads to

$$\Omega(z, \bar{z}) = \frac{\sum\limits_{j,k=0}^{n-1} (\varphi_j \psi_k - \psi_j \varphi_k) \bar{z}^j z^k}{z - \bar{z}} = \sum_{j,k=0}^{n-1} \underbrace{\frac{(\varphi_j \psi_k - \psi_j \varphi_k)}{z - \bar{z}}}_{a_{jk}} \bar{z}^j z^k = \sum_{j,k=0}^{n-1} a_{jk} \bar{z}^j z^k > 0,$$

Following Pick, we consider the Hermitian form

$$H(t) = \sum_{j,k=0}^{n-1} a_{jk} t_j \bar{t}_k, \quad t = (t_0, ..., t_{n-1}) \in \mathbb{C}^n.$$

Further let us consider the linear transform $U_{z_1,...,z_n} : \mathbb{C}^n \to \mathbb{C}^n$ represented relative to the standard basis by the matrix

$$V(z_1, ..., z_n) = \begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \\ \vdots & \ddots & \vdots \\ z_1^{n-1} & \cdots & z_n^{n-1} \end{pmatrix}$$

where we assume that the points $z_1, ..., z_n$ are distinct, thus the matrix is invertible (the determinant is $\prod\limits_{n \geq j > k \geq 1} (z_j - z_k)$, the Vandermonde determinant). Consequently the linear transformation $U_{z_1,...,z_n}$ is a bijection. So for $s \in \mathbb{C}^n$ we have

$$H(s) = H(U_{z_1,...,z_n} s)$$

In other words, the Pick matrix and the matrix (a_{jk}) have the same rank. And hence $\Omega(z, \bar{z}) \;\; > \; 0$.

Next notice that when z tends to a real number x (also $\bar{z} \to x$), we have

$$\Omega(x) = \varphi(x)\psi'(x) - \psi(x)\varphi'(x), \quad (\text{ where } (\cdot)' \text{ stands for the derivative of } \cdot)$$

From this we see that neither φ nor ψ has multiple real zeros, for otherwise both $\varphi(x), \varphi'(x)$ would be zero (or the same for $\psi(x)$), violating $\Omega(x) > 0$. Let $\varphi(x) = 0$ we have

$$\frac{\psi(x)}{\varphi'(x)} = -\frac{\Omega(x)}{[\varphi'(x)]^2} < 0$$

So if φ has $n-1$ zeros $a_1, ..., a_{n-1}$ we have

$$\frac{\psi(z)}{\varphi(z)} = C - \frac{A_1}{z - a_1} - \cdots - \frac{A_{n-1}}{z - a_{n-1}}$$

where $A_1, ..., A_{n-1}$ are positive. In case $\varphi(z)$ has less than $n-1$ zeros then we have

$$\frac{\psi(z)}{\varphi(z)} = C + Az - \frac{A_1}{z - a_1} - \cdots - \frac{A_{n-2}}{z - a_{n-2}}$$

where $A, A_1, ..., A_{n-1}$ are positive. Then we can show that the resulting Pick matrix is positive semi-definite and the Pick matrix is singular.

It remains to prove Claim 1.

Proof of Claim 1. Pick claimed that $D(p_1,...,p_n,p)$ is a sum of $D(p_1,...,p_n)$ multiplied by a factor and

$$-\left|\lambda_1\frac{w-\bar{w}_1}{z-\bar{z}_1}+\lambda_2\frac{w-\bar{w}_w}{z-\bar{z}_w}+\cdots+\lambda_n\frac{w-\bar{w}_n}{z-\bar{z}_n}\right|^2.$$

From this the Claim follows because of the non-negativity of $D(p_1,...,p_n,p)$. So we have to show that

$$D(p_1,...,p_n,p)=C\cdot D(p_1,..,p_n)-\left|\lambda_1\frac{w-\bar{w}_1}{z-\bar{z}_1}+\lambda_2\frac{w-\bar{w}_w}{z-\bar{z}_w}+\cdots+\lambda_n\frac{w-\bar{w}_n}{z-\bar{z}_n}\right|^2,\ C \text{ is a factor.}$$

Using the Laplace' formula after the last row,

$$D(p_1,...,p_n,p)=D(p_1,...,p_n)\frac{w-\bar{w}}{z-\bar{z}}+(-1)^{n+2}D_1\frac{w-\bar{w}_1}{z-\bar{z}_1}+(-1)^{n+3}D_2\frac{w-\bar{w}_2}{z-\bar{z}_2}+$$

$$+\cdots+(-1)^{2n+1}D_n\frac{w-\bar{w}_n}{z-\bar{z}_n}$$

where D_j, $(j=1,...,n)$, is the determinant of $n\times n$ matrix with jth column and the last row of $D(p_1,...,p_n,p)$ deleted:

$$\begin{aligned}D_j=&\begin{vmatrix}\frac{w_1-\bar{w}_1}{z_1-\bar{z}_1} & \cdots & \frac{w_1-\bar{w}_{j-1}}{z_1-\bar{z}_{j-1}} & \frac{w_1-\bar{w}_{j+1}}{z_1-\bar{z}_{j+1}} & \cdots & \frac{w_1-\bar{w}_n}{z_1-\bar{z}_n} & \frac{w_1-\bar{w}}{z_1-\bar{z}}\\ \frac{w_2-\bar{w}_1}{z_2-\bar{z}_1} & \cdots & \frac{w_2-\bar{w}_{j-1}}{z_2-\bar{z}_{j-1}} & \frac{w_2-\bar{w}_{j+1}}{z_2-\bar{z}_{j+1}} & \cdots & \frac{w_2-\bar{w}_n}{z_2-\bar{z}_n} & \frac{w_2-\bar{w}}{z_2-\bar{z}}\\ \cdots & \cdots & \cdots & \cdots & & & \\ \frac{w_n-\bar{w}_1}{z_n-\bar{z}_1} & \cdots & \frac{w_n-\bar{w}_{j-1}}{z_n-\bar{z}_{j-1}} & \frac{w_n-\bar{w}_{j+1}}{z_n-\bar{z}_{j+1}} & \cdots & \frac{w_n-\bar{w}_n}{z_n-\bar{z}_n} & \frac{w_n-\bar{w}}{z_n-\bar{z}}\end{vmatrix},\\ =&(-1)^{n+1}\frac{w_1-\bar{w}}{z_1-\bar{z}}D_{j1}+(-1)^{n+2}\frac{w_2-\bar{w}}{z_2-\bar{z}}D_{j2}+\cdots+(-1)^{2n}\frac{w_n-\bar{w}}{z_n-\bar{z}}D_{jn}\end{aligned}$$

Again we used the Laplace' formula in the last equality but after the last column. By a careful inspection we see that D_{jk} $(k=1,...,n)$ is the the determinant of $(n-1)\times(n-1)$ matrix with the jth column and the kth row of the Pick matrix P_n deleted. Therefore,

$$D(p_1,...,p_n,p)=D(p_1,...,p_n)\frac{w-\bar{w}}{z-\bar{z}}+\begin{pmatrix}\frac{w-\bar{w}_1}{z-\bar{z}_1} & \cdots & \frac{w-\bar{w}_n}{z-\bar{z}_n}\end{pmatrix}\operatorname{Adj}(P_n)\begin{pmatrix}\frac{w_1-\bar{w}}{z_1-\bar{z}}\\ \vdots\\ \frac{w_n-\bar{w}}{z_n-\bar{z}}\end{pmatrix}.$$

Since the Pick matrix has rank $n-1$) there is at least one nonzero $(n-1)\times(n-1)$ minor of P_n. This implies that $\operatorname{Adj}(P_n)\neq 0$. Since

$$P_n\operatorname{Adj}(P_n)=\det(P_n)I=0$$

we conclude that the columns of $\operatorname{Adj}(P_n)$ lie in the kernel of P_n, $\ker(P_n)$. By the rank-nutty Theorem

$$\dim \ker(P_n) = n - \dim R(P_n) = n - (n-1) = 1.$$

So the rank of $\operatorname{Adj}(P_n)$ is 1. Note that $\operatorname{Adj}(P_n)$ is Hermitian as P_n. Thus there is a full rank factorization of $\operatorname{Adj}(P_n)$:

$$\operatorname{Adj}(P_n) = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \cdots & \bar{\lambda}_n \end{pmatrix},$$

which yields

$$\begin{aligned} D(p_1, ..., p_n, p) &= D(p_1, ..., p_n)\frac{w - \bar{w}}{z - \bar{z}} + \\ &+ \begin{pmatrix} \frac{w - \bar{w}_1}{z - \bar{z}_1} & \cdots & \frac{w - \bar{w}_n}{z - \bar{z}_n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} \bar{\lambda}_1 & \bar{\lambda}_2 & \cdots & \bar{\lambda}_n \end{pmatrix} \begin{pmatrix} \frac{w_1 - \bar{w}}{z_1 - \bar{z}} \\ \vdots \\ \frac{w_n - \bar{w}}{z_n - \bar{z}} \end{pmatrix} \\ &= \frac{w - \bar{w}}{z - \bar{z}} \cdot D(p_1, .., p_n) - \left| \lambda_1 \frac{w - \bar{w}_1}{z - \bar{z}_1} + \lambda_2 \frac{w - \bar{w}_w}{z - \bar{z}_w} + \cdots + \lambda_n \frac{w - \bar{w}_n}{z - \bar{z}_n} \right|^2. \end{aligned}$$

Note that at least one $\lambda_j \neq 0$. □

5 Nevanlinna's solutions

Nevanlinna's view on the interpolation theorem is based on the assumption that if the function f_n is in the unit disc then the function f_{n+1} also is in $\mathbb{D}$.

With the same reasoning as in the proof of the Schwarz Lemma there exists a function

$$f_1(z) = f(z)/z.$$

If $|w| = 1$ then $f(z)$ is a constant. But if $|w_1| < 1$ then we get a solution for $f(z_1) = w_1$, which is the function f_1. So with the Schwarz-Pick Lemma we should be able to write a solution for the function f_2 as

$$f_2(z) = \frac{\frac{f_1(z) - w_1}{1 - \overline{w_1} f_1(z)}}{\frac{z - z_1}{1 - \overline{z_1} z}}.$$

As long as there is an expression of the form above with $f_n(z)$ we know there exists a function $f(z)$. For every w_α and z_α we get a new function with $|f_\alpha(z)| < |z_i\alpha|$.

So what happens when $|f_\alpha(z)| = |z_\alpha|$?

The inequality in the Schwarz-Pick Lemma becomes an equality and so $|f(z_\alpha)| = 1$ and by the Maximum Modulus Principle $f(z)$ is uniquely determined by the values of z_α and w_α. If $|w_\alpha| < 1$ there is an unlimited number of functions. This is because the Maximum Modulus Principle says that analytic functions have their maximum value (modulus) on points at the boundary and so therefore if $|w_\alpha| < 1$ then we know by the Schwarz Lemma that there exists a function for $w_\alpha = f(z_\alpha)$. So whenever we have $|w_\alpha| < |z_\alpha|$ there is a function $f(z) = w$.

The resulting function $f_{n+1}(z)$ is given as a linear fractional transformation with the function $f_1(z)$ and can therefore be written as

$$f_n(z) = \frac{a_n(z)f_{n+1}(z) + b_n(z)}{c_n(z)f_{n+1}(z) + d_n(z)}.$$

The terms $a_n(z)$, $b_n(z)$, $c_n(z)$, $d_n(z)$ are polynomials.

6 Parametrization of solutions, Schur's algorithm

Schur's algorithm is a parametric representation of the coefficients in a power series representing an analytic function. So we show that all functions analytic in $\mathbb{D}$ and with values in $\bar{\mathbb{D}}$ can be represented using Schur's algorithm.

If f is an analytic function it can be represented with the power series

$$f(z) = c_0 + c_1 z + c_2 z^2 + ...$$

It is convergent for all $|z| \leq 1$ and so $|c_0| \leq 1$. If $|c_0| = 1$ then $f(z)$ is reduced to the constant c_0. If $|c_0| \leq 1$ we can form the expression, where γ_0 represents c_0,

$$f_1 = \frac{1}{z}\frac{f - \gamma_0}{1 - \bar{\gamma}_0 f} = \frac{c_1 + c_2 z + c_3 z`2 + ...}{1 - \gamma_0\bar{\gamma}_0 - \bar{\gamma}_0 c_1 z - \bar{\gamma}_0 c_2 z^2 - ...}.$$

This gives

$$\gamma_1 := f_1(0) = \frac{c_1}{1 - \bar{c}_0 c_0}.$$

If $|\gamma_1| < 1$ we define

$$f_2 = \frac{1}{z}\frac{f_1 - \gamma_1}{1 - \bar{\gamma}_1 f_1}, \quad \gamma_2 := f_2(0).$$

This gives us a sequence of functions $f_0 = f, f_1, f_2, ...$ in $\mathcal{S}$, determined by the two algorithms

$$f_{n+1} = \frac{f_n - \gamma_n}{z(1 - \bar{\gamma}_n f_n)}$$

or, equivalently

$$f_n = \frac{\gamma_n + z f_{n+1}}{1 + \bar{\gamma}_n z f_{n+1}}.$$

Recall that an analytic function f on $\mathbf{D} = \{z = |z| < 1\}$ is called a Schur function if and only if $\sup\limits_{z\in\mathbb{D}} |f(z)| \leq 1$. We say f is trivial if it is a finite Blaschke product

$$f(z) = e^{i\theta} \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z}, \qquad z_1, ..., z_m \in \mathbb{D}.$$

If $f(z) := w_0$ is a constant then $f(z)$ is a finite Blaschke product of degree 0. If $|w_0| = 1$ f is said to be trivial and if $|w_0| < 1$ non-trivial. In case $f(z) = w_0 \in \mathbb{D}$ f is called a degenerate Schur function. Thus nondegenerated Schur functions are precisely analytic maps of $\mathbb{D}$ onto $\mathbb{D}$. Note that degenerated functions are also trivial.

Schur provided a continued fraction expansion of an arbitrary Schur function of the form

$$\begin{cases} f(z) = & f_0(z) \\ f_{j+1} = & \dfrac{1}{z}\left(\dfrac{f_j(z) - \gamma_j}{1 - \bar{\gamma}_j f_j(z)}\right) \\ \gamma_j := & f_j(0) \end{cases}$$

This is the Schur algorithm. The $\{\gamma_j\}_{j\geq 0}$'s are the Schur parameters.

Now we can formulate the proposition that f_{j+1} above can be expressed as a continued fraction. We solve for f_j in the definition above:

$$f_j(z) = \frac{\gamma_j + z f_{j+1}(z)}{1 + \bar{\gamma}_j f_{j+1}(z)} = \gamma_j + \frac{(1 - |\gamma_j|^2) z f_{j+1}}{1 - \bar{\gamma}_j z f_{j+1}(z)} = \gamma_j + \frac{1 - |\gamma_j|^2}{\frac{1}{z f_{j+1}(z)} + \gamma_j}.$$

Hence

$$f(z) = \gamma_0 + \cfrac{1 - |\gamma_0|^2}{\bar{\gamma}_0 + \cfrac{1}{z\gamma_1 + \cfrac{z(1 - |\gamma_1|^2)}{\bar{\gamma}_1 + \cfrac{1}{z f_2(z)}}}}$$

Proposition. *Nondegenerate Schur functions are maps of* $\mathbb{D}$ *onto* $\overline{\mathbb{D}}$. *The Schur algorithm exploits two ways of mapping a Schur function to another.*

Proof. We know that (i) for $\gamma \in \mathbb{D}$ we have

$$T_\gamma(0) = \frac{w - \gamma}{1 - \bar{\gamma} w}$$

is an invertible analytic homeomorphism of $\mathbb{D}$ to $\mathbb{D}$ mapping γ to zero. So $T_{f(0)}$ is a nondegenerate Schur function which vanishes at zero.

(ii). By the Schwarz Lemma, if f is a Schur function and $f(0) = 0$, then either $\frac{f(z)}{z}$ is a nondegenerate Schur function or a constant on $\partial\mathbb{D}$. $\square$

Corollary. *The Schur papameters satisfy $|\gamma_j| < 1$ for all j if f is nondegenerate.*

As we saw above we get a sequence of Schur parameters $\gamma_0, \gamma_1, ..., \gamma_n$ and Schur functions $f = f_0, f_1, ..., f_n$ calle the Schur iterates of f. At some stage $f_n(z) = \gamma_n$ will lie on $\partial\mathbb{D}$. In that case we stop. If $f_n(z) = \gamma_n \in \mathbb{D}$, we continue, which means $f_{n+1} = f_{n+2} = ... = 0$ and $\gamma_{n+1} = \gamma_{n+2} = ... = 0$. Thus, any Schur function f is associated to either an infinite sequence $\gamma_0, \gamma_1, ... \in \mathbb{D}$ or a finite sequence $\gamma_0, ..., \gamma_n$ whith $\gamma_j \in \mathbb{D}$ for $j < n$ and $\gamma_n \in \partial\mathbb{D}$. It is not hard to se that the case $\gamma_n \in \partial\mathbb{D}$ holds if and only if f is trivial, i.e. a finite Blaschke product.

The Schur algorithm is widely used in many applications,. Here we mention a few: moment problems, robust control, signal processing, circuit theory, and numerical analysis.

7 Some worked examples

In this section we solve the problems mentioned in Section 1 and describe, fairly in detail, how a robust control problem can be solved.

Example 1. (Revisited) As argued in Section 1, we have $e(1) = \frac{1}{2}$. By the Pick Interpolation Theorem, there is a solution e. Since e is required to be stable, the Maximum Modulus Principle ensures that

$$\|e\|_\infty = \sup_{s=i\omega} |e(s)| = \sup_{\mathrm{Re}(s)\geq 0} |e(s)| \geq |e(1)| = \frac{1}{2}.$$

Therefore, the interpolating function with minimum infinity norm is the constant $e = \frac{1}{2}$ and the associated norm is $\|e\|_\infty = \frac{1}{2}$. Remember that $f = g^{-1}(h - e)$, with $g = \frac{s-1}{s+2}$ and $h = \frac{s+1}{s+3}$. So we obtain

$$f = \frac{s+2}{s-1}\left(\frac{s+1}{s+3} - \frac{1}{2}\right) = \frac{1}{2}\frac{s+2}{s+3}$$

Although this is a very simple example it raises the interpolation problem with an additional constraint, that is, with the minimum infinite norm. In this simple example we simply choose the constant function. Next example shows that such choice is not feasible.

Example 2. (Revisited) In this example we shall find an e such that $e(1) = \frac{1}{6} =: h_1$ and $e(2) = \frac{2}{15} =: h_2$ $(s_1 = 1, s_2 = 2)$ with minimum infinity norm. If we imitate the method used in the previous example we would have

$$\|e\|_\infty \geq \max\left\{\frac{1}{6}, \frac{2}{15}\right\}.$$

Since we have two values to interpolate, simply setting $e = \frac{1}{6}$ will not work. The Nevanlinna-Pick Interpolation theory says that there is an interpolation function e with $\|e\|_\infty \leq \gamma$ if and only if the Pick matrix

$$P_2(\gamma) = \begin{pmatrix} \dfrac{\gamma^2 - h_1^2}{2} & \dfrac{\gamma^2 - h_1 h_2}{3} \\ \dfrac{\gamma^2 - h_1 h_2}{3} & \dfrac{\gamma^2 - h_2^2}{4} \end{pmatrix}$$

is positive semidefinite. (See [6] for a general statement). We can show that if $\gamma_1 \geq \gamma_2$ then $P_2(\gamma_1) \geq P_2(\gamma_2)$. So the desired optimal norm is the largest value of γ for which the Pick matrix $P_2(\gamma)$ is singular. The optimal value of γ, γ^* is the square root of the largest eigenvalue of the symmetric matrix pencil

$$\lambda \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{3} \\ \dfrac{1}{3} & \dfrac{1}{4} \end{pmatrix} - \begin{pmatrix} \dfrac{h_1^2}{2} & \dfrac{h_1 h_2}{3} \\ \dfrac{h_1 h_2}{3} & \dfrac{h_2^2}{4} \end{pmatrix}$$

which is $\gamma^* = \sqrt{\frac{49+3\sqrt{89}}{1800}} \approx 0.207233$. The Nevanlinna-Pick Interpolation Theory gives the optimal interpolating function as

$$e = \gamma^* \frac{a - s}{a + s}$$

where

$$a = s_\alpha \frac{\gamma^* + h_\alpha}{\gamma^* - h_\alpha} = \frac{1}{2}\left(9 + \sqrt{89}\right) \approx 9.21699$$

with α either 1 or 2. Obviously, the e obtained here satisfies the interpolation conditions and $\|e\|_\infty = \gamma^*$ because $\left\| \dfrac{a-s}{a+s} \right\|_\infty = \sup\limits_{\omega > 0} \left| \dfrac{a - i\omega}{a + i\omega} \right| = 1$. So the optimal prefilter is

$$f = \gamma^* \frac{s+3}{s+a}.$$

Notice that the optimal interpolating function is a constant multiplied by a stable transfer function with unit magnitude on the imaginary axis, which is a general property of optimal interpolating functions.

By this example we illustrated that an increase in the number of interpolation constraints makes the evaluation of the interpolating function much harder. Despite this, the error function retains the "constant magnitude on the imaginary axis" property associated with constants.

Example 3. (Revisited) Efficient techniques for constructing an interpolating positive real function are proposed by Youla-Saito [11], as follows: Let $(p_1, z_1), ..., (p_n, z_n)$ be given. We consider two cases:

(i) The Pick-matrix A is singular and has rank $r < n$.

(ii) The Pick-matrix is positive definite.

For case (i) we have:

(1) Pick up any non-trivial solution of $Ax = 0$ for example $x = (x_1, ..., x_n)^T$.

(2) Form two rational functions

$$g_1(s) = \sum_{\alpha=1}^{n} = \frac{p_\alpha x_\alpha}{p_\alpha^2 + s}, \quad g_2(s) = \sum_{\alpha=1}^{n} \frac{x_\alpha s}{p_\alpha^2 + s}$$

(3) Let $s_1, ..., s_r$ denote the finite, distinct, common real non-negative roots of $g_1(s) = 0, \;\; g_2(s) = 0$. Then $r \leqq n - 1$.

(4) Set $w_\beta^2 = s_\beta$, $\beta = 1, ..., r$

$$z(s) = cs + \sum_{\beta=1}^{r} \frac{2sa_\beta}{s^2 + {w_\beta}^2}$$

The solution of the $(r+1) \times (r+1)$ linear inhomogenous system

$$\frac{z_k}{p_k} = c + \sum_{\beta=1}^{r} \frac{2a_\beta}{p_r^2 + w_\beta^2}, \quad k = 1, 2, ..., r + 1$$

yields a unique real non-negative determinant of the $r + 1$ unknowns $c, a_1, ..., a_r$. These determine $z(s)$ as the unique Foster interpolating function. Note that we may choose $c = 0$ if $x_1 + ... + x_n \neq 0$.

Next we turn to the second situation where $A > 0$. There are two subcases.

(i) The interpolating positive real function $Z(s)$ need not to be chosen Foster, and

(ii) it must be chosen Foster.

In both cases we reduce the problem to case (i). We omit the details here since it is beyond the scope of this report.

Now we go back to solve the problem stated earlier.

$$A = \begin{pmatrix} 2 & 3/2 & 4/3 \\ 3/2 & 5/4 & 7/6 \\ 4/3 & 7/6 & 10/9 \end{pmatrix}$$

As shown before the rank of A is 2. By the algorithm above we choose a solution to $Ax = 0$ to be $x = (1, -4, 3)^T$. Then,

$$g_1(s) = \frac{1}{1+s} + \frac{-8}{4+s} + \frac{9}{9+s} \quad \text{and} \quad g_2(s) = \frac{s}{1+s} + \frac{-4s}{4+s} + \frac{3s}{9+s}.$$

Solving $g_1(s) = 0$ and $g_2(s) = 0$, i.e.

$$g_1(0) = \frac{2(s-11)s}{(s+1)(s+11)(s+9)} = 0, \quad g_2(s) = -\frac{12s(s-1)}{(s+1)(s+4)(s+9)} = 0$$

which gives,

$$s = 11, \quad s = 0, \quad \text{and} \quad s = 0, \quad s = 1.$$

The only non-negative common root is 0. Thus we get

$$z(s) = cs + \frac{2a_1}{s}.$$

For $s = p_1 = 1$ and $s = p_2 = 2$ we get two interpolating equations

$$\begin{aligned} 2 &= c + 2a_1 \\ 5/2 &= 2s + a_1 \end{aligned} \Leftrightarrow c = 1, a_1 = 1/2$$

So $z(s) = s + 1/3$ is the desired Foster function.

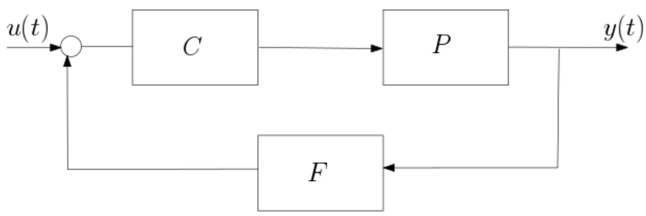
Example 4. (Feedback stabilization of linear dynamical plants with uncertainty in the gain factor. [10])

We start with some necessary definitions:

(i) A rational function whose denominator has degree greater than or equal to the degree of the numerator is called a *proper rational function.*

(ii) A rational function whose numerator and denominator have no common factor and which has no zeros in $\mathbb{C}^+$ is called a *Hurwitz rational function.* Furthermore, a Hurwitz rational function has no zeros in $\bar{\mathbb{C}}^+$ is called *strictly Hurwitz rational function.*

Problem formulation: Let P be our plant such that its input-output transfer function $P(s) = k\bar{P}(s)$, where $\bar{P}$ is a fixed proper rational function with real coefficients and k is the gain factor, which may vary in some interval $[k_{\min}, k_{\max}]$. This problem can be put in the following cybernetic diagram:



The aim of the control design is to stabilize the system. We want to find necessary and sufficient conditions for being able to construct $C(s)$ and $F(s)$ holomorphic in $\overline{\mathbb{C}}^+$ such that $\frac{1}{C(s)}$ and $\frac{1}{F(s)}$ should be strictly Hurwitz, and such that the denominator $1+P(s)C(s)F(s)$ of the closed-loop transfer function will be strictly Hurwitz. The problem is commonly referred to as the *strong stabilisation problem for uncertainty gain factors.* It can be reduced to the problem of finding a fixed rational function $g(s) = \frac{G_1(s)}{G_2(s)}$ with $\frac{1}{g(s)}$ proper and G_1 strict Hurwitz such that $-g + k\overline{P}$ is strict Hurwitz for all $k \in [k_{\min}, k_{\max}]$.

In other words, given a real proper rational function $\bar{P}$ we wish to solve the strong stabilization problem for an uncertain gain factor by finding an interval $[k_{\min}, k_{\max}] \in S$ such that for all $[a, b] \in S$, $k_{max}/k_{min} \geq \frac{b}{a}$ where S is the set of all intervals $[a, b], (b > a > 0)$ such that there exists $g(s)|_{[a,b]}$ is a real strict Hurwitz rational function and $\frac{1}{g(s)|_{[a,b]}}$ proper, such that for all $k \in [a, b]$, $-g(s)|_{[a,b]} + k\bar{P}(s)$ is strict Hurwitz. (For simplicity we drop the explicit reference to the dependence of $g(s)|_{[a,b]}$ on the interval $[a, b]$.) Moreover, it is demanded to construct the corresponding rational function $g(s)$.

To demonstrate the theory developed in [10] suing the Schur algorithm, we compute a concert example by methd proposed in [10]: Consider $\bar{P}(s) = \frac{(s-3)}{(s-1)}$. We want to determine $g(s)$ as a real rational strict Hurwitz such that $-g(s) + k\bar{P}(s)$ is strict Hurwitz for all k in some maximal interval $[k_{\min}, k_{\max}]$, $(k_{\max} > k_{\min} > 0)$. It is shown in [10] that this problem is equivalent to finding a real rational function $h(s)$ such that,

(1) $h(s) \in [k_{\min}, k_{\max}]$ for all $s \in \bar{\mathbb{C}}^+ \bigcup\{\infty\}$;

(2) the zeros of $h(s)$ in $\bar{\mathbb{C}}^+$ are precisely the poles of $\bar{P}(s)$ in $\mathbb{C}^+$ counted by multiplicities;

(3) any zero of $\bar{P}(s)$ in $\bar{\mathbb{C}}^+ \bigcup\{\infty\}$ of multiplicity m is a pole of $h(s)$ of multiplicity at least m.

Therefore we want to have h in $\bar{\mathbb{C}}^+ \bigcup\{\infty\} \to \mathbb{C}^+ \bigcup\{\infty\}/[k_{\min}, k_{\max}]$ such that $h(1) = 0$ and $h(3) = \infty$. The idea is to use the Schur algorithm.

Define first $\psi : \bar{\mathbb{D}} \to \bar{\mathbb{C}}^+ \bigcup\{\infty\}$. $\psi(s) = \frac{1+s}{1-s}$. In particular, $\psi(0) = 1$, the pole of $\bar{P}(s)$. Now define

$$\alpha_1 = \frac{\frac{1}{1} - 1}{\frac{1}{1} + 1} = 0$$
$$\alpha_2 = \frac{\frac{3}{1} - 1}{\frac{3}{1} + 1} = \tfrac{1}{2}$$

where both $\alpha_1$and α_2 is in $\mathbb{D}$. (notice that 1 and 3 in the α's are the pole and the zero pf $\bar{P}$). This gives us $\psi(0) = 1$ and $\psi(1/2) = 3$, both in $\mathbb{C}^+$

So we wish to find a $\tilde{h} : \mathbb{D} \to \mathbb{D}$ such that $\tilde{h}(0) = 0 \in \mathbb{D}$ and $\tilde{h}(1/2) = \alpha \in \mathbb{D}$ and $\tilde{h}(s)$ is closen so that $|\alpha|$ is maximized.

Our task is to make

$$\frac{k_{\max}}{k_{\min}} = \left(\frac{1 + |\alpha|_{\max}}{1 - |\alpha|_{\max}}\right)^2$$

and

$$h(s) = \phi^{-1} \circ \tilde{h} \circ \psi^{-1},$$

where $\phi : \bar{\mathbb{C}}^+ cup\{\infty\}|_{[k_{\min}, k_{\max}]} \to \mathbb{D}$, $(k_{\max} > k_{\min} > 0)$ and $\phi(0) = 0$, $\phi(\infty) = \alpha$, $(0 < |\alpha| < 1)$.

To determine $\tilde{h}$ we apply the Schur algorithm

$$\tilde{h}_1(s) = \frac{\tilde{h}(s) - \beta_1^{(0)}}{1 - \bar{\beta}_1^{(0)} \tilde{h}(s)} \frac{1 - \bar{\alpha}_1 s}{s - \alpha_1}$$

$$\beta_2^{(1)} = \frac{\beta_2^{(0)} - \beta_1^{(0)}}{1 - \bar{\beta}_1^{(0)} \beta_2^{(0)}} \frac{1 - \bar{\alpha}_1 \alpha_2}{\alpha_2 - \alpha_1}$$

In our case $\beta_1^{(0)} = 0$, $\beta_2^{(0)} = \alpha$, $\alpha_1 = 0$, $\alpha_2 = \frac{1}{2}$.

Then

$$\tilde{h}_1(s) = \frac{\tilde{h}(s)}{1 - 0 \cdot \tilde{h}(s)} \frac{1 - 0 \cdot s}{s - 0} = \frac{\tilde{h}(s)}{s}$$

and

$$\beta_2^{(1)} = \alpha \frac{1}{1/2} = 2\alpha.$$

Continue in the recursion,

$$\tilde{h}_2(s) = \frac{\tilde{h}_1(s) - 2\alpha}{1 - 2\bar{\alpha}\tilde{h}_1(s)} \frac{1 - (1/2)s}{s - (1/2)} = \frac{\tilde{h}_1(s) - s\alpha}{1 - 2\bar{\alpha}\tilde{h}_1(s)} \frac{2 - s}{2s - 1}.$$

It can be shown [10] that we get a solution if $|\beta_2^{(1)}| = |2\alpha| \leq 1$ so that $|\alpha| \leq (1/2)$, and we can take $\tilde{h}_2(s)$ to be arbitrary, in particular we take $\tilde{h}_2 = 0$.

Then

$$0 = \frac{\tilde{h}_1(s) - 2\alpha}{1 - 2\bar{\alpha}\tilde{h}_1(s)} \frac{2 - s}{2s - 1} \Leftrightarrow \tilde{h}_1(s) = 2\alpha.$$

Thus $\tilde{h}(s) = 2\alpha s$ is the required function.

Since α is bounded by $1/2$ we can take α real and positive so that $\alpha_{\max} = |\alpha|_{\max} = 1/2$. Then $\tilde{h}(s) = s$ is our solution. By the Schwarz Lemma $|\alpha| \leq 1/2$ so that $|\alpha|_{\max} = 1/2$ and then $\tilde{h}(s) = s$ is a solution. Since $\alpha_{\max} = 1/2$ we have $\frac{k_{\max}}{k_{\min}} = \left(\frac{1+1/2}{1-1/2}\right)^2 = 9$.

In [10] ϕ is defined by the composition $\phi = \phi_4 \circ \phi_3 \circ \phi_2 \circ \phi_1$ where

$$\phi_1(s) = \frac{s-a}{s-b}, \quad \text{mapping } \mathbb{C}\cup\{\infty\}\setminus[a,b] \text{ to } \mathbb{C}\cup\{\infty\}\setminus[-\infty,0], \text{ resulting in } \phi_1(0)=\frac{a}{b}$$

$$\phi_2(s) = \sqrt{s}, \quad \text{mapping } \mathbb{C}\cup\{\infty\}\setminus[-\infty,0] \text{ to } \mathbb{C}^+ \text{ then } \phi_2(a/b) = \sqrt{a/b}$$

$$\phi_3(s) = \gamma\frac{1-s}{1+s} \text{ where } |\gamma|=1, \text{ mapping } \mathbf{C}^+ \text{ to } \mathbb{D} \text{ with } \phi_3(\sqrt{a/b}) = \gamma\left(\frac{1-\sqrt{a/b}}{1+\sqrt{a/b}}\right) =: \alpha'$$

$$\text{note that } \frac{\alpha'}{\gamma} = |\alpha'| \text{ and } 0<|\alpha'|<1$$

$$\phi_4(s) = \gamma'\frac{s-\alpha'}{\bar{\alpha}'s-1}, \quad \text{where } |\gamma'|=1, \text{ mapping } \mathbb{D} \text{ to } \mathbb{D}, \text{ then } \phi_4(\alpha')=0.$$

This amounts to

$$\phi^{-1} = \phi_1^{-1}\circ\phi_2^{-1}\circ\phi_3^{-1}\circ\phi_4^{-1}$$

which gives

$$\phi^{-1}(s) = \frac{a(\alpha+1)^2(s-1)^2-(b(1-\alpha)^2(s+1)^2}{(\alpha+1)^2(s-1)^2-(1-\alpha)^2(s+1)^2}.$$

Since we are only interested in b/a , we might take $a=1$. However $k_{\max}/k_{\min}=9$ thus the maximal interval is $[k_{\min},k_{\max}]=[1,9]$, i.e. $b=k_{\max}=9$. This yields

$$\phi^{-1}(s) = \frac{-9s}{2s^2-5s+2}, \quad \psi^{-1} = \frac{s-1}{s+1}$$

So

$$h(s) = \phi^{-1}\circ\tilde{h}\circ\psi^{-1}(s) = \frac{-9(s-1)(s+1)}{-s^2+9} = \frac{9(s-1)(s+1)}{(s-3)(s+3)}.$$

Next we compute $g(s)$:

$$g(s) = \bar{P}(s)h(s) = \left(\frac{s-3}{s-1}\right)\left(\frac{9(s-1)(s+1)}{(s-3)(s+3)}\right) = \frac{9(s+1)}{s+3},$$

as expected (or the pole -1 and the zero -3 are in $\mathbb{C}^+$). Finally

$$-g(s)+k\bar{P}(s) = -\frac{9(s+1)}{s+3}+\frac{k(s-3)}{s-1} = \frac{(k-9)s^2-9(k-1)}{(s-1)(s+3)}$$

is obviously Hurwitz for all $k\in[1,9]$.

Remark. If in $\tilde{h}(s)=2\alpha s$ we take $\alpha<\frac{1}{2}$ and then do the similar calculations as above to find corresponding h and g, then $-g(s)+k\bar{P}(s)$ is strictly Hurwitz. For example $\alpha=\frac{1}{4}$

then $\tilde{h}(s) = \frac{1}{2}s$. $\frac{b}{a} = \left(\frac{1+\frac{1}{4}}{1-\frac{1}{4}}\right)^2 = \left(\frac{5}{3}\right)^2 = \frac{25}{9}$ and again since we are interested in the ratio $\frac{b}{a}$ we may take $a = 1$, $b = \frac{25}{9}$ which yields

$$\phi^{-1}(s) = \frac{25(s-1)(s+1)}{4s^2 - 17s + 4}.$$

Now

$$h(s) = \phi^{-1} \circ \tilde{h} \circ \psi^{-1}(s) = \frac{25(s-1)(s+1)}{(s-3)(7s+9)} \Rightarrow g(s) = \bar{P}(s)h(s) = \frac{25(s+1)}{7s+9}.$$

It is strict Hurwitz and

$$-g(s) + k\bar{P}(s) = -\frac{25(s+1)}{7s+9} + k\frac{s-3}{s-1} = -\frac{(7k-25)s^2 - 12ks + (25-27k)}{(7s+9)(s-1)}.$$

The discriminant of the numerator is

$$144k^2 - 4(25-27k)(7k-25) = 100(9k^2 - 34k + 25) > 0$$

since $34^2 - 4 \cdot 9 \cdot 25 < 0$. Now $k \in [1, \frac{25}{9}]$ we see that all coefficients of the numerator are all negative so $(25-7k)s^2 + 12ks + (7k - 2s)$ has all zeros in $\mathbb{C}^+$. The zeros are

$$s_\pm = \frac{6k \pm 5\sqrt{9k^2 - 34k + 25}}{7k - 25}.$$

So for all $k \in [1, \frac{25}{9}]$ we get $-g(s) + k\bar{P}(s)$ as the sought function.

Finally it is worthwhile pointing out that Pick-Nevanlinna interpolation theory has been applied to retention-solubility studies in the lungs, [4], somewhat not expected area. In this paper computational procedures for retention-solubility studies are given which determine data feasibility and some extreme properties of lung models compatible with given data. The procedures provided in this paper are analytic and are based on the Pick-Nevanlinna interpolation.

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Appendix

The purpose of this this Appendix is to provide geometric view of the Schwarz Lemma, and some results of linear algebra in the Pick proof in [7].

A1. Circles on the complex plane

A circle in the complex plane can be described by

$$z\bar{z} - \bar{\gamma}z - \gamma\bar{z} - \gamma\bar{\gamma} - \rho^2 = 0 \tag{3}$$

where $\gamma = \alpha + i\beta$ and ρ is the radius from the centre. Multiply with a real constant A and we get

$$Az\bar{z} + Bz + C\bar{z} + D = 0. \tag{4}$$

where B and C are complex conjugates, and D is real. $A = 0$ gives the equation for a straight line. Since the expressions (3) and (4) are quadric functions with complex numbers the circle can be written as a Hermitian matrix

$$\mathfrak{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with the determinant $\Delta = \det(\mathfrak{T}) = AD - BC = AD - |B|^2$. When the determinant is $\Delta = -\rho^2$ or $\Delta = -A\rho^2$ we get a real circle. If $\Delta = 0$ we get a point circle (a point) where $\rho = 0$, and $\Delta > 0$ gives us an imaginary circle. As an example, the imaginary unit circle $z\bar{z} + 1 = 0$.

If we have two different circles C_1 and C_2 so that the their corresponding Hermitian matrices, C_1, C_2 are not proportional we can form the pencil with the two circles

$$C = \lambda_1 C_1 + \lambda_2 C_2$$

where λ_1, λ_2 are real and both not zero. Thus we get the determinant of the pencil

$$|C| = \begin{vmatrix} \lambda_1 A_1 + \lambda_2 A_2 & \lambda_1 B_1 + \lambda_2 B_2 \\ \lambda_1 C_1 + \lambda_2 C_2 & \lambda_1 D_1 + \lambda_2 D_2 \end{vmatrix} = \Delta_1 {\lambda_1}^2 + 2\Delta_{12}\lambda_1\lambda_2 + \Delta_2 {\lambda_2}^2$$

which is a quadratic form of the variables λ_1, λ_2 with real coefficients

$$\Delta_1 = |C_1|,\ \Delta_2 = |C_2|,\ 2\Delta_{12} = A_1 D_2 + A_2 D_1 - B_1 C_2 - B_2 C_1.$$

The two circles are centered at γ_1, γ_2 and with radius ρ_1, ρ_2 so

$$\Delta_1 = -{A_1}^2{\rho_1}^2,\ \Delta_2 = -{A_2}^2{\rho_2}^2,\ 2\Delta 12 = A_1 A_2(\delta^2 - {\rho_1}^2 - {\rho_2}^2).$$

If one of the circles are contained within the other the distance between their centres is $\delta = |\gamma_1 - \gamma_2|$.

At a common point their tangents form an angle ω and by the law of cosines we get

$$\delta^2 = {\rho_1}^2 + {\rho_2}^2 \pm 2\rho_1\rho_2 \cos\omega$$

so we write

$$2\Delta_{12} = \pm A_1 A_2 \rho_1 \rho_2 \cos\omega = -2\sqrt{\Delta}_1\sqrt{\Delta}_2 \cos\omega$$

wich gives

$$\cos\omega = \frac{\Delta_{12}}{\sqrt{\Delta}_1\sqrt{\Delta}_2}. \tag{5}$$

When the circles have points in common the angle ω between their tangents is real and

$$-1 \le \cos\omega \le 1.$$

Together with (5) we then get

$$\Delta_1\Delta_2 - \Delta_{12}^2 \geq 0 \tag{6}$$

and if $\Delta_1 < 0$ this is the condition for the quadratic form (6) to have only non positive values for any λ_1, λ_2.

We note that this illustrates a geometric approach to the algebraic solution of the interpolation problem.

A2. From circle to half plane

Choose a Möbius transformation in the z and w plane:

$$z = \frac{a + bZ}{c + dZ} \tag{7}$$

$$w = \frac{a' + b'W}{c' + d'W} \tag{8}$$

With the Pick matrix of the form

$$\frac{w_\alpha + \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta} \tag{9}$$

we get with (7), (8) and (9)

$$\frac{(c + dZ)(\bar{c} + \bar{d}Z)}{(c' + d'W)(\bar{c}' + \bar{d}'\bar{W})} \frac{(a'\bar{c} + \bar{a}'c') + (\bar{a}'d' + b'\bar{c}\neg)W + a'\bar{d}\neg + \bar{b}'c')\bar{W} + (b'\bar{d}' + \bar{b}'d')W\bar{W}}{|c|^2 - |a|^2 + (c\bar{d} - a\bar{b})\bar{Z} + (\bar{c}d - \bar{a}b)Z + (|d|^2 - |b|^2)Z\bar{Z}} \tag{10}$$

The left hand factor is a product of two conjugate quantities and a non negative real number. The right hand factor is an equation for a circle or a plane depending on the coefficients. The cross ratio $T(w) = (w, w_1, w_2, w_3)$ can be written as $T(w) = \frac{a+bW}{c+dW}$ so the denominator of (10) comes from

$$\frac{a + bW}{c + dW} = -\frac{\bar{a} + \overline{bW}}{\bar{c} + \overline{dW}}$$

which is the same as

$$T(w) = -\overline{T(w)}.$$

In the same way we get for the denominator

$$\frac{a + bZ}{c + dZ} = \frac{\bar{c} + \bar{d}Z}{\bar{a} + \bar{b}Z}$$

and

$$Tz = \overline{T(z)}^{-1}$$

Since the cross ratios $T(w)$ and $T(z)$ are real numbers when the four points a, b, c, d lie on a circle or a straight line we can express

$$\frac{w_\alpha + \bar{w}_\beta}{1 - z_\alpha \bar{z}_\beta}$$

as the real number

$$\frac{K_w(w_\alpha, \overline{w}_\beta)}{K_z(z_\alpha, \overline{z}_\beta)} = p_{\alpha,\beta}.$$

And as the assumption from Schwarz lemma is that there exists a function $f(z_\alpha) = w_\alpha$ we can express that with the value pair p_α.