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Greedoids: Basic Theory and Illustrative Examples

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Greedoids:

Basic Theory and Illustrative Examples

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Abstract

A greedoid is a combinatorial structure which arose from another combinatorial structure, the matroid. In this thesis, we go through some basic greedoid theory supplemented by various examples. After a short introduction to the concept, we look at a greedoid both in terms of a set system and in terms of a formal language. Later on, we explore various classes such as the Gaussian greedoids and the interval greedoids, among which, for example, matroids and antimatroids are included. In the final part, we examine the rank function and the closure operator of a greedoid more closely. Further, the operations known as truncation, restriction and contraction are presented and finally, the relationship between greedoids and optimization problems is briefly discussed.

Acknowledgements

I would like to thank my supervisor Paul Vaderlind for his suggestion of this topic, which gave me the opportunity to discover and learn about this interesting and rewarding field of combinatorics.

Contents

1 Introduction

To introduce the greedoid concept in an instructive manner, it seems natural to start with some words of the matroid concept from which it originated. Matroids were invented in 1935 by Hassler Whitney as a certain combinatorial structure. Later, in 1959, modern matroid theory started when William Thomas Tutte established deep connections between matroid theory, graph theory and matrix algebra. The utility of matroids has then lead to several generalizations of the concept [1]. A formal definition will be presented later in the text but in short one can say that a matroid structure generalizes linear independence in vector spaces.

However, Otakar Borůvka presented a structure equivalent to matroids back in 1926 using an algorithmic approach. This is interesting since the algorithmic way of looking at matroids, which is certainly not the most frequent way, leads us to the greedoid concept. It turned out that a greedy algorithm gives optimal solutions to combinatorial problems satisfying the properties of matroids. However, in 1981, Bernhard Korte and László Lovász observed that often the optimality of a greedy algorithm not needed the combinatorial structure of a matroid. Instead, another combinatorial structure was sufficient, and by combining the words greedy and matroid they named it a greedoid [1]. So, a greedoid is a generalization of a matroid, and as such it can be distinguished from the latter by being modeled on the algorithmic construction of certain sets, thus making the ordering of elements in a set important [2].

With a couple of different graphs as our main guiding examples, this thesis will introduce some basic theory and properties of the greedoid concept. We will however not limit this presentation of greedoids to its connections with graphs, but also include different applicabilities where it is appropriate.

2 Definitions and guiding examples

In this part we will see that a greedoid can be defined in different ways, and we will also prove the equivalence of these definitions. Further, a couple of graphs will be introduced that will recur along the way. The purpose is to visualize clearly how differences among these lead to different classifications of greedoids. We assume that the reader is familiar with basic graph theory terms, and if not, a useful glossary can be found at [4]. Theory and use of notation are taken from [1] and [2] throughout this thesis unless otherwise is stated.

2.1 Greedoids in terms of set systems

As the title here tells us, greedoids can be defined as set systems. Therefore, lets first of all be clear of what we mean with a set system.

Definition 2.1 *We say that a pair $(E, \mathscr{F})$, where E is a nonempty finite set and $\mathscr{F}$ is a family of subsets of E, is a* **set system**. *Further, we will denote the set of all subsets of E by 2^E.*

Example 2.1 Let $T = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \big\{\{a, c\}, \{b, c\}, \{c, d\}\big\}$ be the tree in Figure 1 below. Further, let $\mathscr{F} = \Big\{\{\emptyset\}, \{a, c\}, \{b, c\}, \{c, d\}, \big\{\{a, c\}, \{b, c\}\big\}, \big\{\{a, c\}, \{c, d\}\big\}, \big\{\{b, c\}, \{c, d\}\big\}\Big\}$ be a family of subsets of the set of edges E. Then, the pair $(E, \mathscr{F})$ is a set system.

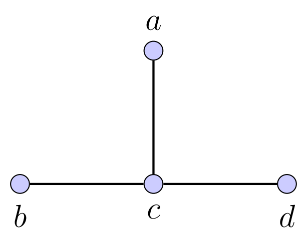


Figure 1

Now that we have become acquainted with the definition as well as an example of a set system, we take a look at what turns a set system into a greedoid.

Definition 2.2 *We say that a* **greedoid** *is a set system* $(E, \mathscr{F})$ *satisfying*

(G1) $\emptyset \in \mathscr{F}$;
(G2) *If* $X, Y \in \mathscr{F}$ *and* $|X| > |Y|$, *then there is an* $x \in X \setminus Y$ *with* $Y \cup x \in \mathscr{F}$.

As can be verified, the pair $(E, \mathscr{F})$ in Example 2.1 satisfies both (G1) and (G2), so not only is it a set system but also a greedoid. However, Definition 2.2 is not the only way to define a greedoid as a set system. It is possible to drop the axiom (G1) but first we have to define what an accessible set system is.

Definition 2.3 *Given a set system* $(E, \mathscr{F})$, *we say that a set in* $\mathscr{F}$ *is* **feasible** *and a set not in* $\mathscr{F}$ *is* **infeasible**. *An* **accessible set system** *is a set system where there exists some element* x *in every nonempty feasible set* X *with* $X \setminus x \in \mathscr{F}$.

From this, we see that for every nonempty feasible set in $\mathscr{F}$ we can successively remove elements and get new feasible sets. For a new feasible set the cardinality will be strictly less than the cardinality of the previous feasible set and since E is finite we will eventually arrive at the conclusion that the empty set must be in $\mathscr{F}$ as well. Thus, (G1) follows from the accessible property and therefore an accessible set system satisfying (G2) meets the criteria of a greedoid. In fact, every greedoid is an accessible set system. We can see this if we start with the empty set, which is feasible, and then apply (G2) repeatedly to get a sequence of feasible sets where the cardinality of these sets increases by one for each repetition. This knowledge is very useful since it turns out that (G2) also can be replaced from the greedoid definition if we assume an accessible set system.

Proposition 2.1 *Let* $(E, \mathscr{F})$ *be an accessible set system. Then the following two statements are equivalent:*

(G2) *If* $X, Y \in \mathscr{F}$ *and* $|X| > |Y|$, *then there is an* $x \in X \setminus Y$ *with* $Y \cup x \in \mathscr{F}$;
(G2') *If* $X, Y \in \mathscr{F}$ *and* $|X| = |Y| + 1$, *then there is an* $x \in X \setminus Y$ *with* $Y \cup x \in \mathscr{F}$.

Proof. (G2) $\Rightarrow$ (G2') is trivial, since if Y has cardinality strictly less than the cardinality of X this includes the case $|Y| = |X| - 1$. To prove (G2') $\Rightarrow$ (G2), let $X, Y \in \mathscr{F}$ with $|X| > |Y|$. Now, either $|X| = |Y| + 1$ or $|X| > |Y| + 1$. In the first case we are done, so we assume that the latter holds. Since $(E, \mathscr{F})$

is accessible there is an $x \in X$ with $X \setminus x \in \mathscr{F}$. Now, either $|X \setminus x| = |Y| + 1$ or $|X \setminus x| > |Y| + 1$, and if the latter holds we can repeat the argument above. However, X has finite cardinality, so eventually we will arrive at a feasible set $Z \subseteq X$ with $|Z| = |Y| + 1$. Now, (G2') says there is an $x \in Z \setminus Y$ with $Y \cup x \in \mathscr{F}$ and since $Z \subseteq X$ we must have $x \in X \setminus Y$ with $Y \cup x \in \mathscr{F}$. Hence, (G2') $\Rightarrow$ (G2). $\square$

We have now looked at some different ways in which a greedoid can be defined as a set system. Before moving on to view greedoids as formal languages we will look at an example where a set system satisfies axioms (G1) and (G2') but fails to be a greedoid.

Example 2.2 Let $G = (V, E)$, where $V = \{a, b, c, d\}$ and $E = \big\{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}\big\}$ be the graph in Figure 2 below, and let $\mathscr{F} = \Big\{\{\emptyset\}, \{a, b\}, \big\{\{a, b\}, \{b, c\}, \{c, d\}\big\}, \big\{\{a, b\}, \{a, d\}, \{b, c\}, \{c, d\}\big\}\Big\}$ be a family of subsets of E. Thus, the pair $(E, \mathscr{F})$ is a set system. The empty set is in $\mathscr{F}$ so (G1) is satisfied. Further, we can add $\{a, b\}$ to the empty set and get a feasible set, and we can add $\{a, d\}$ to $\big\{\{a, b\}, \{b, c\}, \{c, d\}\big\}$ and again get a feasible set. Hence, (G2') is also satisfied. However, if we remove an edge from $\big\{\{a, b\}, \{b, c\}, \{c, d\}\big\}$, no matter which one, we get an infeasible set, showing that $(E, \mathscr{F})$ is not accessible. Since adding any other edge to $\{a, b\}$ results in an infeasible set, (G2) is not satisfied and we can conclude that the set system is not a greedoid.

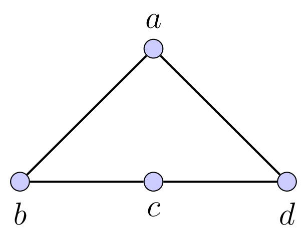


Figure 2

2.2 Greedoids in terms of formal languages

Before we proceed with the definition of a greedoid language we will go through some terminology and notation that will be used. The finite nonempty ground set E that we work in will be called an **alphabet** and its elements will be called **letters**, for which we will use x, y, z from the Latin alphabet. We will denote the set of all sequences of letters in the alphabet by E^* and its elements will be referred to as **words**. Words will be appearing as α, β, γ from the Greek alphabet. A **language** over the alphabet E will be a nonempty set $\mathscr{L} \subseteq E^*$ of words. If no letter is repeated in a word, the word will be called **simple** and in the same way a language will be called simple if every word in it is simple. The number of letters in a word α will be referred to as the **length** of α and will be denoted by $|\alpha|$. Further, the **support** $\widetilde{\alpha}$ of a word α will be the set of distinct letters in α. Thus, if α is simple we have $|\widetilde{\alpha}| = |\alpha|$. Lastly, the **concatenation** of the word α followed by the word β will be denoted $\alpha\beta$ and in the same way the concatenation of the word α followed by the letter x will be denoted αx.

Definition 2.4 *We say that a* **greedoid language** *is a pair* $(E, \mathscr{L})$*, where* E *is an alphabet and* $\mathscr{L}$ *is a simple language over* E*, satisfying*

(L1) $\emptyset \in \mathscr{L}$
(L2) *If* $\alpha\beta \in \mathscr{L}$*, then* $\alpha \in \mathscr{L}$*;*
(L3) *If* $\alpha, \beta \in \mathscr{L}$ *and* $|\alpha| > |\beta|$*, then there exists a letter* x *in* α *with* $\beta x \in \mathscr{L}$*.*

We note that since $\mathscr{L}$ is simple it follows that βx is simple and therefore that x can not be in $\widetilde{\beta}$. The axioms (L1) and (L2) together defines a **hereditary language** while (L3) is referred to as an exchange axiom. The hereditary language has its set system counterpart in the accessible property. The equivalence of the exchange axioms (G2) and (G2') for an accessible set system can analogously be translated to a greedoid language. Likewise, a feasible word has the same meaning in a greedoid language as a feasible set has in a set system.

Example 2.3 Let $E = \{x, y, z\}$ be an alphabet and define a function, that assigns values to the letters corresponding to the alphabetical order according to the Latin alphabet, as follows: $val(x) = 1, val(y) = 2, val(z) = 3$. The set E^* consists of the three possible one-letter words, the nine possible two-letter words and the 27 possible three-letter words. Now, let $\mathscr{L}$ be the language over E where a word α is feasible if it is simple and if its first letter has value less than or equal to 2. That is, $\mathscr{L} = \{\emptyset, x, y, xy, xz, yx, yz, xyz, xzy, yxz, yzx\}$. Further, let $\mathscr{L}'$ be the language over E where a word α is feasible if it is simple and does not contain a letter with value greater than $|\alpha|$. That is, $\mathscr{L}' = \{\emptyset, x, xy, yx, xyz, xzy, yxz, yzx, zxy, zyx\}$. Since the languages are simple, they are greedoid languages if they satisfy (L1), (L2) and (L3). The empty set is in both $\mathscr{L}$ and $\mathscr{L}'$, so (L1) holds. Furthermore, one can verify that both languages satisfies (L3). However, in $\mathscr{L}$, every initial substring of letters in a feasible word form a feasible word. This is not the case in $\mathscr{L}'$. For example, neither z or zx is a feasible word although zxy is feasible. Thus, only $\mathscr{L}$ satisfies (L2), so we can conclude that it is a greedoid language whilst $\mathscr{L}'$ is not. This shows that the ordering of letters is crucial.

Now that the greedoid definitions have been introduced, it is time to look at how they are related to each other.

Proposition 2.2 *The following statements hold true for the relation between a greedoid and a greedoid language:*

(i) *If* $(E, \mathscr{L})$ *is a greedoid language, then* $(E, \mathscr{F}(\mathscr{L}))$ *is a greedoid, where* $\mathscr{F}(\mathscr{L}) = \widetilde{\mathscr{L}} = \{\widetilde{\alpha} : \alpha \in \mathscr{L}\}$.
(ii) *If* $(E, \mathscr{F})$ *is a greedoid, then* $(E, \mathscr{L}(\mathscr{F}))$ *is a greedoid language, where* $\mathscr{L}(\mathscr{F}) = \{x_1...x_k : \{x_1, ..., x_i\} \in \mathscr{F} \; for \; 1 \leq i \leq k\}$.

Proof. (i) Suppose $(E, \mathscr{L})$ is a greedoid language. Since $\widetilde{\alpha}$ is the set of distinct letters in α, $\widetilde{\mathscr{L}}$ is a family of subsets of E. Hence, $(E, \mathscr{F}(\mathscr{L})) = (E, \widetilde{\mathscr{L}})$ is a set system. Assume that a nonempty feasible set $\widetilde{\alpha} \in \widetilde{\mathscr{L}}$ does not contain an element x with $\widetilde{\alpha} \setminus x \in \widetilde{\mathscr{L}}$. Since $\mathscr{L}$ is simple, $|\widetilde{\alpha}| = |\alpha|$ and since $\widetilde{\alpha}$ is nonempty $|\alpha| > 0$. But $(E, \mathscr{L})$ is a greedoid language, so there also exists a $\beta \in \mathscr{L}$ (possibly $\beta = \emptyset$) with $|\alpha| = |\beta| + 1$ and $\widetilde{\beta} \in \widetilde{\mathscr{L}}$. This implies that there exists a letter $x \in \alpha$ with $\beta x \in \mathscr{L}$. Thus, $x \in \widetilde{\alpha}$ and with β simple we must have $|\widetilde{\alpha}| = |\widetilde{\beta} \cup x|$, contradicting the assumption of the feasible set $\widetilde{\alpha}$. Hence, $(E, \mathscr{F}(\mathscr{L})) = (E, \widetilde{\mathscr{L}})$ is an accessible set system. Now, let $\widetilde{\alpha}, \widetilde{\beta} \in \widetilde{\mathscr{L}}$ and $|\widetilde{\alpha}| > |\widetilde{\beta}|$. Then $\alpha, \beta \in \mathscr{L}$ and since $\mathscr{L}$ is simple we have $|\alpha| > |\beta|$, so from (L3) we know there exists a letter $x \in \alpha$ with $\beta x \in \mathscr{L}$. But βx is simple so $x \in \widetilde{\alpha} \setminus \widetilde{\beta}$ and $\widetilde{\beta} \cup x \in \widetilde{\mathscr{L}}$. Hence, $(E, \mathscr{F}(\mathscr{L})) = (E, \widetilde{\mathscr{L}})$ is an accessible set system satisfying (G2), so it is a greedoid.

(ii) Suppose $(E, \mathscr{F})$ is a greedoid. Since $\emptyset \in \mathscr{F}$ we have $\emptyset \in \mathscr{L}(\mathscr{F})$, so (L1) is satisfied. Now, let $\alpha\beta = x_1...x_k \in \mathscr{L}(\mathscr{F})$ where $\alpha = x_1...x_j$ and $\beta = x_{j+1}...x_k$, and assume that $\alpha \notin \mathscr{L}(\mathscr{F})$. We have that $\{x_1, ..., x_i\} \in \mathscr{F}$ for $1 \leq i \leq k$ and since $(E, \mathscr{F})$ is an accessible set system we can remove x_k from $X = \{x_1, ..., x_k\}$ and have $X \setminus x_k \in \mathscr{F}$. If we continue the removal of elements in this manner we can conclude that $X \setminus \{x_k, ..., x_{j+1}\} \in \mathscr{F}$. From here, we see that $\{x_1, ..., x_j\} \in \mathscr{F}$ for $1 \leq i \leq j$, so we must have $\alpha = x_1...x_j \in \mathscr{L}(\mathscr{F})$ contradicting our assumption. Hence, (L2) is satisfied.

Now, let $\alpha = x_1 ... x_i \in \mathscr{L}(\mathscr{F})$, $\beta = y_1 ... y_j \in \mathscr{L}(\mathscr{F})$ with $|\alpha| > |\beta|$. We must then have a set $X = \{x_1, ..., x_{j+1}\} \in \mathscr{F}$ and since $(E, \mathscr{F})$ is a greedoid, it follows from (G2) that there is an $x \in X$ with $\{y_1, ..., y_j\} \cup x \in \mathscr{F}$. Hence, there is an $x \in \alpha$ with $\beta x \in \mathscr{L}(\mathscr{F})$. This means (L3) is satisfied, so $(E, \mathscr{L}(\mathscr{F}))$ is a greedoid language. $\square$

2.3 Graph examples

In this section, we will present some graphs from which we will form greedoids or greedoid languages that will be used illustratively later in this thesis.

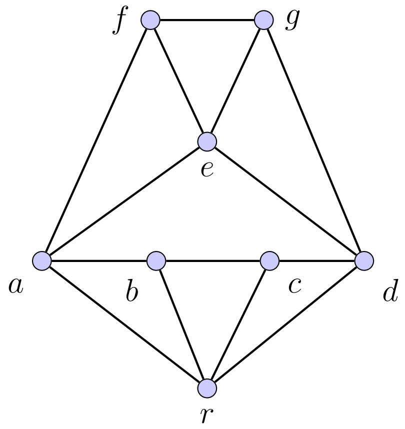
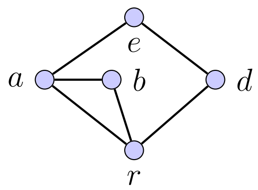
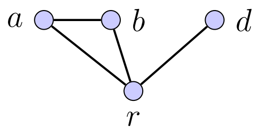


Figure 4

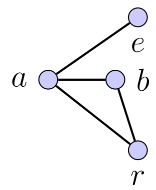
We let E_1 be the set of all vertices, except the vertex r, of the graph in Figure 4. Further, we let $\mathscr{F}_1$ be a family of subsets of E_1, in which the empty set is included and a subset is feasible if it, together with the root r, induce a connected subgraph of the graph in Figure 4. It can be verified that the set system $(E_1, \mathscr{F}_1)$ is accessible. For example, let $X = \{a, b, d, e\}$. This is a feasible set and we can remove any vertex from X and get another feasible set. In figure 5, we find the induced connected subgraphs of the graph in Figure 4 for $X \cup r, X \cup r \setminus e, X \cup r \setminus d, X \cup r \setminus b, X \cup r \setminus a$ depicted.



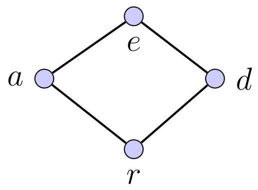
(a) $X \cup r$



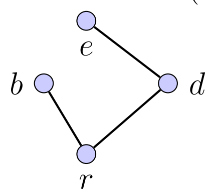
(b) $X \cup r \setminus e$



(c) $X \cup r \setminus d$



(d) $X \cup r \setminus b$

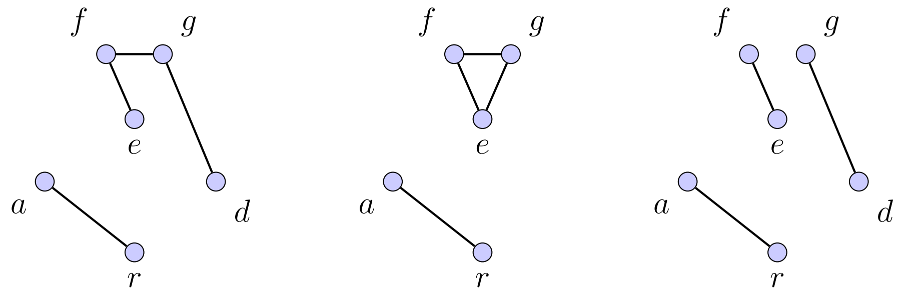


(e) $X \cup r \setminus a$

Figure 5

It can also be verified that this set system satisfies (G2'). For example, with X as before and with $Y = X \setminus e$, we have $|X| = |Y| + 1$ for the feasible sets X and Y. Now, there is an $x \in X \setminus Y$ with $Y \cup x \in \mathscr{F}_1$, namely when x is the vertex e. So, we have that $(E_1, \mathscr{F}_1)$ is a greedoid and from now on we will refer to it as Greedoid 1. Viewed as a greedoid language, $\mathscr{L}(\mathscr{F}_1)$ consists of simple words corresponding to the ordering in which vertices of a feasible set can be visited when starting at r.

Again, we consider the graph in Figure 4, but this time we ignore that r is a root and instead treat it as an ordinary vertex. We let E_2 be the set of all edges of the graph in Figure 4. Further, we let $\mathscr{F}_2$ be a family of subsets of E_2, in which the empty set is included and a subset is feasible if it is the edge set of a forest of the graph in Figure 4. For example, the edge set $X = \big\{\{r, a\}, \{d, g\}, \{e, f\}, \{f, g\}\big\}$ is feasible but the edge set $X' = \big\{\{r, a\}, \{e, f\}, \{e, g\}, \{f, g\}\big\}$ is infeasible. It can be verified that also $(E_2, \mathscr{F}_2)$ is an accessible set system. For instance, we can remove any edge in X and get another feasible set. Similarly, it can be verified that this set system satisfies (G2'). For example, with $Y = \big\{\{r, a\}, \{d, g\}, \{e, f\}\big\}$ we have $|X| = |Y| + 1$ for the feasible sets X and Y. It then holds that there is an $x \in X \setminus Y$ with $Y \cup x \in \mathscr{F}_2$, namely when x is the edge $\{f, g\}$. So, $(E_2, \mathscr{F}_2)$ turns out to be a greedoid as well and it will further on be referred to as Greedoid 2. The graphs derived from the edge sets X, X' and Y can be seen in Figure 6. Viewed as a greedoid language, $\mathscr{L}(\mathscr{F}_2)$ consists of all the simple words that can be formed from edges (letters) of a feasible set.



(a) X (b) X' (c) Y

Figure 6

Now, we associate a direction with every edge of the graph in Figure 4, resulting in the directed graph presented in Figure 7.

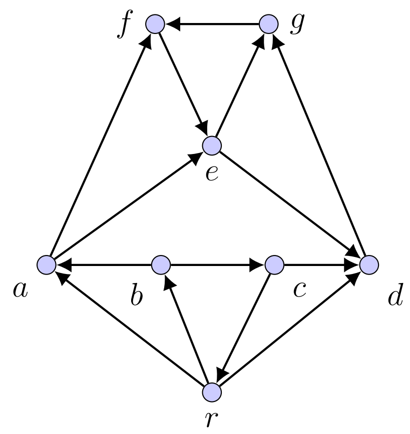


Figure 7

We let E_3 be the set of all directed edges of the graph in Figure 7. Further, we let $\mathscr{F}_3$ be a family of subsets of E_3, in which the empty set is included and a subset is feasible if it is the edge set of a directed subtree rooted at r and such that the edges are directed away from r. In every nonempty feasible set we can identify a leaf vertex of the directed subtree, and by removing it we get another directed subtree rooted at r. Thus, $(E_3, \mathscr{F}_3)$ is an accessible set system. For two feasible sets X, Y with $|X| = |Y|+1$ it follows that the edges in X reach a vertex of the graph in Figure 7 which the edges in Y do not reach. But then we can add the edge in X that reaches this specific vertex to Y and get another feasible set. This shows that the set system satisfies (G2') and therefore is a greedoid. We will refer to $(E_3, \mathscr{F}_3)$ as Greedoid 3 further on. Viewed as a greedoid language, $\mathscr{L}(\mathscr{F}_3)$ consists of simple words corresponding to the ordering in which edges of a feasible set are travelled through when starting at r.

Our final example differ somewhat from the previous ones and this time we consider a bipartite graph. We let E_4 be the disjoint set of vertices to the left of the graph in Figure 8, that is $E_4 = \{u_1, u_2, u_3, u_4, u_5\}$. Further, we let v_1, v_2, v_3, v_4, v_5 be an ordering on the disjoint set of vertices to the right. From this, we obtain $\mathscr{F}_4$ as a family of subsets of E_4, in which the empty set is included and a subset X is feasible if the subgraph induced by X and the first $|X|$ vertices of $\{v_1, v_2, v_3, v_4, v_5\}$ has a perfect matching. Specifically, $\mathscr{F}_4 = \big\{\emptyset, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_5\}, \{u_2, u_5\}, \{u_1, u_2, u_4\}, \{u_1, u_4, u_5\}, \{u_2, u_4, u_5\}, \{u_1, u_2, u_3, u_4\}, \{u_1, u_2, u_4, u_5\}, \{u_1, u_3, u_4, u_5\}, \{u_2, u_3, u_4, u_5\}, \{u_1, u_2, u_3, u_4, u_5\}\big\}$, so it can be verified that $(E_4, \mathscr{F}_4)$ is an accessible set system satisfying (G2').
This set system will be referred to as Greedoid 4 from now on. Viewed as a greedoid language, we have that $\mathscr{L}(\mathscr{F}_4) = \{\emptyset, u_1, u_2, u_1u_2, u_1u_5, u_2u_1, u_2u_5, u_2u_1u_4, u_1u_5u_4, u_2u_5u_4, u_2u_1u_4u_3, u_1u_5u_4u_2, u_1u_5u_4u_3, u_2u_5u_4u_3, u_2u_1u_4u_3u_5\}$.

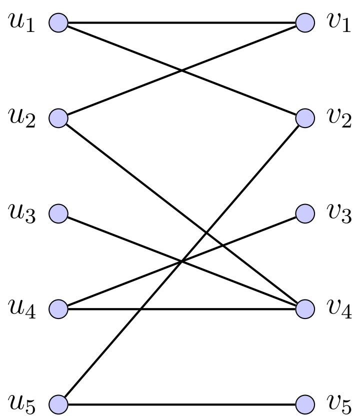


Figure 8

3 Some classes

In this part we will take a look at some classes of greedoids, including fundamental properties that some but not all classes share.

3.1 Interval greedoids

The first class to be explored is the class of interval greedoids. These greedoids are characterized by having the interval property which we now define in terms of a greedoid language.

Definition 3.1 *We say that a greedoid language* $(E, \mathscr{L})$ *has the* **interval property** *if* $\alpha x, \alpha\beta\gamma x \in \mathscr{L}$, *then* $\alpha\beta x \in \mathscr{L}$.

According to [1], interval greedoids behave better than general greedoids in many ways, and sometimes the interval property is even necessary to obtain meaningful results. We will now take a look at which of the greedoids from section 2.3 that possess this property.

Example 3.1 It can be verified that Greedoid 1, Greedoid 2 and Greedoid 3 all have the interval property. Consider Greedoid 1, defined on page 12-13, with the letters in its simple language corresponding to the vertices in feasible sets. For example, since $adg, adefg \in \mathscr{L}(\mathscr{F}_1)$ we must have $adeg \in \mathscr{L}(\mathscr{F}_1)$ for the interval property to hold. This is indeed a feasible word, since it is a possible order to visit the vertices in the induced connected subgraph for the feasible set $\{a, d, e, g\}$ and the root r. More generally, that Greedoid 1 possess the interval property follows from the fact that the corresponding induced subgraphs are connected. For Greedoid 2, since the feasible words αx and $\alpha\beta\gamma x$ correspond to edge sets of forests, $\alpha\beta x$ must also correspond to an edge set of a forest. Regarding Greedoid 3, consider an arbitrary edge x. If this is a possible edge to travel along to an unvisited vertex at an early stage, i.e. if αx is a feasible word, and if it is still a possible edge to travel to the same unvisited vertex at a later stage, i.e. if $\alpha\beta\gamma x$ is a feasible word, then it must also be a possible edge to travel to this unvisited vertex at any stage between those stages already mentioned, i.e. then $\alpha\beta x$ is a feasible word. However, if we consider Greedoid 4 defined on page 15, we see that it does not possess the interval property. We have, for example, that u_1u_2 and $u_1u_5u_4u_2$ are feasible words while $u_1u_5u_2$ is infeasible.

As noted above, three out of four greedoids can be classified as interval greedoids. Yet we can distinguish these three from each other. The class of interval greedoids is a generalization of several other classes and can be further divided as can be seen in the following definition.

Definition 3.2 *We say that a greedoid language* $(E, \mathscr{L})$ *is a* **matroid** *if it has the interval property without lower bounds. That is, if* $\alpha\beta x \in \mathscr{L}$*, then* $\alpha x \in \mathscr{L}$.

Example 3.2 From Example 3.1, we know that $adg \in \mathscr{L}(\mathscr{F}_1)$. However, $ag \notin \mathscr{L}(\mathscr{F}_1)$ since the induced subgraph for $\{a, g\} \cup r$ is not connected, and therefore Greedoid 1 can not be a matroid. Neither is Greedoid 3, defined on page 14-15, a matroid. We can, for example, travel along the edge from r to a, then further along the edge from a to e and finally along the edge from e to g. On the other hand, there are no single directed edge to travel along from a to g, so the interval property without lower bounds is not fulfilled. It can be verified, however, that Greedoid 2 is a matroid. We can also argue that since a feasible word in this greedoid language corresponds to an edge set of a disjoint union of unrooted trees, the removal of any edge will also result in a disjoint union of unrooted trees, and therefore the interval property without lower bounds holds. Greedoid 2 is specifically known as a graphic matroid.

A more common way to define a matroid is through the combinatorial structure known as an independence system.

Definition 3.3 *We say that an* **independence system** *is a set system* $(E, \mathscr{F})$ *satisfying*

(M1) $\emptyset \in \mathscr{F}$;
(M2) *If* $X \subseteq Y \in \mathscr{F}$*, then* $X \in \mathscr{F}$.

Further, the elements of $\mathscr{F}$ *are called* **independent** *and the elements of* $2^E \setminus \mathscr{F}$ *are called* **dependent**.

Definition 3.4 *We say that a* **matroid** *is an independence system satisfying*

(M3) *If* $X, Y \in \mathscr{F}$ *and* $|X| > |Y|$*, then there is an* $x \in X \setminus Y$ *with* $Y \cup x \in \mathscr{F}$.

We see that (M1) and (G1) are identical and this also applies to (M3) and (G2). Thus, from this perspective we get a matroid from a greedoid by requiring (M2). The independent sets are the matroid counterpart of the feasible sets for greedoids. To show that greedoids is not merely a concept of graphs we now look at two other examples.

Example 3.3 Consider a matrix A over an arbitrary field and let E denote the set of columns of A. Further, let $\mathscr{F}$ be all subsets of E in which the columns are linearly independent over this field. Clearly, $(E, \mathscr{F})$ satisfies (M1) and (M2) and hence, is an independence system. From linear algebra we are familiar with the fact that every set of linearly independent vectors can be completed to a basis. Using this fact on the independent sets of $\mathscr{F}$ we see that (M3) holds, so we can conclude that $(E, \mathscr{F})$ is a matroid.

Example 3.4 Let E be a finite set and let the independent sets of $\mathscr{F}$ be all subsets of E with cardinality less than or equal to some positive integer k. Again, (M1) and (M2) obviously holds, so $(E, \mathscr{F})$ is an independence system. Further, for $X, Y \in \mathscr{F}$ with $|X| > |Y|$ there exists at least one element $x \in X \setminus Y$. Now, since $|X| \leq k$ we must have $|Y \cup x| \leq k$ and therefore $Y \cup x$ must be an independent set. Thus, (M3) holds, so $(E, \mathscr{F})$ is a matroid.

After these two examples, it is time to leave the world of matroids and refer interested readers to available literature on the subject. At this point, it seems suitable to proceed with the case where the interval property is without upper bounds.

Definition 3.5 *We say that a greedoid language* $(E, \mathscr{L})$ *is an* **antimatroid** *if it has the interval property without upper bounds. That is, if* $\alpha x, \alpha\beta \in \mathscr{L}$ *with* $x \notin \widetilde{\beta}$*, then* $\alpha\beta x \in \mathscr{L}$*. In terms of set systems, we say that a greedoid* $(E, \mathscr{F})$ *is an antimatroid if* $X, Y \in \mathscr{F}, X \subseteq Y, x \in E \setminus Y$ *and* $X \cup x \in \mathscr{F}$*, then* $Y \cup x \in \mathscr{F}$.

Example 3.5 We know that a feasible set of Greedoid 1 together with the root r induce a connected subgraph of the graph in Figure 4 on page 12. Thus, if we consider $\mathscr{L}(\mathscr{F}_1)$ and let αx and $\alpha\beta$ be two feasible words, where the letter x is not a part of the latter, it follows, since the corresponding graphs are connected, that $\alpha\beta x$ must also be a feasible word. Since the interval property without upper bounds holds we can conclude that Greedoid 1 is an antimatroid. This particular type of a greedoid is known as a vertex search greedoid. Regarding Greedoid 2, which we now know is a matroid, there are plenty of examples where the interval property without upper bounds does not hold. Let, for example, α be the edge between r and a, x be the edge between a and b and β be the edge between r and b. Clearly, αx and $\alpha\beta$ are feasible words but the concatenation of $\alpha\beta$ followed by x produces a cycle of the edge set, which shows that Greedoid 2 can not be an antimatroid. In the same manner we can find counterexamples that Greedoid 3 is not an antimatroid. We get one such example if we let α equal the edge from r to b followed by the edge from b to c, β equal the edge from r to d and finally x equal the edge from c to d.

The following proposition is helpful when we want to check if a set system is an antimatroid.

Proposition 3.1 *Let $(E, \mathscr{F})$ be a set system. Then the following statements are equivalent:*

(i) *$(E, \mathscr{F})$ is an antimatroid.*
(ii) *$(E, \mathscr{F})$ is an accessible set system and $\mathscr{F}$ is closed under union.*
(iii) *$\emptyset \in \mathscr{F}$ and for $X, Y \in \mathscr{F}$ such that $X \not\subseteq Y$, there is an $x \in X \setminus Y$ such that $Y \cup x \in \mathscr{F}$.*

Proof. (i) $\Rightarrow$ (ii) Let $(E, \mathscr{F})$ be an antimatroid. Then it is also a greedoid and every greedoid is an accessible set system. Further, let $X, Y, Z \in \mathscr{F}$ with $X \subseteq Z \subseteq X \cup Y$. Now, if $Y \setminus Z = \emptyset$, we must also have $X \cup Y \subseteq Z$, so $X \cup Y \in \mathscr{F}$. On the other hand, if $Y \setminus Z \neq \emptyset$, we can repeatedly remove elements from Y until we get a feasible set Y' with $Y' \subseteq Z$. We know there exists an element $y \in Y \setminus Z$ with $Y' \cup y \in \mathscr{F}$ and since $(E, \mathscr{F})$ is an antimatroid, we must have $Z \cup y \in \mathscr{F}$. Now, either $X \cup Y \subseteq Z \cup y$ or else there exists an element $y' \in Y \setminus (Z \cup y)$ and since E is a finite set we can repeat the same argument and eventually conclude that $X \cup Y \in \mathscr{F}$.

(ii) $\Rightarrow$ (iii) Let $(E, \mathscr{F})$ be a set system and suppose there exists some element x in every nonempty feasible set X with $X \setminus x \in \mathscr{F}$. Further, suppose that for all $X, Y \in \mathscr{F}$ we also have $X \cup Y \in \mathscr{F}$. Now, let $A, B \in \mathscr{F}$ with $A \not\subseteq B$. Since A is a nonempty feasible set we can repeatedly remove elements and get other feasible sets. Let $A' \in \mathscr{F}$ arise this way, where A' has least possible cardinality such that $A' \not\subseteq B$ still hold. Then there exists an $x \in A' \setminus B \subseteq A \setminus B$ and since $\mathscr{F}$ is closed under union we must have $B \cup A' = B \cup x \in \mathscr{F}$.

(iii) $\Rightarrow$ (i) Let $(E, \mathscr{F})$ be a set system and suppose statement (iii) is true. If $X, Y \in \mathscr{F}$ with $|X| > |Y|$, then $X \not\subseteq Y$, so both (G1) and (G2) are satisfied and therefore $(E, \mathscr{F})$ is a greedoid. Further, suppose that $A, B \in$

$\mathscr{F}, A \subseteq B, x \in E \setminus B$ and $A \cup x \in \mathscr{F}$. Now consider the set $(A \cup x) \setminus B$. This must be equal to the set $\{x\}$, so $(A \cup x) \nsubseteq B$ and clearly there is an $x \in (A \cup x) \setminus B = \{x\}$ with $B \cup x \in \mathscr{F}$. Since the interval property without upper bounds is satisfied, we conclude that $(E, \mathscr{F})$ is an antimatroid. $\square$

We will use Proposition 3.1 in the next example where we show that a poset greedoid is an antimatroid. First we need some definitions, which are taken from [3].

Definition 3.6 *We say that a* **partially ordered set** (**poset**) *is an ordered pair* $(P, \leq)$ *of a set* P *and a binary relation* $\leq$ *contained in* $P \times P$ *called the partial order on* P*, such that*

(i) *The relation* $\leq$ *is reflexive, i.e. each element of the ordered set is related to itself.*
(ii) *The relation* $\leq$ *is antisymmetric, i.e. if p is related to q and q is related to p, then p must equal q.*
(iii) *The relation* $\leq$ *is transitive, i.e. if p is related to q and q is related to r, then p is related to r.*

Definition 3.7 *Let* $(P, \leq)$ *be a poset. We say that a subset* $I \subseteq P$ *is an* **ideal** *of* P *if* $y \in I$ *and* $x \leq y$ *implies* $x \in I$.

Example 3.6 Let $P = (E, \leq)$, where E is finite, be a poset and let $\mathscr{F}$ be the set of ideals of E. Then $(E, \mathscr{F})$ is called the poset greedoid of P. For an arbitrary ideal $X \subseteq E$, we can remove a maximal element $x \in X$ and get a new ideal $X \setminus x \in \mathcal{F}$. Thus, $(E, \mathscr{F})$ is an accessible set system. Further, for two ideals $X, Y \in \mathscr{F}$, we see that $X \cup Y$ satisfies the definition of an ideal as well. Hence, we can conclude from Proposition 3.1 that $(E, \mathscr{F})$ is an antimatroid.

At this point we know that Greedoid 1 and Greedoid 2 are interval greedoids, and where the former is also an antimatroid while the latter is a matroid. As for Greedoid 3, we know it is also an interval greedoid. Nevertheless, it is not a matroid nor an antimatroid. However, it has many similarities with Greedoid 2 which becomes apparent if we remove the directions from the edges of the graph in Figure 7 on page 14 and stop treating r as a root. In fact, Greedoid 2 and Greedoid 3 both belong to a class called local poset greedoids, which we will return to later. A subclass to the local poset greedoids are the branching greedoids, and Greedoid 3 is known as a directed branching greedoid. According to [1], this is a class of well behaved greedoids that often serve as standard examples.

3.2 Gaussian greedoids

Not all greedoids possess the interval property, as could be seen in Example 3.1. One class in which the greedoids are not interval greedoids in general is the class called Gaussian greedoids. A formal definition of this class will be postponed to section 4.2, where we have acquired the necessary prerequisites. Instead, we will now consider a subclass of it with close connections to matrices.

Definition 3.8 *Let $M = (m_{ij})$ be an $m \times n$ matrix over an arbitrary field. Further, let $M_{\{1,...,|X|\},X}$ be an $|X| \times |X|$ submatrix of M, consisting of the columns in a subset X of the set of columns of M and the $|X|$ first rows of M. We say that a set system $(E, \mathscr{F})$, where $E = \{1, ..., n\}$ and $\mathscr{F} = \{A \subseteq E$: the submatrix $M_{\{1,...,|A|\},A}$ is invertible$\}$, is a* **Gaussian elimination greedoid**.

The possible sets of column indices for the pivot elements when performing Gaussian elemination form the **maximal feasible sets** of this greedoid, hence the name Gaussian elimination greedoid. A feasible set is maximal if it is not a proper subset of any feasible set, and such a set is also called a **basis**. Next, we will look at an example of a Gaussian elimination greedoid and verify that it really is a greedoid.

Example 3.7 Let a 5×5 matrix M be given by

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have that $E = \{1, 2, 3, 4, 5\}$ and $\mathscr{F} = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 5\}, \{2, 5\}, \{1, 2, 4\}, \{1, 4, 5\}, \{2, 4, 5\}, \{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\big\}$. From this, we can verify that $(E, \mathscr{F})$ is an accessible set system satisfying (G2'), so it is indeed a greedoid. The feasible sets here can of course be obtained in many different ways, including calculating determinants and finding inverses. Another way is to, with a slightly different label, reuse the feasible sets of $\mathscr{F}_4$ on page 15. Why is that?

We have already concluded that Greedoid 4 can not be an interval greedoid. In fact, it is a type of greedoid known as bipartite matching greedoid or medieval marriage greedoid, where the latter name refers to that the oldest daughter must be married first, and so on. Every such greedoid is a Gaussian elimination greedoid, and in this case M in Example 3.7 is the incidence matrix of the graph in Figure 8 on page 15. Here, the rows correspond to $v_1, ..., v_5$, the columns correspond to $u_1, ..., u_5$ and the entry in place (i, j) is 1 if v_i and u_j are connected, for $1 \leq i, j \leq 5$.

4 Structural properties and optimization

In this part we will take a closer look at the rank function, the closure operator and examine the meaning of rank and closure feasibility. Further, we will examine some basic operations on greedoids and briefly explore its relation with optimization.

4.1 Rank function

With the rank function of a greedoid, we can associate the size of a maximal feasible subset of every set. We will first formally define the rank function and then state and prove a theorem which shows that it uniquely determines a greedoid.

Definition 4.1 *For a subset $X \subseteq E$ in a greedoid $(E, \mathscr{F})$ we define the* **rank function** *as* $r(X) = \max\{|A| : A \subseteq X, A \in \mathscr{F}\}$. *We say that the* **rank** *of the greedoid $(E, \mathscr{F})$ is equal to* $\max\{|A| : A \in \mathscr{F}\}$.

Theorem 4.1 *A function $r : 2^E \mapsto \mathbb{N}$ is the rank function of a greedoid if and only if for all $X, Y \subseteq E$ and all $x, y \in E$:*

(i) $r(X) \leq |X|$,
(ii) $X \subseteq Y$ *implies* $r(X) \leq r(Y)$,
(iii) $r(X) = r(X \cup x) = r(X \cup y)$ *implies* $r(X) = r(X \cup x \cup y)$.

Further, the rank function uniquely determines the greedoid.

Proof. Suppose that r is the rank function of a greedoid $(E, \mathscr{F})$. Then both (i) and (ii) are consequences of Definition 4.1. For (iii), the case where $x = y$ is trivial so we look at the case where $x \neq y$. Now, let $r(X) = r(X \cup x) = r(X \cup y)$. This implies that $X, X \cup x$ and $X \cup y$ all have the same maximal feasible subset A. Now, assume that $r(X) < r(X \cup x \cup y)$. Then for B, the maximal feasible subset of $X \cup x \cup y$, we must have $|B| > |A|$. Since $(E, \mathscr{F})$ is a greedoid, (G2) holds. Thus, either $X \cup x$ or $X \cup y$ is a feasible set with cardinality strictly larger than $|X|$. But this contradicts $r(X) = r(X \cup x) = r(X \cup y)$, so (iii) holds. Now, suppose (i),(ii) and (iii) holds for r and let $(E, \mathscr{F})$ be a set system with $\mathscr{F} = \{X \subseteq E : r(X) = |X|\}$. Since $r(\emptyset) = |\emptyset| = 0$ implies $\emptyset \in \mathscr{F}$, (G1) holds. Now, let $X, Y \in \mathscr{F}$ with $|X| > |Y|$, and assume that (G2) does not hold, i.e. there is no $x \in X \setminus Y$ with $Y \cup x \in \mathscr{F}$. Then we can apply (iii) for all the $|X| - |Y|$ elements in

$X \setminus Y$ and arrive at the conclusion $r(Y) = r(Y \cup (X \setminus Y)) = r(X)$. But since X and Y are feasible sets we must have $|X| = |Y|$, contradicting our assumption. Thus (G2) holds, which shows that $(E, \mathscr{F})$ is a greedoid. At last, since X is feasible if and only if $r(X) = |X|$, we have that r uniquely determines the greedoid. $\square$

Example 4.1 We examine the rank function for Greedoid 4 defined on page 15. This greedoid is of rank 5 with E_4 as the only maximal feasible set. Let $X = \{u_2, u_5\}$ and $Y = \{u_1, u_2, u_4, u_5\}$. We have that $r(X) = |X| = 2$, so (i) holds in this case. Further, $X \subseteq Y$ and $r(Y) = |Y| = 4$, so (ii) holds in this case as well. Moreover, we have that $r(X) = r(X \cup u_1) = r(X \cup u_3) = 2$ since $\{u_1, u_2, u_5\}, \{u_2, u_3, u_5\} \notin \mathscr{F}_4$ and that $r(X) = r(X \cup u_1 \cup u_3) = 2$, which shows that (iii) holds in this case. One can also verify that all feasible sets $A \subseteq E$ satisfies $r(A) = |A|$ while all infeasible sets $A' \subseteq E$ satisfies $r(A') < |A'|$.

Corollary 4.1 *The rank function r of a greedoid is the rank function of an antimatroid if and only if for all $X \subseteq E$ and all $x, y \in E, x \neq y$:*

$$r(X \cup x) > r(X) \text{ and } r(X \cup y) > r(X) \text{ implies } r(X \cup x \cup y) > r(X) + 1.$$

Proof. Suppose r is the rank function of an antimatroid and let $A \subseteq X$ be a maximal feasible subset of $X \in E$. Further, suppose $r(X \cup x) > r(X)$. Now, since $(E, \mathscr{F})$ is an antimatroid, Proposition 3.1 tells us $\mathscr{F}$ is closed under union, and therefore $A \cup x$ is a feasible set with $|A \cup x| = |A| + 1$. If we also suppose that $r(X \cup y) > r(X)$, we get by the same reasoning that $A \cup y$ is a feasible set with $|A \cup y| = |A| + 1$. Again, since $\mathscr{F}$ is closed under union, $A \cup x \cup y$ must be a feasible set satisfying $|A \cup x \cup y| = |A| + 2$. Now, from Theorem 4.1 we have that $|A| + 2 = |A \cup x \cup y| \leq r(X \cup x \cup y)$ and with $r(X) + 1 = |A| + 1$ we get $r(X \cup x \cup y) > r(X) + 1$. This time, suppose that r is the rank function of a greedoid and that the statement in Corollary 4.1 is satisfied for all $X \subseteq E$ and all $x, y \in E, x \neq y$. Moreover, let X be a feasible set with $r(X) = |X|$ and let $r(X \cup x) > r(X)$ for $x \in E$. Then $X \cup x$ must be a feasible set with cardinality $|X| + 1$. Clearly, $X \subseteq X \cup x$. Now, let $y \in E \setminus (X \cup x)$ be such that $r(X \cup y) > r(X)$. Then $X \cup y$ must be a feasible set with cardinality $|X| + 1$. Since $r(X \cup x) > r(X)$ and $r(X \cup y) > r(X)$ implies that $r(X \cup x \cup y) > r(X) + 1 = |X| + 1$ we have that also $X \cup x \cup y$ is a feasible set. This means $(E, \mathscr{F})$ has the interval property without upper bounds. Hence, it is an antimatroid. $\square$

Example 4.2 Consider Greedoid 1 defined on page 12-13, and let $X \subseteq E_1$ be an arbitrary set. From Definition 4.1, we know that $r(X)$ is the size of a maximal feasible subset of X. Hence, such a maximal feasible subset, together with the root, induce a connected subgraph of the graph in Figure 4. Let x and y be two distinct vertices of this graph with $r(X \cup x) > r(X)$ and $r(X \cup y) > r(X)$. Then we must have $x, y \notin X$ and further, $r(X \cup x)$ and $r(X \cup y)$ must give rise to two different maximal feasible subsets. Thus, if we can add either x or y to a connected subgraph and get a new connected subgraph, then we can as well add both x and y and get a new connected subgraph. The rank function of its corresponding maximal feasible subset must then satisfy $r(X \cup x \cup y) > r(X \cup x), r(X \cup y) > r(X)$ which implies $r(X \cup x \cup y) > r(X) + 1$. Since the rank function uniquely determines the greedoid we can conclude from Corollary 4.1 that $(E_1, \mathscr{F}_1)$ is an antimatroid.

We will now revisit the matroids. For any set system $(E, \mathscr{F})$ it follows from greedoid axiom (G2) that for any subset $X \subseteq E$ all maximal feasible subsets of X have the same cardinality. On the other hand, for an accessible set system $(E, \mathscr{F})$, the fact that for any subset $X \subseteq E$ all maximal feasible subsets of X have the same cardinality does not imply (G2). However, for an independence system $(E, \mathscr{F})$, axiom (M3) is equivalent with the fact that for any subset $X \subseteq E$ all maximal feasible subsets of X have the same cardinality.

Example 4.3 If we consider Greedoid 2 defined on page 13, we know that $\emptyset \in \mathscr{F}_2$, so (M1) is satisfied. Further, if Y is a feasible set, it corresponds to a disjoint union of unrooted trees and every subset of Y will also correspond to a disjoint union of unrooted trees, so (M2) is satisfied as well. Now, take an arbitrary subset $X \subseteq E_2$ and consider the subgraph associated with this subset. This subgraph consists of, say, k vertices. If the subgraph is connected, the maximal feasible subsets of X will have cardinality $k - 1$, according to a famous theorem about the number of edges and vertices in a tree. If the subgraph consists of two components, the maximal feasible subsets of X will have cardinality $k - 2$. If three components, cardinality $k - 3$ and so on. In any case the maximal feasible subsets of X will have the same cardinality. From this, we can once again conclude that Greedoid 2 is a matroid.

Before closing this section, we will introduce the concept of rank feasibility.

Definition 4.2 *For a subset $X \subseteq E$ in a greedoid $(E, \mathscr{F})$ we define the* **basis rank** *as $\beta(X) = \max\{|X \cap A| : A \in \mathscr{F}\}$. Further, we say that X is*

rank feasible *if* $\beta(X) = r(X)$.

We know that for a subset $X \subseteq E$, if X is feasible we have $r(X) = |X|$. Further, the basis rank of X is bounded from above by the cardinality of X and, since X is a feasible set, bounded from below by the cardinality of the intersection with itself. Thus, $\beta(X) = |X|$. On the other hand, if X is infeasible we have $r(X) = |A| < |X|$. The basis rank of X is then bounded from below by $|X \cap A| = |A|$ and from above by $\min(|X|, |B|)$, where B is a basis. So, in any way, we have $r(X) \leq \beta(X)$ for all $X \subseteq E$. If we denote the family of all rank feasible sets by $\mathscr{R}$ we have, since $\mathscr{F} = \{X \subseteq E : r(X) = |X|\}$, $\mathscr{F} \subseteq \mathscr{R}$. Equality between $\mathscr{F}$ and $\mathscr{R}$ must hold if $E \in \mathscr{F}$ since then $\beta(X) = |X|$ for all $X \subseteq E$. Regarding the equality $r = \beta$, the following proposition tells us when it occurs.

Proposition 4.1 We have that $r(X) = \beta(X)$ for all $X \subseteq E$ if and only if $(E, \mathscr{F})$ is a matroid.

Proof. Suppose $r(X) = \beta(X)$ for all $X \subseteq E$ of a greedoid $(E, \mathscr{F})$. For a feasible set B we have that $r(B) = |B|$ and $\beta(B) = |B|$. Now, if $A \subseteq B$ with $A \notin \mathscr{F}$, then $r(A) < |A|$ while $\beta(A) = |A|$ since A is a subset of B. Since this contradicts the assumption, we must have $A \in \mathscr{F}$. Thus, (M2) holds which shows that $(E, \mathscr{F})$ is a matroid. Instead, suppose now that $(E, \mathscr{F})$ is a matroid and let $A \subseteq E$ be an arbitrary set. We know that $r(A) = |A'|$ for some $A' \in \mathscr{F}$. If A' is a basis, then $\beta(A) = |A'|$. Otherwise, since $(E, \mathscr{F})$ is a matroid, A' can be completed to a basis B and thus, $A' \subseteq B$. But then we have $\beta(A) = |A \cap B| = |A' \cap B| = |A'|$, so $r(A) = \beta(A)$. This shows that only when a greedoid is a matroid, all sets in 2^E are rank feasible. $\square$

4.2 Closure operator

In this section we will, with the help of the rank function, define the closure operator of a greedoid and then have a look at some of its behaviour.

Definition 4.3 *For a subset $X \subseteq E$ in a greedoid $(E, \mathscr{F})$ we define the* **(rank) closure operator** $\sigma : 2^E \mapsto 2^E$ *as* $\sigma(X) = \{x \in E : r(X \cup x) = r(X)\}$. *Further, we say that a set $X \subseteq E$ is* **closed** *if* $X = \sigma(X)$.

Example 4.4 We examine the closure operator for Greedoid 4 defined on page 15. Let $X = \{u_3, u_4, u_5\}$, $Y = \{u_2, u_4, u_5\}$ and $Z = \{u_1, u_2, u_5\}$, then

$r(X) = 0$, $r(Y) = 3$ and $r(Z) = 2$. The two elements in $E_4 \setminus X$ are both feasible singleton sets so we can not add any of them to X without increasing the rank. Thus, $X = \sigma(X)$ and therefore closed. Similarly, if we add u_1 or u_3 to Y we will get another feasible set and thus increase the rank, so Y is also a closed set. However, Z is not a closed set since $\sigma(Z) = Z \cup u_3$.

Before we proceed any further with the closure operator, it is time to go back to the greedoid classes, as was treated earlier. We now have all the necessary tools to define the Gaussian greedoid from section 3.2.

Definition 4.4 *Let $M_i = (E, \mathscr{F}_i), i = 0, 1, ..., m$ be a sequence of matroids satisfying:*

(i) *if $A \subseteq E$ is closed in M_{i-1}, then it is closed in M_i, for $1 \leq i \leq m$,*
(ii) *the rank of M_i is equal to i, for $0 \leq i \leq m$.*

We say that a greedoid $(E, \mathscr{F})$ of rank m, where $\mathscr{F} = \{A \subseteq E : A$ is a basis of $M_{|A|}\}$, is a **Gaussian greedoid**.

Example 4.5 Recall the Gaussian elimination greedoid $(E, \mathscr{F})$ in Example 3.7, which is Greedoid 4 transcoded. Let $M_i = (E, \mathscr{F}_i), i = 0, 1, ..., 5$ be a sequence of set systems with $\mathscr{F}_0 = \{\emptyset\}$, $\mathscr{F}_1 = \mathscr{F}_0 \cup \big\{\{1\}, \{2\}\big\}$, $\mathscr{F}_2 = \mathscr{F}_1 \cup \big\{\{1, 2\}, \{1, 5\}, \{2, 5\}\big\}$, $\mathscr{F}_3 = \mathscr{F}_2 \cup \big\{\{1, 2, 4\}, \{1, 4, 5\}, \{2, 4, 5\}\big\}$, $\mathscr{F}_4 = \mathscr{F}_3 \cup \big\{\{1, 2, 3, 4\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}\big\}$, $\mathscr{F}_5 = \mathscr{F}_4 \cup \{1, 2, 3, 4, 5\}$. The M_i:s have the interval property without lower bounds, so this is a sequence of matroids. Further, it can be verified that property (i) of Definition 4.4 holds. For example $\{3, 4, 5\}$ is closed in M_1 and we can check that it is closed in $M_2, ..., M_5$ as well. Moreover, we have rank $M_i = i$ for all i. Now, since $(E, \mathscr{F})$ is a greedoid of rank 5 and its feasible sets is exactly those that satisfy the condition in Definition 4.4, we know that it is a Gaussian greedoid.

We will now state a theorem that shows the characterization of closure operators of greedoids. However, the proof will be omitted but interested readers can find it in [2].

Theorem 4.2 *A function $\sigma : 2^E \mapsto 2^E$ is the closure operator of a greedoid if and only if for all $X, Y, Z \subseteq E$ and $x, y \in E \setminus X$:*

(i) $X \subseteq \sigma(X)$,
(ii) $\sigma(\sigma(X)) = \sigma(X)$,

(iii) *if* $\sigma(X) = \sigma(Y)$, *then* $\sigma(X) = \sigma(X \cup Y)$,
(iv) *if* $X \subseteq Y \subseteq Z$ *and* $\sigma(X) = \sigma(Z)$, *then* $\sigma(X) = \sigma(Y)$,
(v) *if* $\sigma(X \cup y) = \sigma(X \cup x \cup y)$ *but* $\sigma(X \cup x) \neq \sigma(X \cup x \cup y)$, *then there exists a* $z \in X \cup x$ *with* $\sigma((X \cup x) \setminus z) = \sigma(X \cup x)$.

We see that the closure operator of a greedoid is extensive, (i), and idempotent, (ii). However $X \subseteq Y$ does not imply $\sigma(X) \subseteq \sigma(Y)$ so it is, in general, not monotone and therefore not a closure operator in the formal sense. Next, we will examine the closure operator of our vertex search greedoid.

Example 4.6 Consider Greedoid 1 defined on page 12-13 and let $X = \{b\}$. We then have $\sigma(X) = \{b, e, f, g\}$, which shows that (i) holds for this set. Further, we can not add a, c or d to $\{b, e, f, g\}$ without increasing the rank function, so (ii) holds for this set as well. Now, if we let $Y = \{a, b\}$ we find out that $\sigma(Y) = \{a, b, g\}$. This shows that $X \subseteq Y$ but $\sigma(X) \nsubseteq \sigma(y)$. Hence, the closure operator is not monotone. If we instead consider the sets $\{a, b, d\}$ and $\{a, b, e\}$ we can check that they are both closed sets. However their intersection equals $\{a, b\}$ which we have seen is not a closed set.

We close this section with another feasibility concept, namely closure feasibility.

Definition 4.5 *Let* $(E, \mathscr{F})$ *be a greedoid. We say that a subset* $X \subseteq E$ *is* **closure feasible** *if* $X \subseteq \sigma(A)$ *implies* $X \subseteq \sigma(B)$ *for all* $A \subseteq B \subseteq E$.

If we denote the family of all closure feasible sets by $\mathscr{C}$, it can be shown that $\mathscr{C} \subseteq \mathscr{R}$ and that $\mathscr{C}$ is closed under union. Further, it can be shown that $\mathscr{C} = \mathscr{R}$ if and only if $(E, \mathscr{F})$ is an interval greedoid, and in this case $(E, \mathscr{C}) = (E, \mathscr{F})$ is an antimatroid. For proofs, see [2].

4.3 Operations

In this section we will introduce the three operations truncation, restriction and contraction, and further, apply these to our greedoid examples. We start with defining truncation.

Definition 4.6 *Let* $(E, \mathscr{F})$ *be a greedoid of rank* r. *Then the* **k-truncation**, *where* $0 \leq k \leq r$, *is defined as* $\mathscr{F}^{(k)} = \{X \in \mathscr{F} : |X| \leq k\}$.

Example 4.7 Recall the Gaussian elimination greedoid $(E, \mathscr{F})$ in Example 3.7 and the sequence of matroids $M_i = (E, \mathscr{F}_i)$, $0 \leq i \leq 5$ in Example 4.5. Then $\mathscr{F}_i = \mathscr{F}^{(i)}$, that is, the i-truncation defined above.

We have that the empty set is included in every k-truncation. Further, (G2) implies that if $X, Y \in \mathscr{F}^{(k)} \subseteq \mathscr{F}$ and $|X| > |Y|$ then there is an $x \in X \setminus Y$ with $Y \cup x \in \mathscr{F}^{(k)}$, which shows that $(E, \mathscr{F}^{(k)})$ is a greedoid as well.

Example 4.8 Consider Greedoid 3 defined on page 14-15. We have that the 2-truncation is equal to $\mathscr{F}_3^{(2)} = \big\{\emptyset, \{(r,a)\}, \{(r,b)\}, \{(r,d)\}, \{(r,a),(a,e)\}, \{(r,a),(a,f)\}, \{(r,a),(r,b)\}, \{(r,a),(r,d)\}, \{(r,b),(b,a)\}, \{(r,b),(b,c)\}, \{(r,b),(r,d)\}, \{(r,d),(r,g)\}\big\}$. It can be verified that (G1) and (G2) are satisfied, so $(E, \mathscr{F}_3^{(2)})$ is a greedoid.

Next, we move on to the operation known as restriction, which also gives rise to a new greedoid. That (G1) and (G2) hold in the restricted set system is obvious from the definition below.

Definition 4.7 *Let $(E, \mathscr{F})$ be a greedoid and let $T \subseteq E$ be an arbitrary subset. Then the* **restriction** *of $(E, \mathscr{F})$ to T is defined as the set system $(T, \mathscr{F}_T)$, where $\mathscr{F}_T = \{X \in \mathscr{F} : X \subseteq T\}$.*

In Example 3.6, we became acquainted with the poset greedoid. We saw that it was a subclass to the class of antimatroids and it turns out that it is also a subclass of the previously mentioned class called local poset greedoids. In fact, we have that $(E, \mathscr{F})$ is a local poset greedoid if and only if the restriction of $(E, \mathscr{F})$ to any feasible set is a poset greedoid.

Example 4.9 Again, consider Greedoid 3 and let $T = \{(r,a),(r,b),(a,e))\}$. The restriction of $(E_3, \mathscr{F}_3)$ to T is the greedoid $(T, \mathscr{F}_T)$, where $\mathscr{F}_T = \big\{\emptyset, \{(r,a)\}, \{(r,b)\}, \{(r,a),(r,b)\}, \{(r,a),(a,e)\}, \{(r,a),(r,b),(a,e)\}\big\}$. The feasible sets are precisely the ideals of T for the poset $(T, \subseteq)$. It can be verified that the restriction of Greedoid 3 to any feasible set is a poset greedoid. Therefore, as mentioned in section 3.1, Greedoid 3 belongs to the class of local poset greedoids.

We know from before that matroids also belong to the class of local poset greedoids, and are precisely those that satisfy the interval property without lower bounds. A way to distinguish which local poset greedoids that are also directed branching greedoids is given by the following proposition, where we refer to [2] for the proof.

Proposition 4.2 *A local poset greedoid $(E, \mathscr{F})$ is a directed branching greedoid if and only if for all $A, B \in \mathscr{F}$:*

$$\sigma(A) \cap \sigma(B) \subseteq \sigma(A \cup B) \subseteq \sigma(A) \cup \sigma(B).$$

Example 4.10 Consider Greedoid 2 defined on page 13. We have seen that this is a matroid and it can therefore not satisfy the condition in Proposition 4.2. For example, let $A = \big\{\{r,a\},\{r,b\},\{r,c\},\{r,d\},\{e,g\},\{f,g\}\big\}$ and $B = \big\{\{r,a\},\{r,b\},\{a,e\},\{a,f\},\{c,d\},\{d,g\}\big\}$. Then we get that $\sigma(A) = A \cup \big\{\{a,b\},\{b,c\},\{c,d\},\{e,f\}\big\}$, $\sigma(B) = B \cup \big\{\{a,b\},\{e,f\}\big\}$ and $\sigma(A \cup B) = E_2$. So far so good, but $\sigma(A) \cup \sigma(B) = E_2 \setminus \{d,e\}$ which shows that $\sigma(A \cup B) \nsubseteq \sigma(A) \cup \sigma(B)$.

The next proposition makes use of the restriction operation to show the relationship between interval greedoids and antimatroids.

Proposition 4.3 *Let $(E, \mathscr{F})$ be a greedoid. It is an interval greedoid if and only if the restriction of $(E, \mathscr{F})$ to each feasible set $X \in \mathscr{F}$ is an antimatroid.*

Proof. Suppose $(E, \mathscr{F})$ is an interval greedoid and let T be an arbitrary feasible set. The restriction of $(E, \mathscr{F})$ to T is the greedoid $(T, \mathscr{F}_T)$, where $\mathscr{F}_T = \{X \in \mathscr{F} : X \subseteq T\}$. Suppose that for $A, B, C \in \mathscr{F}_T$ we have $A \subseteq B \subseteq C$, and for $x \in T \setminus C$ we have $A \cup x \in \mathscr{F}_T$ and $C \cup x \in \mathscr{F}_T$. Then, since $\mathscr{F}_T \subseteq \mathscr{F}$, we have $A, B, C \in \mathscr{F}, x \in E \setminus C, A \cup x \in \mathscr{F}$ and $C \cup x \in \mathscr{F}$. But this implies that $B \cup x \in \mathscr{F}$ since $(E, \mathscr{F})$ has the interval property, and since $B \cup x \subseteq C \cup x \subseteq T$ we also have $B \cup x \in \mathscr{F}_T$. Thus, $(T, \mathscr{F}_T)$ inherits the interval property. Now, if $A, B \in \mathscr{F}_T, A \subseteq B, x \in T \setminus B, A \cup x \in \mathscr{F}_T$ implies $B \cup x \in \mathscr{F}_T$, we know that $(T, \mathscr{F}_T)$ is an antimatroid. Well, $(T, \mathscr{F}_T)$ is accessible and every feasible set is a subset of T, so $C \cup x \in \mathscr{F}_T$ for every C and x such that $B \subseteq C, x \in T \setminus C$. Further, $x \in T \setminus C$ implies $x \in T \setminus B$ for every C such that $B \subseteq C$. This shows that $(T, \mathscr{F}_T)$ has the interval property without upper bounds, so it is an antimatroid. On the other hand, suppose that $(T, \mathscr{F}_T)$ is an antimatroid where T is a feasible set in the greedoid $(E, \mathscr{F})$. We know that every antimatroid is also an interval greedoid, so the interval property holds for $(T, \mathscr{F}_T)$ for all $T \in \mathscr{F}$. But $\mathscr{F} = \{\cup \mathscr{F}_T : T \in \mathscr{F}\}$, so the interval property holds for all feasible sets. Thus, $(E, \mathscr{F})$ is an interval greedoid. $\square$

The third and final operation that will be presented, contraction, is defined as follows.

Definition 4.8 *Let $(E, \mathscr{F})$ be a greedoid and let $B \subseteq E$ be a feasible set. Then the* **contraction** *of B is defined as the set system $(E \setminus B, \mathscr{F}/B)$, where $\mathscr{F}/B = \{X \subseteq E \setminus B : X \cup B \in \mathscr{F}\}$.*

From the definition we see that $\emptyset \in \mathscr{F}/B$. Further, if $X, Y \in \mathscr{F}/B$ with $|X| > |Y|$ we also have $|X \cup B| = |X| + |B| > |Y| + |B| = |Y \cup B|$. Since $X \cup B$ and $Y \cup B$ are feasible sets there exists an $x \in (X \cup B) \setminus (Y \cup B)$ such that $Y \cup B \cup x \in \mathscr{F}$, which implies that there exists an $x \in X \setminus Y$ such that $Y \cup x \in \mathscr{F}/B$. Hence, $(E \setminus B, \mathscr{F}/B)$ is also a greedoid. We close this section with an example of the contraction operation.

Example 4.11 Consider Greedoid 4 defined on page 15. We have that $B = \{u_1, u_4, u_5\}$ is a feasible set and the contraction of B is equal to the set system $(E_3 \setminus B, \mathscr{F}_4/B)$, where $E_3 \setminus B = \{u_2, u_3\}$ and $\mathscr{F}_4/B = \big\{\emptyset, \{u_2\}, \{u_3\}, \{u_2, u_3\}\big\}$. We can see that this set system satisfies the greedoid axioms.

4.4 Optimization

In this final section of the thesis we will briefly look at how greedoids are related to optimization problems via the greedy algorithm. We let $(E, \mathscr{L})$ be a simple hereditary language over a finite alphabet E, in which we call a maximal feasible word **basic**. The optimization problem to solve is as follows: Given an objective function $\omega : \mathscr{L} \mapsto \mathbb{R}$, we want to find a basic word α that maximizes $\omega(\alpha)$. The greedy algorithm, with which we tackle this problem, seeks the best option at each stage in hope to find the optimal solution. It can be described as follows:

(1) Set $\alpha = \emptyset$.
(2) Choose an $x \in E$ with $\alpha x \in \mathscr{L}$ such that $\omega(\alpha x) \geq \omega(\alpha y)$ for all $y \in E$ with $\alpha y \in \mathscr{L}$.
(3) Set $\alpha = \alpha x$.
(4) If α is basic, then stop. Otherwise, go to (2).

The suitable question to ask now is: Will the greedy algorithm produce an optimal solution in this setting? The rest of the thesis is devoted to answer that question. First, we need to specify a certain relationship between the objective function and the language.

Definition 4.9 *We say that an objective function* $\omega : \mathscr{L} \mapsto \mathbb{R}$ *is* **compatible** *with* $\mathscr{L}$ *if it, for* $\alpha x \in \mathscr{L}$ *such that* $\omega(\alpha x) \geq \omega(\alpha y)$ *for every* $\alpha y \in \mathscr{L}$, *satisfies*

(i) $\alpha\beta x\gamma \in \mathscr{L}$ *and* $\alpha\beta z\gamma \in \mathscr{L}$ *imply that* $\omega(\alpha\beta x\gamma) \geq \omega(\alpha\beta z\gamma)$,
(ii) $\alpha x\beta z\gamma \in \mathscr{L}$ *and* $\alpha z\beta x\gamma \in \mathscr{L}$ *imply that* $\omega(\alpha x\beta z\gamma) \geq \omega(\alpha z\beta x\gamma)$.

What this definition says is that if x is the best option after α then x should also be the best option at every later stage (i), and it should always be a better option to choose x before z than vice versa (ii). If so, the objective function is compatible with the language. After this definition it is possible to characterize greedoid languages algorithmically.

Theorem 4.3 Let $(E, \mathscr{L})$ be a simple hereditary language. Then it is also a greedoid language if and only if the greedy algorithm produces an optimal solution for every compatible objective function on $\mathscr{L}$.

Proof. For interval greedoids, see [1]. For greedoids in general, see [2].

Now, let us study the objective function a bit closer. Let $u : E \mapsto \mathbb{R}$ be some given weight function. Then a **linear** objective function $\omega : \mathscr{L} \mapsto \mathbb{R}$ is a function of the form $\omega(x_1x_2...x_n) = \sum_{i=1}^{n} u(x_i)$. All linear objective functions are compatible if $(E, \mathscr{L})$ is a matroid but not necessarily if $(E, \mathscr{L})$ is a greedoid. Recall the family of all rank feasible sets $\mathscr{R} \subseteq 2^E$ of a greedoid $(E, \mathscr{F})$. Further, let a **level set** of the weight function u be a set of the form $\{x \in E : u(x) \geq c\}$. The linear objective function described above is called $\mathscr{R}$**-compatible** if the level set $\{x \in E : u(x) \geq c\} \in \mathscr{R}$ for any given constant $c \in \mathbb{R}$. It can then be proven that for a greedoid $(E, \mathscr{F})$, the greedy algorithm produces an optimal solution for every $\mathscr{R}$-compatible objective function. From Proposition 4.1, we know that all sets in 2^E are rank feasible if and only if $(E, \mathscr{F})$ is a matroid. Thus, $(E, \mathscr{F})$ is a matroid if and only if every linear objective function is $\mathscr{R}$-compatible.

We have now reached the end of this glimpse into greedoid theory. For more of optimization on greedoids and of greedoids in general, [1] and [2] are highly recommended.

References

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