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MASTER'S THESIS

On Minimal Non-$(2,1)$-Colorable Graphs

Author:
Ruth Bosse

Supervisors:
Docent Jörgen Backelin
Prof. Dr. Martin Skutella
Dr. Torsten Mütze

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Department of Mathematics

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Declaration of Authorship

I, Ruth Bosse, declare that this thesis, titled "On Minimal Non-$(\mathbf{2},\mathbf{1})$-Colorable Graphs" and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at Technical University of Berlin and in the context of an exchange with Stockholm University.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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- All Figures in this document are mine. They are created by means of the TEX-Paket TikZ of Till Tantau, see [19].

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Abstract

A graph is $(2,1)$-colorable if it allows a partition of its vertices into two classes such that both induce graphs with maximum degree at most one. A non-$(2,1)$-colorable graph is minimal if all proper subgraphs are $(2,1)$-colorable. We prove that such graphs are 2-edge-connected and that every edge sits in an odd cycle. Furthermore, we show properties of edge cuts and particular graphs which are no induced subgraphs. We demonstrate that there are infinitely many minimal non-$(2,1)$-colorable graphs, at least one of order n for all $n \geq 5$. Moreover, we present all minimal non-$(2,1)$-colorable graphs of order at most seven. We consider the maximum degree of minimal non-$(2,1)$-colorable graphs and show that it is at least four but can be arbitrarily large. We prove that the average degree is greater than $8/3$ and give sufficient properties for graphs with average degree greater than $14/5$. We conjecture that all minimal non-$(2,1)$-colorable graphs fulfill these properties.

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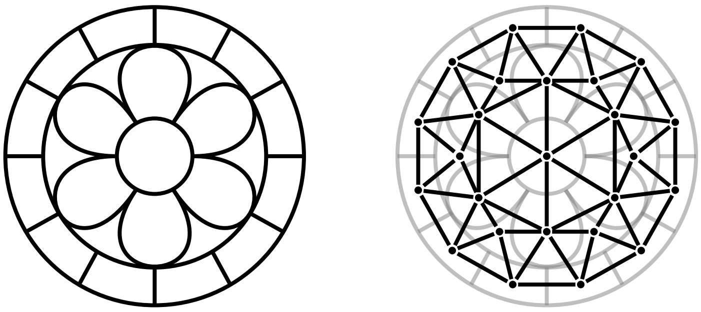
Chapter 1

Introduction

Is it possible to dye the mandala in Figure 1.1 (a) in four colors such that no adjacent regions have the same color? Can the organizer of a conference about cultural diversity invite exactly one speaker per regarded country if they have scheduled various talks, each comparing two of the countries and given by the corresponding two speakers, and no speach shall be given by speakers of the same gender? What is the minimum amount of time, sports classes of one hour need in total, if some of them require the same room?



(a) Mandala (b) Corresponding graph

FIGURE 1.1: Mandala

1.1 Graph Colorings

All these questions can be answered by coloring graphs. To see that, we first need to cast our examples into a graph setting. In the mandala, two regions shall not get the same color if they are neighboring. We can illustrate the mandala by a planar graph in such a way that vertices represent regions and edges adjacencies between them, see Figure 1.1 (b). No two adjacent vertices shall be colored alike. Hence, our question asks if the graph is 4-colorable.

Regarding the second question, consider a graph with the countries discussed in the conference as vertices. We join two vertices if and only if there is a speech about the two corresponding countries. We color a vertex in one color if the representative of this country is female and in the other color if he is male. Our question is answered in the affirmative if and only

if there is a coloring such that every edge is dichromatic.
Two sports classes cannot take place at the same time if and only if they require the same room. As in the other examples, we can also map this into the framework of graph coloring. In fact, we can think of each sports class being represented by a vertex in the graph and connect them if their room is the same. In a proper coloring of this graph, each color represents a time slot. Hence, we need as many hours for our schedule as we need colors in our graph.

But how many colors do we actually need? Of course, this number is bounded from above by the number of vertices in the graph (i.e., in our sport example the total number of sports classes). The more interesting question is that of the minimum number of colors (or hours, in our example) that are required. This number is called the *chromatic number*. For $k \geq 3$, calculating the chromatic number of a graph is NP-complete, see, e.g., [14]. One can verify whether a given coloring is valid in quadratic running time by checking each edge. A polynomial time reduction from 3-SAT gives the NP-hardness. The best known exact algorithm applies inclusion-exclusion and zeta transformation. It decides whether a graph is k-colorable (we call it also *k-partite*) in running time $\mathcal{O}(2^n n)$, see [4]. The problem is easier to solve for 2-colorability. A graph is 2-colorable (or *bipartite*) if and only if it contains no odd cycle, see, e.g., [9]. This can be checked in linear time using breadth-first search or depth-first search.
A fast procedure to color a graph with a bounded number of colors is following greedy algorithm: regard all vertices in a fixed order and pick for every vertex the first color which is not already used in its neighborhood. A vertex of degree d receives at most color $d + 1$. Therefore, the algorithm does not need more than $\Delta + 1$ colors in total, where Δ denotes the *maximum degree* over all vertices. In complete graphs, we need a different color for each vertex and in odd cycles, we need three colors. It follows in both cases that the chromatic number is $\Delta + 1$. Brooks' Theorem [6] shows that the chromatic number of any other connected graph is at most the maximum degree.

The probably best-known theorem in the field of graph theory is the Four Color Theorem. Its statement was already conjectured in 1852 by Francis Guthrie but remained open for more than hundred years. Guthrie asked if four or less colors are sufficient to color the countries of any map such that no neighboring countries have the same color. We saw above, in the example of the mandala, that this is equivalent to the question if any planar graph can be 4-colored. After a sequence of proof attempts, the conjecture was finally shown by Appel and Haken in 1976 [2, 3]. This was the first major proof using the help of computers and hence was initially not accepted by all mathematicians. Figure 1.2 shows a 4-coloring of our mandala.
From this theorem, we can derive that any outerplanar graph is 3-colorable. A graph is outerplanar if it has a planar drawing such that all vertices belong to the outer face. To see the 3-colorability, we add one vertex to the outerplanar graph and join it to every other vertex. The new vertex must have a different color from any vertex in the outerplanar graph. Thus, this graph requires one color more than the outerplanar graph. The graph is still planar as we can draw the new vertex in the outer face. Therefore,

it needs at most four colors and the outerplanar graph cannot need more than three colors. Moreover, Grötzsch [12] proved already in 1959 that also planar graphs without triangles as subgraphs are 3-partite.

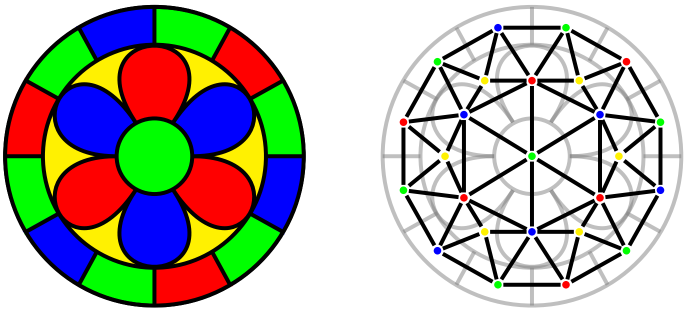


FIGURE 1.2: 4-Colored mandala

Referring back to the conference about different cultures, what happens with our graph if the speakers accept to give one gender-equal talk? What happens in the case of scheduling sports classes, if the rooms in the gym have sufficient space for two or more contemporaneous courses?
In terms of the conference, each vertex might sit in one monochromatic edge. The fitness center can offer more courses per room at once. Vertices of the same color represent simoultaneous courses. Hence, any vertex can have j same-colored neighbors if the rooms are big enough for $j+1$ courses.

We characterize a coloring as *defective* or *j-improper*, if it is such that every vertex has at most j monochromatic edges. We refer to j-improper k-colorings as *(k, j)-colorings*. The minimum number k such that a graph G is (k, j)-colorable is called its *j-defective chromatic number* $\chi_j(G)$. Defective colorings were introduced almost simoultaneously by Andrews and Jacobson [1], Harary and Jones [13] and Cowen, Cowen and Woodall [7]. They are defined for all integers $j \geq 0$ and $k \geq 1$. Hence, proper colorings are the special case of defective colorings where $j = 0$. We denote $\chi_0(G)$ by $\chi(G)$.
The problem (k, j)-COLORING asks whether a given graph is (k, j)-colorable. As previously seen, $(k, 0)$-COLORING is NP-complete for $k \geq 3$ and quadratic for $k = 2$. Also (k, j)-COLORING is in NP since checking the neighbors of each vertices in a colored graph can be done in quadratic running time. In addition, Cowen, Goddard and Jesurum [8] showed by a reduction from $(k, 0)$-COLORING that it is NP-hard to determine whether a graph is (k, j)-colorable for all $k \geq 3$ and $j \geq 1$. Furthermore, they proved the NP-completeness of $(2, 1)$-COLORABILITY for graphs of maximum degree four and for planar graphs of maximum degree at most five by means of polynomial time reduction from 3-SAT. This problem is reducable to $(2, j)$-COLORABILITY and to $(3, 1)$-COLORABILITY for planar graphs for all $j \geq 1$. It is especially interesting to see that $(2, j)$-COLORING is fast for $j = 0$ but cannot be solved efficiently for all $j \geq 1$.
We saw by dint of the greedy algorithm that the chromatic number is bounded by the maximum degree Δ. Also the defective chromatic number is bounded in terms of the maximum degree. Gerencsér [10] showed for 1-improper colorings, that any graph G with maximum degree Δ fulfills

$\chi_1(G) \leq \lfloor \Delta/2 \rfloor + 1$. This result was extended by Lovász [16] to $(k, \lfloor \Delta/k \rfloor)$-colorability for all $k \geq 1$.
Applying the notation of defective colorings, the Four Color Theorem states that planar graphs are $(4,0)$-colorable. Cowen, Cowen and Woodall [7] proved that planar graphs are moreover $(3,2)$-colorable and that outerplanar graphs are $(2,2)$-colorable. Planar graphs even allow a $(3,2)$-coloring without monochromatic cycles, i.e., where all monochromatic connected components are paths, as shown by Poh [17] and Goddard [11].

Defective colorings are introduced to allow monochromatic star graphs with at most j leaves. In our example from above, where sports classes are scheduled, this would correspond to permitting $j+1$ concurrent classes in one room. In the graph setting, these classes induce not only a monochromatic star but even a monochromatic *clique* (a complete subgraph). Hence, an optimal schedule has at most k hours if we can color the graph in k colors without a monochromatic clique of order $j+2$. Let us apply this idea to an arbitrary graph F. A coloring without a monochromatic copy of F is called an *F-coloring*. The defective colorings are the special case of F-colorings where F is the star of order $j+2$.
The study of F-colorings is amongst others motivated by Ramsey theory. The classical problem in this field asks for the minimum number of people one must invite such that at least r will know each other or at least s will not know each other. Let us map this question onto the following graph: every person is represented by a vertex and every two vertices are adjacent. Now, let us color the edges of this graph. The edge between two persons receives one color if the persons know each other and the other color if not. An r-clique of the first color represents r persons which know each other and an s-clique of the second color represents s persons which do not know each other. Therefore, our question asks for the minimum size of a complete graph such that any coloring of its edges in two colors contains either an r-clique of one color or an s-clique of the other. Ramsey [18] proved the existence of such a minimum size for any two integers r and s. There are various generalizations of the classical problem within Ramsey theory. They treat for example higher numbers of colors, forbidden monochromatic subgraphs (w.r.t. the edge coloring) which are no cliques or sets of forbidden monochromatic subgraphs, e.g., the set of all cycles.

1.2 Extremal Graphs

We saw above that there is a fast algorithm deciding whether a graph is 2-colorable or not. This algorithm employs the fact that a graph cannot be 2-colored if and only if it contains an odd cycle as a subgraph. We refer to a graph as *minimal* with respect to a certain property if it fulfills this property but no proper subgraph does. Similarly, a graph is *maximal* w.r.t. a property if the graph itself has the property but no proper supergraph does. Consider the set of all graphs together with the subgraph relation. This is a partially ordered set with the empty graph as its least element and without any maximal elements. Both, the set of minimal and the set of maximal graphs w.r.t. some property form an antichain. Consider a property which is closed under taking subgraphs and the minimal graphs which do not fulfill it. Also these graphs form an antichain in the partially ordered set. As the set is closed downwards, precisely the graphs in and above this antichain do not fulfill the property. With these notations, the odd cycles are the minimal non-2-colorable graphs. They form an antichain and precisely their supergraphs are not bipartite. Also defective colorablility is closed under taking subgraphs. Therefore, the minimal non-(k, j)-colorable graphs form an antichain for any k and j. Every non-(k, j)-colorable graph contains at least one of them.

How do the maximal (k, j)-colorable graphs look like? Considering this question, we see that such graphs do not exist. We could always add an isolated vertex and the graph would remain (k, j)-colorable. We might instead ask for the graphs where we can not add an edge without loosing (k, j)-colorability. In other words, we only consider graphs of the same order. Hence, this partially ordered set is bounded from above. We call a graph *edge-maximal* w.r.t. some property if the graph itself fulfills this property but no proper supergraph of the same order does. It is well-known that the edge-maximal k-colorable graphs are the *complete k-partite graphs*. These are the graphs with a vertex partition into k classes such that any two vertices are adjacent if and only if they are in different partition classes, see, e.g., [9].
We want to extend this idea to $(2, 1)$-colorability. Let us partition the vertices into two classes, one per color. As our graph shall be maximal, we join any two vertices, which are not in the same class, by an edge. Within the partition classes, the degree is bounded by one. To obtain maximality, both classes contain a disjoint union of 2-cliques and possibly one additional isolated vertex. Figure 1.3 illustrates these graphs.

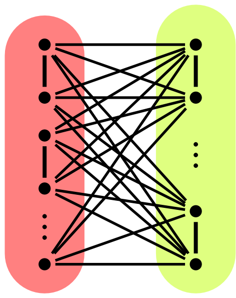


FIGURE 1.3: Edge-maximal $(2, 1)$-colorable graphs

1.3 About this Thesis

We studied minimal non-$(2, 0)$-colorable graphs, maximal k-colorable graphs and maximal $(2, 1)$-colorable graphs. This leads to the question, how minimal non-$(2, 1)$-colorable graphs look like. We will henceforth refer to the class of all these graphs as $\mathcal{G}$. We saw that $(2, 0)$-COLORING can be solved in quadratic running time, but already $(2, 1)$-COLORING is NP-complete, even for bounded maximum degree and planar graphs. This indicates that $\mathcal{G}$ might be intricate and encourages its analysis.

A second motivation to study minimal non-$(2,1)$-colorable graphs arises from an article from Borodin, Kostochka and Yancey [5], published in 2013. The authors showed that all non-$(2,1)$-colorable graphs have at least one subgraph with average degree greater than $14/5$. Already in 1994, Kurek and Riciński [15] proved the existence of a subgraph with average degree at least $8/3$. A non-$(2,1)$-colorable graph contains a subgraph G which belongs to $\mathcal{G}$. We wondered if G is one of the subgraphs with average degree at least $8/3$. Is it even possible to bound the average degree of G by $14/5$? These questions inspired us to analyze the average degree of minimal non-$(2,1)$-colorable graphs.

1.4 Our Results

The thesis at hand proves various graph invariants fulfilled by the graphs in $\mathcal{G}$. These are local restrictions, e.g., that no bivalent vertices are adjacent, and global properties, such as 2-edge-connectivity. One main result improves this conclusion and shows that every edge even sits in an odd cycle, see Theorem 7. Moreover, we present subsets of $\mathcal{G}$. First, we display all graphs with at most seven vertices. Secondly, we demonstrate infinite subsets of $\mathcal{G}$. The existence of such subsets directly implies the infinity of $|\mathcal{G}|$ which we also conclude from the NP-completeness of $(2,1)$-COLORING assuming $\mathsf{P} \neq \mathsf{NP}$.
It follows directly from the previously mentioned results of Gerencsér and Lovász, that the maximum degree of minimal non-$(2,1)$-colorable graphs is at least four. We display an infinite subset of $\mathcal{G}$ which contains only graphs of maximum degree four. On the other hand, we employ an infinite subset of $\mathcal{G}$ to show that the maximum degree of a minimal non-$(2,1)$-colorable graph can be arbitrarily large, see Theorem 5. Some infinite subsets belong entirely to the planar graphs. Nevertheless, there are non-planar graphs in $\mathcal{G}$, see Theorem 9. Furthermore, we show that there is a $G \in \mathcal{G}$ of order n for all $n \geq 5$, see Theorem 13.
In a final step, we study the average degree of minimal non-$(2,1)$-colorable graphs. In Theorem 14, we show that the graphs in $\mathcal{G}$ have average degree strictly greater than $8/3$. As mentioned above, a recent publication of Borodin, Kostochka and Yancey raised the question if this lower bound can be improved to $14/5$. We identified sufficient properties for an average degree greater than $14/5$, see Theorem 15. We analyze if $G \in \mathcal{G}$ fulfills them.

1.5 Outline of this Thesis

In Chapter 2, we introduce the notations and definitions used in this work. In Chapter 3, we characterize the structure of the graphs in $\mathcal{G}$. This includes basic properties about connectivity and vertices of small degree and restrictions for their subgraphs. Chapter 4 presents all graphs in $\mathcal{G}$ which have less than eight vertices and proves the completeness of this set. In Chapter 5, we prove that $\mathcal{G}$ has unbounded maximum degree by presenting a subset of $\mathcal{G}$ with this property. In Chapter 6, we study odd cycles. We prove that any edge in $G \in \mathcal{G}$ belongs to an odd cycle and present sets of minimal non-$(2,1)$-colorable graphs with one central odd cycle. In Chapter 7, we investigate the infimum for the average degree of $\mathcal{G}$. We show that this infimum is at least $8/3$ and at most $14/5$. In Section 7.4, we conjecture that the infimum is $14/5$ and reduce this conjecture to weaker statements.

Chapter 2

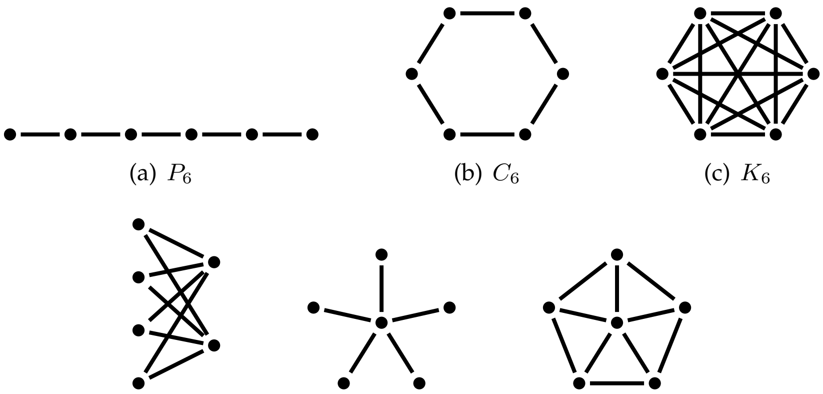
Preliminaries

We denote the vertex set of a graph G by $V(G)$ and the edge set by $E(G)$. The edge $\{x, y\}$ is usually written as xy. The function n maps a graph onto its order and the function m onto its number of edges.

Let V' be a vertex subset and v a vertex, E' an edge subset and e an edge of a graph $G = (V, E)$. The graph $G[V']$ is the subgraph of G induced by V'. The graph $G[E']$ contains the edges in E' and all vertices which belong to one of these edges. We write $G - V'$ for the graph $G[V \backslash V']$ and $G - G'$ for $G - V(G')$. The graph $G - E'$ denotes $(V, E \backslash E')$ and for a set F of pairs of vertices in G, $G + F$ is the graph $(G, E \cup F)$. In case of singletons, we shorten $G - v := G - \{v\}$, $G - e := G - \{e\}$ and $G + f := G + \{f\}$. For $e = vw$, $G - \{v, w\}$ means $G - v - w$, not $G - e$.

The *complement graph* of G is $\overline{G} := (V, \overline{E})$ where two vertices are adjacent if and only if they are not adjacent in G. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs. Their *union* is $G_1 \cup G_2 := (V_1 \cup V_2, E_1 \cup E_2)$. If $V_1 \cap V_2 = \emptyset$, this is a *disjoint union*, denoted by $G_1 \dot{\cup} G_2$. Their *intersection* is defined as $G_1 \cap G_2 := (V_1 \cap V_2, E_1 \cap E_2)$. The union of two vertex-disjoint graphs together with edges between any two vertices v_1 and v_2 such that $v_1 \in V_1$ and $v_2 \in V_2$ is called *graph join* $G_1 + G_2$.

Following special types of graphs play an important role in this thesis:
A *k-path* P_k is a graph with k vertices $v_1, \dots, v_k$ and edges $v_i v_{i+1}$ for all $1 \leq i \leq k - 1$. We write $P_k = v_1 v_2 \dots v_k$. The *length* of the path P_k is $k - 1$, the number of its edges. We call a path *odd* if its length is odd and *even* otherwise. A path which is a subgraph of P_k is called *subpath* of P_k.
A *k-cycle* C_k is the 2-regular connected graph with k vertices. We write $C_k = v_1 v_2 \dots v_k$. For an odd number k, C_k is called *odd* and for an even k, it is called *even*. A graph is called *cyclic* if it contains a cycle as a subgraph and *acyclic* otherwise.
A *k-clique* K_k is the graph on k vertices where all pairs of vertices are adjacent. We call this graph *complete*.
A *complete bipartite graph* K_{n_1,n_2} has a partition of its vertex set into two classes of size n_1 and n_2 such that two vertices are adjacent if and only if they belong to the different classes.
A *k-star* S_k is the tree on k vertices with one *central vertex* of degree $k - 1$.
A *k-wheel* W_k is the graph join of a cycle C_{k-1} and a graph of order 1.



(a) P_6 (b) C_6 (c) K_6

(d) $K_{4,2}$ (e) S_6 (f) W_6

FIGURE 2.1: Special graphs of order six

The degree $d_G(v)$ of a vertex is the size of its neighborhood $N_G(v)$. For a subgraph G' of G, $d_{G'}(v)$ denotes the size of $N_{G'}(v) := N_G(v) \cap V(G')$. The set of neighbors of $V' \subseteq V$ in $V \backslash V'$ is $N_G(V')$. Similarly, $N_G(G')$ are the neighbors of $V(G')$ outside G'. If there is no chance for confusion, we omit the index G. The maximum degree of a graph is denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. The average degree is $\mathrm{ad}(G) := 2m(G)/n(G)$. The maximum average degree over all subgraphs $H \subseteq G$ is denoted by $\mathrm{mad}(G)$. A vertex of degree zero is called *isolated* and a vertex of degree one is a *leaf*.

For any vertex coloring c of G, we define the *impropriety* of a vertex as the number of its monochromatic edges and the impropriety of c as the maximum over all improprieties of vertices in G. Let G' be a subgraph of G. We call a $(2,1)$-coloring c' of G' *extendable* to G if there is a $(2,1)$-coloring c of G such that $c|_{G'} = c'$. The terms "Coloring" and "colorability" are abbreviations for "$(2,1)$-coloring" and "$(2,1)$-colorability" unless otherwise stated.

Chapter 3

Structural Properties

In this chapter, we consider the structure of minimal non-(k, j)-colorable graphs, primarily the case $j = 1$ and $k = 2$. The main points of interest are connectivity results and induced subgraphs which cannot occur.

3.1 Connectivity and Minimal Degree

Lemma 1. *All minimal non-(k, j)-colorable graphs are connected.*

Proof. A non-(k, j)-colorable graph contains a connected component which is non-(k, j)-colorable. If the graph were disconnected, this component were a proper subgraph. □

Lemma 2. *Let G be minimal non-(k, j)-colorable for a $k \geq 2$. Then G has no separating edge, i.e., deleting any edge does not increase the number of components.*

Proof. For a contradiction, assume that G contains a separating edge $e = vw$. The graph $G - e$ has a (k, j)-coloring c since G is minimal. The vertices v and w are not connected in $G - e$. Thus, we can assume $c(v) \neq c(w)$. It follows that c is also a (k, j)-coloring of G which leads to contradiction. □

An edge which is not separating belongs to a cycle. Together with Lemma 1, this gives following results:

Corollary 1. *Minimal non-(k, j)-colorable graphs are 2-edge-connected for $k \geq 2$, i.e., removing any edge does not distroy the connectivity.*

Corollary 2. *For all $k \geq 2$, every vertex in a minimal non-(k, j)-colorable graph has degree at least two.*

Corollary 2 also follows from the following stronger result:

Theorem 1. *The minimal degree $\delta(G)$ of a minimal non-(k, j)-colorable graph with $k \geq 2$ is at least k.*

Proof. Assume that G contains a vertex v with at most $k-1$ neighbors. As G is minimal, $G-v$ has a (k,j)-coloring. In this coloring, one color is not used in the neighborhood of v. Coloring v in this color and all other vertices as in the coloring of $G-v$ gives a (k,j)-coloring of G. □

3.2 Configurations

In this section, we consider local structures which cannot occur in minimal non-$(2,1)$-colorable graphs.

Definition 1. A *configuration* in G is a triple $(H, \deg, V_H)$ such that H is an induced subgraph of G, the *configuration subgraph*. Moreover, V_H is a vertex subset of H and $\deg$ a function assigning a non-negative integer to each vertex in $V(H)\backslash V_H$. Vertices in V_H fulfill $d_G(v) \geq d_H(v)$, we call these vertices *unbounded*. Vertices in $V(H)\backslash V_H$ have degree $d_G(v) = \deg(v)$, we call them *bounded*. We define k_v as $\deg(v) - d_H(v)$ for all $v \in V(H)\backslash V_H$. For a configuration C in G, we denote by $G\backslash C$ the graph $G-(H-V_H)$, where C is *deleted*.

The illustration of configurations is as follows: we draw the subgraph H together with k_v additional vertices for each $v \in V(H)\backslash V_H$. These vertices are drawn distinct and non-adjacent even if bounded vertices might have common neighbors outside H or these neighbors can be connected by an edge. The additional vertices are joined to v by dashed edges. With three short and thin edges, we symbolize that a vertex can have further neighbors outside the configuration. An example is shown in Figure 3.1. The picture illustrates the configuration

$$((\{v,x,y,z\}, \{vx, vy, vz, yz\}), \{\deg : v \mapsto 3, x \mapsto 2, y \mapsto 2\}, \{z\}) .$$

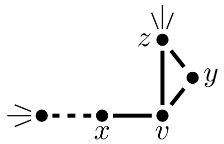


FIGURE 3.1: Example of a configuration

A configuration is a set of induced subgraphs with specified vertices which are allowed to have neighbors outside the subgraph. In this section, we show local properties of minimal non-$(2,1)$-colorable graphs. These properties can be described as configurations which do not appear.

Lemma 3. *Every edge in a minimal non-$(2,1)$-colorable graph contains a vertex of degree at least 3.*

This equals the fact that the configurations $(K_2, \deg, \emptyset)$ with $\deg(v) \leq 2 \ \forall v \in V(K_2)$ do not occur.

Proof. Let $e = vw$ be an edge in a graph $G \in \mathcal{G}$. We know that v and w have degree at least two, see Corollary 2. Suppose that both vertices are bivalent. The graph $G' := G - \{v, w\}$ is a proper subgraph of G and hence permits a $(2,1)$-coloring c'. Coloring all vertices in $V \backslash \{v, w\}$ as in c' and the vertices v and w in the other color from their neighbor in G' gives a $(2,1)$-coloring of G. This contradicts the fact that G is non-$(2,1)$-colorable. □

Lemma 4. *In a minimal non-$(2,1)$-colorable graph, every vertex of degree three has a neighbor of degree at least three.*

Proof. Assume for a contradiction that there is a vertex v in $G \in \mathcal{G}$ with $N(v) = \{v_1, v_2, v_3\}$ and $d(v_i) \leq 2$ for all $i \in \{1,2,3\}$. By Corollary 2, $d(v_i) = 2$. Let w_i be the second neighbor of v_i. By minimality, the graph $G - v$ has a $(2,1)$-coloring c. We can assume w.l.o.g. that $c(v_i) \neq c(w_i)$ because the vertices v_i are leaves in $G - v$. One color occurs at most once in $N_G(v)$. Coloring v in this color gives a 1-improper 2-coloring of G. □

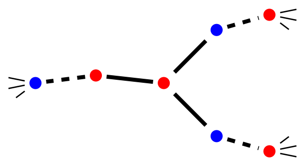


FIGURE 3.2: Configuration which does not appear in minimal non-$(2,1)$-colorable graphs

Less formally, we can say that a trivalent vertex is not surrounded by bivalent vertices. This statement can be extended to bipartite graphs. Kurek and Ruciński showed the weaker extension to trees, see Lemma 3 in [15].

Lemma 5. *Let G be a minimal non-$(2,1)$-colorable graph and V_d the set of all d-valent vertices. Every component of $G - V_2$ contains either an odd cycle or a vertex of degree at least four.*

Proof. For a contradiction, assume that there is a bipartite component G' of $G - V_2$ with $V(G') \subseteq V_3$. Let V_2' be the set of all vertices of degree two whose neighbors are both in $V(G')$. The graph G is minimal non-$(2,1)$-colorable. Thus, there is a $(2,1)$-coloring c of the graph $G'' := G - G' - V_2'$. It holds $d_{G''}(v) = 1$ for the vertices v in $N(G') \backslash V_2'$. We can assume w.l.o.g. that their edges are dichromatic in c.
Let $\widehat{G}$ be the graph which results from G if we replace every vertex v in V_2' by two leaves such that each neighbor of v is adjacent to one of the leaves. Consider following coloring $\hat{c}$ of $\widehat{G}$: leaves receive color 1 and the other vertices in $\widehat{G} - G'$ the same color as in c. We extend this coloring successively to a supergraph $\widehat{G}'$ of $\widehat{G} - G'$. All vertices in G', which have two neighbors of the same color, receive the other color. Let $\widehat{G}'$ be the maximal graph where this is possible.
All vertices $v \in G'$ have three neighbors in $\widehat{G}$. Hence, if we color v, at most one neighbor is not colored yet. We see by induction, that, except for the

last vertex, the third neighbor is always uncolored and therefore, no vertex in $V(G') \cap V(\widehat{G}')$ is in a monochromatic edge. Induction also gives that the uncolored subgraph stays connected. If $\widehat{G}' = \widehat{G}$, i.e., if we color the entire subgraph G' by this procedure, the last vertex has impropriety at most one. Hence, this is a $(2,1)$-coloring of $\widehat{G}$.
If $\widehat{G}' \neq \widehat{G}$, no vertex in $V(G') \cap V(\widehat{G}')$ is in a monochromatic edge. The neighborhood of the uncolored subgraph $\widehat{G}'' = \widehat{G} - \widehat{G}'$ consist of leaves, vertices in $N(G') \backslash V_2'$ and vertices in $V(G') \cap V(\widehat{G}')$. All these vertices are not in monochromatic edges. Moreover, a vertex in $\widehat{G}''$ is not adjacent to two vertices of the same color as it would belong to $\widehat{G}'$ in this case. Thus, if we extend $\hat{c}$ by coloring the vertices in $\widehat{G}''$ properly, the whole graph contains no monochromatic P_3 and $\hat{c}$ is a $(2,1)$-coloring of $\widehat{G}$. This is possible since $\widehat{G}'' \subseteq G'$ is bipartite.
The following derives a $(2,1)$-coloring of G from any $(2,1)$-coloring of $\widehat{G}$ where all leaves have color 1. Color $G - V_2'$ as in $\widehat{G}$. In $\widehat{G}$, $v \in V_2'$ is replaced by leaves of color 1. As these leaves are not in $\widehat{G}$, we can color v in color 1 if not v and its two neighbors would form a P_3 of color 1. If so, both neighbors have color 1 and we can give color 2 to v. This is a 1-improper 2-coloring of G und thus, gives the sought contradiction. □

Figure A.1 in Appendix A illustrates this coloring.

We can strengthen this result as follows:

Lemma 6. *Let G and V_d be as in Lemma 5. If a component in $G - V_2$ consists only of vertices in V_3, it contains at least two odd cycles.*

Proof. Assume that there is such a component G' with exactly one odd cycle $C = v_1 v_2 \dots v_k$. Let $\widehat{G}$ be as in the proof of Lemma 5. We construct a $(2,1)$-coloring $\hat{c}$ of $\widehat{G}$ in a similar manner as above. Again, we give color 1 to the leaves and $(2,1)$-color the other vertices in $\widehat{G} - G'$ such that no neighbor of G' is in a monochromatic edge. We color a supergraph $\widehat{G}'$ of $\widehat{G} - G'$ such that every vertex with two neighbors in one color gets the other color and choose $\widehat{G}'$ maximal.
The graph $\widehat{G}'$ does not contain the odd cycle C as each v_i has only one neighbor outside C. We call this neighbor w_i. Let us color $\widehat{G}'' := \widehat{G} - \widehat{G}'$ as follows: if all w_i are in $\widehat{G}'$, there is a $j \leq k-1$ such that $\hat{c}(w_j) = \hat{c}(w_{j+1})$. Color v_j and v_{j+1} in the other color from w_j. Otherwise, let w_j be a vertex which does not belong to $\widehat{G}'$. If w_{j+1} is in $\widehat{G}'$, color v_j and v_{j+1} in the other color from w_{j+1}. If w_{j+1} is not in $\widehat{G}'$, color v_j and v_{j+1} in color 1. The graph $\widehat{G}'' - v_j v_{j+1}$ is bipartite. We 2-color it properly, extending the coloring of v_j and v_{j+1}.
This coloring $\hat{c}$ of $\widehat{G}$ has no monochromatic P_3 in the subgraphs $\widehat{G}'$ and $\widehat{G}''$. Consider an edge xy with $x \in V(\widehat{G}')$ and $y \in V(\widehat{G}'')$. The edge $v_j v_{j+1}$ is the only monochromatic edge in $\widehat{G}''$. If $y \in \{v_j, v_{j+1}\}$, x is its neighbor outside the cycle. We see that y received the other color from x. Similarly to the proof above, no vertex in $N_{\widehat{G}}(\widehat{G}'')$ is in a monochromatic edge. As

$x \in N_{\widehat{G}}(\widehat{G}'')$, the coloring $\hat{c}$ is a $(2,1)$-coloring of $\widehat{G}$. We can construct a $(2,1)$-coloring of G as in the proof of Lemma 5. □

3.3 Edge Cuts

Definition 2. An *edge cut* of a graph G is a set of edges $E' \subseteq E$ such that $G - E'$ is disconnected. *Edge-connectivity* $\lambda(G)$ is the minimum size of an edge cut of G. We call edges in E' *cut edges* and their vertices *edge cut vertices*.

Let G be a minimal non-$(2,1)$-colorable graph. We know from Lemma 2, that G has no separating edge and thus, $\lambda(G) \geq 2$.

First, we treat the case $\lambda(G) = 2$. Let $E' = \{e_1, e_2\}$ be an edge cut of G and G_x and G_y the components of $G - E'$, the *cut sets*. The subgraphs G_x and G_y are connected. Otherwise, either G were disconnected or both cut edges were separating. Let $e_i = x_i y_i$ with $x_i \in V(G_x)$ and $y_i \in V(G_y)$.

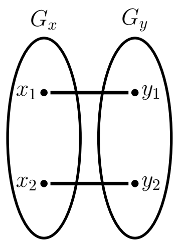


FIGURE 3.3: Graph with an edge cut of size two

Definition 3. Let G be a $(2,1)$-colorable graph with specified vertices x_1 and x_2. We call G *enforced same-colored w.r.t. x_1 and x_2* if $c(x_1) = c(x_2)$ in any $(2,1)$-coloring of G and *enforced different-colored w.r.t. x_1 and x_2* if $c(x_1) \neq c(x_2)$ in any $(2,1)$-coloring of G.

Lemma 7. *It holds w.l.o.g., that the subgraph G_x is enforced same-colored w.r.t. x_1 and x_2 and the subgraph G_y is enforced different-colored w.r.t. y_1 and y_2.*

Proof. Suppose that both cut sets admit a coloring where the edge cut vertices have the same color. We call these colorings c_x and c_y. As the colors are symmetric, we can assume $c_x(x_1) = c_x(x_2) = 1$ and $c_y(y_1) = c_y(y_2) = 2$. Then $c_x \cup c_y$ is a $(2,1)$-coloring of G. In a similar way, let both subgraphs have a $(2,1)$-coloring with different-colored edge cut vertices, say c'_x and c'_y. Then we can assume $c'_x(x_i) \neq c'_y(y_i)$ for both cut edges e_i and $c'_x \cup c'_y$ is a $(2,1)$-coloring of G. As both components are $(2,1)$-colorable, the symmetry of G_x and G_y gives the lemma. □

Remark 1. The case $x_1 = x_2$ is possible.

Remark 2. Lemma 7 holds for all non-$(2,1)$-colorable graphs with an edge cut of size two if the cut sets are $(2,1)$-colorable.

Corollary 3. *The cut sets contain at least five vertices.*

Proof. The 4-clique is $(2,1)$-colorable and neither enforced same-colored nor enforced different-colored w.r.t. x_1 and x_2 for any two vertices x_1 and x_2 in K_4. Every graph with less than five vertices is a subgraph of K_4 and thus, also fulfills this property. □

We can conclude the following lemma from this corollary:

Lemma 8. *Let v and w be adjacent vertices of degree three. Then all vertices in $N(v) \cap N(w)$ have degree greater than two.*

Proof. Let $N(v) = \{w, x, y\}$ and X be the set of bivalent vertices in $N(v) \cap N(w)$. We assume for a contradiction that $X \neq \emptyset$. If $X = \{x, y\}$, the connectivity of G gives $V(G) = \{v, w, x, y\}$. Since this graph is $(2,1)$-colorable, we can assume $X = \{x\}$. Let z be the other neighbor of w. The edges vy and wz form an edge cut of G such that one cut set a 3-clique. This contradicts Corollary 3. □

Figure 3.4 shows $(2,1)$-colorings of the configuration with $X = \{x\}$. One of them colors the vertices v and w in the same color and one in different colors. We call this configuration C_1'.

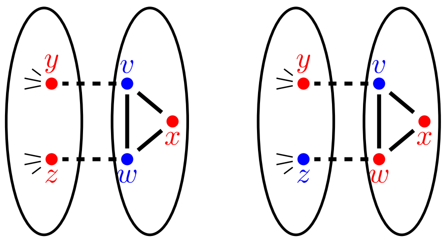


FIGURE 3.4: Configuration C_1'

Now, consider a graph H with an edge cut E' of size two. We denote the cut edges as in Lemma 7 and the cut sets by H_x and H_y. Let H_x be enforced same-colored w.r.t. x_1 and x_2 and H_y be enforced different-colored w.r.t. y_1 and y_2. Furthermore, let both cut sets are minimal with this property, i.e., any proper subgraphs of H_x admits a $(2,1)$-coloring c such that $c(x_1) \neq c(x_2)$ and similarly for H_y.

Lemma 9. *The graph H is minimal non-$(2,1)$-colorable iff in any $(2,1)$-coloring of $H - E'$, at least one vertex of each cut edge is in a monochromatic edge.*

This is equivalent to the fact that, either in every coloring of one cut set, both edge cut vertices are in monochromatic edges, or, that for an $i \in \{1, 2\}$, the vertices x_i and y_{3-i} are in monochromatic edges in every coloring of the entire graph $H - E'$.

Proof. Assume that there is a $(2,1)$-coloring of H. As H_x is enforced same-colored w.r.t. x_1 and x_2 and H_y is enforced different-colored w.r.t. y_1 and y_2, one cut edge is monochromatic and thus, in a monochromatic P_3. We want

to show that H is minimal, i.e., that any proper subgraph of H is $(2,1)$-colorable. As $(2,1)$-colorability is closed under taking subgraphs, it is sufficient to show this for all maximal proper subgraphs. These are the graphs in $\{H - e \mid e \in E(G)\}$. First, let e be an edge in one of the components, say w.l.o.g. $e \in E(H_x)$. The graph H_x is minimal enforced same-colored w.r.t. x_1 and x_2. Thus, $H_x - e$ is neither enforced same-colored nor enforced different-colored w.r.t. x_1 and x_2. By Remark 2, the graph $H - e$ is $(2,1)$-colorable. Now, consider $H - e_i$ for $i \in \{1, 2\}$. In this graph, the cut edge e_{3-i} is a separating edge and we can $(2,1)$-color H_x and H_y such that this edge is dichromatic.
Conversely, assume that there is a $(2,1)$-coloring c of $H - E'$ such that w.l.o.g. x_1 and y_1 are in no monochromatic edge. If c is no $(2,1)$-coloring of H, the edge e_2 belongs to a monochromatic P_3. Interchanging colors in H_y leads to $c(x_2) \neq c(y_2)$ and thus to a $(2,1)$-coloring of H. □

Remark 3. Let $E' = \{e_1, \dots, e_l\}$ be an edge cut of size $l \in \mathbb{N}$ with $e_i = x_i y_i$ for all $i \leq l$. In this case, the cut sets G_x and G_y might be disconnected. It follows from the same arguments that there are no $(2,1)$-colorings c_x and c_y of G_x and G_y such that $c_x(x_i) = c_y(y_i)\ \forall i \leq l$ or $c_x(x_i) \neq c_y(y_i)\ \forall i \leq l$.

Remark 4. Lemma 7 and Remark 3 hold for any defective 2-coloring: if the cut sets could be $(2,j)$-colored such that all cut edges were dichromatic, no cut edge were in a monochromatic S_{j+1} and we had a $(2,j)$-coloring of G.

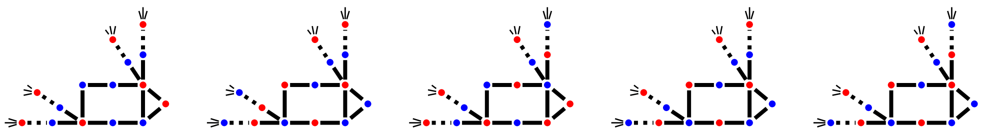
We apply our results to forbid certain configurations in minimal non-$(2,1)$-colorable graphs.

Corollary 4. *Let C be a configuration with $V_H = \emptyset$ and $k_v \leq 1$ for any vertex v in H. If C admits a $(2,1)$-coloring for any partition of the vertices with $k_v = 1$ into two color classes, then C does not occur in a minimal non-$(2,1)$-colorable graph.*

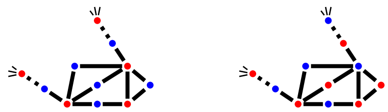
Proof. In a configuration without unbounded vertices, the dashed edges form an edge cut. The vertices of H are in the edge cut iff $k_v \geq 1$. In C, all these vertices belong to exactly one cut edge. Any 2-coloring of the edge cut vertices is extendable to C. A graph $G \in \mathcal{G}$ does not contain C because the subgraph $G \backslash C$ were $(2,1)$-colorable and we could extend this to G by a $(2,1)$-coloring of C where each cut edge is dichromatic. □

Figure 3.5 displays six configurations $C'_i = (H_i, \deg_i, V_{H_i})$ which cannot occur in any minimal non-$(2,1)$-colorable graph. The picture shows that these configurations fulfill the conditions of Corollary 4. This proves Lemma 10. We use the lemma in Chapter 7.4.

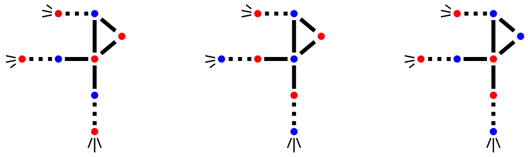
Lemma 10. *A minimal non-$(2,1)$-colorable graph does not contain the configurations $C'_2, C'_3, C'_4, C'_5, C'_6$ and C'_7 shown in Figure 3.5.*



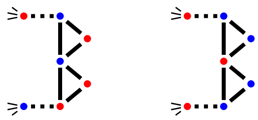
(a) Configuration C_2'



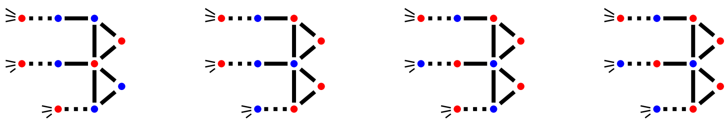
(b) Configuration C_3'



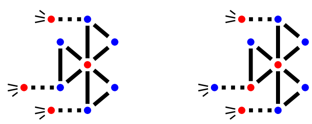
(c) Configuration C_4'



(d) Configuration C_5'



(e) Configuration C_6'



(f) Configuration C_7'

FIGURE 3.5: Forbidden configurations

3.4 Unique Violation of $(2,1)$-Colorability

Let G be a graph in $\mathcal{G}$. We show that we can 2-color its vertices such that only one path violates the requirements of a $(2,1)$-coloring.

Lemma 11. *Every minimal non-$(2,1)$-colorable graph admits a 2-coloring of its vertices with exactly one monochromatic subgraph with more than two vertices. This subgraph is a path of length at most three.*

Proof. Let G be a minimal non-$(2,1)$-colorable graph with an arbitrary edge $e = vw$. By minimality, $G - e$ has a $(2,1)$-coloring. If we color the vertices of G analogously, every monochromatic connected subgraph of order at least three contains e. Thus, there is only one such subgraph. Both, v and w, have at most one neighbor of the same color and this neighbor has no further monochromatic edges. Hence, the unique monochromatic subgraph with at least three vertices is a P_3 or a P_4. $\square$

Lemma 12. *A minimal non-$(2,1)$-colorable graph G with $\delta(G) \leq 3$ permits a 2-coloring such that only one vertex has impropriety two.*

Proof. Let v be a vertex with $d(v) \leq 3$. By minimality, the graph $G-v$ admits a $(2,1)$-coloring such that one color occurs only once in $N_G(v)$. Giving v this color leads to a coloring which fulfills the requirements. $\square$

Chapter 4

Graphs of Small Order

We saw different properties of the graphs in $\mathcal{G}$. Some of these graphs are presented in this chapter. It displays all minimal non-$(2,1)$-colorable graphs with at most seven vertices.

4.1 Graphs with a Central Vertex

Definition 4. Let G be a graph with a vertex v of degree $n(G)-1$. The vertex v is called *central vertex*.

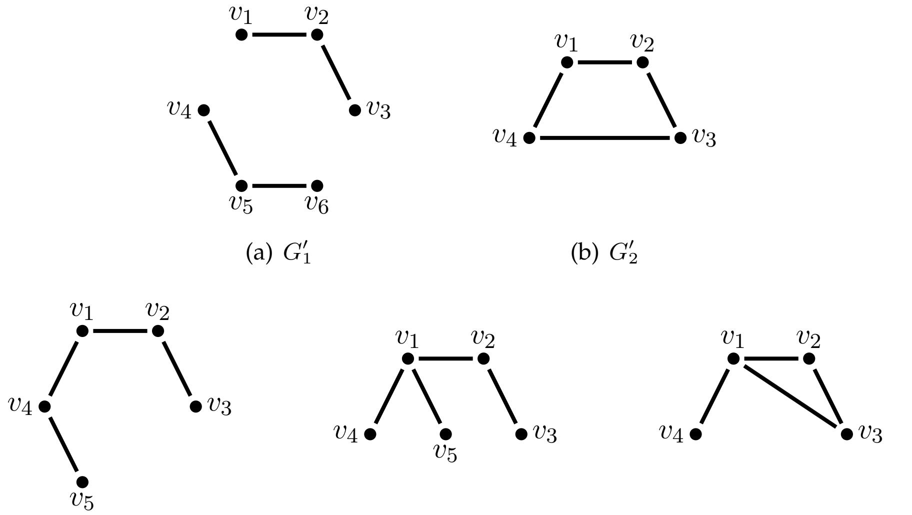
Lemma 13. *Consider a graph G with a central vertex v. The graph G is $(2,1)$-colorable if and only if there is a vertex set of size $n(G)-2$ in $G-v$ which does not contain a P_3.*

Proof. If such a set exists, color it in color 1 and the two remaining vertices in color 2. Conversely, any 1-improper 2-coloring of G colors at most one vertex in $G-v$ in the same color as v. Thus, $n(G)-2$ vertices in $G-v$ are monochromatic and hence contain no P_3. $\square$

For any minimal non-$(2,1)$-colorable graph G with a central vertex v, the graph $G' := G-v$ is minimal with the property that any vertex set of size $n(G')-1$ contains a P_3. We denote this property by $(*)$.

Let $n := n(G')$ and $V(G') := \{v_1,\ldots,v_n\}$. For all $i \leq n$, let W_i be the set $V(G')\backslash\{v_i\}$. Any graph G' which fulfills $(*)$ has at least four vertices and contains P_3, say w.l.o.g. $P_3 = v_1v_2v_3$. It follows that all $G'[W_i]$ with $i \geq 4$ contain a P_3. The set W_2 shall also fulfill this property. Up to isomorphy, Figure 4.1 shows all minimal graphs with the edges v_1v_2 and v_2v_3 where $G'[W_2]$ contains a P_3.

The graphs G_1' and G_2' fulfill $(*)$ with minimality. In G_1'', G_2'' and G_3'', only the set W_1 does not induce a P_3. A supergraph of G_1'' fulfills $(*)$ and does not contain a subgraph isomorphic to G_1' or G_2' if either $v_3v_5 \in E(G')$ or $v_2v_4 \in E(G')$, compare Figure 4.2. Every supergraph of G_2'' or G_3'' such that

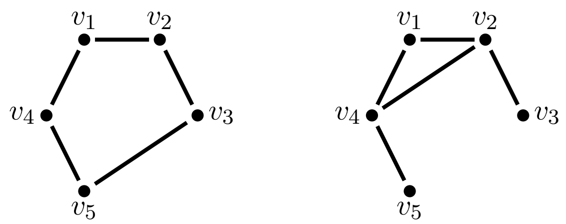


(a) G'_1 (b) G'_2

(c) G''_1 (d) G''_2 (e) G''_3

FIGURE 4.1: Graphs with $P_3 \subseteq G'[W_2]$

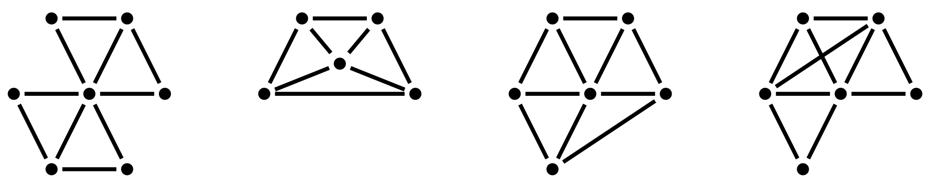
$G'[W_1]$ contains a P_3 is a supergraph of G'_j for a $j \leq 4$. Thus, the graphs G'_j are the only minimal graphs with property $(*)$.



(a) G'_3 (b) G'_4

FIGURE 4.2: Further minimal graphs with property $(*)$

Figure 4.3 shows the graphs $G_j := G'_j + v$ where v is a central vertex. These are the graphs which are minimal with the property that there is a central vertex and that G is not $(2,1)$-colorable. We will see later that the graph G_4 has a proper minimal non-$(2,1)$-colorable subgraph but G_1, G_2 and G_3 do not. Hence, the graphs G_1, G_2 and G_3 in Figure 4.3 are precisely the minimal non-$(2,1)$-colorable graphs with a central vertex.



(a) G_1 (b) G_2 (c) G_3 (d) G_4

FIGURE 4.3: Graphs with a central vertex

No Generalization to the Maximum Degree

A central vertex in a graph in $\mathcal{G}$ has at most six neighbors. Every vertex v is a local central vertex in $G[N(v) \cup \{v\}]$. However, considering local central vertices gives no upper bound of the maximum degree. A vertex in a minimal non-$(2,1)$-colorable graph can have more than six neighbors. An example is shown in Figure 4.4. Furthermore, Chapter 5 presents minimal non-$(2,1)$-colorable graphs with arbitrarily large maximum degree.

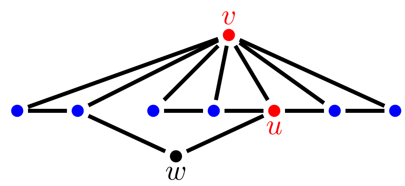


FIGURE 4.4: Graph in $\mathcal{G}$ with maximum degree seven

Lemma 14. *The graph in Figure 4.4 is minimal non-$(2,1)$-colorable.*

Proof. Let v be the vertex of degree seven, w the vertex with distance two to v and u the middle vertex of the P_5 induced by $N(v)$. We color the vertex v w.l.o.g. in color 1. Assume that there is a $(2,1)$-coloring of G. At most one vertex in $N(v)$ has the same color as v. This is the vertex u as G contains no monochromatic P_3. We denote this unique $(2,1)$-coloring of $G - w$ by c. The vertex w is adjacent to a vertex in a monochromatic edge of each color. Thus, c is not extendable to G and G not $(2,1)$-colorable.
The graph is minimal if $G-e$ is $(2,1)$-colorable for any edge e. If e is an edge of w or a monochromatic edge in c incident to an edge of w, c is extendable to $G-e$. If e is in the P_5 induced by the neighbors of v, we can color another vertex in $N(v)$ in color 1. Thus, both neighbors of w have color 2 what is extendable to $G - e$. If $e = vx$ for an $x \neq u$, color x in color 1. If x belongs to the P_5 in $N(v)$, then there is a coloring of $G - e - w$ where u has color 2. Otherwise, w is not adjacent to a vertex in a monochromatic edge of color 2. Both is extendable to $G - e$ which shows that G is indeed minimal. □

4.2 Graphs with Five Vertices

Graphs with less than five vertices are $(2,1)$-colorable as this allows color classes of size at most two.

Theorem 2. *The wheel graph W_5 is the only minimal non-$(2,1)$-colorable graph on five vertices.*

Proof. All graphs in $\mathcal{G}$ fulfill $\delta(G) \geq 2$ and have no adjacent vertices of degree two, see Corollary 2 and Lemma 3. Furthermore, any trivalent vertex has a neighbor of degree at least three, see Lemma 4. Figure 4.5 displays all graphs with five vertices and these poperties. The picture presents

$(2,1)$-colorings of the graphs in the first row. We proved the non-$(2,1)$-colorability of the wheel graph W_5 in Section 4.1. The picture shows that both isomorphic types of maximal proper subgraphs are $(2,1)$-colorable. Thus, we have minimality. The further graphs in Figure 4.5 are supergraphs of W_5. □

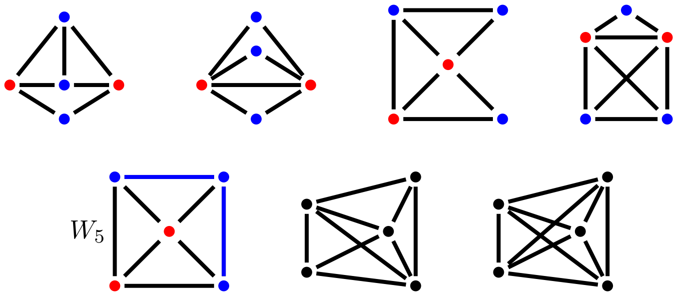


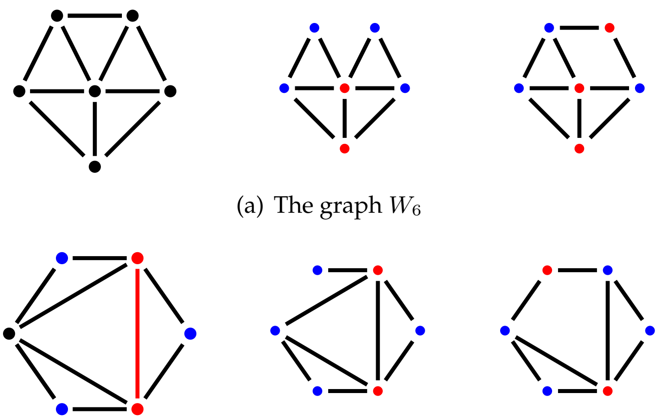
FIGURE 4.5: Graphs of order five

4.3 Graphs with Six Vertices

Two graphs of order six are minimal non-$(2,1)$-colorable. They are presented in Figure 4.6. First, we prove their minimality. Secondly, we show that there are no further graphs with six vertices in $\mathcal{G}$.

4.3.1 Minimal Non-$(2,1)$-Colorable Graphs of Order Six

Lemma 15. *The graphs W_6 and $\widetilde{G}$ in Figure 4.6 are minimal non-$(2,1)$-colorable.*



(a) The graph W_6

(b) The graph $\widetilde{G}$

FIGURE 4.6: Graphs of order six in $\mathcal{G}$

Proof. The graph W_6 is not $(2,1)$-colorable as shown in Section 4.1. It contains two isomorphic types of edges. Figure 4.6 shows a $(2,1)$-coloring of

$G - e$ for both. Now, consider the graph $\widetilde{G}$. The triangle induced by the tetravalent vertices contains a monochromatic edge in any 2-coloring. Every bivalent vertex is adjacent to at least one vertex of this edge. Thus, if the coloring is 1-improper, all bivalent vertices receive the other color. The third tetravalent vertex is adjacent to two vertices of both colors. Hence, $\widetilde{G}$ has no $(2,1)$-coloring. Figure 4.6 shows that $\widetilde{G} - e$ is $(2,1)$-colorable for all edges e. □

4.3.2 Completeness

Lemma 16. *Every minimal non-$(2,1)$-colorable graph G with six vertices fulfills $8 \le m(G) \le 12$.*

Proof. Every vertex in a graph in $\mathcal{G}$ has degree at least two and the neighbors of a bivalent vertex have degree at least three. Thus, at most four vertices in G are bivalent. In this case, only $K_{2,4}$ fulfills these properties and $m(K_{2,4}) = 8$. If less than four vertices are bivalent, the graph has at least eight edges. In a minimal non-$(2,1)$-colorable graph on six vertices, every subgraph on five vertices may not contain W_5. All graphs with five vertices and at least nine edges are supergraphs of W_5, as shown in Figure 4.5. Thus, $G - v$ has at most eight edges for all $v \in V(G)$. The vertex v has degree at most five in G. If G had all these 13 edges, the graph $G - w$ had at least $13 - 4 = 9$ edges for all w with $d_G(w) \le 4$. Thus, G has at most 12 edges. □

Now, let us consider the structure of W_5 to find all graphs G of order six which are edge-maximal with the property $W_5 \not\subseteq G$. Every minimal non-$(2,1)$-colorable graph is a subgraph of such a graph.

Lemma 17. *A graph G with $n(G) \ge 5$ contains no W_5 if and only if any set of five vertices in its complement graph $\overline{G}$ induces a supergraph of P_3.*

Proof. Let G' be an induced subgraph of G with $n(G') = 5$. We claim that G' is a supergraph of W_5 if and only if $\delta(G') \ge 3$. If $W_5 \subseteq G'$, then $\delta(G') \ge \delta(W_5) = 3$. On the other hand, $\delta(G') \ge 3$ implies $m(G') \ge 8$. The only graph with five vertices, eight edges and minimal degree at least three is W_5. Moreover, the graphs K_5 and $K_5 - e$ are the graphs of order five with more than eight edges. Both have minimal degree at least three and contain W_5. Thus, G' contains no W_5 if and only if there is a vertex v with $d_{G'}(v) \le 2$. This vertex v has degree at least 2 in $\overline{G'}$. Therefore, v is the inner vertex of a P_3 in $\overline{G'}$. If $\Delta(\overline{G'}) \le 1$, the complement graph $\overline{G'}$ contains no P_3. This proves the lemma for graphs of order five. As W_5 is no subgraph of G if and only if $W_5 \not\subseteq G'$ for all $G' \subseteq G$ with $n(G') = 5$, the lemma follows. □

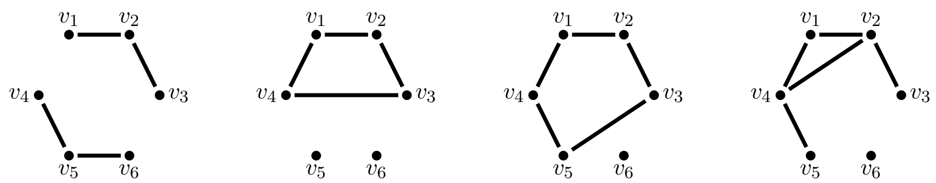
This result enables us to show the completeness of the set in Lemma 15.

Theorem 3. *The graphs W_6 and $\widetilde{G}$ are the only minimal non-$(2,1)$-colorable graphs with six vertices.*

Proof. All minimal non-$(2,1)$-colorable graphs of order six are contained in an edge-maximal graph G of order six with $W_5 \nsubseteq G$. Thus, we want to find all minimal graphs $\overline{G}$ on six vertices fulfilling the property of Lemma 17. Apart from the number of vertices in $\overline{G}$, this equals property $(*)$ in Section 4.1. The graphs satisfying $(*)$ are the graphs G_i' with $i \leq 4$ in Figure 4.1 and Figure 4.2. As all G_i' have at most six vertices, the minimal graphs of order six which fulfill the condition of Lemma 17, are the graphs $\overline{H}_i := G_i' + V'$ where V' is a set of $6 - n(G_i')$ isolated vertices. These graphs are shown in Figure 4.7.

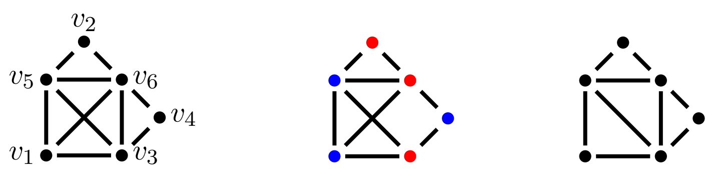
Any minimal non-$(2,1)$-colorable graph of order six is a subgraph of one complement graph $H_i := \overline{\overline{H}}_i$. The complements of the graphs $\overline{H}_1$ and $\overline{H}_2$ are $(2,1)$-colorable: in H_1, we can color the vertices v_1, v_2 and v_3 in color 1 and the vertices v_4, v_5 and v_6 in color 2. In H_2, the vertices v_1, v_2, v_3 and v_4 receive one color and the further vertices the other. Therefore, they contain no minimal non-$(2,1)$-colorable subgraph.

The complement of $\overline{H}_3$ is the graph W_6 which is a minimal non-$(2,1)$-colorable graph as shown in Subsection 4.3.1. The graph H_4 is shown in Figure 4.8 (a). The graph $\widetilde{G}$ is a proper subgraph of H_4. Thus, H_4 is not $(2,1)$-colorable but does not fulfill minimality either. Let us consider subgraphs of H_4 with six vertices. These are the graphs $H_4 - E'$ for an edge set $E' \subseteq E(H_4)$. The wheel graph W_6 is the only graph on six vertices with a central vertex and no subgraph of H_4. Hence, there is an edge $e \in E'$ with $v_6 \in e$. It holds $\delta(G) \geq 2$ and $H_4 - v_3v_6$ is $(2,1)$-colorable, compare Figure 4.8 (b). Therefore, $v_1v_6 \in E'$. The graph $H - v_1v_6$ is the minimal non-$(2,1)$-colorable graph $\widetilde{G}$ and hence the only such subgraph of H_4. □



(a) $\overline{H}_1$ (b) $\overline{H}_2$ (c) $\overline{H}_3$ (d) $\overline{H}_4$

FIGURE 4.7: Minimal graphs of order six with property $(*)$



(a) H_4 (b) $H_4 - v_3v_6$ (c) $H_4 - v_1v_6$

FIGURE 4.8: H_4 and its considered subgraphs

We can conclude following:

Lemma 18. *Let G be a graph with at least six vertices which is no supergraph of W_5, W_6 and $\widetilde{G}$. Then for any induced subgraph G' with $n(G') = 6$, the complement graph $\overline{G'}$ contains either two vertex-disjoint paths of length two or a 4-cycle.*

Proof. We know that if $W_5 \not\subseteq G$, the complement of G' is a supergraph of a graph $\overline{H}_i$, $i \leq 4$. The graphs $\overline{H}_1$ and $\overline{H}_2$ are two vertex-disjoint P_3 (write P_3^2) and $C_4 + \{v_5, v_6\}$ for isolated vertices v_5 and v_6. Thus, if $\overline{H}_i \subseteq \overline{G'}$ for an $i \leq 2$, the lemma holds.
Now, consider the case $\overline{H}_3 \subseteq \overline{G'} \Leftrightarrow G' \subseteq H_3 \simeq W_6$. The graph G does not contain W_6. Hence, G' is a proper subgraph of H_3 and $\overline{G'}$ a proper supergraph of $\overline{H}_3$. The graph $\overline{H}_3$ is a 5-cycle with an isolated vertex v_6. It has two minimal proper supergraphs $\overline{H}_3 + e$. Either e is a chord in C_5 or $v_6 \in e$. If e is a chord, $\overline{H}_3 + e$ contains C_4 and if $v \in e$, it contains P_3^2.
It remains to treat $\overline{H}_4 \subseteq \overline{G'} \Leftrightarrow G' \subseteq H_4$. The graph $\widetilde{G}$ is a subgraph of H_4 and does not arise in G. Thus, $G' \neq H_4$. We consider the complement graph $\overline{G'}$ which is a proper supergraph of $\overline{H}_4$. It contains $\overline{H}_4$ and an additional edge. All such graphs are shown in Figure 4.9. The complement of the first graph is $\widetilde{G}$ and the other graphs contain either C_4 or P_3^2.
Hence, a graph G' of order six which does not contain the graphs W_5, W_6 and $\widetilde{G}$ fulfills $C_4 \subseteq \overline{G'}$ or $P_3^2 \subseteq \overline{G'}$. This gives the lemma as the graphs W_5, W_6 and $\widetilde{G}$ have at most six vertices. □

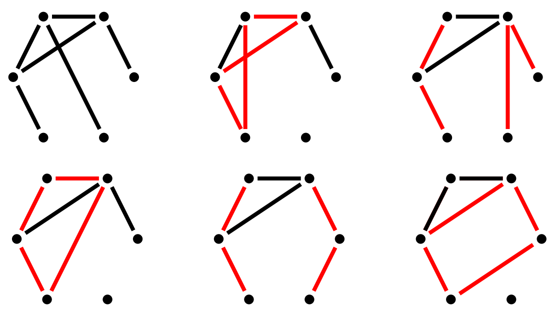


FIGURE 4.9: $\overline{H}_4 + e$

4.4 Graphs with Seven Vertices

In this section, we present the four minimal non-$(2,1)$-colorable graphs of order seven. We prove that they belong to $\mathcal{G}$ but no further graph with seven vertices does.

4.4.1 Minimal Non-$(2,1)$-Colorable Graphs of Order Seven

Lemma 19. *The graph G_1 is minimal non-$(2,1)$-colorable.*

Proof. The graph has no $(2,1)$-coloring by Section 4.1. The picture shows $(2,1)$-colorings for all $G_1 - e$. □

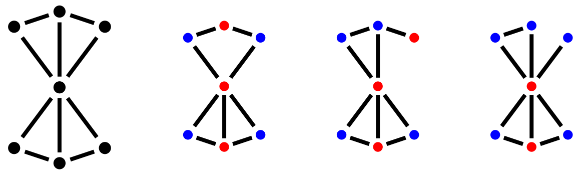


FIGURE 4.10: The graph G_1

Lemma 20. *The graph G_2 is minimal non-$(2,1)$-colorable.*

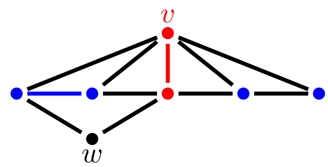


FIGURE 4.11: The graph G_2

Proof. Let v be the vertex of degree five and w the vertex of distance two to v. Say w.l.o.g. that v has color 1. There is a unique $(2,1)$-coloring of $G-w$. In this coloring, the middle vertex of the P_5 which is induced by $N(v)$, has color 1. For both colors, the vertex w is adjacent to a vertex in a monochromatic edge. Thus, the coloring of $G-w$ is not extendable to G.
Now, consider $G_2 - e$ for an edge $e \in E(G_2)$. If $w \in e$, we can extend the unique $(2,1)$-coloring of $G_2 - w$. If $e = vx$ for an $x \in N(v)$, color v, x and one neighbor of v in color 1 such that $G_2 - \{v, w\}$ does not contain a monochromatic P_3 and the neighbors of w receive the same color. This is extendable to a $(2,1)$-coloring of $G_2 - e$. If e is an edge of the P_5 induced by the neighborhood of v, color v, w and one vertex in $N(v) \backslash N(w)$ in color one and the other vertices in color 2 such that every P_3 contains both colors. □

Lemma 21. *The graph G_3 is minimal non-$(2,1)$-colorable.*

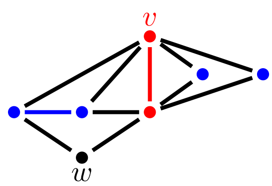
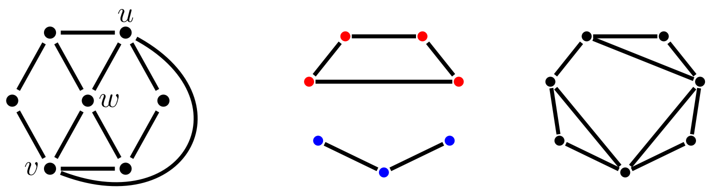


FIGURE 4.12: The graph G_3

Proof. Let again v be the vertex of degree five and w the other vertex which is not in the neighborhood of v. Similar to G_2, there is a unique $(2,1)$-coloring c of $G_3 - w$ because exactly one vertex belongs to all P_3 in $N(v)$. The vertex w is in an edge incident to a monochromatic edge for both colors. Thus, we cannot extend this coloring to G_3.
To show the minimality of G_3, we consider all $G_3 - e$. If $w \in e$ or e is one of the monochromatic edges in c, the coloring c of $G_3 - e - w$ is extendable

to $G_3 - e$. If e is another edge of v, color the vertices in e and one neighbor of v in color 1 such that there is no monochromatic P_3 in $G - \{v, w\}$ and the neighborhood of w is monochromatic. Such a coloring is extendable to $G_3 - e$. If e joins two neighbors of v and is not considered yet, we color v and w in color 1 and the vertices in $N(w)$ in color 2. This coloring can be extended to $G_3 - e$ □

Lemma 22. *The graph G_4 is minimal non-$(2,1)$-colorable.*



(a) G_4 (b) H (c) $\overline{G}_4$

FIGURE 4.13: The graph G_4

Remark 5. Adjacent vertices x and y with $P_3 \subseteq G[N(\{x, y\})]$ have different colors in any $(2,1)$-coloring.

Proof. Consider the vertices u, v and w as denoted in the picture. The graph G_4 has no $(2,1)$-coloring as any two of these vertices fulfill the property of Remark 5.
To show minimality, we look at the graph H in Figure 4.13 (b). Its complement admits a $(2,1)$-coloring as shown in the picture. Every subgraph of $\overline{H}$ is also $(2,1)$-colorable. Adding an arbitrary edge to the complement of G_4 (see Figure 4.13 (c)) gives a supergraph of H. Thus, every proper subgraph of G_4 is a subgraph of $\overline{H}$ and hence $(2,1)$-colorable. □

4.4.2 Completeness

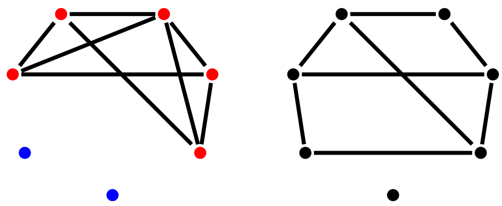
Lemma 23. *Every graph with seven vertices which does not contain W_5, W_6 and $\widetilde{G}$ as a subgraph has at most 15 edges.*

Proof. Let G be such a graph and v a vertex in G. By Lemma 18, the graph $\overline{G - v}$ contains either C_4 or P_3^2 and therefore at least four edges. There is a vertex w of degree at least two in $\overline{G - v}$. The graph $\overline{G - w}$ has also at least four edges and therefore, $\overline{G}$ has at least six edges. This gives $m(G) \leq 15$. □

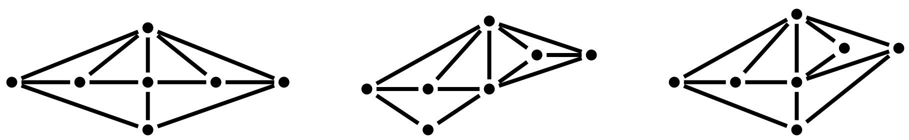
The upper bound for $m(G)$ can be improved as follows:

Lemma 24. *A minimal non-$(2,1)$-colorable graph with seven vertices has at most eleven edges.*

Proof. We employ the property shown in Lemma 18 and hence, consider complementary graphs. A minimal non-$(2,1)$-colorable graph G contains no W_5, W_6 or $\widetilde{G}$. For all vertices v, the graph $\overline{G} - v$ contains C_4 or P_3^2. By inspection, we see that seven graphs of order seven with at most nine edges are edge-minimal with this property: $C_4 \dot{\cup} P_3$, C_7, G_1' and G_2' in Figure 4.14 (a) and G_3', G_4' and G_5' whose complements are shown in Figure 4.14 (b).
The complements of C_4 and of P_3 have maximum degree one. Hence, coloring the vertices of $\overline{C_4}$ in color 1 and the vertices of $\overline{P_3}$ in color 2 is a $(2,1)$-coloring of $\overline{C_4 \dot{\cup} P_3}$. Moreover, Figure 4.13 shows that the complement of G_4 is a proper supergraph of C_7 and therefore, G_4 a proper subgraph of $\overline{C_7}$. It follows that $\overline{C_7}$ is non-$(2,1)$-colorable, but not minimal. One can check that all non-$(2,1)$-colorable subgraphs of $\overline{C_7}$ are supergraphs of G_4. Furthermore, the complement of G_1' is $(2,1)$-colorable as shown in the picture. The graph G_2' consists of one isolated vertex and a connected compontent with two stable sets of size three. Thus, its complement is a proper supergraph of the graph G_1 in Figure 4.10. Therefore, it is not minimal non-$(2,1)$-colorable. One can verify that every subgraph of its complement is either $(2,1)$-colorable or a supergraph of G_1. Figure 4.14 (b) displays the complements of the graphs G_3', G_4' and G_5'. They are proper supergraphs of G_2 or G_3 and thus not minimal non-$(2,1)$-colorable.
Hence, the complement of any graph with seven vertices and at most nine edges is not minimal non-$(2,1)$-colorable. As the 7-clique has 21 edges, all minimal non-$(2,1)$-colorable graphs of order seven fulfill $m(G) \leq 11$. □



(a) The graphs G_1' and G_2'



(b) Complements of G_3', G_4' and G_5'

FIGURE 4.14: Lemma 24

Theorem 4. *The graphs G_1, G_2, G_3 and G_4 are the only minimal non-$(2,1)$-colorable graphs of order seven.*

Proof. By inspecting the edge-minimal (w.r.t. the property that every graph $\overline{G} - v$ contains C_4 or P_3^2) graphs of order seven, we see that the complement of every graph with more than nine edges is $(2,1)$-colorable. Therefore, the proof of Lemma 24 shows that all minimal non-$(2,1)$-colorable graphs with seven vertices are either G_1 or G_4 or a proper subgraph of a graph in Figure 4.14 (b). Analyzing all their subgraphs shows that G_2 and G_3 are the only further minimal non-$(2,1)$-colorable graphs. □

Chapter 5

Unbounded Maximum Degree

The odd cycles are the minimal non-$(2,0)$-colorable graphs. All graphs in this set have maximum degree two. In the introduction, we concluded from Gerencsér [10] and Lovász [16], that minimal non-$(2,1)$-colorable graphs have maximum degree at least four. In Section 4.1, we displayed a minimal non-$(2,1)$-colorable graph with maximum degree seven. In fact, the degree of a vertex in a minimal non-$(2,1)$-colorable graph can be arbitrarily large. We prove this by presenting a set $\mathcal{G}_\Delta$ of minimal non-$(2,1)$-colorable graphs with unbounded maximum degree.

5.1 The Graph Set $\mathcal{G}_\Delta$

The general structure of the graphs in $\mathcal{G}_\Delta$ is shown in Figure 5.1. The subgraphs G_i for $i \in \{1,2,3,4\}$ are described below. Their choice allows an arbitrarily large degree of the vertex v.

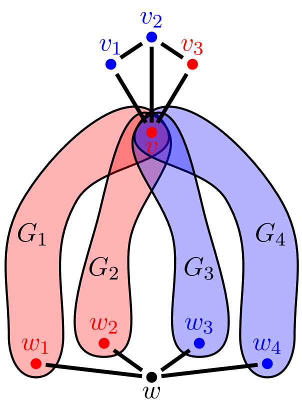


FIGURE 5.1: Structure of graphs in $\mathcal{G}_\Delta$

Let G be a graph in $\mathcal{G}_\Delta$. We call the subgraph induced by the vertices v, v_1, v_2 and v_3 the *basic flag* G_b of G. In any $(2,1)$-coloring of G_b, the vertex v is in a monochromatic edge. All neighbors of v in a G_i have thus the other color. We will choose graphs G_i such that any $(2,1)$-coloring c of $G - w$ fulfills $c(w_1) = c(w_2) = c(v)$ and $c(w_3) = c(w_4) \neq c(v)$. It follows that G is non-$(2,1)$-colorable.

5.2 The Sets $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$

Let $\mathcal{G}'$ be the set of all triples (G', v', w') such that G' is a graph and v' and w' are distinct vertices in $V(G')$. The sets $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$ are subsets of $\mathcal{G}'$. Later on, we will define $\mathcal{G}_\Delta$ by means of $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$.

We specify $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$ via a mapping which connects triples in $\mathcal{G}'$:

Definition 5. Let $f : \mathcal{G}' \times \mathcal{G}' \to \mathcal{G}'$ be the function which maps (G'_1, v'_1, w'_1) and (G'_2, v'_2, w'_2) onto (G', v', w') such that G' is the union of G'_1 and G'_2, in which $V(G'_1)$ and $V(G'_2)$ intersect exactly in $v' := v'_1 = v'_2$, together with an additional vertex w'. The vertex w' has neighborhood $\{w'_1, w'_2\}$.

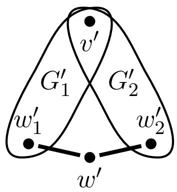
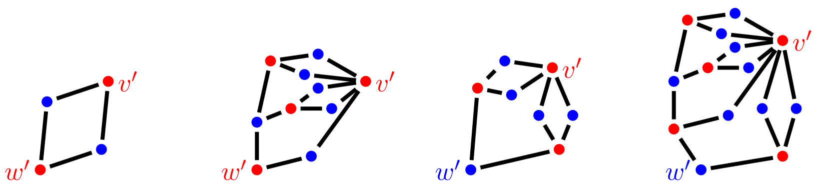


FIGURE 5.2: The function f

We characterize $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$ recursively: the recursion starts with (P_2, v', w') in $\mathcal{G}^{\neq}$ where $V(P_2) = \{v', w'\}$. Furthermore, a triple (G', v', w') is in $\mathcal{G}^{\neq}$ if and only if there are triples (G'_1, v'_1, w'_1) and (G'_2, v'_2, w'_2) in $\mathcal{G}^=$ such that $f((G'_1, v'_1, w'_1), (G'_2, v'_2, w'_2)) = (G', v', w')$. Complementary, $\mathcal{G}^=$ is the image of $\mathcal{G}^{\neq} \times \mathcal{G}^{\neq}$ under f. Figure 5.3 shows examples for graphs in $\mathcal{G}^=$ and $\mathcal{G}^{\neq}$.



(a) Triples in $\mathcal{G}^=$ (b) Triples in $\mathcal{G}^{\neq}$

FIGURE 5.3: $\mathcal{G}^{\neq}$ and $\mathcal{G}^=$

We define $\mathcal{G}_\Delta$ as the set with the structure of Figure 5.1 and (G_1, v, w_1), $(G_2, v, w_2) \in \mathcal{G}^=$ and $(G_3, v, w_3), (G_4, v, w_4) \in \mathcal{G}^{\neq}$.

Lemma 25. *The set $\mathcal{G}_\Delta$ has unbounded maximum degree.*

Proof. Both sets $\mathcal{G}^{\neq}$ and $\mathcal{G}^=$ contain triples (G', v', w') such that $d_{G'}(v') > n$ for any $n \in \mathbb{N}$. Thus, there are graphs $G \in \mathcal{G}_\Delta$ with arbitrarily large maximum degree $\Delta(G) \geq d_G(v) > d_{G_i}(v)$. $\square$

5.3 Homo- and Heterochromatic Graphs

Definition 6. Let G be a graph and $v \in V(G)$. A coloring of G such that v is in no monochromatic edge is called *v-isolated*.

Definition 7. Consider a graph with two specified vertices v and w which allows a v-isolated $(2,1)$-coloring.
We call the graph *v-w-homochromatic* if following properties hold:
(1) Every v-isolated $(2,1)$-coloring c fulfills $c(v) = c(w)$.
(2) Every v-isolated $(2,1)$-coloring is also a w-isolated $(2,1)$-coloring.
(3) G admits a $(2,1)$-coloring c' with $c'(v) \neq c'(w)$.
The graph is *v-w-heterochromatic* if:
(1) Every v-isolated $(2,1)$-coloring c fulfills $c(v) \neq c(w)$.
(2) Every v-isolated $(2,1)$-coloring is also a w-isolated $(2,1)$-coloring.
(3) G admits a $(2,1)$-coloring c' with $c'(v) = c'(w)$.

In this case, we slightly adjust the definition of minimality:

Definition 8. A graph G is *minimal v-w-homochromatic* if it is v-w-homochromatic but every proper subgraph H with $v, w \in V(H)$ has a v-isolated $(2,1)$-coloring with $c(v) \neq c(w)$. *Minimal v-w-heterochromaticity* is defined symmetrically.

Lemma 26. *Graphs G' with $(G', v', w') \in \mathcal{G}^{=}$ are minimal v'-w'-homochromatic and graphs G' with $(G', v', w') \in \mathcal{G}^{\neq}$ are minimal v'-w'-heterochromatic.*

Proof. The graph P_2 is minimal v'-w'-heterochromatic. Let G' be a graph with distinct vertices $v', w' \in V(G')$ and

$$(G', v', w') = f((G'_1, v'_1, w'_1), (G'_2, v'_2, w'_2))$$

for either

$$(G'_1, v'_1, w'_1), (G'_2, v'_2, w'_2) \in \mathcal{G}^{=} \quad \text{(case A)}$$

or

$$(G'_1, v'_1, w'_1), (G'_2, v'_2, w'_2) \in \mathcal{G}^{\neq} \quad \text{(case B)}.$$

In case A, (G', v', w') is in $\mathcal{G}^{\neq}$ and in case B, (G', v', w') is in $\mathcal{G}^{=}$. By the induction hypothesis, the graphs G'_i are v'_i-w'_i-homochromatic in case A and v'_i-w'_i-heterochromatic in case B for $i \in \{1, 2\}$. If c is a v'-isolated $(2,1)$-coloring of G', the coloring $c|_{G'_i}$ is a v'_i-isolated $(2,1)$-coloring of G'_i. By induction, any v'-isolated $(2,1)$-coloring c of G' fulfills therefore $c(w'_1) = c(w'_2)$. As the vertices w'_1, w' and w'_2 induce a P_3, $c(w') \neq c(w'_i)$. Thus, there are v'-isolated $(2,1)$-colorings of G' and property (2) is fulfilled. Property (1) holds as $c(v') = c(w'_i) \neq c(w')$ in case A and $c(v') \neq c(w'_i) \neq c(w')$ in case B.
Now, let us show property (3). Let c_1 be a $(2,1)$-coloring of G'_1 which witnesses property (3) for G'_1 and c_2 a v'_2-isolated $(2,1)$-coloring of G'_2. The vertex sets $V(G'_1)$ and $V(G'_2)$ intersect only in v'. The colors are symmetric and v' is in one monochromatic edge of G'_1 and in none of G'_2. Thus, $c := c_1 \cup c_2$ is a $(2,1)$-coloring of $G' - w'$. In this coloring, it holds $c(w'_1) \neq c(w'_2)$ and w'_2 is in no monochromatic edge as c_2 is w'-isolated by property (2). Hence,

$c(w') := c(w_2')$ gives a $(2,1)$-coloring of G' which fufills $c(w') = c(v')$ if $(G', v', w') \in \mathcal{G}^{\neq}$ and $c'(w') \neq c'(v')$ if $(G', v', w') \in \mathcal{G}^{=}$.
The only condition left to be proven is minimality. For this purpose, we show by induction that the v'-isolated $(2,1)$-coloring of a graph G' in $\mathcal{G}^{=}$ or $\mathcal{G}^{\neq}$ is unique apart from interchanging colors. Obviously, this holds for P_2. Now, consider a graph G' with $(G', v', w') = f((G_1', v_1', w_1'), (G_2', v_2', w_2'))$ and either $(G', v', w') \in \mathcal{G}^{\neq}$ or $(G', v', w') \in \mathcal{G}^{=}$. If c is a v'-isolated $(2,1)$-coloring of G', the colorings $c|_{G_i'}$ are v_i'-isolated $(2,1)$-colorings of G_i'. By induction, these colorings are unique. As G_1' and G_2' intersect exactly in v', there is a unique v'-isolated coloring of $G' - w'$. It colors w_1' and w_2' alike. Thus, choosing the other color for w' gives the only v'-isolated $(2,1)$-coloring of G'. With this result, we can show the minimality of G'. First, consider the case $w' \in e$, w.l.o.g. say $e = w'w_1'$. We color all vertices but w' as in the unique v'-isolated coloring of G'. The vertex w_2' is in no monochromatic edge. Hence, coloring w' in either color gives a $(2,1)$-coloring of G'. For the second case, assume w.l.o.g. that $e \in E(G_1')$. We apply induction and suppose that there is a v_1'-isolated $(2,1)$-coloring c_1 of $G_1' - e$ such that $c_1(w_1') \neq c_1(v')$ if $(G_1', v_1', w_1') \in \mathcal{G}^{=}$ and $c_1(w_1') = c_1(v')$ if $(G_1', v_1', w_1') \in \mathcal{G}^{\neq}$. Furthermore, there is a v_2'-isolated $(2,1)$-coloring c_2 of G_2' such that $c_2(w_2') = c_2(v')$ if $(G_2', v_2', w_2') \in \mathcal{G}^{=}$ and $c_2(w_2') \neq c_2(v')$ if $(G_2', v_2', w_2') \in \mathcal{G}^{\neq}$. If we color w' in $c_2(w_2')$ and the other vertices in $c_1 \cup c_2$, we obtain a v'-isolated $(2,1)$-coloring c of $G' - e$ because w_2' is in no monochromatic edge. It holds $c(w') = c(v')$ if $(G', v', w') \in \mathcal{G}^{\neq}$ and $c(w') \neq c(v')$ if $(G', v', w') \in \mathcal{G}^{=}$ which gives minimality. □

5.4 Minimal Non-$(2,1)$-Colorability of Graphs in $\mathcal{G}_{\Delta}$

Lemma 27. *Let G be a graph with the structure in Figure 5.1. Moreover, the subgraphs G_1 and G_2 are minimal v-w_i-homochromatic and the subgraphs G_3 and G_4 minimal v-w_i-heterochromatic. Then G is minimal non-$(2,1)$-colorable.*

Proof. To prove this Lemma, we have to show that G is not $(2,1)$-colorable but $G - e$ is so for any $e \in E(G)$. Assume that there is a $(2,1)$-coloring c of G. The vertex v is in a monochromatic edge of G_b and thus, $c|_{G_i}$ is v-isolated for all $i \leq 4$. It follows $c(w_1) = c(w_2) \neq c(w_3) = c(w_4)$. The vertices $\{w_1, w, w_2\}$ and the vertices $\{w_3, w, w_4\}$ induce a P_3. Therefore, G is not $(2,1)$-colorable.
First, let us study the graphs $G - e$ with $w \in e$. Color the subgraph $G - w$ as above. No neighbor of w is in a monochromatic edge and the degree of w in $G - e$ is three. Hence, one color occurs only once in $N_{G-e}(w)$ and coloring w in this color gives a 1-improper 2-coloring of $G - e$. If e is an edge in the basic flag G_b, the graph $G_b - e$ admits a v-isolated $(2,1)$-coloring. We color G_2, G_3 and G_4 also v-isolated. By property (3) for G_1, there is a non-v-isolated $(2,1)$-coloring of G_1 which colors w_1 in the other color from v. The union of all these colorings is a $(2,1)$-coloring of $G - e - w$ as G_b and the G_i intersect just in v and v has only a same-colored neighbor in G_1. We follow from property (2) for G_2 that $c(w) := c(w_2)$ provides a $(2,1)$-coloring of $G - e$. The only remaining case to look at is $e \in E(G_j)$ for a

$j \leq 4$. The graph G_j is minimal v_j-w_j-homochromatic or minimal v_j-w_j-heterochromatic. Thus, if $j \in \{1, 2\}$, there is a v-isolated coloring of $G_j - e$ such that w_j does not have the color of v and if $j \in \{3, 4\}$, there is a v-isolated coloring of $G_j - e$ such that w_j has the color of v. It follows that there is a $(2, 1)$-coloring of $G - e - w$ such that three neighbors of w are colored alike. The fourth neighbor does not belong to a monochromatic edge. Therefore, this is extendable to $G - e$. □

Theorem 5. *The set $\mathcal{G}$ has unbounded maximum degree.*

Proof. Graphs in $\mathcal{G}_\triangle$ fulfill the properties of Lemma 27 and hence $\mathcal{G}_\triangle$ is a subset of $\mathcal{G}$. Together with Lemma 25, this shows the theorem. □

5.5 Infinity of $\mathcal{G}$

The set $\mathcal{G}_\triangle$ is an infinite subset of $\mathcal{G}$. This gives the following theorem:

Theorem 6. *There are infinitely many minimal non-$(2, 1)$-colorable graphs.*

As mentioned in the introduction, there is a polynomial reduction from 3-SAT to $(2, 1)$-COLORABILITY. Already the NP-completeness enables us to conclude that there are infinitely many minimal non-$(2, 1)$-colorable graphs if P $\neq$ NP holds. To see this, we consider that one can test in polynomial time if a graph contains a certain subgraph. This can be simply done by brute force (cf. also [20]). If $\mathcal{G}$ were finite, we could check polynomially if a certain graph contains any graph in $\mathcal{G}$ as a subgraph. If and only if this is the case, the graph is not $(2, 1)$-colorable. Hence, we could solve $(2, 1)$-COLORING in polynomial time which would contradict P $\neq$ NP.

Cowen, Goddard and Jesurum [8] proved that even for planar graphs and graphs with maximum degree four, it is NP-complete to determine whether or not a graph is $(2, 1)$-colorable. Recall that graphs with maximum degree smaller than four are not $(2, 1)$-colorable. Both, planarity and upwards bounded maximum degree, are closed under taking subgraphs. Hence, a planar graph is $(2, 1)$-colorable if and only if none of its subgraphs is a planar graph in $\mathcal{G}$. The respective holds for graphs of maximum degree at most four. Therefore, even the set of planar graphs in $\mathcal{G}$ and the set of graphs with maximum degree four in $\mathcal{G}$ are infinite if P $\neq$ NP holds. In Subsection 6.2.1, we present an infinite set of minimal non-$(2, 1)$-colorable graphs which are all planar and have maximum degree four. Thus, also the infinity of this set is proven independently of P vs. NP, see Corollary 5.

Chapter 6

Extension of Odd Cycles

This chapter shows properties of $\mathcal{G}$ and presents families of minimal non-$(2,1)$-colorable graphs. Most of these families contain one central odd cycle C_k and graphs which are glued to it, i.e., intersect with C_k in at least one vertex, but in no edge. The number k denotes the length of the odd cycle and $k' := (k-1)/2$.

6.1 Composition of Odd Cycles

In this section, we prove that the graphs in $\mathcal{G}$ only consist of odd cycles.

Theorem 7. *In a minimal non-$(2,1)$-colorable graph, every edge belongs to an odd cycle.*

Let G be a graph in $\mathcal{G}$ with an arbitrary edge $e = \{v, w\}$. By Lemma 2, e is not separating and hence lies in a cycle. Assume for a contradiction, that e only belongs to even cycles. It follows that every v-w-path in $G - e$ is odd.

Claim 1. *There is a $(2,1)$-coloring c of $G - e$ with the property $c(v) \neq c(w)$.*

Since such a coloring is also a $(2,1)$-coloring of G, indeed this yields the sought contradiction. Therefore, a proof of Claim 1 shows Theorem 7.

We define E' as the set of all edges which belong to a v-w-path of $G - e$ and G' as the graph $G[E']$.

Lemma 28. *The graph G' is bipartite.*

Proof. Assume for a contradiction that there is an odd cycle C in G'. Let P be a v-w-path which contains an edge of C. Let v' be the first and w' the last vertex of P which lie in the cycle. We call P_v the v-v'-subpath and P_w the w'-w-subpath of P. Let P_1 and P_2 be the two v'-w'-paths in C. As C is odd, one of these paths is odd and one is even. It follows that the path unions $P_v \cup P_1 \cup P_w$ and $P_v \cup P_2 \cup P_w$ are v-w-paths in G' whereof one is even. Together with e, this path forms an odd cycle in C which contradicts the assumption. $\square$

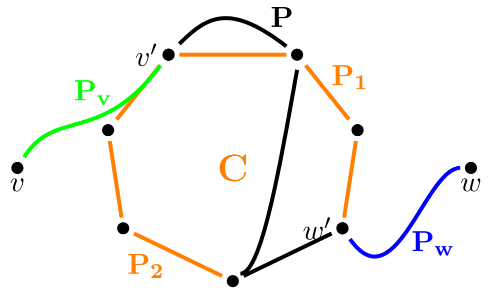
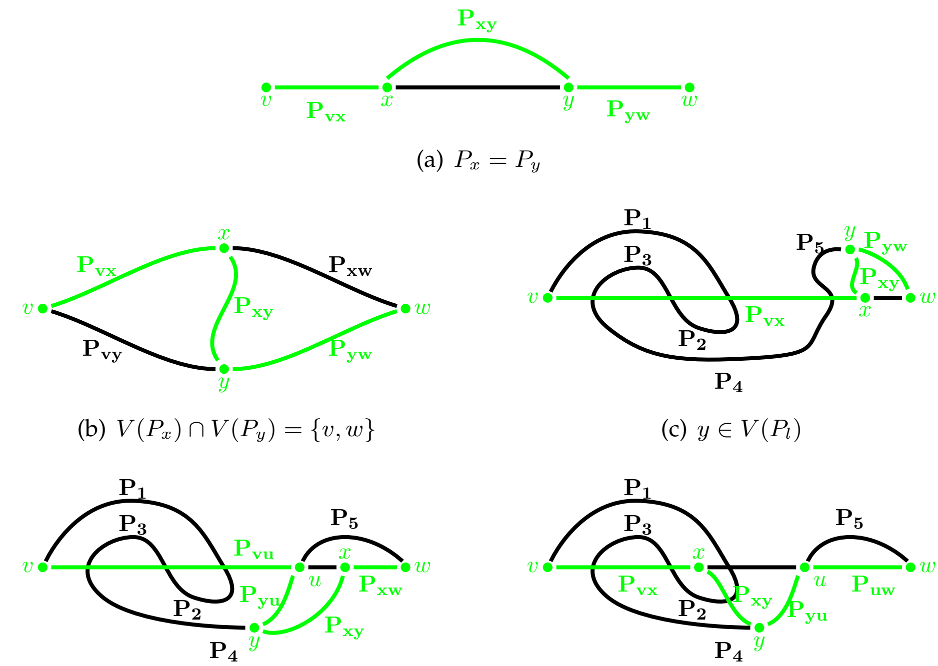


FIGURE 6.1: Proof of Lemma 28

Now, consider the further edges. Let E'' be the edge set $E(G)\setminus(E' \cup \{e\})$ and $C_1, \ldots, C_k$ the components of $G[E'']$ for a $k \geq 0$.

Lemma 29. *Any component C_i intersects with G' in exactly one vertex v_i.*

Proof. The intersection is not empty as otherwise, G were disconnected. Suppose that there are two distinct vertices x and y in $V(C_i) \cap V(G')$. Let P_{xy} be an x-y-path in C_i. If there are more than two vertices in $V(C_i) \cap V(G')$, choose x and y such that all inner vertices of P_{xy} do not belong to G'. Let P_x and P_y be v-w-paths containing x and y. Let P_{vx} be the subpath of P_x with the end vertices v and x and P_{xw} the x-w-subpath of P_x. Define P_{vy} and P_{yw} analogously for P_y. If there is a v-w-path containing x and y, we choose $P_x = P_y$. This is in particular the case if x or y is one of the vertices v and w. If $P_x = P_y$, we can assume w.l.o.g. that $x \in V(P_{vy})$ by the symmetry of x and y. This is shown in Figure 6.2 (a). Then $P_{vx} \cup P_{xy} \cup P_{yw}$ is a v-w-path which contains edges of E''. This contradicts the choice of E''.
Thus, neither P_x contains y nor P_y contains x. Let $u_0, \ldots, u_l$ be the vertices in $V(P_x) \cap V(P_y)$ and P_i the u_{i-1}-u_i-subpath of P_y. It holds $l \geq 1$ as $u_0 = v$ and $u_l = w$. Let j be the integer $1 \leq j \leq l$ such that $y \in V(P_j)$. As $y \notin V(P_x)$, the vertex y is in inner vertex of P_j.
Consider at first the case $l = 1$, i.e., $V(P_x) \cap V(P_y) = \{v, w\}$. In this case, $P_{vx} \cup P_{xy} \cup P_{yw}$ is a v-w-path as shown in Figure 6.2 (b).
Let $l \geq 2$. This situation is illustrated in the Subfigures (c), (d) and (e). The straight path represents P_x and the oscillating path represents P_y. For the case that y is either in $V(P_1)$ or in $V(P_l)$, it is sufficient to consider $y \in V(P_l)$ as the vertices v and w are symmetric. The inner vertices of P_{yw} do not intersect with P_x and $P_{vx} \cup P_{xy} \cup P_{yw}$ is again a v-w-path.
Now, consider the case that P_{vy} and P_{yw} both intersect with P_x, i.e., $l \geq 3$ and $2 \leq j \leq l-1$, see Figure 6.2 (d) and (e). Denote the vertex u_j by u and the y-u-subpath of P_y by P_{yu}. The paths P_{yu} and P_x intersect only in u. Call P_{vu} the v-u-subpath of P_x and P_{uw} the u-w-subpath of P_x. If $x \in V(P_{uw})$, the union $P_{vu} \cup P_{yu} \cup P_{xy} \cup P_{xw}$ is a v-w-path. The paths P_{yu} and P_{xy} are here considered in the other direction, i.e., as an u-y-path and as a y-x-path. If x lies in P_{vu}, the union $P_{vx} \cup P_{xy} \cup P_{yu} \cup P_{uw}$ is a v-w-path.
All these v-w-paths contain P_{xy}, i.e., edges of E'' and thus contradict the choice of E''. □



(a) $P_x = P_y$

(b) $V(P_x) \cap V(P_y) = \{v, w\}$

(c) $y \in V(P_l)$

(d) $y \in V(P_j),\ j \notin \{1, l\}$ and $x \in V(P_{uw})$

(e) $y \in V(P_j),\ j \notin \{1, l\}$ and $x \in V(P_{vu})$

FIGURE 6.2: Proof of Lemma 29

Proof of Theorem 7. We show Theorem 7 by means of a coloring of $G - e$ which fulfills the properties of Claim 1 and employ the Lemmas 28 and 29. Let us color all vertices of the bipartite graph G' in a proper 2-coloring c'. This coloring fulfills $c'(v) \neq c'(w)$ as all v-w-paths are odd. Every component C_i is a proper subgraph of G and thus obtains a $(2,1)$-coloring c_i. Let v_i be the unique vertex in $V(G') \cup V(C_i)$. We assume w.l.o.g. that $c_i(v_i) = c'(v_i)$. Therefore, the coloring $c(v) := c' \cup \bigcup_{i \leq k} c_i$ is well-defined. A vertex v_i does not belong to a monochromatic edge in c' and only to one component C_i. Hence, c is a 1-improper 2-coloring of $G - e$ with $c(v) \neq c(w)$. □

6.2 Addition of $\mathbf{P_3}$

This section describes minimal non-$(2,1)$-colorable graphs which contain an odd cycle $C_k = v_1 \dots v_k$ and k paths of length two. These paths are glued to C_k in their end vertices. We identify v_i and $v_{i'}$ iff $i \equiv i' \mod k$.

6.2.1 Odd Cycle with k Triangles

Consider the following graphs G_k for any odd integer $k \geq 3$:

$$V(G_k) = \{v_1, \ldots, v_k, w_1, \ldots, w_k\}$$
$$E(G_k) = \{\{v_i, v_{i+1}\}, \{v_i, w_i\}, \{v_i, w_{i+1}\} \mid 1 \leq i \leq k\}$$

We denote this set of graphs by $\mathcal{G}_\mathcal{T}$. Figure 6.3 shows the graph G_7:

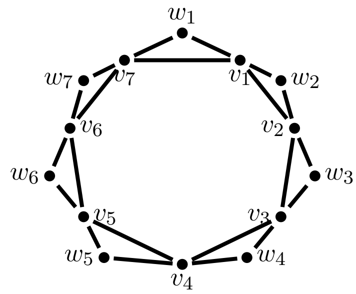


FIGURE 6.3: The graph $G_7 \in \mathcal{G}_\mathcal{T}$

Lemma 30. *The graphs $G_k \in \mathcal{G}_\mathcal{T}$ are not $(2, 1)$-colorable.*

Proof. Assume for a contradiction that c is a $(2, 1)$-coloring of G_k. The vertices $v_1, \ldots, v_k$ induce an odd cycle $C_k \subseteq G_k$. Every 2-coloring of C_k contains a monochromatic edge e. By symmetry, we can assume w.l.o.g. that $e = v_1 v_k$ and $c(v_1) = c(v_k) = 1$. As c is 1-improper, all vertices in $N(\{v_1, v_k\})$ have color 2. We claim that for all i with $i \geq 2$, the edge $v_i w_i$ is monochromatic and the edge $v_{i-1} v_i$ dichromatic.
We show this claim by induction. The vertices v_2 and w_2 are in $N(\{v_1, v_k\})$ and thus have both color 2 whereas v_1 has color 1. Consider an integer i with $i \geq 3$. The edge $v_{i-1} w_{i-1}$ is monochromatic by the induction hypothesis. As c is 1-improper, all vertices in $N(\{v_{i-1}, w_{i-1}\})$ have the other color. Both v_i and w_i are in this neighborhood. Hence, $v_i w_i$ is monochromatic and $v_i v_{i-1}$ dichromatic. It follows in particular $c(w_k) = c(v_k) = 1$. This contradicts the fact that all vertices in $N(\{v_1, v_k\})$ have color 2. □

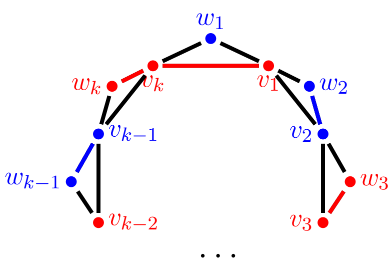


FIGURE 6.4: Non-$(2, 1)$-colorability of the graphs in $\mathcal{G}_\mathcal{T}$

Theorem 8. *The set $\mathcal{G}_\mathcal{T}$ is a subset of $\mathcal{G}$.*

Proof. By Lemma 30, all $G_k \in \mathcal{G}_\mathcal{T}$ are non-$(2,1)$-colorable. We have to show for all $e \in E(G_k)$, that $G_k - e$ admits a $(2,1)$-coloring. By isomorphy, it is sufficient to consider the cases $e_1 := v_1 v_k$ and $e_2 := v_k w_k$. We color the vertices v_i and w_i with even index i and the vertex w_1 in color 1 and all other vertices in color 2. In G_k, only the edges $v_i w_i$, $i \geq 2$, and $v_1 v_k$ are monochromatic. Among the monochromatic edges, only e_1 and e_2 are incident. Thus, neither $G_k - e_1$ nor $G_k - e_2$ contains a monochromatic P_3. □

Corollary 5. *There are infinitely many planar minimal non-*$(2,1)$*-colorable graphs with maximum degree four.*

6.2.2 Non-Planar Graphs in $\mathcal{G}$

All previously studied minimal non-$(2,1)$-colorable graphs are planar. Figure 6.5 shows a graph G which extends the C_7 by paths of length two. We will show that this graph is not planar and minimal non-$(2,1)$-colorable. Thus, $\mathcal{G}$ is no subset of the planar graphs.

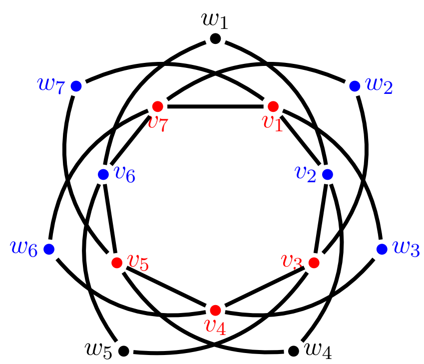


FIGURE 6.5: Graph G consisting of the C_7 and seven 3-paths

Lemma 31. *The graph G is minimal non-*$(2,1)$*-colorable.*

Proof. Suppose that there is a $(2,1)$-coloring of G. By symmetry and as the vertices v_i induce a C_7, we assume w.l.o.g. that v_1 and v_7 both receive color 1. Thus, all vertices in $N(\{v_1, v_7\}) = \{v_2, v_6, w_2, w_3, w_6, w_7\}$ have color 2. Each of the vertices v_3, v_4 and v_5 has two neighbors in this set and hence, has color 1. This contradicts the assumption as v_3, v_4 and v_5 induce a P_3.
There are two isomorphic types of graphs $G - e$ for $e \in E(G)$. Figure 6.6 shows 1-improper 2-colorings for both. Therefore, the graph fulfills minimality. □

Lemma 32. *The graph G is not planar.*

Proof. This follows from Kuratowski's Theorem (see, e.g., [9]) since G contains $K_{3,3}$ as a topological minor: the graph $G - \{w_4, w_5, w_6, w_7\}$ is a subdivion of $K_{3,3}$. The partition classes are $\{v_1, v_3, v_6\}$ and $\{v_2, v_4, v_7\}$. □

Theorem 9. *Minimal non-*$(2,1)$*-colorability does not imply planarity.*

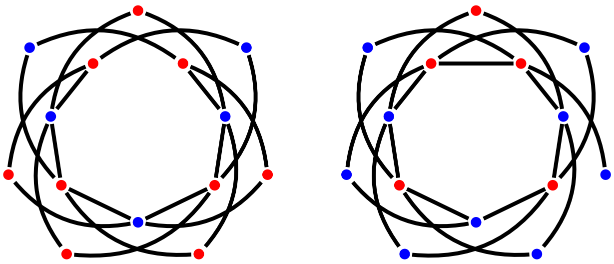


FIGURE 6.6: $(2,1)$-Colorings of $G - e$

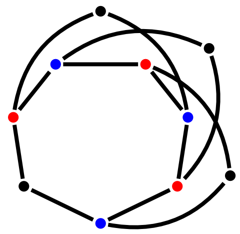


FIGURE 6.7: Non-planarity of G

6.3 Addition of Building Blocks

6.3.1 Building Blocks

Definition 9. A *building block* is a graph G with a specified *base* $b \in V(G)$ such that:
(1) G is 1-improper 2-colorable.
(2) G has no b-isolated $(2,1)$-coloring.
(3) Every proper subgraph of G has a b-isolated $(2,1)$-coloring.

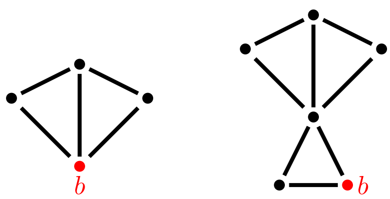


FIGURE 6.8: Examples of building blocks

Following [5], the first graph in Figure 6.8 is called a *flag*. It is the building block with the smallest order. Let G be a graph with a vertex v and B a building block with base b. We say B is *glued* to v and v *holds* B if we unite B and G in such a way that they intersect exactly in $v = b$. The second graph in Figure 6.8 is a flag glued to a triangle.

Corollary 6. *Let B_1 and B_2 be building blocks. If we glue B_1 to the base of B_2, we obtain a minimal non-$(2,1)$-colorable graph.*

Proof. We call b the vertex which is the base of B_1 and of B_2. In any 2-coloring such that all vertices but b have impropriety at most one, the vertex b has two monochromatic edges by property (2) in Definition 9. Minimality follows directly from property (3). □

Remark 6. Let G be a graph with a vertex v and B a building block glued to v. In any $(2,1)$-coloring of $G \cup B$, the vertices in $N_G(v)$ have the other color from v because v is in a monochromatic edge of B.

6.3.2 Odd Cycles with Building Blocks

Let $\mathcal{G}_\mathcal{B}$ be the set of all graphs which consist of an odd cycle $C_k = v_1 \ldots v_k$ and $k' + 1$ building blocks such that each v_i with an odd index i holds one building block. Figure 6.9 shows the example where k is seven and the building blocks are flags.

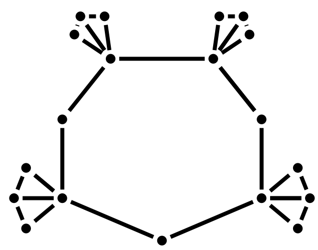


FIGURE 6.9: Example of a graph in $\mathcal{G}_\mathcal{B}$

Lemma 33. *The graphs in $\mathcal{G}_\mathcal{B}$ are non-$(2,1)$-colorable.*

Proof. Assume for a contradiction that there is such a coloring. Every edge of C_k is incident to a vertex which holds a building block. Thus, they are all dichromatic by Remark 6. This contradicts the fact that C_k is an odd cycle and hence not bipartite. □

Theorem 10. *The set $\mathcal{G}_\mathcal{B}$ is a subset of $\mathcal{G}$.*

Proof. Let G be a graph in $\mathcal{G}_\mathcal{B}$. We need to show that every proper subgraph of G is $(2,1)$-colorable. The graph $G - e$ is $(2,1)$-colorable if e belongs to C_k as a proper 2-coloring of the path $C_k - e$ is extendable to $G - e$. Let e belong to a building block B which is glued to a vertex v_i. Color one edge of C_k which contains v_i and no further base of a building block monochromatic and all other edges of C_k dichromatic. The graph $B - e$ is a proper subgraph of B and thus obtains a v_i-isolated $(2,1)$-coloring. All other building blocks receive an arbitrary $(2,1)$-coloring. The union of these coloring is well-defined and a $(2,1)$-coloring of G. □

6.3.3 Examples of Building Blocks

Recall the graphs $G_k \in \mathcal{G}_\mathcal{T}$ in Subsection 6.2.1 which consist of an odd cycle and k triangles. Let G'_k be the graph $G_k - \{w_1, w_k\}$. Figure 6.10 shows these graphs for $k \leq 7$.

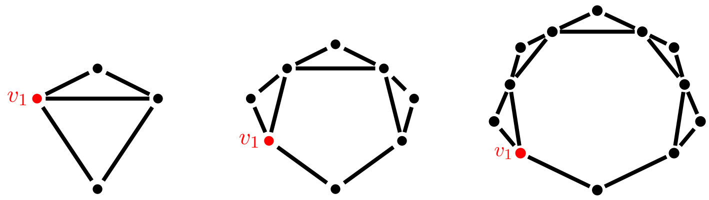


FIGURE 6.10: The graphs G'_k for $k \leq 7$

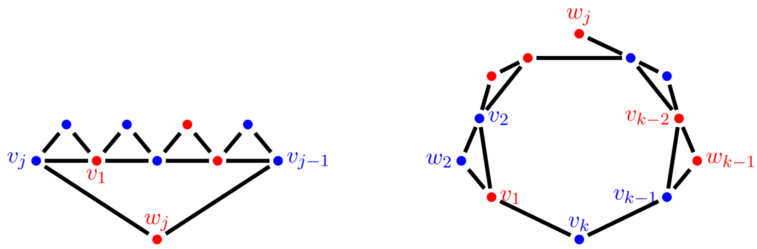
Lemma 34. *The graphs G'_k are building blocks with base v_1.*

Proof. The $(2,1)$-colorability of G'_k follows directly from Theorem 8 since G'_k is a proper subgraph of $G_k \in \mathcal{G}_\mathcal{T}$. Let us show that in any $(2,1)$-coloring, v_1 lies in a monochromatic edge. Assume for a contradiction that there is a $(2,1)$-coloring c of G'_k such that v_1 is in no monochromatic edge, say $c(v_1) = 1$ and $c(v_2) = c(v_k) = c(w_2) = 2$. Similarly to the proof of Lemma 30, it follows by induction for all i with $2 \leq i \leq k-1$, that $v_i w_i$ is monochromatic and $v_{i-1}v_i$ dichromatic. Thus, w_{k-1} and v_{k-1} both have color 2 since $k-1$ is even. Also v_k is a neighbor of v_{k-1} and has color 2. Hence, c is no $(2,1)$-coloring.
To prove minimality, we show a v_1-isolated $(2,1)$-coloring of $G'_k - e$ for any edge e in G'_k. First, consider the case $e = v_{j-1}v_j$. The following coloring is even a $(2,1)$-coloring of the supergraph $G_k - e$: color the vertices in the path $C_k - e$ alternating and the vertex w_j in the other color from its neighbors v_{j-1} and v_j. Say w.l.o.g. that v_1 has color 1. A vertex w_i with $i \neq j$ receives color 2 if and only of its distance to v_1 is odd. Figure 6.11 (a) shows an example. In this coloring, no monochromatic edges are incident and all neighbors of v_1 have color 2. If $w_j \in e$ for a $j \in \{2, \ldots, k-1\}$, color $v_{k-1}v_k$ monochromatic and all other edges of the cycle dichromatic. For $i < j$, color w_i in the color of v_i and for $i > j$, color w_i in the color of v_{i-1}. The vertex w_j receives the other color from its neighbor. Figure 6.11 (b) demonstrates an example. There is no monochromatic P_3 in $G'_k - e$ and the edges of v_1 are dichromatic. $\square$

In a similar manner, the graphs in $\mathcal{G}_\mathcal{B}$ contain building blocks as proper subgraphs. Consider the set $\mathcal{G}'_\mathcal{B}$ of all graphs $G' := G - (B - v_1)$ such that $G \in \mathcal{G}_\mathcal{B}$ and B is the building block glued to v_1. These graphs are illustrated in Figure 6.12.

Lemma 35. *The graphs in $\mathcal{G}'_\mathcal{B}$ are building blocks with base v_1.*

Proof. Every graph $G' \in \mathcal{G}'_\mathcal{B}$ is a proper subgraph of a $G \in \mathcal{G}_\mathcal{B}$ and thus $(2,1)$-colorable. In any such coloring, the vertices which hold a building



(a) $G_5 - v_4v_5$ (b) $G_7' - v_3w_4$

FIGURE 6.11: Minimality of the building blocks G_k'

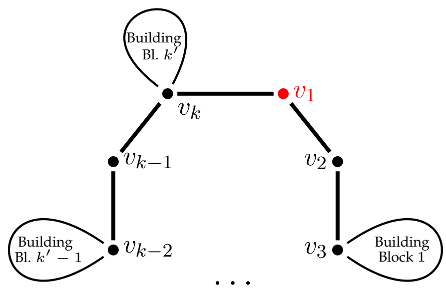


FIGURE 6.12: The graphs in $\mathcal{G}_{\mathcal{B}}'$

block are in no monochromatic edge of C_k. Hence, v_1v_2 is monochromatic and the coloring not v_1-isolated.

Let us consider $G' - e$ for an edge e of C_k. The proper 2-coloring of the path $C_k - e$ is extendable to $G' - e$. The vertex v_1 is in no monochromatic edge because all its neighbors belong to C_k. If e belongs to a building block B' glued to v_i, the graph $B' - e$ obtains a v_i-isolated coloring by the minimality of building blocks. We color the edge $v_{i-1}v_i$ monochromatic and the remaining edges of C_k dichromatic. This coloring is extendable to $G' - e$ and the vertex v_1 in no monochromatic edge since $i - 1 \geq 2$. □

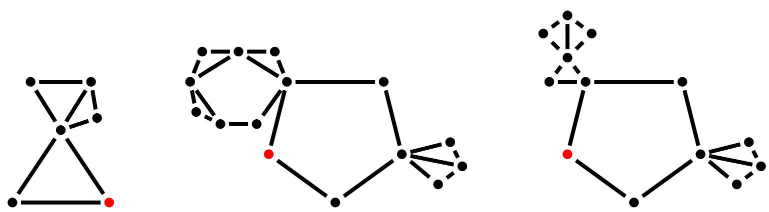


FIGURE 6.13: Examples of building blocks in $\mathcal{G}_{\mathcal{B}}'$

6.4 Combination of P_3 and Building Blocks

This Section presents subsets of $\mathcal{G}$ which contain both, building blocks and paths with three vertices.

6.4.1 Odd Cycles with Triangles and Building Blocks

Let G be a graph which consists of an odd cycle $C_k = v_1 \dots v_k$, building blocks $B^1, \dots, B^l$ for an $l \in \{1, \dots, k'\}$ and 3-paths $P^1, \dots, P^{k-2l}$. Furthermore, the graph fulfills the following:
(1) The building blocks are glued to distinct vertices of C_k.
(2) The end vertices of the P^i are glued to neighboring vertices of C_k.
We call them *triangles on* C_k and a continous set of such triangles a *sequence*.
Let $v_{i'}$ and $v_{i''}$ be bases of building blocks such that $i' < i''$ and there is no base on the $v_{i'}$-$v_{i''}$-path $P := v_{i'}v_{i'+1} \dots v_{i''}$.
(3) The length l_P of P is at least two.
(4) There are exactly $l_P - 2$ triangles on P. The two edges which do not belong to triangles are incident.

We call the set of these graphs $\mathcal{G}_{\mathcal{BT}}$. Figure 6.14 shows some examples:

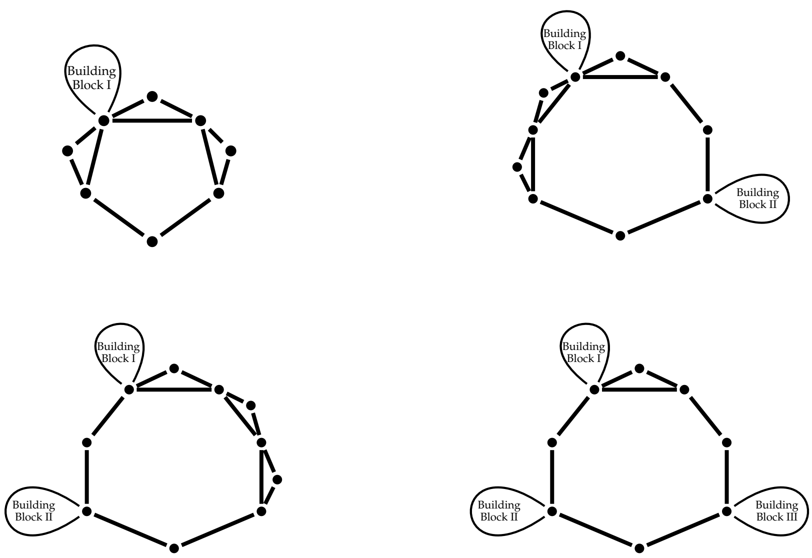


FIGURE 6.14: Graphs in $\mathcal{G}_{\mathcal{BT}}$

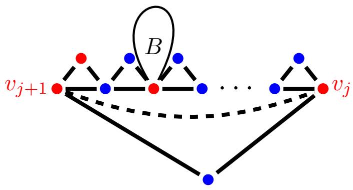
Lemma 36. *The graphs in $\mathcal{G}_{\mathcal{BT}}$ are not $(2,1)$-colorable.*

Proof. Assume there is a $(2,1)$-coloring of G. We show for each edge $e = v_iv_{i+1}$ in C_k, that it is dichromatic. This leads to contradiction as k is odd. If either v_i or v_{i+1} holds a building block, it has a same-colored neighbor outside C_k and thus, e is dichromatic. Otherwise, let $v_{i'}$ and $v_{i''}$ be bases of building blocks such that there is no base on the $v_{i'}$-$v_{i''}$-path that contains v_i. The two consecutive edges of this path that do not belong to triangles, are w.l.o.g. in the v_i-$v_{i''}$-subpath. Hence, every edge in the $v_{i'}$-v_i-subpath is in a triangle. One can see by induction along the $v_{i'}$-v_i-subpath, that in any $(2,1)$-coloring of G, the vertex v_i has the same color as its bivalent neighbor in the triangle on $v_{i-1}v_i$. Thus, v_i is in a monochromatic edge outside C_k and e is dichromatic. □

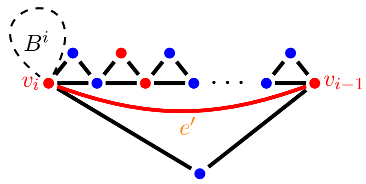
Theorem 11. *The set $\mathcal{G}_{\mathcal{BT}}$ is a subset of $\mathcal{G}$.*

Proof. It remains to show that $G - e$ is $(2,1)$-colorable for all $G \in \mathcal{G}_{\mathcal{BT}}$ and $e \in E(G)$. The proof is illustrated in Figure 6.15.
Case 1: $e = v_j v_{j+1} \in E(C_k)$. Let c be a proper 2-coloring of the k-path $C_k - e$. It holds $c(v_j) = c(v_{j+1})$. If there is a triangle on e in G, color its bivalent vertex in the other color from v_j and v_{j+1}. Extend this coloring to the building blocks. Let V_b be the set of all bivalent vertices in the triangles on $C_k - e$. Any maximal sequence of triangles contains at most one base of a building block. Color a vertex $w \in V_b$ in the same color as this base v_i if and only if its distance to v_i is even. One of the vertices v_j and v_{j+1} is either in no sequence or in a sequence which contains no base. Let us assume w.l.o.g. that this is the vertex v_j. There is no further sequence without a base. Color a vertex $w \in V_b$ which belongs to this sequence in the same color as v_j if and only if its distance to v_j is even.
Let us show that no vertex has two same-colored neighbors in c. This holds for the vertices in the building blocks which are not the base. The path $C_k - e$ is colored alternating and thus, no vertex in V_b has more than one same-colored neighbor. If there is a bivalent vertex in the triangle on e in G, it has impropriety zero. Moreover, no vertex v_i of the cycle has a same-colored neighbor in $C_k - e$. If v_i is a base and adjacent to vertices in V_b, these vertices have the other color since their distance to v_i is one and hence odd. Therefore, v_i is only in the monochromatic edge of the building block. If v_i is no base and has two neighbors outside C_k, these neighbors belong to V_b and to the same sequence of triangles. Their distance to the base respectively to v_j differs exactly by one. Thus, they have different colors and v_i is only in one monochromatic edge. Hence, no vertex has more than one neighbor of the same color. Furthermore, it holds that v_j is in no monochromatic edge.
Case 2: $e \in E(B^i)$ for $i \leq l$. Let v_i be the base of B^i and $e' := v_{i-1}v_i$. Consider the $(2,1)$-coloring of $G - e'$ introduced in Case 1. If we color $G - e$ analogously, any monochromatic P_3 contains the edge e'. The vertex v_{i-1} is the vertex v_j of Case 1. Hence, it has no neighbor of the same color in $G - \{e, e'\}$. By the minimality of building blocks, we can recolor the vertices in $B^i - e$ such that v_i is in no monochromatic edge of $G - \{e, e'\}$ either. This gives a $(2,1)$-coloring of $G - e$.
Case 3: $w \in e$ for a $w \in V_b$. We consider w.l.o.g. $G - w$. Let v_i and v_{i+1} be the neighbors of w. Let $v_{i'}$ and $v_{i''}$ be bases of building blocks such that $i' \leq i < i''$ and that there is no base between $v_{i'}$ and $v_{i''}$. The two edges which do not belong to triangles are w.l.o.g. in the v_{i+1}-$v_{i''}$-subpath. It follows $i'' \geq i + 3$. Let $e' := v_{i+1}v_{i+2}$ and consider the coloring of $G - e'$ treated in Case 1. By symmerty, we can choose v_{i+2} as the vertex v_j of Case 1 which is in no monochromatic edge of $G - e'$. The vertex v_{i+1} is in the monochromatic

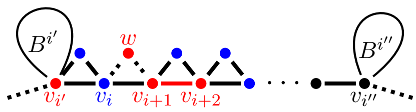
edge wv_{i+1}. Thus, it has no same-colored neighbor of $G - w - e'$. As neither v_{i+1} nor v_{i+2} is in a monochromatic edge of $G-w-e'$, this is a $(2,1)$-coloring of $G - w$. □



(a) Case 1: $e = v_j v_{j+1}$



(b) Case 2: $e \in E(B^i)$



(c) Case 3: $w \in e$

FIGURE 6.15: Proof of the minimality

6.4.2 Generalization to 3-Paths and Building Blocks

Remark 7. Let $e = vw$ be an edge in a $(2,1)$-colored graph and v a vertex of color 1. Furthermore, let e' be a monochromatic edge of color 2. If there is a vertex x adjacent to a vertex of e and to a vertex of e', it has color 1. Since v, w and x induce a P_3, w has color 2, i.e., e is dichromatic.

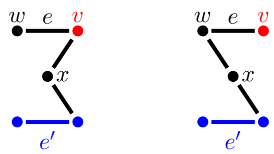


FIGURE 6.16: Dichromaticity of e

We can apply Remark 7 to construct minimal non-$(2,1)$-colorable graphs. For this purpose, replace the triangles in a graph $G \in \mathcal{G}_{\mathcal{BT}}$ by suitable 3-paths with one end vertex in a monochromatic edge. These end vertices might not belong to the cycle C_k. Figure 6.17 shows examples of such graphs. One can prove minimal non-$(2,1)$-colorability in a similar way as in Subsection 6.4.1.

6.4.3 Paths with 3-Paths and Building Blocks

In this section, we consider a graph set $\mathcal{G}_{\mathcal{P}}$. It contains the graphs which are a sequence of $k-1$ triangles on the path $P := v_1 v_2 \ldots v_k$, united with two

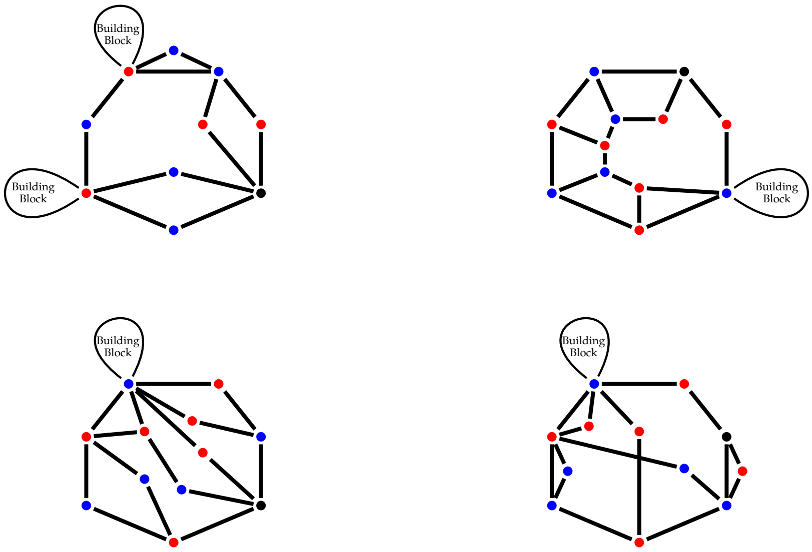


FIGURE 6.17: Generalization of $\mathcal{G}_{\mathcal{BT}}$

building blocks B_1 and B_2 which are glued to v_1 and v_k. We call the bivalent vertices in the triangles $w_2, \ldots, w_k$. Figure 6.18 shows these graphs.

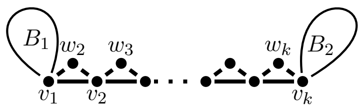


FIGURE 6.18: Graphs in $\mathcal{G}_{\mathcal{P}}$

Lemma 37. *The graph $B := G - (B_2 - v_k)$ is a building block with base v_k for all $G \in \mathcal{G}_{\mathcal{P}}$.*

Proof. We show by induction that any $(2,1)$-coloring c of B fulfills $c(v_i) = c(w_i)$ for all $i \geq 2$. By the definition of building blocks, the vertex v_1 is in a monochromatic edge of B_1. Thus, $c(v_1) \neq c(v_2)$ and $c(v_1) \neq c(w_2)$, i.e., $c(v_2) = c(w_2)$. Assume $c(v_{j-1}) = c(w_{j-1})$ for a $j \geq 2$. As w_{j-1} is a neighbor of v_{j-1}, all further vertices in $N(v_{j-1})$ have the other color. Both v_j and w_j are in $N(v_{j-1})$ and therefore, $c(v_j) = c(w_j)$. It follows in particular that $c(v_k) = c(w_k)$ and that B is 1-improper 2-colorable.
It remains to show the minimality. First, consider $B - e$ for a edge e in B_1. We can color all vertices v_i and w_{i+1} in color 1 iff i is even and $B_1 - e$ in a v_1-isolated coloring. This is a $(2,1)$-coloring of $B - e$ such that all edges of v_k are dichromatic. If $w_j \in e$, we consider w.l.o.g. $B - w_j$. The graph $B - w_j - v_{j-1}v_j$ has two components. The component containing v_1 has the same structure as B and thus admits a $(2,1)$-coloring c_1. The second component is a sequence of triangles. We again color the v_i and the w_{i+1} in color 1 iff i is even. Hence, no edge of v_k is monochromatic. Then $c_1 \cup c_2$ fulfills the requirements as we can assume w.l.o.g. $c_1(v_{j-1}) \neq c_2(v_j)$. Lastly,

if $e = v_{j-1}v_j$, color the components of $B - e - w_j$ as in Case 2. Assume w.l.o.g. that $c_1(v_{j-1}) = c_2(v_j)$, consider $c_1 \cup c_2$ and color w_j in the other color from its neighbors. $\square$

Together with Corollary 6, this gives the following theorem:

Theorem 12. *The graphs in $\mathcal{G}_{\mathcal{P}}$ are minimal non-$(2,1)$-colorable.*

We can extend this set of minimal non-$(2,1)$-colorable graphs by Remark 7. Figure 6.19 shows examples of graphs where not all 3-paths create triangles.

FIGURE 6.19: Extension of $\mathcal{G}_{\mathcal{P}}$

6.5 Order of Minimal Non-$(2,1)$-Colorable Graphs

This chapter presents a variety of graphs in $\mathcal{G}$. The order of these graphs depends on the size of the central odd cycle or the central path and partly on the choice and number of building blocks. We want to employ this to find graphs of every order greater than four in $\mathcal{G}$.

Theorem 13. *The set of minimal non-$(2,1)$-colorable graphs contains a graph with n vertices if and only if $n \geq 5$.*

First consider following graph G with eight vertices:

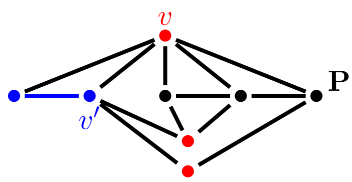


FIGURE 6.20: Graph G with $n(G) = 8$

Lemma 38. *The graph in Figure 6.20 is minimal non-$(2,1)$-colorable.*

Proof. Let v and v' be as in the picture and denote the P_3 in $G[N(v)]$ by P. Assume that G is $(2,1)$-colorable. As P is dichromatic in such a coloring, one vertex $x \in V(P)$ has the same color as v, say color 1. All further neighbors of v have color 2. Since v' has a same-colored neighbor in $N(v)$, the vertices in $G - v - N(v)$ have color 1. One of them is adjacent to x which contradicts the condition that x has not more than one neighbor of color 1.

To show minimality, we consider $G - e$ for all edges $e \in E(G)$. Consider the vertex-coloring of $G - P$ shown in the figure. Coloring all vertices in P blue gives a $(2,1)$-coloring of $G - e$ if $e \in E(P)$. Any vertex y in P has exactly two red neighbors. Thus, if $y \in e$ and $e \notin E(P)$, coloring y red and $V(P)\backslash\{y\}$ blue gives a $(2,1)$-coloring. If $e = vz$ for a $z \notin V(P)$, let us recolor z and the vertex drawn on the lowest position. This coloring is extendable to $G - e$ by coloring the rightmost vertex red and the others blue. If e is one of the remaining edges, i.e., incident to v' but not to v, we can recolor a vertex of distance 2 to v and extend this coloring to $G - e$. □

Proof of Theorem 13. The graphs $G_k \in \mathcal{G}_{\mathcal{T}}$ have $2k$ vertices for an odd integer $k \geq 3$. A graph in $\mathcal{G}_{\mathcal{BT}}$ with exactly two flags as building blocks and hence $k - 4$ paths of length two has $k + 2 \cdot 3 + k - 4 = 2k + 2$ vertices for an odd $k \geq 5$. The graphs in $\mathcal{G}_{\mathcal{P}}$ have $2k + n(B_1) + n(B_2) - 1$ vertices. Choosing flags as building blocks gives graphs of order $2k + 7$ for $k \in \mathbb{N}$.
It follows that there are minimal non-$(2,1)$-colorable graphs with n vertices for all $n \geq 9$. Chapter 4 shows minimal non-$(2,1)$-colorable graphs with $5, 6$ and 7 vertices and proves that there are no such graphs with less than five vertices. Together with Lemma 38, this proves the theorem. □

Chapter 7

Average Degree

A graph with relatively few edges is called *sparse*. If all subgraphs of a graph G are sufficiently sparse, G obtains a $(2,1)$-coloring. Borodin, Kostochka and Yancey showed in their article "On 1-improper 2-coloring of sparse graphs" [5] that any non-$(2,1)$-colorable graph has a subgraph with average degree greater than $14/5$. This leads to the question whether one of these subgraphs is also non-$(2,1)$-colorable. This would imply that a minimal non-$(2,1)$-colorable graph has average degree at least $14/5$ as its unique non-$(2,1)$-colorable subgraph is the graph itself.
In this chapter, we analyze the average degree of minimal non-$(2,1)$-colorable graphs and study their lower bounds. As an introduction, we present a coloring of a connected graph with n vertices and $n+1$ edges.

7.1 Coloring of Graphs with $\mathbf{n+1}$ Edges

Let G be a connected graph with n vertices, $n+1$ edges and a spanning tree T. Let e_1 and e_2 be the edges outside T and c a coloring of G without monochromatic edges in T. This is no $(2,1)$-coloring of G if and only if e_1 and e_2 are monochromatic and incident. If so, let $e_1 := v_1v_2$ and $e_2 := v_2v_3$ and root the tree T in v_1. Let T_v be the subtree of T with root v. We denote the parent of v by v^p for all $v \neq v_1$. If $v_3 \notin V(T_{v_2})$, color the vertices in T_{v_2} in the other color from c and all remaining vertices as in c. This is a $(2,1)$-coloring of G as only the edge $v_2v_2^p$ is monochromatic. If $v_3 \in V(T_{v_2})$, color only the vertices in $T_{v_2} - T_{v_3}$ differently from c and all other vertices as in c. Only the edges $v_2v_2^p$ and $v_3v_3^p$ are monochromatic. They have different colors as $c(v_2) = c(v_3)$ but just v_2 sits in $T_{v_2} - T_{v_3}$.

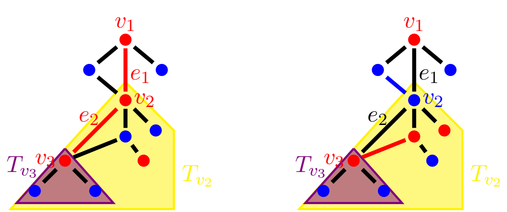


FIGURE 7.1: Coloring c and $(2,1)$-coloring if $v_3 \in V(T_{v_2})$

7.2 Lower Bound for the Average Degree

We prove that minimal non-$(2,1)$-colorable graphs have average degree greater than $8/3$. First, we show that a graph, which fulfills some properties of minimal non-$(2,1)$-colorable graphs, cannot have average degree smaller than $8/3$. Afterwards, we consider all graphs with these properties and average degree exactly $8/3$. We prove that these graphs obtain a $(2,1)$-coloring. Hence, the average degree of a minimal non-$(2,1)$-colorable graph exceeds $8/3$.

For this purpose, let H be a connected graph without separating edges and hence, minimal degree at least two. We denote the set of d-valent vertices in H by V_d. Let the set V_2 be independent and any component of $H - V_2$ whose vertices are in V_3 contain at least two odd cycles. We denote these properties by $(*)$.

Lemma 39. *The graph H has average degree at least $8/3$.*

We apply discharging for the proof of Lemma 39. This method of proof works with an initial charge on each vertex. Afterwards, we discharge between the vertices, i.e., transfer some of the initial charge, and use the fact that the sum over all charges stays the same.

Proof. Each vertex gets initial charge $d(v)$. In the discharging, every bivalent vertex receives charge $1/3$ from its neighbors. Hence, bivalent vertices have final charge $2 + 2 \cdot 1/3 = 8/3$ since both neighbors have degree at least three.
Let v be a vertex of degree at least 3 and H' the component of $H - V_2$ which contains v. We set $n' := n(H')$, $m' := m(H')$ and $V' := V(H')$. Let V_2' be the set of bivalent vertices whose neighbors are both in V'. The neighborhood of H' has the size

$$|N(H')| = \sum_{v \in V'} d(v) - 2m' - |V_2'|.$$

We show that the total final charge $\mathrm{ch}(H')$ of H' is sufficient for its vertices. That means that $\mathrm{ch}(H') \geq 8/3n'$. The total final charge is

$$\mathrm{ch}(H') := \sum_{v \in V'} d(v) - \frac{1}{3}|N(H')| - \frac{1}{3}|V_2'| = \frac{2}{3}\left(\sum_{v \in V'} d(v) + m'\right).$$

If $V(H') \subseteq V_3$, it contains two odd cycles and hence, $m' \geq n' + 1$. This gives

$$\mathrm{ch}(H') \geq \frac{2}{3}\left(3n' + (n' + 1)\right) > \frac{8}{3}n'.$$

Otherwise, there is a vertex of degree at least four in H'. As H' is connected, $m' \geq n' - 1$ and thus,

$$\mathrm{ch}(H') \geq \frac{2}{3}\left((3n' + 1) + (n' - 1)\right) = \frac{8}{3}n'.$$

Therefore, the graph H has in total charge at least $8/3n(H)$. We conclude that its average degree is at least $8/3$. □

Lemma 40. *A graph H with the properties $(*)$ and average degree exactly $8/3$ is $(2,1)$-colorable.*

Proof. Let V_d be the set of d-valent vertices in H and X_i the set of all vertices which have i bivalent neighbors. The proof above shows that H only has average degree $8/3$ if every component of $H' := H - V_2$ is a tree with exactly one vertex in V_4 and all other vertices in in V_3. In particular, $\Delta(H) = 4$ and thus, $X_i = \emptyset$ for all $i \geq 5$. A vertex in X_3 has degree 4 in H as, otherwise, it were isolated in H' and hence, contradicted the last property in $(*)$. It follows that such a vertex is unique in its component of H'. Vertices in X_4 are isolated in H'. Paths of length two whose inner vertex is bivalent are called *double-edges* and its end vertices 2*-neighbors*. A double-edge is monochromatic if their end vertices have the same color and dichromatic otherwise. Let c'_1 be a 2-coloring of H' such that all components in $H' - X_4$ are colored properly and the vertices in X_4 have at most two different-colored 2-neighbors.
Let us deduce a coloring c' of H' where no vertex has more than two dichromatic double-edges. If this holds for c'_1, set $c' := c'_1$. Otherwise, let $x_1, \ldots, x_k$ be the vertices with more than two dichromatic double-edges and C_{x_i} their respective component in H'. It holds $x_i \in X_3$ and $d_{C_{x_i}}(x_i) = 1$. The components C_{x_i} are considered one after the other. We call c'_{i+1} the coloring obtained after treating C_{x_i}. All vertices outside C_{x_i} have the same color in c'_i and c'_{i+1}. Furthermore, the vertex x_i has color $\overline{c'_i(x_i)}$ in c'_{i+1}. Let y_i^1 be the unique neighbor of x_i in C_{x_i}. If $y_i^1 \in X_2$, the component C_{x_i} consists only of x_i and y_i^1. In c'_{i+1}, we color y_i^1 in $\overline{c'_i(y_i^1)}$ if and only if both double-edges of y_i^1 are dichromatic in c'_i. If y_i^1 sits in X_0, we color all vertices in $C_{x_i} - x_i$ as in c'_i.
Trivalent vertices in X_1 have degree two in H'. If $y_i^1 \in X_1$, let $P := y_i^1 \ldots y_i^{l-1}$ be a maximal path of vertices in X_1. Let y_i^m be the first vertex in P whose double-edge is monochromatic in c'_i. In c'_{i+1}, we color the vertices $y_i^1, \ldots,$ y_i^{m-1} differently from c'_i and all remaining vertices of $C_{x_i} - x_i$ as in c'_i. If no such y_i^m exists, color all vertices in P in the other color from c'_i and let y_i^l be the trivalent neighbor of y_i^{l-1} which does not belong to P. This vertex is either in X_0 or in X_2. If it is in X_2 and has two dichromatic double-edges in c'_i, color y_i^k in c'_{i+1} differently from c'_i. Otherwise, color y_i^k in c'_{i+1} as in c'_i. All further vertices of $C_{x_i} - x_i$ receive the same color as in c'_i. Let P' be the path of vertices with different colors in c'_i and c'_{i+1}. Figure 7.2 shows an example.
Subsequently, we remove the vertices x_j with $j > i$, which do not have three different-colored 2-neighbors in c'_{i+1}, from the sequence $x_{i+1}, \ldots, x_k$. Only vertices whose 2-neighbors are all different-colored in c'_i fulfill $c'_{i+1} \neq c'_i$. Thus, a monochromatic double-edge in any coloring c'_i is monochromatic in all colorings c'_j with $j \geq i$. Hence, all vertices with three different-colored 2-neighbors in c'_{i+1} are in the sequence $x_{i+1}, \ldots, x_k$. We can choose $c' := c'_{k+1}$. Now, let us consider a vertex v which is in a monochromatic edge of c'. It is either an end vertex of P' or the unique neighbor of P' in C_{x_i} for an $i \leq k$. If $v \in X_0$, it has no 2-neighbors. If v belongs to X_3, all double-edges

are monochromatic in c'. If v is in X_1, its double-edge is monochromatic since it holds $c'_{i+1}(v) \neq c'_i(v)$ if and only if the double-edge is dichromatic in c'_i. If $v \in X_2$, it has at most one different-colored 2-neighbor. If there is a different-colored 2-neighbor, we call v *evil*. An example is the vertex y_i^4 in Figure 7.2.

Let c be the following coloring of H: inner vertices of monochromatic double-edges in c' receive the other color from their neighbors. Let H'' be the graph induced by the dichromatic double-edges in c'. All non-bivalent vertices outside H'' are colored as in c'. As no vertex has more than two dichromatic double-edges in c', H'' has maximum degree two. Hence, it consists of disjoint paths and cycles. The cycles have an even number of double-edges as in c', the colors of the (in H) non-bivalent vertices alternate.

Let Y be a component of H''. If none of the vertices in Y sits in a monochromatic edge in c', color the (in H) non-bivalent vertices as in c' and the bivalent vertices alternating along the path or cycle. If a vertex a in H'' sits in a monochromatic edge of c', it is evil. Hence, only one of its double-edges is dichromatic and Y is a path with end vertex a. Let b be the second end vertex of Y. We color the non-bivalent vertices in $Y - b$ as in c' and the bivalent vertices alternating such that a is in no monochromatic edge of Y. Color b as in c' if and only if it is not evil.

If no component of H'' contains more than one evil vertex, all vertices w in H' fulfill $c(w) = c'(w)$ and the bivalent vertices have at most one same-colored neighbor. All vertices of degree at least three have no same-colored neighbor in V_2 if they are in a monochromatic edge in c' and at most one otherwise. Therefore, it is a $(2,1)$-coloring of H.

If there are paths in H'' with two evil end vertices, one of them has different colors in c' and in c. An evil vertex is a leaf in H' and sits in a monochromatic edge in c'. Hence, its component in H' is properly 2-colored in c. Moreover, its edge in Y is dichromatic in c. Thus, it has only one same-colored neighbor in c. This is the inner vertex of its double-edge which is monochromatic in c'. It follows that the only vertices in H, whose number of monochromatic edges is higher than in the case without components with two evil vertices, are inner vertices of monochromatic double-edges with two evil end vertices. Hence, all centers of monochromatic S_3 in c are bivalent vertices with two neighbors of the same color. Recoloring these vertices gives a $(2,1)$-coloring of H. This is illustrated in Figure 7.3. □

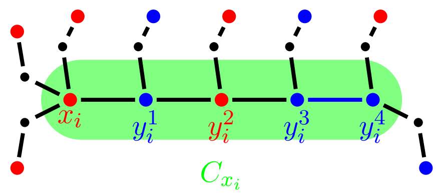
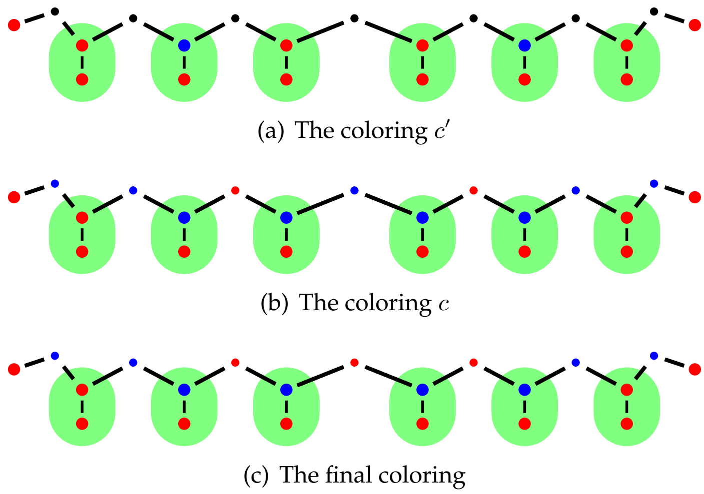


FIGURE 7.2: Coloring c_{i+1}: $P' = x_i y_i^1 y_i^2 y_i^3$

We showed in Chapter 3 that minimal non-$(2,1)$-colorable graphs fulfill the properties $(*)$. Therefore, Lemma 39 and Lemma 40 give following theorem:

Theorem 14. *The average degree of a minimal non-$(2,1)$-colorable graph is strictly greater than $8/3$.*



(a) The coloring c'

(b) The coloring c

(c) The final coloring

FIGURE 7.3: Paths in H'' with two evil endvertices

Corollary 7. *A minimal non-$(2,1)$-colorable graph G fulfills*

$$m(G) \geq \left\lceil \frac{4}{3}n(G) + \frac{1}{3} \right\rceil = n(G) + \left\lceil \frac{1}{3}(n(G)+1) \right\rceil .$$

It follows that any minimal non-$(2,1)$-colorable graph G with $n(G) \geq 6$ has at least $n(G)+3$ edges. This holds for all graphs in $\mathcal{G}$ as W_5 is the only graph with less than six vertices and it has eight edges. Figure 7.4 shows graphs of order n with $n+3$ edges. We proved in Chapter 4 that they are minimal non-$(2,1)$-colorable.

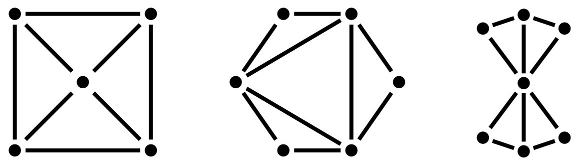


FIGURE 7.4: Graphs in $\mathcal{G}$ with $m(G) = n(G) + 3$

7.3 $(2,1)$-Density of Minimal Graphs

Definition 10. The $(2,1)$-*density of minimal graphs* m^* is the maximum number such that all graphs in $\mathcal{G}$ have average degree at least m^*.

Consider the minimal non-$(2,1)$-colorable graphs in the set $\mathcal{G}_{\mathcal{BT}}$, introduced in Subsection 6.4.1, with $k' = (k-1)/2$ flags as building blocks. This set of graphs is presented in [5]. The graphs have $k + 3k' + 1 = 5k' + 2$ vertices and $k + 5k' + 2 = 7k' + 3$ edges. Thus, they have average degree

$$\frac{2(7k'+3)}{5k'+2} \xrightarrow{k' \to \infty} \frac{14}{5}.$$

It follows that there are minimal non-$(2,1)$-colorable graphs with average degree arbitrarily close to $14/5$. Together with Theorem 14, this gives

$$\frac{8}{3} \le m^* \le \frac{14}{5}.$$

We can replace the flag in the considered graphs by any building block B. Such a graph has average degree

$$\frac{2((m(B)+2)k'+3)}{(n(B)+1)k'+2} \xrightarrow{k'\to\infty} \frac{2(m(B)+2)}{n(B)+1}.$$

The flag F fulfills $7n(F)-5m(F)=3$. So far, we only know building blocks with $7n(B)-5m(B) \le 3$. However, if B would fulfill $7n(B)-5m(B) > 3$, then it follows

$$\frac{2(m(B)+2)}{n(B)+1} < \frac{14(m(B)+2)}{5m(B)+3+7} = \frac{14}{5}.$$

In this case, the average degree were smaller than $14/5$ for all sufficiently large k'.

7.4 Conjecture: $\mathrm{ad}(\mathbf{G}) > \mathbf{14/5}$

The already mentioned article "On 1-improper 2-coloring of sparse graphs" [5] shows that graphs with maximum average degree at most $14/5$ are $(2,1)$-colorable. We apply similar ideas aiming for a proof of $\mathrm{ad}(G) > 14/5$ for minimal non-$(2,1)$-colorable graphs. This statement is not shown yet. However, the proof of Conjecture 1 follows if the Conjectures 2 and 3 hold.

Conjecture 1. *Every minimal non-*$(2,1)$*-colorable graph has average degree greater than* $14/5$.

We prove that graphs with specified properties have average degree greater than $14/5$, see Theorem 15 in Subsection 7.4.5. We show that these properties hold or follow from Conjectures 2 and 3 for the graphs we consider in this section.

To simplify the calculations, we define the *potential* of a graph as

$$p(G) := 7n(G) - 5m(G).$$

A graph has non-negative potential iff its average degree is at most $14/5$.

7.4.1 Order on the Set of Graphs

As in [5], we employ a partial order on the set of all graphs.

Definition 11. We call a vertex v *unimportant* if either $d(v) \leq 1$ or $d(v) = 2$ and v sits in a triangle. All other vertices are *important*. We denote by $n_i(G)$ the number of important vertices in G.

Definition 12. The configurations in Figure 7.5 are called *order-configurations*. For a graph G, let $o_1(G)$ be the number of order-configurations C_1^O in G, $o_2(G)$ the number of order-configurations C_2^O in G and $o(G)$ the sum $o_1(G) + o_2(G)$.

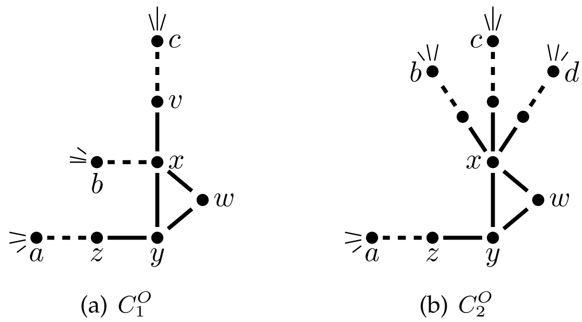


FIGURE 7.5: The order-configurations

Definition 13. Let H and G be graphs. We call H *smaller* than G and G *greater* than H, write $H \prec G$, if one of following conditions holds:
(1) $n(H) < n(G)$
(2) $n(H) = n(G) \;\wedge\; n_i(H) < n_i(G)$
(3) $n(H) = n(G) \;\wedge\; n_i(H) = n_i(G) \;\wedge\; o(H) < o(G)$
(4) $n(H) = n(G) \;\wedge\; n_i(H) = n_i(G) \;\wedge\; o(H) = o(G) \;\wedge\; o_1(H) < o_1(G)$

The binary relation "$\prec$" defines a strict partial order on the set of all finite simple graphs. The empty graph is its least element.

It would be nice to prove Conjecture 1 by contradiction. Thus, assume that there is a graph which is minimal non-$(2,1)$-colorable and has average degree at most $14/5$. This is equivalent to a non-negative potential. If furthermore, the graph is smallest with respect to the partial order "$\prec$", we call it *critical*. Throughout this section, we assume that G is a critical graph.

As G is a minimal non-$(2,1)$-colorable graph, it fulfills the properties of Chapter 3 such as connectivity, the absence of separating edges and hence, minimum degree at least two. No bivalent vertices are adjacent and any induced subgraph, whose vertices are trivalent in G and whose neighbors are bivalent in G, contains two odd cycles. Moreover, the configurations in Figure 3.4 and Figure 3.5 in Section 3.3 do not occur.

7.4.2 Flags and Superflags

We call the configuration

$$F := (K_4 - xy, \{deg : x \mapsto 2, y \mapsto 2, z \mapsto 3\}, \{v\})$$

a *flag*, compare Subsection 6.3.1. A flag is a building block with base v. The configuration obtained by gluing a flag to a triangle with a bivalent vertex, which does not hold the flag, is called a *superflag*, as in [5]. Only the base of the flag and one vertex of the triangle have neighbors outside the superflag. This second unbounded vertex is called the *secondary base*.

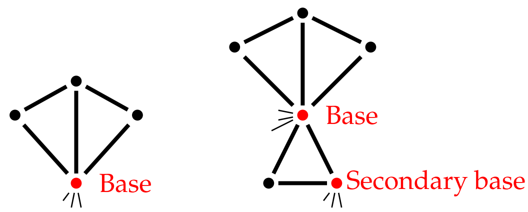


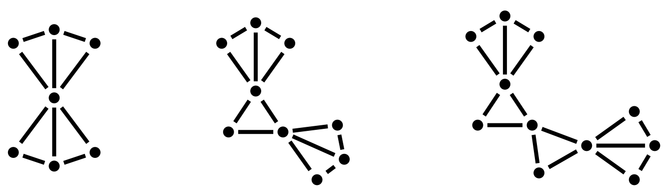
FIGURE 7.6: Flag and superflag

Remark 8. In any $(2,1)$-coloring of a superflag, the secondary base has the same color as its bivalent neighbor.

Remark 9. A flag has potential three and a superflag potential two. The graph which is a flag glued to a vertex of H has potential $p(H) - 4$.

Lemma 41. *In a minimal non-$(2,1)$-colorable graph H with $p(H) \geq 0$, flags and superflags are either vertex-disjoint or intersect in an entire flag.*

Proof. Let A and B be flags or superflags such that their intersection $A \cap B$ is no flag. Suppose that $A \cap B \neq \emptyset$. A vertex v in $A \cap B$ is either the base or the secondary base in A and in B. If we color $A \cup B$ such that all vertices except for v have impropriety at most one, v has two same-colored neighbors by Remark 8 and as flags are building blocks. Hence, $A \cup B$ is not $(2,1)$-colorable. The minimality of H gives $H = A \cup B$. There are three graphs which are minimal with the property that they consist of exactly two (super)flags. These graphs are shown in Figure 7.7. As $p(H_1) = -1$, $p(H_2) = -2$ and $p(H_3) = -3$, all of them have negative potential. This gives the sought contradiction. □



(a) H_1 (b) H_2 (c) H_3

FIGURE 7.7: Intersecting flags and superflags

Lemma 42. *In any minimal non-*$(2,1)$*-colorable graph, the base* v *of a flag has degree at least five.*

Proof. Flags are $(2,1)$-colorable and hence, $d(v) \geq 4$. If the base had one edge outside the flag, this were a separating edge and inconsistent with Lemma 2. □

Lemma 43. *In any minimal non-*$(2,1)$*-colorable graph, the secondary base of a superflag has degree at least three.*

Proof. This follows directly from Lemma 3 as the secondary base has a bivalent neighbor. □

7.4.3 Open Conjectures

We shall reduce Conjecture 1 to following two conjectures:

Conjecture 2. *Let* G *be a critical graph with a trivalent vertex* v *whose set of bivalent neighbors* V_2 *is not empty. Then* $V_2 \cup \{v\}$ *has at most two neighbors.*

Remark 10. The set V_2 has size at most two by Lemma 4.

We call a triangle with a bi- and a trivalent vertex which does not belong to a flag a *semiflag*. The third vertex has degree at least four by Lemma 8.

The property of Conjecture 2 does not hold for all minimal non-$(2,1)$-colorable graphs. Figure 7.8 shows two examples of graphs in $\mathcal{G}$ which do not fulfill this property. The red vertices are trivalent. Together with their bivalent neighbors, they have a neighborhood of size three. However, these graphs are not critical as their potential is -3.

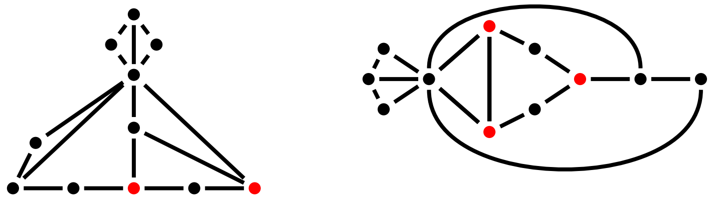


FIGURE 7.8: Minimal non-$(2,1)$-colorable graphs which do not fulfill the poperty of Conjecture 2

We will see in Lemma 47 in the next subsection, that if $|V_2| = 2$ and the vertices in V_2 share the second neighbor a, the vertex a also is a neighbor of v. Hence, the vertices in $V_2 \cup \{v, a\}$ induce a flag with base a. Together with Conjecture 2, we conclude that in critical graphs, any trivalent vertex with a bivalent neighbor sits in a flag or a semiflag.

Definition 14. We define a *supercross* as the configuration which consists of a tetravalent vertex - the *center* - and its neighbors who are all bivalent.

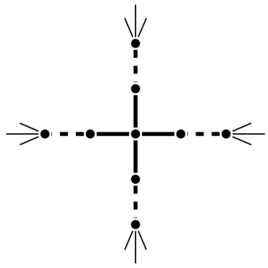
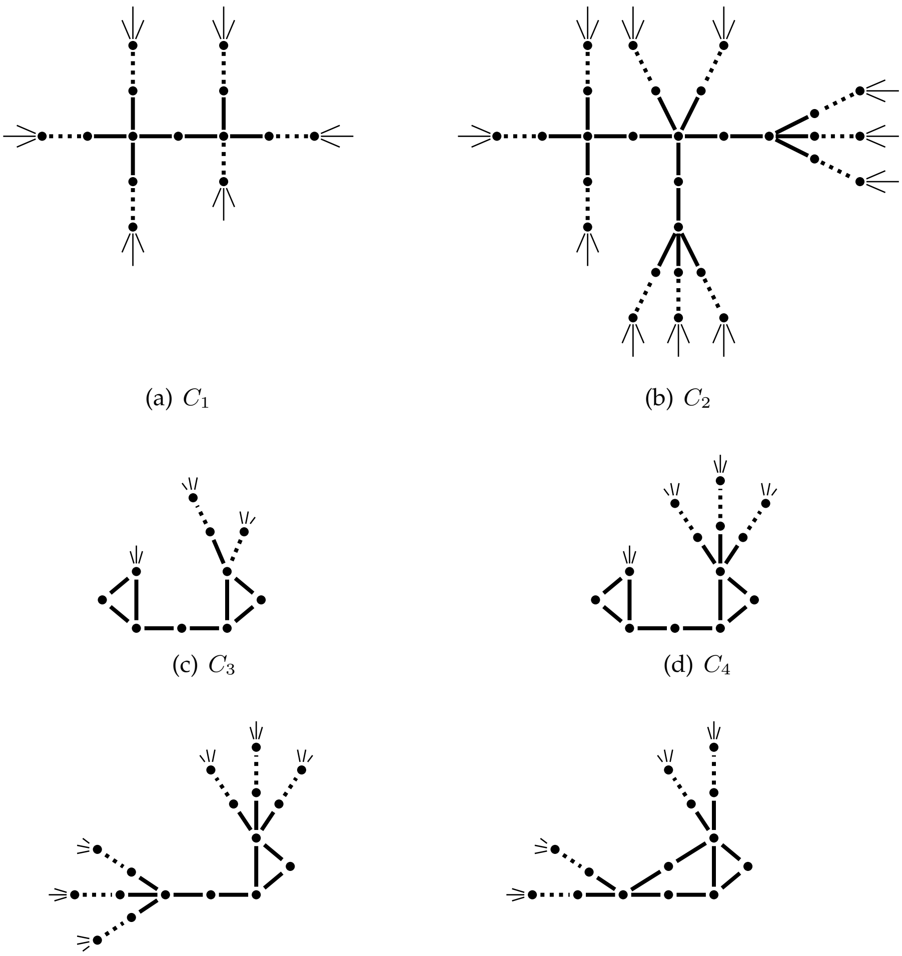


FIGURE 7.9: Supercross

The average degree of a supercross is $12/5$. Thus, its neighbors need to have sufficient large degree. Therefore, we have to forbid some structures in G.

Conjecture 3. *A critical graph G does not contain any of the configurations shown in Figure 7.10.*



(a) C_1 (b) C_2

(c) C_3 (d) C_4

(e) C_5 (f) C_6

FIGURE 7.10: Conjecture 3: Forbidden configurations

To bound the average degree, we have to show that vertices of small degree imply high degree on other vertices. Instead of proving the Conjectures 2 and 3, it is sufficient to show that vertices close to the edge or the configurations have sufficiently large degree.

7.4.4 Forbidden Subgraphs

In the following, we show that some subgraphs do not occur in the critical graph G. The applied method of proof was used by Borodin, Kostochka and Yancey.

We want to show that a graph A is not an induced subgraph of G in such a way that only vertices in a specified subset V_A have neighbors in $G - A$. We call the vertices in V_A *frontier vertices*. For a contradiction, assume that such a subgraph A exists. Let A' be a graph and $\sigma : V_A \to V(A')$ an injection such that $xy \in E(A)$ if and only if $\sigma(x)\sigma(y) \in E(A')$. The set $V'_A := \sigma(V_A)$ are the frontier vertices of A'. We define G' as the graph which occurs when we replace A in G by A' such that a vertex in $G' - A'$ is adjacent to a vertex $\sigma(x) \in V'_A$ if and only if it is adjacent to x in G. We denote this replacement by $\varphi : G \mapsto G'$ and $G - (A - V_A)$ by $\widehat{G}$. The graph $\widehat{G}$ is also a subgraph of G'. The mapping φ is illustrated in Figure 7.11:

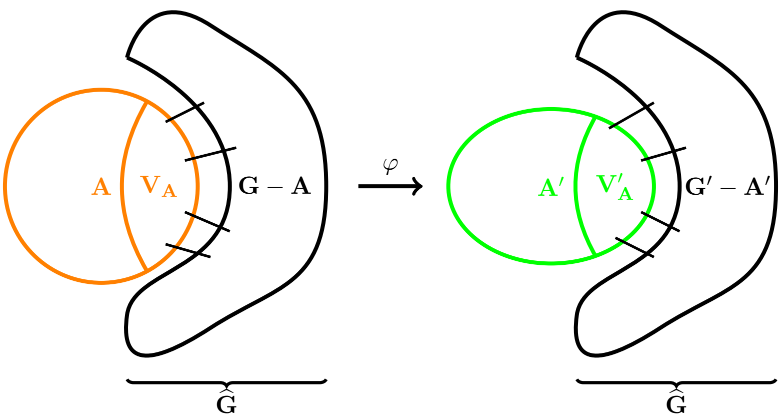


FIGURE 7.11: Mapping $\varphi : G \mapsto G'$

Lemma 44. *With the notations from above, let A be an induced subgraph of a critical graph G. Then A' fulfills at most three of following conditions:*
(1) $p(A) \leq p(A')$
(2) $G' \prec G$
(3) *A $(2,1)$-coloring of $\widehat{G}$ is extendable to G iff it is extendable to G'.*
(4) *Any proper minimal non-$(2,1)$-colorable subgraph H' of G' fulfills $A' \subseteq H'$.*

Proof. Assume for a contradiction that G contains A and that a graph A' with the properties $(1)-(4)$ exists. The graph G' has potential $p(G)-p(A)+p(A')$. This is at least $p(G) \geq 0$ by property (1). The graph G is a smallest (w.r.t. $\prec$) minimal non-$(2,1)$-colorable graph with non-negative potential. Since G' is smaller as G and $p(G') \geq 0$, either G' is $(2,1)$-colorable or it contains a proper non-$(2,1)$-colorable subgraph. The graph G' has no $(2,1)$-coloring c' because $c'|_{\widehat{G}}$ were extendable to G by property (3). Therefore, G'

contains a proper non-$(2,1)$-colorable subgraph. Let H' be a minimal such graph. By property (4), this graph contains A'. The graph $H := \varphi^{-1}(H')$ is a proper subgraph of G. Hence, H has a $(2,1)$-coloring c. This leads to contradiction as $c|_{\widehat{H}}$ is extendable to H' by property (3). □

Borodin, Kostochka and Yancey study non-$(2,1)$-colorable graphs in general. They consider a graph G which is smallest (w.r.t. a certain partial order $\prec$) among all non-$(2,1)$-colorable graphs with non-negative potential. In this case, any graph G' with $p(G') \geq 0$ and $G' \prec G$ has a $(2,1)$-coloring. Thus, showing that for any $(2,1)$-coloring c of G', $c|_{\widehat{G}}$ is extendable to G already gives the contradiction. The conditions (1), (2) (for their order) and the if-part of condition (3) are therefore sufficient.

Nevertheless, one subgraph A' used in the proofs of [5] fulfills all properties of Lemma 44. This graph is employed in Lemma 47. For the further subgraphs A in this subsection, we found graphs A' fulfilling the requirements. All statements in this subsection apply to a critical graph G. We employ the notations from Lemma 44.

Lemma 45. *The base of a superflag has a neighbor outside the superflag.*

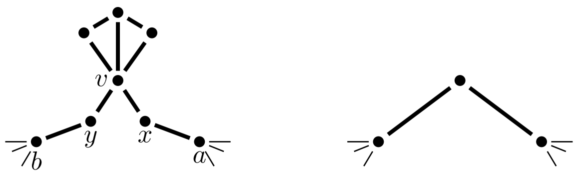
Proof. A superflag B whose base has no neighbors outside B is an induced subgraph where only the secondary base is a frontier vertex. We claim that a flag F with the base as frontier vertex fulfills the properties of Lemma 44. It holds $p(F) = 3 > p(B) = 2$. Furthermore, $G' \prec G$ because $n(G') = n(G) - 2$. Superflags and flags are building blocks. Thus, for G and G' holds that a $(2,1)$-coloring of $\widehat{G}$ is extendable iff the frontier vertex is in no monochromatic edge. Let H' be a proper minimal non-$(2,1)$-colorable subgraph of G'. As every proper subgraph of G is $(2,1)$-colorable, it holds $H' \not\subseteq G$. Since H' is minimal, it contains the entire flag F. Therefore, by Lemma 44, the base of any superflag in G has degree at least six. □

(a) $B \subseteq G$ (b) $F \subseteq G'$

FIGURE 7.12: Superflags are no building blocks

Lemma 46. *Let F be a flag with base v and $d(v) = 5$. Then one neighbor of v outside F has degree at least three.*

Proof. Let x and y be the neighbors of v outside F. By the way of contradiction, assume $N(x) = \{v, a\}$ and $N(y) = \{v, b\}$. As G is minimal non-$(2,1)$-colorable, these four vertices are distinct. Our proof applies Lemma 44 and shows that the P_3 with its end vertices as frontier vertices fulfills the required conditions. The subgraph A is induced by the vertices x, y, a, b and the vertices in F, compare Figure 7.13. The vertices a and b are drawn non-adjacent, which does not hold in general. The graphs P_3 and A both have potential 11 and G is greater than G' by their order. Any $(2,1)$-coloring c of A fulfills $c(x) = c(y) \neq c(v)$. Thus, a coloring of $\widehat{G}$ is not extendable to G if and only if $c(a) \neq c(b)$ and both vertices are in monochromatic edges. The same holds for G'. Condition (4) of Lemma 44 is also fulfilled as any subgraph H' of G' which has minimal degree two and is no subgraph of G contains v' and its edges. Therefore, G cannot contain the subgraph A. □



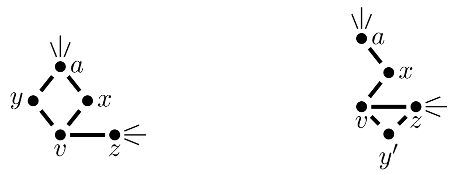
(a) $A \subseteq G$ (b) $P_3 \subseteq G'$

FIGURE 7.13: Pentavalent base of a flag

Remark 11. The graph in Figure 6.9 is a minimal non-$(2,1)$-colorable graph which contains A as a subgraph. Thus, it cannot be critical. In fact, its potential is -2.

Lemma 47. *Let $N(v) = \{x, y, z\}$ and $N(x) = N(y) = \{v, a\}$. Then $a = z$, i.e., the vertices v, x, y and z induce a flag with base z.*

Proof. For a contradiction, we assume that $a \neq z$, i.e., G contains the subgraph $A := G[\{v, x, y, z, a\}]$ as shown in Figure 7.14 (a). The graph A possibly contains the edge az which is not drawn. Replacing the vertex y by a vertex y' which is adjacent to v and z gives a graph A' with $n(A') = n(A)$ and $m(A') = m(A)$. Thus, $p(A') = p(A)$. The vertex y is important and y' is unimportant. All other vertices keep their importance, hence, $G' \prec G$. In both graphs, the non-frontier vertices form a P_3. Therefore, a coloring of $\widehat{G}$ is extendable iff either a and z have different colors or one of them is in no monochromatic edge. A proper subgraph H' of G' which is minimal non-$(2,1)$-colorable contains y', as any proper subgraph of G is $(2,1)$-colorable. It holds $\delta(H') \geq 2$ and no adjacent vertices in H' have both degree two. Thus, the whole subgraph A' is in H'. Together with Lemma 44, we can conclude that the graph A is no induced subgraph of G. □



(a) $A \subseteq G$ (b) $A' \subseteq G'$

FIGURE 7.14: Forbidden subgraph

Together with Conjecture 2, we have following statement:

Corollary 8. *In a critical graph, any trivalent vertex with a bivalent neighbor belongs to a flag or a superflag.*

A configuration represents a set of induced subgraphs with specified frontier vertices. Recall the order-configurations C_1^O and C_2^O in Figure 7.5 in Subsection 7.4.1. We consider all induced subgraphs which they represent and call the vertices as in the picture.

Lemma 48. *The order-configurations do not occur in G if the Conjectures 2 and 3 hold.*

Proof. For a contradiction, we assume that G contains C_1^O. First, consider the subgraphs with $a = b$. They are in no minimal non-$(2,1)$-colorable graph, as any coloring of $G \backslash C_1^O$ were extendable to G. The same holds for $a = c$. Thus, $a \notin \{b, c\}$. Let A be the configuration subgraph of C_1^O together with the vertices a, b and c and their induced edges and A' the graph which arises when we replace the vertex w by a vertex w' with $N(w') = \{a, z\}$. The vertices a, b and c might be adjacent and b and c even identified. The graphs G and G' have the same number of vertices, edges and important vertices. This implies $p(G') = p(G) \geq 0$. If the vertex a has degree three in G, its neighbors outside C_1^O form a semiflag by Corollary 8. Hence, the graph G would contain the configuration C_3 in Figure 7.10 which is conjectured not to happen. Thus, $d_G(a) \geq 4$ and G contains the first order-configuration once more than G'. As G' contains C_2^O at most once more than G, we have $o(G') \leq o(G)$ and $o_1(G') < o_1(G)$. This gives $G' \prec G$. In both graphs G and G', a coloring c of $\widehat{G}$ is not extendable if and only if $c(a) \neq c(b) = c(c)$ and these three vertices are in monochromatic edges of c. A subgraph of G' which is no subgraph of G contains w' and therefore, $A' \subseteq H'$ by $\delta(H') \geq 2$ and Lemma 3. This shows that the configuration C_1^O does not occur in G by means of Lemma 44.
In a similar manner, we prove that G does not contain C_2^O. If C_2^O is in G, we consider the graph G' which arises from replacing w by a vertex w' which is adjacent to a and z. Again, $p(G') = p(G) \geq 0$ and both graphs G and G' have the same order and the same number of important vertices. If $d(a) = 3$, the two further neighbors build a semiflag by Corollary 8. This is the configuration C_4 in Figure 7.10 and believed not to occur in G. Therefore, $d_G(a) \geq 4$ and $o_1(G') = o_1(G)$. If a is not the center of a supercross, then $o_2(G') < o_2(G)$. This gives $o(G') < o(G)$ and thus, $G' \prec G$. Suppose a were in a supercross. The configurations C_5 and C_6 in Conjecture 3 do not appear and hence, w.l.o.g. either a and b are adjacent or $a = b = c$. Both cases are not possible, compare C_2' and C_3' in Lemma 10. Therefore, the graph G' is smaller than G. Let A be the configuraton subgraph of C_2^O together with the vertices a, b, c and d and their edges. These are the frontier vertices whereof some might be identified or adjacent. Both graphs G and G' fulfill that a coloring of $\widehat{G}$ is not extendable if and only if all frontier vertices are in monochromatic edges and a has the same color as exactly two other frontier vertices. Any proper minimal non-$(2,1)$-colorable subgraph H' of G' which is no subgraph of G contains w' and by the properties in Chapter 3 the entire graph A'. The requirements of Lemma 44 are fulfilled and hence, C_2^O is not in G. ∎

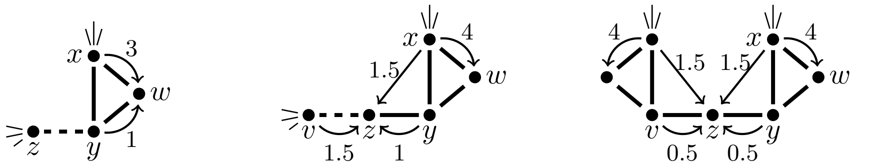
7.4.5 Proof by Discharging

In this section, we consider a graph H which fulfills some of the properties showed and conjectured for G. First, we prove that H has average degree at least $14/5$. Afterwards, we study all graphs with those properties which have average degree exactly $14/5$. We show a $(2,1)$-coloring of these graphs. Therefore, H has average degree greater than $14/5$. The properties of H hold for critical graphs if the Conjectures 2 and 3 are true. Hence, these conjectures yield Conjecture 1.

Let H be a graph which fulfills the following properties:
(i) Connectivity and absence of separating edges.
(ii) No adjacent bivalent vertices.
(iii) Any trivalent vertex has a neighbor of degree at least three.
(iv) H does neither contain the configurations C_1', C_4', C_5', C_6' and C_7' in the Lemmas 8 and 10, nor the order-configurations, nor the configurations C_1 and C_2 of Conjecture 3.
(v) (Super)flags only intersect in an entire flag.
(vi) Any trivalent vertex with a bivalent neighbor belongs to a (semi)flag.
(vii) The base of a flag has degree at least five. If so, it has at most one bivalent neighbor outside the flag.
$(viii)$ In a superflag, the base has degree at least six and the secondary base degree at least three.

Now, let us prove that H has average degree at least $14/5$. We apply the method of discharging. Let $\mu(v) := 5d(v) - 14$ be the initial charge of each vertex $v \in V(H)$. The total charge of all vertices in H is non-negative iff the average degree of H is at least $14/5$. The following rules give the final charge $\mu^*(v)$:
(1) In a flag, the vertices of degree two get each charge 4 from the base.
(2) In a superflag, the base gives charge 4 to all bivalent vertices and charge 1.5 to the secondary base.
(3) In a semiflag on the vertices w, x and y whereof w is bivalent and y trivalent, we shift as follows:
(3.1) If z, the neighbor of y outside the triangle, has degree at least three, w gets charge 3 from x and charge 1 from y.
(3.2) If z has degree two, x gives charge 4 to w and charge 1.5 to z. Furthermore, z receives charge 1 from y and charge 1.5 from its second neighbor.
In the case that both neighbors of z are trivalent and in a semiflag, x charges as above but z only receives charge 0.5 from each neighbor.
Figure 7.15 illustrates this discharging rule.
Consider the case that x is the base of a flag, i.e., the flag and the semiflag build a superflag. If $d(z) \geq 3$, then we charge as in a superflag, i.e., the bivalent vertex w receives charge 4 from x and no charge from y and y receives charge 1.5 from x. If $d(z) = 2$, we charge among rule (3.2).
(4) The bivalent vertices in a supercross get charge 1.5 from the center and charge 2.5 from their second neighbor (or from x and y together if the conditions of Rule (3.2) are fulfilled).
(5) Vertices of degree two which do not belong to a flag, a superflag, a supercross or a configuration as in (3), receive charge 2 from both neighbors.

These discharging rules are well-defined as superflags and flags only intersect in an entire flag and no vertex sits in two supercrosses, see configuration C_1.



(a) Configuration C_7: $d(z) \geq 3$

(b) Config. C_8: $d(z) = 2$ It holds $v \neq x$.

(c) Configuration C_9: Special case of C_8

FIGURE 7.15: Discharging rule (3)

We want to show that all vertices in H have non-negative final charge. First, let us consider vertices which belong to C_7 or C_8.

Lemma 49. *If a vertex belongs to exactly one configuration isomorphic to C_7 or C_8 and to no flag or superflag, it has final charge at least zero.*

Proof. We call the vertices as in the picture. First, consider the configuration C_7. The vertex w receives in total charge 4 from its neighbors. Thus, $\mu^*(w) = 5 \cdot 2 - 14 + 4 = 0$. The vertex y has initial charge 1 and therefore final charge 0. Outside the configuration, only bivalent neighbors receive charge. They get charge 2.5 if they are in a supercross and charge at most 2 otherwise.
The vertex x has degree at least four as the configuration C_1' is not in H. If $d(x) = 4$, its initial charge is 6. The absence of C_4' implies that one neighbor of x outside C_7 is not bivalent and thus gets no charge from x. Therefore, $\mu^*(x) \geq 5 \cdot 4 - 14 - 3 - 2.5 = 0.5$. If $d(x) \geq 5$,

$$\mu^*(x) \geq 5d(x) - 14 - 3 - 2.5(d(x) - 2) = 2.5d(x) - 12 \geq 0.5.$$

If the vertex z has degree three, either none of its neighbors is bivalent and therefore, $\mu^*(z) = 1$, or the two neighbors form a semiflag by property (vi). In this case, x is in two configurations isomorphic to C_7. If $d(z) = 4$ and all neighbors outside C_7 are bivalent, none of them can be in a supercross as this would contain C_1. Thus, it gives at most charge $\max\{3 \cdot 2, 2 \cdot 2.5\} = 6$ and has final charge at least 0. If $d(z) \geq 5$, the final charge is at least

$$\mu^*(z) = 5d(z) - 14 - 2.5(d(z) - 1) = 2.5d(x) - 11.5 \geq 1.$$

In C_8, the vertices w, y and z have final charge 0. If $d(x) = 4$, it has no bivalent neighbor outside C_8 as this were C_1^O. Hence, $\mu^*(x) = 6 - 4 - 1.5 = 0.5$. If $d(x) = 5$, at most two neighbors outside C_8 have degree 2 since C_2^O is not in H. It follows $\mu^*(x) \geq 11 - 4 - 1.5 - 2 \cdot 2.5 = 0.5$. If $d(x) \geq 6$, the vertex has final charge

$$\mu^*(x) \geq 5d(x) - 14 - 4 - 1.5 - 2.5(d(x) - 2) = 2.5d(x) - 14.5 \geq 0.5.$$

Consider v, the second neighbor of z. If it is trivalent, it sits in a semiflag since $d(z) = 2$. In this case, v belongs to two configurations C_8. If $d(v) = 4$ and all its neighbors are bivalent, it has final charge $6 - 4 \cdot 1.5 = 0$ by discharging rule (4). If $d(v) = 4$ and two further neighbors are bivalent, none of them is in a supercross, as this gave C_1. Thus, $\mu^*(v) \geq 6-1.5-2\cdot 2 = 0.5$. If at most one further neighbor is bivalent, $\mu^*(v) \geq 6-1.5-2.5 = 2$. If v has five bivalent neighbors, the absence of C_2 shows that at most two of its neighbors are in supercrosses. Therefore, v gives at most charge $2\cdot 2.5+2\cdot 2 = 9$ to the vertices outside C_8. It follows $\mu^*(v) \geq 11-9-1.5 = 0.5$. If $d(v) = 5$ and v has a neighbor of degree at least three, $\mu^*(v) \geq 11 - 3 \cdot 2.5 - 1.5 = 2$. If v has degree greater than five,

$$\mu^*(v) \geq 5d(v) - 14 - 1.5 - 2.5(d(v) - 1) = 2.5d(v) - 13 \geq 2.$$

□

Remark 12. The vertex x has strictly postive final charge.

Lemma 50. *A vertex which belongs to at least two configurations isomorphic to C_7 or C_8 and to no flag or superflag has non-negative final charge.*

Proof. First, we treat the vertices w, y and z in configuration C_7. As $d(x) \geq 4$, the vertex w does not belong to any further configuration isomorphic to C_i for an $i \in \{7, 8\}$. The vertex z has a neighbor of degree 3 in the configuration and gives no charge. The same would happen if z and zy were in no configuration C_7. Thus, if z belongs to another configuration C_i, we can treat z as in Lemma 49. The vertex y can only belong to a second configuration C_i if z is trivalent and in a semiflag. In the second configuration, y does not give charge to any other vertex. Therefore, $\mu^*(y) \geq 0$.
Now, consider the configuration C_8. The vertex w again cannot belong to any other configuration C_i. The vertex v gives charge 1.5 to its bivalent vertex in the configuration. If v were in no C_8, it gave charge at least 1.5 to a bivalent neighbor. Hence, we can treat v as in Lemma 49. No bivalent vertices are adjacent in G. Thus, the vertices y and z are only in a further configuration C_i if v is trivalent and in a semiflag. This is the special case of rule (3.2), i.e., C_9 in Figure 7.15. One can see that any vertex without neighbors outside C_9 has non-negative final charge.
Lastely, let us consider x, i.e., the case that a vertex belongs to at least two semiflags. Let k be the number of its semiflags and $l := d(x) - 2k$ the number of its further neighbors. If $k = 2$, the absence of C_5' gives $d(x) \geq 5$. If both triangles belong to a configuration of type C_7, the vertex x has final charge at least

$$\mu^*(x) \geq 5d(x) - 14 - 2 \cdot 3 - 2.5l = 2.5d(x) - 10 \geq 2.5.$$

If one triangle belongs to C_8 and $d(x) = 5$, the neighbor of x outside the configuration has degree greater than two as otherwise, this were the configuration C_6'. Thus, $\mu^*(x) \geq 11 - 2 \cdot (4 + 1.5) = 0$. If $d(x) \geq 6$, it follows

$$\mu^*(x) \geq 5d(x) - 14 - 2 \cdot (4 + 1.5) - 2.5l = 2.5d(x) - 15 \geq 0.$$

If $k = 3$, the final charge of x is

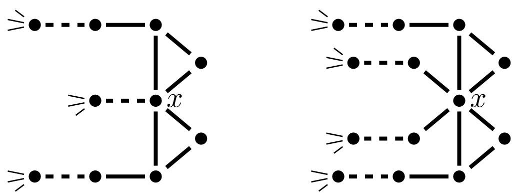
$$\mu^*(x) \geq 5d(x) - 14 - 3 \cdot (4 + 1.5) - 2.5l = 2.5d(x) - 15.5 \geq 2.$$

The last inequality follows from $l \geq 1$, see configuration C_7'. If $k \geq 4$,

$$\mu^*(x) \geq 5(2k + l) - 14 - (4 + 1.5)k - 2.5l = 4.5k + 2.5l - 14 \geq 4.$$

□

Remark 13. In a semiflag, which does not intersect with a flag or a superflag, the vertex with degree at least four has only final charge zero if it belongs to one of the configurations in Figure 7.16. If this vertex x fulfills $\mu^*(x) = 0$ in C_{11}, the bivalent neighbors of x, that have a neighbor outside the configuration subgraph, are in supercrosses.



(a) C_{10} (b) C_{11}

FIGURE 7.16: Configurations with $\mu^*(x) = 0$

Lemma 51. *Vertices in flags and superflags have non-negative final charge.*

Proof. Consider a flag on the vertices x, y, z and v whereof x is the vertex of degree three, y and z are bivalent and v is the base. It holds $\mu^*(x) = \mu(x) = 1$ and $\mu^*(y) = \mu^*(z) = 0$. If the flag does not belong to a superflag and thus, v in particular not to a semiflag, the base has final charge at least 0.5 because by property (vii), either $d(v) = 5$ and $\mu^*(v) \geq \mu(v) - 2 \cdot 4 - 2.5 = 0.5$ or the base has degree at least six and hence,

$$\mu^*(v) \geq 5d(v) - 14 - 2 \cdot 4 - 2.5(d(v) - 3) = 2.5d(v) - 14.5 \geq 0.5.$$

Now, let the base v belong to a superflag, i.e., to $k \geq 1$ triangles with a bivalent vertex, which might be semiflags. The base gives charge $4 + 1.5 = 5.5$ to each such configuration. In the case $k = 1$, the fact $d(v) \geq 6$ gives

$$\mu^*(v) \geq 5d(v) - 14 - 2 \cdot 4 - (4 + 1.5) - 2.5(d(v) - 5) = 2.5d(v) - 15 \geq 0.$$

If v is in $k \geq 2$ triangles with a bivalent vertex, we have

$$\begin{aligned} \mu^*(v) &\geq 5d(v) - 14 - 2 \cdot 4 - (4 + 1.5)k - 2.5(d(v) - 3 - 2k) \\ &= 2.5d(v) - 14.5 - 0.5k \geq 4.5k - 7 \geq 2. \end{aligned}$$

The second inequality follows from $d(v) \geq 2k + 3$. The vertices of degree two in the superflag receive charge 4 in every case and thus, all have final charge 0. If the secondary base w has degree three, it is the trivalent vertex of a semiflag. Hence, it receives at least as much charge as if the semiflag would not belong to a superflag and therefore, has non-negative final charge. If the secondary base has degree at least four and sits in no semiflag outside the superflag, it has final charge

$$\mu^*(w) \geq 5d(w) - 14 + 1.5 - 2.5(d(w) - 2) = 2.5d(w) - 7.5 \geq 2.5.$$

If the secondary base is in $k \geq 1$ semiflags outside the superflag, it has final charge

$$\begin{aligned}\mu^*(v) &\geq 5d(v) - 14 + 1.5 - (4 + 1.5)k - 2.5(d(v) - 2 - 2k) \\ &= 2.5d(v) - 7.5 - 0.5k \geq 4.5k - 2.5 \geq 2.\end{aligned}$$

This follows from $d(v) \geq 2k + 2$. □

Remark 14. The trivalent vertex in a (super)flag has positive final charge.

With these lemmas, we show that no vertex in H has negative final charge and hence, the average degree of H is at least $14/5$.

Lemma 52. *Every vertex v in a graph H with the properties $(i) - (viii)$ has non-negative final charge.*

Proof. If the vertex v belongs to a flag, a superflag, a configuration C_7 or C_8, it has non-negative final charge by the Lemmas 49, 50 and 51. Now, consider a vertex which does not belong to any of these configurations. Such a vertex gives only charge to bivalent vertices. If $d(v) = 2$, it gets in total charge 4 from its neighbors. As no two vertices of degree two are adjacent, v has final charge 0. If v has degree three, none of its neighbors is bivalent by property (vi). Thus, v has final charge 1. Let $d(v) = 4$. Either v gives only charge 1.5 to every neighbor or there is a neighbor of v which has degree at least three. If v gives charge 2.5 to a neighbor, it has at most two neighbors of degree two as C_1 does not arise in H. Thus,

$$\mu^*(v) \geq \mu(v) - \max\{4 \cdot 1.5, 3 \cdot 2, 2 \cdot 2.5\} = 0.$$

If v has degree five and gives at most charge 2 to every neighbor, it fulfills

$$\mu^*(v) \geq \mu(v) - 2d(v) = 11 - 10 = 1.$$

If v has degree five and gives charge 2.5 to a neighbor, then the absence of C_2 implies that it gives in total at most charge

$$\max\{3 \cdot 2.5 + 1 \cdot 2, 2 \cdot 2.5 + 3 \cdot 2\} = 11 = \mu(v).$$

Vertices of degree at least six fulfill

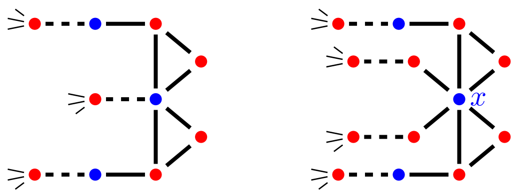
$$\mu^*(v) \geq 5d(v) - 14 - 2.5d(v) \geq 1.$$

□

Let us consider a graph $\widetilde{H}$ with the properties $(i)-(viii)$ such that each vertex has final charge 0. We will show that such a graph is $(2,1)$-colorable.

Lemma 53. *A graph $\widetilde{H}$ with the properties $(i)-(viii)$ and $\mu^*(v) = 0$ for all $v \in V(\widetilde{H})$ is $(2,1)$-colorable.*

Proof. The Remarks 12, 13 and 14 show that the graph $\widetilde{H}$ contains neither flags, nor superflags, nor C_7. If it contains C_8, this configuration belongs to C_{10} or C_{11} in Figure 7.16. The proof of Lemma 52 shows that any vertex v in $\widetilde{H}$, which is not in C_{10} or C_{11}, has degree two, four or five. Let $\widetilde{H}'$ be the subgraph of $\widetilde{H}$ which arises by deleting all configurations C_{10} and C_{11}.
Claim: Coloring the vertices in $\widetilde{H}'$ in color 1 if they are bivalent in $\widetilde{H}$ and in color 2 otherwise gives a $(2,1)$-coloring of $\widetilde{H}'$ which is extendable to $\widetilde{H}$.
First, we show that this is a $(2,1)$-coloring of $\widetilde{H}'$. By property (ii), $\widetilde{H}[V_2]$ is a stable set, i.e., the vertices in the first color class are independent. Furthermore, the final charge of a vertex v of degree five is only zero if all its neighbors are bivalent because $\mu(v) = 11 > 4 \cdot 2.5$. Thus, every vertex of degree five has no neighbor of color 2. If $d(v) = 4$ and $\mu^*(v) = 0$, the vertex v has at least three neighbors of degree two. Hence, in $\widetilde{H}'$, each component in color class 2 contains at most two vertices.
If the configuration C_{10} occurs in $\widetilde{H}$, we $(2,1)$-color a vertex in its configuration subgraph in color 1 if and only if it belongs to a dashed edge. For all these vertices, the neighbor outside the configuration subgraph is not bivalent and has thus color 2, see Figure 7.17 (a). If C_{11} is in $\widetilde{H}$, color the vertices as in Figure 7.17 (b). The neighbors of the configuration subgraph have all degree at least four and hence, color 2. We know from Remark 13 that the vertices of distance two to x, which do not belong to the configuration subgraph, are centers of supercrosses. Thus, they have no further neighbor in $V_4 \cup V_5$ and the colorings in the picture extend above coloring to the entire graph $\widetilde{H}$. $\square$



(a) C_{10} (b) C_{11}

FIGURE 7.17: $(2,1)$-Coloring of C_{10} and C_{11}

Theorem 15. *Non-$(2,1)$-colorable graphs with the properties $(i)-(viii)$ have average degree strictly greater than $14/5$.*

A critical graph G is non-$(2,1)$-colorable. We showed and conjectured that it fulfills the properties of H. This gives the following:

Theorem 16. *If Conjectures 2 and 3 hold, every minimal non-$(2,1)$-colorable graph has average degree strictly greater than $14/5$.*

Proof. We can conclude from Conjecture 2 and Conjecture 3, that a critical graph G fulfills the properties $(i) - (viii)$. By Theorem 15, G has average degree greater than $14/5$. This contradicts $p(G) \geq 0$. Hence, there are no critical graphs. The binary relation "$\prec$" is a partial order on the set of all graphs. This poset has a least element. The set of all minimal non-$(2,1)$-colorable graphs with non-negative potential contains no smallest elements. Hence, it is the empty set. It follows that minimal non-$(2,1)$-colorable graphs have negative potential and their average degree is strictly greater than $14/5$. $\square$

7.5 Remark on "On 1-Improper 2-Coloring of Sparse Graphs"

In the previous section, we employed different ideas and results of the article "On 1-improper 2-coloring of sparse graphs" [5] from Borodin, Kostochka and Yancey. This is an important paper in the field of improper colorings. It proves that the infimum of the maximum average degree of non-$(2,1)$-colorable graphs is $14/5$ which was conjectured in 1994 by Kurek and Ruciński [15]. The writing of the thesis at hand required a careful study of this paper. Doing so, we noticed a few formal incorrectnesses. Apart from minor printing errors, we detected two mathematical mistakes which are presented in this section. Nevertheless, all main statements of the paper hold. The incorrectnesses have no substantial impacts on the proofs. Moreover, we give suitable rectifications.

7.5.1 On the Partial Order

Comparable to the proof in Section 7.4, the authors work with a partial order on the set of all graphs. This order is defined differently from our order $\prec$. To prevent confusion, we denote this order by $\prec'$ albeit the article uses $\prec$. It is stated in the paper that any proper subgraph H of G fulfills $H \prec' G$. We assume that the authors refer to proper induced subgraphs as there are non-induced subgraphs which are greater than G. This subsection introduces the order $\prec'$, gives examples for greater subgraphs but also a proof that proper induced subgraphs are indeed smaller. All conclusions in the paper follow already from this restricted statement.

Flags, superflags, bases and secondary bases are defined as in Subsection 7.4.2. However, unimportant and important vertices are characterized slightly different and the authors additionally call some vertices semi-important:

Definition 15. We call a vertex v *unimportant* if $d(v) \leq 1$ or $d(v) = 2$ and v is in a triangle or $d(v) = 3$ and v is in a flag. A vertex of degree two which is not in a triangle is called *semi-important*. All other vertices are *important*. The numbers $n_u(G), n_s(G)$ and $n_i(G)$ denote the number of unimportant, semi-important and important vertices in a graph G.

Definition 16. A graph H is *smaller* than a graph G (and G *greater* than H), write $H \prec' G$, if one of following conditions holds:
(1) $n_i(H) < n_i(G)$
(2) $n_i(H) = n_i(G) \ \wedge \ n_s(H) < n_s(G)$
(3) $n_i(H) = n_i(G) \ \wedge \ n_s(H) = n_s(G) \ \wedge \ n_u(H) < n_u(G)$
(4) $n_i(H) = n_i(G) \ \wedge \ n_s(H) = n_s(G) \ \wedge \ n_u(H) = n_u(G) \wedge$
$\sum_{v \in V(H)} d(v)^2 > \sum_{v \in V(G)} d(v)^2$

Following proper non-induced subgraphs are greater w.r.t. $\prec'$:

- $G - E'$ for any edge subset $E' \neq \emptyset$ such that the importance of all vertices is the same in G and $G - E'$ (by condition (4)).
- $G - xy$ if x and y are the neighbors of an unimportant bivalent vertex and do not have degree three in G. In $G - xy$, the bivalent vertex is semi-important and x and y stay (un)important.
- $G - e$ for an edge e in a flag which contains the base. The trivalent vertex has a higher importance in $G - e$ and all other vertices at least the same.

Figure 7.18 illustrates the second and the third case. The yellow vertices are unimportant, the orange vertices semi-important and the red vertices important.

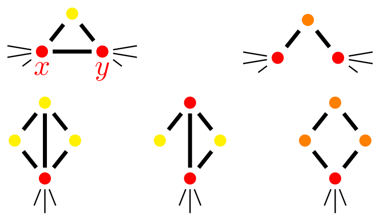


FIGURE 7.18: Greater non-induced subgraphs

Nevertheless, the restiction to proper subgraphs gives $H \prec' G$:

Lemma 54. *Any proper induced subgraph H of G fulfills $H \prec' G$.*

Proof. We may assume w.l.o.g. that $n(H) = n(G) - 1$. We claim that deleting one vertex either does not increase the importance of any vertex or that the deleted vertex was important and we just increased the importance of one other vertex from unimportant to semi-important. As $n(H) < n(G)$, this claim implies the lemma. For all vertices $v \in V(H)$, it holds $d_H(v) \leq d_G(v)$. If a vertex has degree at most one in G or is semi-important, it is not more important in H. If v is already important in G, it cannot be more important in H either.
Thus, we only have to consider bivalent vertices in a triangle and trivalent vertices in a flag. Their importance can only be higher in H if the graph H

does not contain the triangle respectively the flag. Removing another vertex of the triangle decreases the degree of the regarded vertex. Hence, this vertex has degree one in H and is unimportant in both graphs. Let us consider a trivalent vertex v in a flag. Its importance can only be influenced if H does not contain one of the bivalent vertices in the flag or the base. In the first case, v is bivalent in H and sits in a triangle. Therefore, it is also unimportant in H. In the second case, v becomes semi-important. However, the base is an important vertex in G and thus $H \prec' G$. □

Let us now analyze why Lemma 54 is sufficient to conclude the statements in the paper. The article treats graphs with potential at least zero. Differently from Section 7.4, the potential of a graph is defined as

$$\rho_G := \min_{H \subseteq G,\ H \text{ induced}} 7n(H) - 5m(H).$$

A graph has non-negative potential if and only if its maximum average degree is at most $14/5$. Similarly to the considerations in this thesis, the authors work with a graph G which is non-$(2,1)$-colorable, has non-negative potential and is smallest w.r.t. $\prec'$ with these properties.

Lemma 55. *Every subgraph $H \subseteq G$ has non-negative potential. If furthermore $n(H) < n(G)$, the graph H is $(2,1)$-colorable.*

Proof. The potential is chosen minimal over all subgraphs. This gives directly $\rho_H \geq \rho_G \geq 0$. Let $H_i := G[V(H)]$ be the induced subgraph on the vertices in H. If $n(H) < n(G)$, i.e., $H_i \neq G$, the choice of G and Lemma 54 yield that H_i is $(2,1)$-colorable and hence, also its subgraph H is so. □

If $V(H) = V(G)$, H is not necessarily smaller than G and hence, possibly non-$(2,1)$-colorable. Thus, G might have proper non-$(2,1)$-colorable subgraphs. Nevertheless, Lemma 55 is sufficient to show that G fulfills the properties specified in Chapter 3 of [5], as for instance 2-edge-connectivity. Also the further statements in the article can be concluded from this lemma.

7.5.2 Lemma 25

In Lemma 25, the authors state that any two distinct vertices in G, which are bases or secondary bases and do not belong to the same superflag, have distance at least three. In the proof, a graph consisting of two flags or superflags A_1 and A_2 and a shortest path P between their bases or secondary bases is considered. It is shown that this graph has potential at most three if P has length smaller than three. The authors assume for a contradiction, that G contains such a graph as a subgraph. Lemma 13 in the paper says that every non-empty proper induced subgraph with potential at most two is a superflag. The graph $A_1 + P + A_2$ is $(2,1)$-colorable and hence a proper subgraph of G. Furthermore, it is induced, non-empty and no superflag. Thus, it has potential three. Lemma 23 states that a subset of potential three is either a flag of has at least $n(G) - 1$ vertices. Only flags and

$A_1 + P + A_2 = G$ are considered in the article. One might add the case $n(A_1 + P + A_2) = n(G) - 1$ to the proof: we know from Lemma 13, that $A_1 + P + A_2$ has potential three. This is only possible if A_1 and A_2 are flags and P has length two. Suppose there is a vertex v outside $A_1 + P + A_2$. As $\rho_G \geq 0$ and $\delta(G) \geq 2$, the vertex v has exactly two neighbors. These neighbors are vertices of P. Both such graphs are $(2,1)$-colorable, compare Figure 7.19. Hence, the lemma holds.

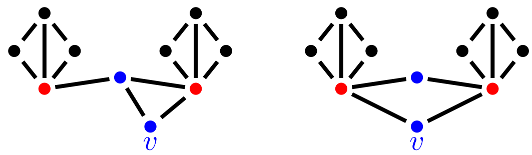


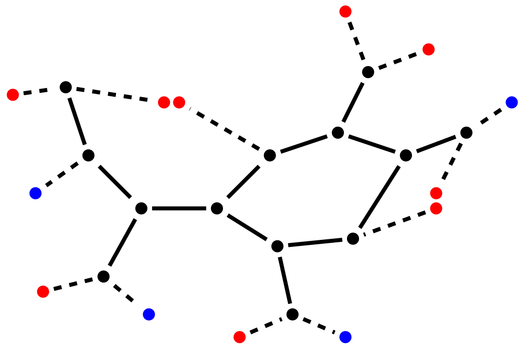
FIGURE 7.19: Lemma 25, Case $A_1 + P + A_2 = G - v$

Appendix A

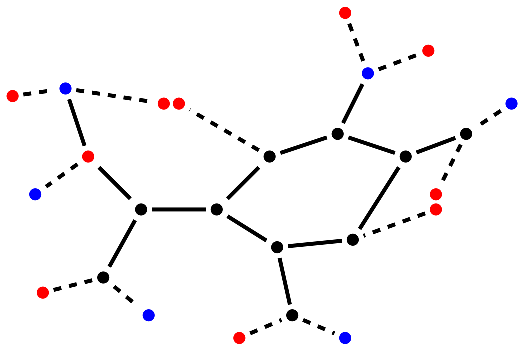
Illustration of Lemma 5

Figure A.1 demonstrates the proof of Lemma 5. The picture shows G' together with its neighborhood $N(G')$ in $\widehat{G}$ or G. Due to the comparatively high number of vertices, we do not use the usual representation of configurations. G' is drawn as a graph with continous edges. Furthermore, the vertices in $N(G')$ are joint to their neighbors in G' by dashed edges. Recall that these vertices are bivalent and that their edges to vertices outside G' are dichromatic in $\hat{c}$. Two leaves in $\widehat{G}$ which replace the same vertex in $V_2' \subseteq V(G)$ are drawn close to each other.

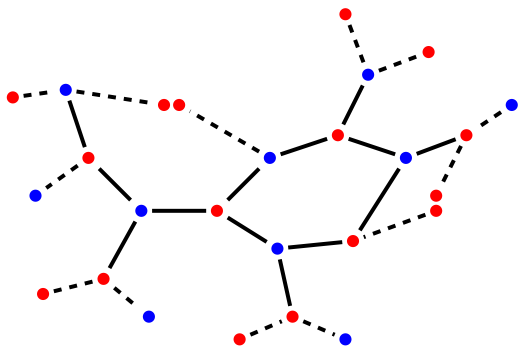
Subfigure (a) shows the coloring $\hat{c}|_{\widehat{G}-G'}$. Red represents color 1 and blue color 2. The black vertices are $V(G')$. The second picture shows the colors of all vertices in $\widehat{G}'$. We can see that the vertices which are colored in the second subfigure but not in the first are in no monochromatic edge. A vertex is black in Subfigure (b) if no color occurs more than once in its neighborhood. The Subfigures (c) and (d) show the $(2,1)$-colorings of $\widehat{G}$ and G.



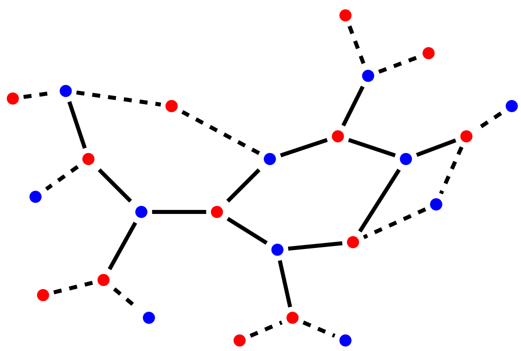
(a) Coloring of $\widehat{G} - G'$



(b) Coloring of $\widehat{G}'$



(c) Coloring of $\widehat{G}$



(d) Coloring of G

FIGURE A.1: Exemplification of the proof of Lemma 5

List of Symbols

G	Graph	9
$V(G)$	Vertex set	9
$E(G)$	Edge set	9
$n(G)$	Number of vertices; $\|V(G)\|$	9
$m(G)$	Number of edges; $\|E(G)\|$	9
$d_G(v), d(v)$	Degree of v	10
$N_G(v), N(v)$	Neighborhood of v	10
$N_G(V'), N(V')$	Neighborhood of $V' \subseteq V(G)$	10
$N_G(G'), N(G')$	Neighborhood of $V(G')$ for $G' \subseteq G$	10
$\delta(G)$	Minimum degree	10
$\Delta(G)$	Maximum degree	10
$\mathrm{ad}(G)$	Average degree	10
$\mathrm{mad}(G)$	Maximum average degree	10
$G[V']$	Graph induced by $V' \subseteq V(G)$	9
$G[E']$	Graph with edge set $E' \subseteq E(G)$	9
$G - V'$	Graph induced by $V(G)\backslash V'$	9
$G - G'$	Graph induced by $V(G)\backslash V(G')$	9
$G - E'$	Graph without $E' \subseteq E(G)$	9
$G + F$	Graph with added edge set F	9
$G - v$	Graph induced by $V(G)\backslash\{v\}$	9
$G - e$	Graph without $e \in E(G)$	9
$G + f$	Graph with added edge f	9
$\overline{G}$	Complement graph	9
$G_1 \cup G_2$	Graph union	9
$G_1 \dot{\cup} G_2$	Disjoint graph union	9
$G_1 \cap G_2$	Graph intersection	9
$G_1 + G_2$	Graph join	9
P_k	Path on k vertices	9
C_k	Cycle on k vertices	9
K_k	Clique on k vertices	9
K_{n_1,n_2}	Complete bipartite graph	9
S_k	Star on k vertices	9
W_k	Wheel on k vertices	9
$\chi(G)$	Chromatic number	3
$\chi_j(G)$	j-Defective chromatic number	3
$\mathcal{G}$	Set of minimal non-$(2,1)$-colorable graphs	5
m^*	$(2,1)$-Density of minimal graphs	57
$\lambda(G)$	Minimum size of an edge cut	15
$\mathbb{N}$	Natural numbers (including 0)	17

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