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Introduction to nested radicals

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Abstract

The study of nested radicals do not date back very long in history. Although those were encountered early on as in Viète's definition of π, they have not been studied for their own sake until the 1800s by Galois or even early 1900s by Ramanujan. The denesting of nested radicals differs a lot between finite and infinite radicals. This paper will show an overall image of what nested radicals are and how to denest those. We will show a relatively broad spectrum of nested radicals which can offer the reader a general view of the topic.

Contents

1 Introduction

$$\sqrt{3\sqrt{3\sqrt{3...}}} = 3$$

$$\sqrt{6+\sqrt{6+\sqrt{6+...}}} = 3$$

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+..}}}} = 3$$

Nested radicals come in many intriguing forms, those involve addition, subtraction, multiplication, roots to different powers, alternating radicals, etc. This has baffled mathematicians throughout time, starting with Viète who defined π using an infinite product that involves nested radicals. From this point onward, we have seen numerous well-known mathematicians work with nested radicals. Among those we find Srinivasa Ramanujan, Aaron Herschfeld, George Pólya, Évariste Galois and Edward Kasner. Even though many mathematicians have researched the subject, Ramanujan has caught a lot of attention and is very well-known within the field, he is also often referred to in studies involving nested radicals. Nested radicals may be solved using different tools and theorems. Those involve using algebraic methods such as induction, manipulating the expression and more, but also using trigonometry, convergence theories, and more.

1.1 Background

Nested radicals derive from a background of geometry, trigonometry and mathematical analysis in general. Tools obtained within those fields are necessary for solving or denesting the radicals. When working with nested radicals one needs to have an understanding of convergence, fields, trigonometrical identities, Maclaurin's formula etc. Depending on the type of radical one must also understand theorems that are specific for nested radicals such as Landau's algorithm and Herschfeld's convergence theorem.

The rise of nested radicals as we know it was, as earlier mentioned, through Viète's formula defining π. This was very much through geometry and bisecting angles within the unit circle which will be demonstrated in Section 5.3. Viète did not only introduce nested radicals, but also infinite products which some nested radicals can be rewritten to. Infinite products as well as infinite sums are also used as tools for comparison when solving nested radicals, this due to the fact that those are well-known and researched and have many theorems supporting their solutions.

After Viète's definition of π, many mathematicians have picked up nested radicals and worked on new definitions of π and other constants. Those include either nested radicals or infinite products or a combination of those.

Even though Viète wrote the first published nested radical, it is mostly Ramanujan who has been known as the mathematician who started researching nested radicals. This is due to the fact that the radicals per se were not the main focus for Viète or any other mathematician that has touched on them. For Ramanujan on the other hand, nested radicals were the focus. He worked with both finite and infinite radicals involving roots to the power of 2, 3 and even general n. Ramanujan was, and still is, very well-known within the fields of nested radicals and in many other mathematical domains. After Ramanujan's intriguing publications, many mathematicians started working with nested radicals as well as questioning Ramanujan's methods. This lead to the developing of countless theorems concerning denesting and convergence of nested radicals. Some of those will be discussed in this paper. If the reader has further interest in those studies, references will be available.

1.2 Definitions of central concepts

We've been discussing nested radicals and the denesting of those as if those were well known concepts, but those are only well-known to the ones who study them. We will therefore thoroughly define these concepts below.

Definition 1. Nested radical

A nested radical is a root expression nested within other root expressions. Infinitely nested radicals are nested radicals that are infinitely locked within each other, whereas finitely nested radicals contain a finite amount of roots.

Now that we have defined nested radicals, we need to understand what it means to denest those and find their nesting depth.

Definition 2. Nesting depth [8]

Let A, B be expressions in $+, -, \times, \div, \sqrt{\ }$ and rational numbers. Then depth is defined recursively as follows:

$depth(A) = 0$ if A is a rational number.

$depth(A \pm B) = depth(AB) = depth(\frac{A}{B}) = \max(depth(A), depth(B))$

$depth(\sqrt[n]{A}) = 1 + depth(A)$

It is of essence to note the difference between the nesting depth of an expression and the amount of radical signs the expression has. We shall view this example to get a better grip on the difference between the two.

$$\sqrt{2 + \sqrt{3 + \sqrt{5}}}$$

is an expression with a nesting depth 3, whereas

$$\sqrt{2} + \sqrt{3} + \sqrt{5}$$

is of depth 1.

For a better understanding of the definition of denesting, which is crucial for the understanding of this paper, an example of the statements will be demonstrated.

Example 1. We view the following example

$$\sqrt{2} + \sqrt{3 + \sqrt{5}}.$$

We will be focusing on one part at a time.

$$depth(5) = 0$$

$$depth(\sqrt{5}) = 1$$

$$depth(3 + \sqrt{5}) = \max(depth(3), depth(\sqrt{5})) = 1$$

$$depth(\sqrt{3 + \sqrt{5}}) = 1 + depth(3 + \sqrt{5}) = 2$$

$$depth(\sqrt{2} + \sqrt{3 + \sqrt{5}}) = \max(depth(\sqrt{2}), depth(\sqrt{2} + \sqrt{3 + \sqrt{5}})) = 2.$$

Definition 3. Denesting
Denesting is simplifying a nested radical so that it decreases in nesting depth.

Among the theories that have been groundbreaking for untangling nested radials, we have Galois theory which has been central for the evolution of the studies of nested radicals. Many mathematicians relied on the theory for a long time, among those the well-known Ramanujan. Galois theory is still used by mathematicians today, although some have chosen to alter the theory or come up with their own theories. The issue those see with Galois theory is the field in which the denesting occurs and the roots of unity within it. More reading on the topic can be found in the following source [2]. Other very well-known theories within the field have been Kummer's, Zippel's and Landau's. For more insight on those theories I would suggest reading the following references [5], [8], [7], [10].

1.3 Field of use

Nested radicals occur within many fields in mathematics. They are often found in a finite form, but at times they can also be infinitely nested. We often find nested radicals in the fields of trigonometry which will be shown below. We will first start by introducing a couple of terms necessary to the demonstration and then move on to showing an example of how nested radicals can occur.

Definition 4. Root of unity
A root of unity is a complex number a such that

$$a^n = 1$$

with n being a positive integer.

These are known as de Moivre numbers and are used to get an approximation of aspects of a circle through increasing the number of sides in a polygon.

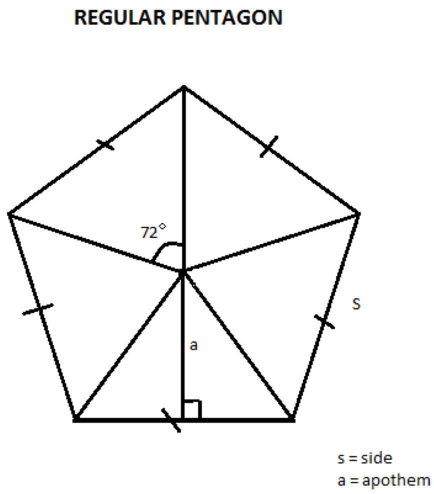

Figure 1: Regular pentagon

Definition 5. Regular Pentagon
A regular pentagon is a polygon that is made up of 5 equal sides.

Figure 1 shows a regular pentagon and gives an idea on how to calculate the area of it. This is obtained through dividing the pentagon in 5 equal triangles.

The area of a regular pentagon follows $a \cdot s \cdot \frac{5}{2}$, where a is the apothem which is the line drawn from the midpoint of one of the sides to the center of the pentagon, and s is the side. This area is calculated through addition of the areas of the triangles that make up the pentagon as shown in Figure 1.

Let us calculate the area of a regular pentagon with the sides of length 1. We need to first find out the length of the apothem.

$$a = \frac{1}{2\tan\frac{\pi}{5}} = \frac{1}{2\sqrt{5-2\sqrt{5}}}$$

This shows that we already get a nested radical before even calculating the area.

$$area = 1 \cdot \frac{1}{2\tan\frac{\pi}{5}} \cdot \frac{5}{2} = \frac{5}{4\sqrt{5-2\sqrt{5}}}$$

As shown above, nested radicals with depth two are often used in the calculations of regular pentagons. This comes to show that nested radicals are often seen within the fields of geometry. Nested radicals are also often a result of trigonometric functions which is shown in the example of the regular pentagon and will also be shown in Theorem 5.3.

Nested radicals do not only arise in studies of trigonometry and geometry but also within algebra. Consider the general quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for $ax^2 + bx + c = 0$.

If $a = 1$, $b = 3$ and $c = \sqrt{5}$ we get that

$$x = \frac{-3 \pm \sqrt{9 - 4\sqrt{5}}}{2}.$$

These are merely a couple examples of how nested radicals can occur which comes to show the value in studying those.

2 Finitely nested radicals

Finitely nested radicals are interesting to tackle due to the frequency of their appearance within other fields where it is beneficial to be able to simplify those. Some examples of how to tackle finitely nested radicals will be demonstrated although there are many shapes those can take. The field of finitely nested radicals has not been researched as much as others, and data are not very easy to find.

Denesting techniques will differ depending on the nested radical that is dealt with, the techniques will therefore be demonstrated in the subsections prior to the solutions.

2.1 Nested radicals of depth 2 containing square roots

We will begin by denesting simple finite nested radicals such as $\sqrt{a + b\sqrt{c}}$.

Since we are looking to reduce the depth of the nested radical which in this example is 2, we can start by equating to a nested radical with a depth one unit smaller which in our case is depth 1.

Theorem 1. [7]

Let $a, b, c \in \mathbb{Q}$ and c not a complete square.

The following denesting $\sqrt{a + b\sqrt{c}} = \sqrt{d} + \sqrt{e}$ is possible, if and only if d and e are rational roots to the quadratic equation $x^2 - ax + \frac{b^2 c}{4} = 0$.

Proof.

$$\sqrt{a + b\sqrt{c}} = \sqrt{d} + \sqrt{e}$$

$$\leftrightarrow a + b\sqrt{c} = (\sqrt{d} + \sqrt{e})^2$$

$$\leftrightarrow a + b\sqrt{c} = d + 2\sqrt{de} + e$$

This implies that $a = d + e$ and $b\sqrt{c} = 2\sqrt{de}$. This step involves setting an equality between the rational and the irrational parts of the equations.

$$b\sqrt{c} = 2\sqrt{de}$$

$$\leftrightarrow b^2c = 4de$$

Let us go back to the quadratic equation we started with and solve for x.

$$x^2 - ax + \frac{b^2c}{4} = 0$$

$$x - \frac{a}{2} = \pm\sqrt{\frac{a^2 - b^2c}{4}}$$

$$x = \frac{a \pm \sqrt{a^2 - b^2c}}{2}$$

Let us now substitute our values for a, b and c. We get that

$$x = \frac{d + e \pm \sqrt{(d+e)^2 - 4de}}{2}$$

$$x = \frac{d + e \pm \sqrt{(d-e)^2}}{2}$$

$$x_1 = d$$

$$x_2 = e$$

These are the exact roots we needed to obtain in order for this denesting to occur.

Going in the converse direction, given that d, e are roots to the quadratic equation we get, through the rational root theorem, that

$$d + e = a$$

$$de = \frac{bc^2}{4}$$

$$\rightarrow \sqrt{a + b\sqrt{c}} = \sqrt{d} + \sqrt{e}.$$

The theorem has now been proven. □

Example 2. We will now apply the knowledge we obtained from the general formula to the following example:

$$\sqrt{5 + 2\sqrt{6}} = \sqrt{d} + \sqrt{e}.$$

Using the denesting technique in Theorem 1, we get that $5 = d + e$ and $6 = de$, which gives us that $d = 2$ and $e = 3$. This leads to following expression:

$$\sqrt{5 + 2\sqrt{6}} = \sqrt{2} + \sqrt{3}.$$

2.2 Nested radicals of depth 2 containing cube roots

Theorem 2. *Let $a, b \in \mathbb{Q}$ with b not a perfect square.*

The denesting of

$$\sqrt[3]{a+\sqrt{b}} = x + \sqrt{y}$$

where $x, y \in \mathbb{R}$ and y is not a perfect square can occur if and only if x is a rational root to the equation

$$1 = \frac{(a-x^3)(a+8x^3)^2}{27bx^3}$$

and $y = \frac{a-x^3}{3x}$.

Proof.

$$\sqrt[3]{a+\sqrt{b}} = x + \sqrt{y}$$

Squaring both sides gives us

$$a + \sqrt{b} = x^3 + y\sqrt{y} + 3x^2\sqrt{y} + 3xy$$

This shows that $\sqrt{y}$ cannot be rational due to the fact that the expression on the left hand side contains an irrational number being $\sqrt{b}$ which means that the right hand side must also contain an irrational number for the expressions to be equal.

We rewrite the above as an expression of $\sqrt{y}$

$$\sqrt{y} = \frac{a+\sqrt{b}-x^3-3xy}{3x^2+y}.$$

If we were to rewrite this in a more general form, we could state that

$$\sqrt{y} = p + q\sqrt{b}$$

$$\leftrightarrow y = p^2 + bq^2 + 2pq\sqrt{b}$$

where $p, y \in \mathbb{Q}$. Again we get an irrational term on one side of the equation which cannot possibly be equated to a rational number. This must mean that

$$2pq\sqrt{b} = 0.$$

Since we know that $b \neq 0$, either p or q must be 0. If $q = 0$, then $y = p^2$ while y cannot be a perfect square. This leaves us with the only possibility, that $p = 0$. This gives that

$$\sqrt{y} = q\sqrt{b}.$$

We make this substitution for $\sqrt{y}$:

$$a + \sqrt{b} = x^3 + bq^3\sqrt{b} + 3x^2q\sqrt{b} + 3bxq^2.$$

We identify the rational and irrational parts of the equation.

$$a = x^3 + 3bxq^2 \leftrightarrow q = \sqrt{\frac{a - x^3}{3bx}}$$

The positive root is the one of interest here because $q > 0$

$$\sqrt{b} = bq^3\sqrt{b} + 3x^2q\sqrt{b} \leftrightarrow 1 = bq^3 + 3x^2q$$

Substituting the expression for q, we get that

$$1 = b\sqrt{\frac{(a - x^3)^3}{27b^3x^3}} + 3x^2\sqrt{\frac{a - x^3}{3bx}}$$

$$\leftrightarrow 1 = \sqrt{\frac{a - x^3}{3bx}}\left(\frac{a - x^3}{3x} + 3x^2\right)$$

$$\leftrightarrow 1 = \sqrt{\frac{a - x^3}{3bx}}\left(\frac{a + 8x^3}{3x}\right).$$

We square both sides and get

$$\leftrightarrow 1 = \frac{(a - x^3)(a + 8x^3)^2}{27bx^3.}$$

The theorem has now been proven. □

Example 3. In order to understand and test Theorem 2.2, we will study the following example.

$$\sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2} = 1$$

We start with the first part of the equation

$$\sqrt[3]{\sqrt{5} + 2}.$$

Here $a = 2$ and $b = 5$. We now need to find the value of x.

Let us start by expanding our expression and then substitute for a and b.

$$1 = \frac{(a - x^3)(a + 8x^3)^2}{27bx^3}$$

$$\leftrightarrow 64x^9 - 48ax^6 + (27b - 15a^2)x^3 - a^3 = 0$$

$$\leftrightarrow 64x^9 - 96x^6 + 75x^3 - 8 = 0$$

According to the well-known rational root theorem, if $r = \frac{s}{t}$ is a root to this equation, then $s|8$ and $t|64$ which gives us the following options

$$s = \pm 1, \pm 2, \pm 3, \pm 4, \pm 8$$

and

$$t = \pm 1, \pm 2, \pm 4, \pm 8, \pm 32, \pm 64.$$

After testing those combinations, we get that $x = \frac{1}{2}$ solves our polynomial equation. This gives us

$$y = \frac{2 - \frac{1}{2^3}}{3 \cdot \frac{1}{2}} = \frac{5}{4}.$$

Therefore the denesting we get is

$$\sqrt[3]{\sqrt{5} + 2} = \frac{1 + \sqrt{5}}{2}.$$

This also happens to be the golden ratio. For more information on that, see Section 5.1.

Similarly, we treat the second part of the equation $\sqrt[3]{\sqrt{5} - 2}$ where $a = -2$ and $b = 1$.

Through substitution, we get the following polynomial

$$64x^9 + 96x^6 + 75x^3 + 8 = 0$$

We find, through the same procedure, that $x = -\frac{1}{2}$ and $y = \frac{5}{4}$. The denesting obtained is

$$\sqrt[3]{\sqrt{5} - 2} = \frac{\sqrt{5} - 1}{2}$$

.

This gives us

$$\sqrt[3]{\sqrt{5} + 2} - \sqrt[3]{\sqrt{5} - 2} = \frac{1 + \sqrt{5}}{2} - \frac{\sqrt{5} - 1}{2} = 1$$

The denesting is thereby complete.

2.3 Ramanujan's theory on finitely nested radicals

Ramanujan came up with this theory in order to denest finitely nested radicals involving square roots of cube roots.

Theorem 3. [12]

For arbitrary m and n we get, assuming that the square root is defined, that

$$\sqrt{m\sqrt[3]{4m - 8n} + n\sqrt[3]{4m + n}} = \pm \frac{\sqrt[3]{(4m + n)^2} + \sqrt[3]{4(m - 2n)(4m + n)} - \sqrt[3]{2(m - 2n)^2}}{3}$$

.

Proof. We start by squaring both sides which produces an equivalent equation.

$$\begin{aligned} 9m\sqrt[3]{4m-8n}+9n\sqrt[3]{4m+n} = &\sqrt[3]{(4m+n)^4}+\sqrt[3]{16(m-2n)^2(4m+n)^2} \\ &+\sqrt[3]{4(m-2n)^4}+2\sqrt[3]{4(4m+n)^3(m-2n)} \\ &-2\sqrt[3]{2(4m+n)^2(m-2n)^2}-2\sqrt[3]{8(m-2n)^3(4m+n)} \end{aligned}$$

$$\begin{aligned} \leftrightarrow 9m\sqrt[3]{4m-8n}+9n\sqrt[3]{4m+n} = &\sqrt[3]{(4m+n)^4}+\sqrt[3]{4(m-2n)^4} \\ &+2\sqrt[3]{4(4m+n)^3(m-2n)}-2\sqrt[3]{8(m-2n)^3(4m+n)} \end{aligned}$$

$$\begin{aligned} \leftrightarrow 9m\sqrt[3]{4m-8n}+9n\sqrt[3]{4m+n} = &(4m+n)\sqrt[3]{4m+n}+(m-2n)\sqrt[3]{4m-8n} \\ &+2(4m+n)\sqrt[3]{4m-8n}-4(m-2n)\sqrt[3]{4m+n} \end{aligned}$$

$$\begin{aligned} &\leftrightarrow 9m\sqrt[3]{4m-8n}+9n\sqrt[3]{4m+n} = \\ &(4m+n-4m+2n)\sqrt[3]{4m+n}+(m-2n+8m+2n)\sqrt[3]{4m-8n} \\ &\leftrightarrow 9m\sqrt[3]{4m-8n}+9n\sqrt[3]{4m+n} = 9n\sqrt[3]{4m+n}+9m\sqrt[3]{4m-8n} \end{aligned}$$

The fact that we have identical expressions on both sides proves Ramanujan's theorem. □

The following theorem helps determine if a nested radical of type $\sqrt{\sqrt[3]{a}+\sqrt[3]{b}}$ can be denested.

Theorem 4. [12]

Let α, $\beta \in \mathbb{Q}$, and $\alpha, \beta \neq 0$ such that $\frac{\alpha}{\beta}$ is not a perfect cube in $\mathbb{Q}$. Then, $\sqrt{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$ can be denested if and only if there are integers m, n such that $\frac{\alpha}{\beta} = \frac{(4m-8n)m^3}{(4m+n)n^3}$.

Rather than proving this theorem, I will demonstrate an example showing how the theorem works.

Example 4. If $m = n = 1$, we get that

$$\frac{\alpha}{\beta} = \frac{(4-8)1^3}{(4+1)1^3} = \frac{-4}{5}.$$

This shows that $\sqrt{\sqrt[3]{5}-\sqrt[3]{4}}$ can be denested.

Using Theorem 3, we can find the solution to this nested radical.

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \pm\frac{\sqrt[3]{25}-\sqrt[3]{20}-\sqrt[3]{2}}{3}$$

We now need to chose so that we get a positive answer.

$$\sqrt{\sqrt[3]{5}-\sqrt[3]{4}} = \frac{\sqrt[3]{20}-\sqrt[3]{25}+\sqrt[3]{2}}{3}$$

3 Infinitely nested radicals involving products

Infinitely nested radicals do not arise as often as the finitely nested radicals, although Viète came across one while finding a formula for π and he was not the only one to encounter those. In general though, infinitely nested radicals are studied more for their own sake as an intriguing subject. Denesting those can still be helpful within the fields of computer science and more. When denesting infinitely nested radicals, as when we deal with infinity in general (infinite sums, infinite products), we have to study the convergence of those. We will therefore start by understanding the concept of convergence and its close relation to denesting, and later move on to denesting different types of infinitely nested radicals.

3.1 Convergence

Definition 6. A sequence a_n, where $n \in \mathbb{N}$, converges if its limit $\lim_{n\to\infty} a_n$ exists. A sequence that does not converge is said to diverge.

In order to tackle infinitely nested radicals, we have to fully understand convergence which is a phenomenon that derives from limits. When we tackle infinitely nested radicals, we are actually finding the limit to which the expression converges. We must therefore understand the concept of limits. We also need to understand convergence and know the different theorems for convergence such as the monotone convergence theorem, the direct comparison test etc.

Theorem 5. Criterion for convergence of infinite products in $\mathbb{R}$ [13]
$\prod_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} \log(a_n)$ *converges (for all* $a_n > 0$*) to a finite number or* $-\infty$.

Proof. If $\prod_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} \prod_{k=1}^{n} a_k = A$ where A is finite.
If $A \neq 0$ then

$$a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot ... = A$$

$$\leftrightarrow \log(a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot ...) = \log A$$

$$\leftrightarrow \log a_1 + \log a_2 + \log a_3 + \log a_4 + ... = \log A$$

Since the product can be rewritten to the sum, the theorem has been proven.
If $A = 0$ we get that $log(A) = -\infty$. □

Theorem 5 also leads to the following, put

$$a_n = 1 + p_n.$$

Then $\prod_{n=1}^{\infty}(1 + p_n)$ for all $p_n \geq 0$ converges if and only if $\sum_{n=1}^{\infty} \log(p_n)$ converges.

We will now present an example in order to test its convergence.

Example 5. We consider the nested radical

$$\sqrt{1\sqrt{2\sqrt{3...}}}$$

This can be rewritten to

$$\prod_{n=1}^{\infty} n^{\frac{1}{2^n}}$$

The product converges if

$$\sum_{n=1}^{\infty} \log(n^{\frac{1}{2^n}})$$

converges.

According to the laws of logarithms we get that

$$\sum_{n=1}^{\infty} \log(n^{\frac{1}{2^n}}) = \sum_{n=1}^{\infty} \frac{\log(n)}{2^n}$$

This sum converges by d'Alambert's ratio test as follows

$$\frac{\frac{\log(n+1)}{2^{n+1}}}{\frac{\log(n)}{2^n}}$$

$$= \frac{\log(n+1)}{2^{n+1}} \cdot \frac{2^n}{\log(n)}$$

$$= \frac{\log(n+1)}{2\log(n)} = \frac{\log(n) + \log(1+\frac{1}{n})}{2\log(n)}$$

$$= \frac{1}{2} + \frac{\log(1+\frac{1}{n})}{2\log(n)} \to \frac{1}{2}$$

This shows that the sum converges meaning that the product also converges by Theorem 5.

Definition 7. The sequence a_n is said to be monotonic when it is increasing meaning that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$, or decreasing meaning that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$.

Theorem 6. Monotone convergence theorem

The sequence a_n converges if it is monotonic and bounded.

Proof. We divide this theorem into two cases.

Case I: If a sequence is increasing and bounded (in case of an increasing sequence we are looking to find if it is bounded from above), it converges to its supremum.

Case II: Similarly, if a sequence is decreasing and bounded from below, it converges to its infimum.

Let us prove case I:

We will let a_n (for all $n \leq 1$) be an increasing sequence and let s be the supremum of a_n.

We shall now prove that $\lim_{n\to\infty} a_n = s$, and that for all $\epsilon > 0$, there exists a given $N \in \mathbb{N}$ so that for any $n \geq N$, then $|a_n - s| < \epsilon \leftrightarrow |a_n| < \epsilon + |s|$.

If $s = \sup a_n$ for all $n \geq 1$, then there exists N such that

$$s - \epsilon < a_N \leq s$$

$$s - \epsilon < a_N \leq a_n \leq s$$

for $n \in \mathbb{N}$ and $n \geq N$.

Thus $a_n \to s$ and the proof is complete.

A similar approach is used to prove case II and will therefore not be demonstrated. $\square$

3.2 Infinitely nested radicals of type $\sqrt{a\sqrt{a\sqrt{a...}}}$

Infinitely nested radicals involving products can always be rewritten to infinite products which simplifies the process due to the fact that more research has been done on infinite products and infinite sums. In this section, we will be focusing on nested radicals of the type $\sqrt{a\sqrt{a\sqrt{a...}}}$, starting with a specific example, and later move on to establishing a general formula.

Example 6. We will now solve for the following nested radical

$$\sqrt{3\sqrt{3\sqrt{3....}}}$$

We will be using different methods to obtain our results as well as show that nested radicals can be denested in many ways.

Method I:

We start by naming the given expression y.

$$y = \sqrt{3\sqrt{3\sqrt{3...}}}$$

We move on to rewriting the square roots to exponents.

$$y = 3^{\frac{1}{2}}3^{\frac{1}{4}}3^{\frac{1}{8}}3^{\frac{1}{16}}...$$

$$y = 3^{\sum_{n=1}^{\infty}\frac{1}{2^n}}$$

The sum we get is $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$ which converges to 1.

Therefore:

$$y = \sqrt{3\sqrt{3\sqrt{3...}}} = 3^1 = 3.$$

Method II: We begin by testing the convergence of the radical which convergence by Theorem 5. Again we name the given expression y.

$$y = \sqrt{3\sqrt{3\sqrt{3...}}}$$

We now rewrite the expression in the following matter since the radical is infinite and one less nest does not affect the results since 1 cannot be compared to infinity.

$$y = \sqrt{3y}$$

Solve for y:

$$y = \sqrt{3y}$$

$$y^2 = 3y$$

$$y_1 = 0, y_2 = 3$$

This method gives us two possibilities, although only one of those is possible. 0 is not a possible solution. First observe that the radical does not involve a 0 as a multiple. We may then go on to observing the convergence of this radical, which is demonstrated in the previous example showing that the exponential function converges to 1 meaning our nested radical cannot converge to 0.

We can now, using the tools we obtained through the example, establish a general formula.

Theorem 7. *Given $a \geq 0$ we get the following formula:*

$$\sqrt{a\sqrt{a\sqrt{a...}}} = a.$$

This can be proven using the same methods as used for previous examples and will therefore not be demonstrated. The procedure merely involves replacing 3 with a in the demonstrated example.

4 Infinitely nested radicals involving addition

Infinitely nested radicals involving addition are tougher to tackle than those involving products due to the fact that they usually cannot be rewritten but merely compared to infinite sums and products. Those have been of highest interest for mathematicians such as Ramanujan and Kasner. To tackle those we need to understand the concept of upper and lower bounds, as well as a couple of other theories that will be demonstrated below. We also need to get tools to evaluate the convergence of the radicals before we tackle nested radicals of this type.

A sequence that does not converge to a limit can still have a so-called limit superior or a limit inferior as long as it is bounded.

4.1 Definitions

Definition 8. Limit superior and limit inferior. [1]

Limit superior $\overline{\lim}_{n\to\infty} a_n$.
For a sequence a_n, where $n \in \mathbb{N}$, that is bounded above, we define

$$\overline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \sup(a_m : m \geq n).$$

Limit inferior $\underline{\lim}_{n\to\infty} a_n$.
For a sequence that is bounded below, we define

$$\underline{\lim_{n\to\infty}}\, a_n = \lim_{n\to\infty} \inf(a_m : m \geq n).$$

We will now define a power series in order to later understand the geometrical series formula.

Definition 9. Power series.
A power series is an expression of form

$$\sum_{n=0}^{\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3...$$

Given that c is a complex constant, a_n the n^{th} complex coefficient and x a complex variable.

Power series often take the following form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3...$$

although the proper definition is as presented in Definition 9.

It is known that there exists a number $0 \leq r \leq \infty$, also know as the radius of convergence, such that a power series
converges if $|x-c| < r$ and
diverges if $|x-c| > r$.

Definition 10. Radius of convergence.
When a series converges, we call the radius of largest disk within the series the radius of convergence.

Theorem 8. The geometrical series formula [1]

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3... = \frac{1}{1-x}$$

for all $|x| < 1$.

Proof. We start by writing out the sum and multiplying it by x.

$$\sum_{n=0}^{\infty} x^n = S_\infty = 1 + x + x^2 + x^3 ...$$

$$x \cdot S_\infty = x + x^2 + x^3 + x^4 ...$$
$$S_\infty - x \cdot S_\infty = 1$$
$$\leftrightarrow S_\infty(1-x) = 1$$
$$\leftrightarrow S_\infty = \frac{1}{1-x}$$

This shows that the geometrical series converges towards the given expression. □

4.2 Convergence

The Direct comparison test is a method that is used to test for convergence of infinite series.

Theorem 9. Direct comparison test

If $0 \leq a_n \leq b_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converges then* $\sum_{n=1}^{\infty} a_n$ *must also converge.*

It follows that, if $0 \leq a_n \leq b_n$ *and* $\sum_{n=1}^{\infty} a_n$ *diverges then* $\sum_{n=1}^{\infty} b_n$ *must also diverge.*

Theorem 10. Herschfeld's convergence theorem [4]

Let $a_n \geq 0$*. The nested radical*

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + ...}}}$$

converges if and only if the following limit superior exists and is not ∞*:*

$$\overline{\lim_{n\to\infty}}\, a_n^{\frac{1}{2^n}}.$$

Proof. Let us put

$$u_n = \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + ... + \sqrt{a_n}}}}.$$

If u_n converges, then $\overline{\lim}_{n\to\infty} a_n^{\frac{1}{2^n}}$ must also converge meaning its limit superior must be finite due to the following, in accordance with definition 8.

$$\sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + ... + \sqrt{a_n}}}} \geq \sqrt{0 + \sqrt{a_2 + \sqrt{a_3 + ... + \sqrt{a_n}}}}$$
$$\geq \sqrt{0 + \sqrt{0 + \sqrt{0 + ... + \sqrt{a_n}}}} = a_n^{\frac{1}{2^n}}$$

Conversly, if $\overline{\lim}_{n\to\infty} a_n^{\frac{1}{2^n}} < +\infty$, we choose $P > 0$ such that for all $n > 0$, we have that $a_n^{\frac{1}{2^n}} \leq P$. This leads to $a_n \leq P^{2^n}$, meaning that

$$u_n \leq \sqrt{P^2 + \sqrt{P^4 + \sqrt{P^6 + ... + \sqrt{P^{2^n}}}}} = P\sqrt{1 + \sqrt{1 + \sqrt{1 + ... + \sqrt{1}}}}.$$

Since

$$\begin{aligned} 2 = \sqrt{2+2} = \sqrt{2 + \sqrt{2+2}} = ... \\ = \sqrt{2 + \sqrt{2 + \sqrt{2 + ... + \sqrt{2+2}}}} > \sqrt{1 + \sqrt{1 + .. + \sqrt{1}}} \end{aligned}$$

we get $u_n < 2P$ for all $n > 0$.

Since u_n is not decreasing, the sequence converges by the Monotone Convergence Theorem. □

4.3 Nested radicals of type $\sqrt{a + \sqrt{a + \sqrt{a...}}}$

We will now deal with infinitely nested radicals involving addition of type $\sqrt{a + \sqrt{a + \sqrt{a...}}}$ which converge by Theorem 10.

Many can be fooled by the fact that

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + ...}}} = 2$$

and make the assumption that

$$\sqrt{a + \sqrt{a + \sqrt{a + ...}}} = a.$$

The solution to the following shows that it is not quite this simple. I will now provide a demonstration of a specific example and then go on to solving the general radical.

Example 7. We will solve $\sqrt{3 + \sqrt{3 + \sqrt{3 + ...}}}$

Lets us re-use the method of substitution.

$$y = \sqrt{3 + \sqrt{3 + \sqrt{3 + ...}}}$$

By squaring both sides we get that

$$y^2 = 3 + \sqrt{3 + \sqrt{3 + ...}}$$

By subtracting the original expression from the square expression we get

$$y^2 - y = 3$$

$$y^2 - y - 3 = 0$$

$$(y - \frac{1}{2})^2 - \frac{13}{4} = 0$$

$$y = \frac{1}{2} \pm \sqrt{\frac{13}{4}}$$

We again get two possible solutions although we will rule out the negative solution due to the fact that we are merely dealing with addition of positive terms which cannot converge towards a negative number. Consequently our solution is

$$\sqrt{3 + \sqrt{3 + \sqrt{3 + ...}}} = \frac{1 + \sqrt{13}}{2} \approx 2.3028$$

We will now present a general formula for such nested radicals

Theorem 11. *The formula*

$$\sqrt{a + \sqrt{a + \sqrt{a + ...}}} = \frac{1 + \sqrt{1 + 4a}}{2}$$

holds for all $a > 0$.

Proof. Using the same procedure as done in our example, we square both sides and subtract expressions to obtain the formula.

$$y = \sqrt{a + \sqrt{a + \sqrt{a + ...}}}$$

$$y^2 = a + \sqrt{a + \sqrt{a + ...}}$$

$$y^2 - y = a$$

$$y^2 - y - a = 0$$

$$(y - \frac{1}{2})^2 - \frac{1}{4} - a = 0$$

$$y = \frac{1 \pm \sqrt{1 + 4a}}{2}$$

As earlier established we will only consider the positive root which leaves us with:

$$\sqrt{a + \sqrt{a + \sqrt{a + ...}}} = \frac{1 + \sqrt{1 + 4a}}{2}$$

for all $a > 0$ which proves the theorem. □

4.4 Nested radicals of type $\sqrt{a+b\sqrt{a+b\sqrt{a+...}}}$

We now consider nested radicals involving addition involving both a and b.

Theorem 12.

$$\sqrt{a+b\sqrt{a+b\sqrt{a+b\sqrt{a+...}}}} = b\frac{\sqrt{1+4\frac{a}{b^2}}}{2}$$

where $a, b > 0$

Proof. The following proof is based on reference [9].

Proving this theorem involves successively multiplying b into the square roots.

$$\sqrt{a+b\sqrt{a+b\sqrt{a+b\sqrt{a+...}}}}$$

$$= \sqrt{a+\sqrt{ab^2+b^3\sqrt{a+b\sqrt{a+...}}}}$$

$$= \sqrt{a+\sqrt{ab^2+\sqrt{ab^6+b^7\sqrt{a+...}}}}$$

$$= \sqrt{a+\sqrt{ab^2+\sqrt{ab^6+\sqrt{ab^{14}+...}}}}$$

$$= \sqrt{\frac{a}{b^2}b^2+\sqrt{\frac{a}{b^2}b^4+\sqrt{\frac{a}{b^2}b^8+\sqrt{\frac{a}{b^2}b^{16}+...}}}}$$

$$= b\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+...}}}}$$

The solution to the following can be related back to Theorem 11 as

$$b\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+\sqrt{\frac{a}{b^2}+...}}}} = b\frac{1+\sqrt{1+4\frac{a}{b^2}}}{2}$$

This proves the theorem above knowing that the radical converges according to Theorem 10. $\square$

5 Well-known nested radicals

In this section, we will be demonstrating and calculating a couple of the most well-known nested radicals. Those are especially interesting because they give an insight into how such complex radicals can be solved using different methods as well as giving insight on how well-known mathematicians solved those.

5.1 The golden ratio

The golden ratio, also known as ϕ, compares two quantities created by dividing a distance in two uneven parts with a larger segment a and a smaller part b when the ratio between a and b (named ϕ) equals the ratio of the sum of those and the larger segment a.

In other words

$$\phi = \frac{a}{b} = \frac{a+b}{a}$$

$$\leftrightarrow \phi = \frac{a}{b} = 1 + \frac{b}{a}$$

$$\leftrightarrow \phi = 1 + \frac{1}{\phi}$$

$$\leftrightarrow \phi^2 - \phi - 1 = 0$$

$$\leftrightarrow (\phi - \frac{1}{2})^2 - \frac{5}{4} = 0$$

$$\leftrightarrow \phi = \frac{1+\sqrt{5}}{2}$$

The negative root is not an option since it generates a negative answer.

The golden ratio can be written in many ways. Among those, as the infinitely nested radical

$$\phi = \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}}$$

This derives from the equation $\phi^2 = 1 + \phi$.

Using Theorem 11, we can solve for this nested radical. We substitute $a = 1$ into the following formula.

$$\sqrt{a + \sqrt{a + \sqrt{a + ...}}} = \frac{1+\sqrt{1+4a}}{2}$$

$$\rightarrow \sqrt{1 + \sqrt{1 + \sqrt{1 + ...}}} = \frac{1+\sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}$$

As we see, we get the same solution as earlier obtained when solving for the golden ratio, proving that this nested radical does converge towards the golden ratio.

5.2 Kasner's number

The Kasner radical is defined as

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+....}}}}$$

The limit of this radical does exist in accordance with Theorem 10 but is not as neat as other limits in this paper. This radical converges towards the Kasner, number after the mathematician Edward Kasner who studied the radical.

Kasner is well-known for the term "googol" that he used as a notion for a finite number 10^{100}. Kasner did not stop there, he called an even larger finite number "googolplex" which is 10^{googol} and "googolplexian" being $10^{googolplex}$. This is also where Google got their name, they merely misspelled googol. (Chapter 1 & 2 in [6]).

Example 8. Approximation of Kasner's number with 6 decimals by reference [4].

$$\sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+...}}}} = 1.757933...$$

To prove this we use a method commonly used when solving for nested radicals which involves finding an upper and a lower bound by using the tools we have obtained from Theorem 11.

$$\sqrt{1+\sqrt{1+\sqrt{1+...}}} < \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+...}}}} < \sqrt{2+\sqrt{2^2+\sqrt{2^3+...}}}$$

$$\leftrightarrow \frac{1+\sqrt{5}}{2} \approx 1.618 < \sqrt{1+\sqrt{2+\sqrt{3+\sqrt{4+...}}}} < 2\cdot(1+\frac{\sqrt{5}}{2}) \approx 3.236$$

This proves that the radical has both and upper and a lower bound which means that it converges. The solution is obtained through getting the upper and the lower bound closer to each other.

We now try getting a more precise approximation.

$$K > \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{10+\sqrt{10+\sqrt{10+...}}}}}}$$

$$= \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{\frac{1+\sqrt{41}}{2}}}}} = 1.757933...$$

The equality above applies due to Theorem 11.

$$K < \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{10+\sqrt{10^2+\sqrt{10^{2^2}+...}}}}}}$$

$$= \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{10+10\sqrt{1+\sqrt{10^{2^2}+...}}}}}}$$

$$= \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{10\sqrt{1+\sqrt{1+...}}}}}}$$

$$= \sqrt{1+\sqrt{2+...+\sqrt{9+\sqrt{10\sqrt{\frac{1+\sqrt{5}}{2}}}}}} = 1.757933...$$

This gives the approximation that $k = 1.757933...$

5.3 Viète's formula for π

The following formula was discovered by Viète in the 1500s:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} ...$$

The right-handed side can be rewritten in terms of limits as

$$\lim_{n\to\infty} \prod_{k=1}^{n} \frac{a_k}{2}$$

where $a_n = \sqrt{2+a_{n-1}}$ and $a_1 = \sqrt{2}$.

Figure 2 shows figuratively, the process in which Viète went about finding a formula for π. The image shows the bisecting of an angle starting with a 90 degree or $\frac{\pi}{2}$ radian angle.

Proof. The proof is based on reference [11].

We begin with the well-known double angle formula.

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

We repeat the procedure.

$$\sin(x) = 2 \cdot 2 \cdot \cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{4}\right) \cdot \sin\left(\frac{x}{4}\right)$$

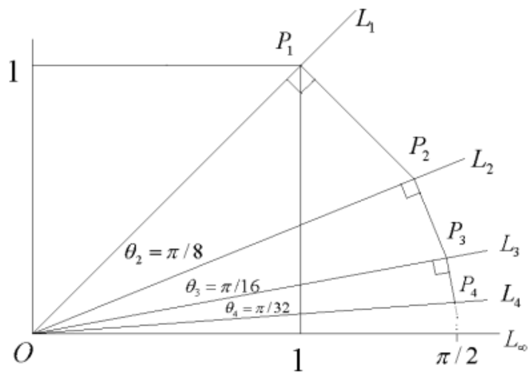

Figure 2: Approximating π
Source: Osler [11]

$$\sin(x) = 2 \cdot 2 \cdot 2 \cdot \cos\left(\frac{x}{2}\right) \cdot \cos\left(\frac{x}{4}\right) \cdot \cos\left(\frac{x}{8}\right) \cdot \sin\left(\frac{x}{8}\right)$$

Through induction we can prove the general formula

$$\begin{aligned} \sin(x) &= 2^n \cdot \cos\left(\tfrac{x}{2}\right) \cdot \cos\left(\tfrac{x}{4}\right) \cdot ... \cdot \cos\left(\tfrac{x}{2^n}\right) \cdot \sin\left(\tfrac{x}{2^n}\right) \\ &= 2^n \cdot \sin\left(\frac{x}{2^n}\right) \prod_{k=1}^{n} \cos\left(\frac{x}{2^k}\right) \\ &= \lim_{n\to\infty} 2^n \cdot \sin\left(\frac{x}{2^n}\right) \cdot \lim_{n\to\infty} \prod_{k=1}^{n} \cos\left(\frac{x}{2^k}\right). \end{aligned}$$

We develop the sine function in a Taylor series.

$$\begin{aligned} \sin(x) &= \left[\lim\nolimits_{n\to\infty} 2^n \left(\sum\nolimits_{r=1}^{\infty} \tfrac{(-1)^{r-1}}{(2r-1)!} \cdot \left(\tfrac{x}{2^n}\right)^{2r-1}\right)\right] \cdot \prod\nolimits_{k=1}^{\infty} \cos\left(\tfrac{x}{2^k}\right) \sin(x) \\ &= \left[\lim_{n\to\infty} \frac{2^n x}{2^n \cdot 1!} - \frac{2^n x^3}{2^{3n} \cdot 3!} + \frac{2^n x^5}{2^{5n} \cdot 5!} - ...\right] \cdot \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right) \end{aligned}$$

The first limit converges to x due to all terms but the first going to 0 when $n \to \infty$.

Hence

$$\sin(x) = x \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right)$$

$$\leftrightarrow \frac{\sin(x)}{x} = \prod_{k=1}^{\infty} \cos\left(\frac{x}{2^k}\right).$$

We substitute $x = \frac{\pi}{2}$.

$$\frac{\sin\left(\frac{\pi}{2}\right)}{\frac{\pi}{2}} = \prod_{k=1}^{\infty} \cos\left(\frac{\frac{\pi}{2}}{2^k}\right).$$

Since $\sin\left(\frac{\pi}{2}\right) = 1$, we get

$$\leftrightarrow \frac{2}{\pi} = \prod_{k=1}^{\infty} \cos\left(\frac{\pi}{2^{k+1}}\right)$$

We use the half-angle formula.

$$\leftrightarrow \frac{2}{\pi} = \prod_{k=1}^{\infty} \frac{\sqrt{2 + 2\cos\left(\frac{\pi}{2^k}\right)}}{2}$$

Expanding the product gives us

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} ...$$

This proves Viète's formula as well as give and insight on how he might have come to this conclusion. $\square$

5.4 Ramanujan's nested radicals

We will start by working with what could be Ramanujan's most famous nested radical although he has worked with many throughout his lifetime. We will tackle this radical using different methods and later move on to a less famous radical that Ramanujan worked with.

5.4.1 Ramanujan I

We will examine Ramanujan's famous nested radical

$$\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + ...}}}$$

We start by checking if the radical converges.

$$\begin{aligned}\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + ...}}} &= \sqrt{1 + \sqrt{2^2 + 2^2 \cdot 3\sqrt{1 + ...}}} \\ &= \sqrt{1 + \sqrt{2^2 + \sqrt{2^4 \cdot 3^2 + ...}}}\end{aligned}$$

The above radical converges in accordance with Theorem 10.

Method I

This method is similar to Ramanujan's initial method, although it is perfected with the help of Herschfeld [4] that proved the existence of a limit to this radical.

We start by finding the finite radicals

$$3 = \sqrt{9} = \sqrt{1+8} = \sqrt{1+2\cdot 4}$$

$$= \sqrt{1+2\sqrt{16}} = \sqrt{1+2\sqrt{1+15}} = \sqrt{1+2\sqrt{1+3\cdot 5}}$$

$$\sqrt{1+2\sqrt{1+3\sqrt{25}}} = \sqrt{1+2\sqrt{1+3\sqrt{1+4...+n\sqrt{1+(n+1)(n+3)}}}}$$

We now go on to comparing this finite radical which we know is exactly equal to 3 with the infinite radical we are looking to denest.

$$u_1 = \sqrt{1+2}$$

$$u_2 = \sqrt{1+2\sqrt{1+3}}$$

$$u_3 = \sqrt{1+2\sqrt{1+3\sqrt{1+4}}}$$

We see from comparing the two that each part of this infinite nested radical is smaller than the one in the finite radical. Meaning that the upper bound of our nested radical must be 3. We now need to find our lower bound.

We need to show that, given a small arbitrary positive $\epsilon < 3$, there exists an integer N such that, for all $n > N$,

$$u_n = \sqrt{1+2\sqrt{1+3\sqrt{1+...+n\sqrt{1}}}} > 3-\epsilon.$$

We set $3-\epsilon = 3r$, where $0 < r = 1-\frac{\epsilon}{3} < 1$.

We will show that

$$u_n > 3r = r\sqrt{1+2\sqrt{1+...n\sqrt{1+(n+1)(n+3)}}}.$$

Rewriting the above inequality gives

$$\sqrt{1+2\sqrt{1+3\sqrt{1+...+n\sqrt{1}}}} > \sqrt{r^2+2\sqrt{r^4+...n\sqrt{r^{2^n}[1+(n+1)(n+3)]}}}.$$

It is given from the definition of r that $1 > r^{2^i}$, $i = 1,2,3...$ There exists an integer N which for all $n > N$ leads to

$$1 > r^{2^n}[1+(n+1)(n+3)] = r^{2^n}(n+2)^2$$

because $r^{2^n}(n+2)^2 \to 0$ when $n \to \infty$. This comes to show that both the upper and lower bound are equal meaning that the nested radical is exactly equal to 3.

Method II / General formula

This proof derives from reference [12].

We start by rewriting the given radical in a more general form.

$$f(x) = \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1 + ..}}}}$$

for all $x > 0$. The function f satisfies

$$f(x) = \sqrt{1 + xf(x+1)}.$$

Hence,

$$f(x) \leq \sqrt{(1+x)f(x+1)}.$$

which is due to the fact that we are adding the entire sequence rather than 1 and $f(x) \geq 1$.

Since $f(x) \geq 1$, we have

$$f(x) \leq \sqrt{(1+x)f(x+1)} \leq \sqrt{1+x}\sqrt{\sqrt{(2+x)f(x+2)}} \leq ...$$

This inequality is obtained using the same reasoning as for $f(x) \leq \sqrt{(1+x)f(x+1)}$.

Therefore

$$f(x) \leq \prod_{k=1}^{\infty}(k+x)^{\frac{1}{2^k}}$$

$$\leftrightarrow \sqrt{1 + x\sqrt{1 + (x+1)\sqrt{1 + (x+2)\sqrt{1...}}}} \leq \sqrt{(1+x)\sqrt{(2+x)\sqrt{(3+x)...}}}$$

We are now looking to show that the increasing sequence $f(x)$ is bounded above to fulfill the criteria for convergence in accordance with the Monotone Convergence Theorem.

We see that

$$\prod_{k=1}^{\infty}(k+x)^{\frac{1}{2^k}} \leq \prod_{k=1}^{\infty}(2kx)^{\frac{1}{2^k}}$$

for $x \geq 1$ because we can rewrite the products as

$$\sqrt{(1+x)\sqrt{(2+x)\sqrt{(3+x)...}}} \leq \sqrt{2x\sqrt{4x\sqrt{6x....}}}$$

We see that

$$\prod_{k=1}^{\infty}(2kx)^{\frac{1}{2^k}} = 2x\prod_{k=1}^{\infty}k^{\frac{1}{2^k}}$$

due to

$$\prod_{k=1}^{\infty}(2x)^{\frac{1}{2^k}} = \sqrt{2x\sqrt{2x\sqrt{2x...}}} = 2x$$

in accordance with Theorem 7.

We then get

$$2x\prod_{k=1}^{\infty}k^{\frac{1}{2^k}} < 2x\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}}$$

which is implied by $k < 2^{k-1}$ for all $k \geq 1$.

We want to prove that

$$2x\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}} \leq 4x$$

which is equivalent to

$$\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}} \leq 2.$$

Let us focus on the exponents in order to grasp the given inequality. Those form the series

$$\sum_{k=1}^{\infty}\frac{k-1}{2^k}.$$

Referring back to Theorem 8.

$$f(k) = 1 + k + k^2 + k^3... = \frac{1}{1-k}$$

for all $|k| < 1$, hence

$$f'(k) = 1 + 2k + 3k^2 + ... = \frac{1}{(1-k)^2}$$

$$kf'(k) = k + 2k^2 + 3k^3 + ... = \frac{k}{(1-k)^2}$$

If we set $k = \frac{1}{2}$ in $kf'(k)$, we get a series that is equal to:

$$\sum_{k=1}^{\infty}\frac{k}{2^k} = \frac{\frac{1}{2}}{(\frac{1}{2})^2} = 2.$$

We see that

$$\sum_{k=1}^{\infty}\frac{k-1}{2^k} = \sum_{k=1}^{\infty}\frac{k}{2^k} - \sum_{k=1}^{\infty}\frac{1}{2^k}.$$

Since we've already shown that the second series converges to 1, we get that

$$\sum_{k=1}^{\infty}\frac{k-1}{2^k} = 2 - 1 = 1.$$

Thus,

$$f(x) \leq \prod_{k=1}^{\infty}(k+x)^{\frac{1}{2^k}} \leq \prod_{k=1}^{\infty}(2kx)^{\frac{1}{2^k}} = 2x\prod_{k=1}^{\infty}k^{\frac{1}{2^k}} < 2x\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}} = 4x.$$

We have proven that $f(x) < 4x$, therefore we get $f(x+1) < 4(x+1)$, leading to

$$f(x) = \sqrt{1+xf(x+1)} < \sqrt{1+4x(x+1)} = 2x+1$$

for all $x \geq 0$. We've established that $f(x) < 2x+1$, giving us an upper value. This also implies the inequality $f(x) < 4x+1$ which will be our starting point to finding a more precise upper bound.

To find this upper bound, we introduce some constant $a > 0$ where $f(x) = ax+1$.

We start by acknowledging that $f(x+1) \leq a(x+1)+1$,
meaning that

$$f(x) \leq \sqrt{1+(a+1)x+ax^2} \leq \sqrt{1+(a+1)x+(\frac{a+1}{2}x)^2} \leq 1+\frac{a+1}{2}x.$$

We go on with our initial estimate being $a = 4$, we get the inequality $f(x) \leq ax+1$ recursively for $a = \frac{5}{2}, \frac{7}{4}, \frac{11}{8}...$ which converges to 1. This means that $f(x) \leq 1+x$ for all $x \geq 0$.

Since $f(x) \leq f(x+1)$, we get that $f(x) \geq \sqrt{1+xf(x)}$ due to $f(x) = \sqrt{1+xf(x+1)}$ which gives $f(x) \geq 1+\frac{x}{2}$, this due to the following

$$f(x) \geq \sqrt{1+xf(x)}$$

$$\leftrightarrow f^2(x) \geq 1+xf(x)$$

$$\leftrightarrow f^2(x) - xf(x) - 1 \geq 0$$

$$\leftrightarrow f(x) \geq \frac{x}{2} + \sqrt{\frac{x^2}{4}+1}$$

And since $x \geq 0$, $f(x) \geq 1+\frac{x}{2}$ also applies.

If $a > 0$ satisfies the inequality $f(x) \geq 1+ax$ for all $x \geq 0$ then $f(x) \geq 1+\sqrt{a}x$. This is due to

$$f(x) = \sqrt{1+xf(x+1)} \geq \sqrt{1+x(1+a(x+1))}$$

We now need to show that

$$\sqrt{1+x+ax^2+ax} \geq 1+\sqrt{a}$$

$$\leftrightarrow 1+x+ax^2+ax \geq 1+2\sqrt{a}x+ax^2$$

$$\leftrightarrow 1+a \geq 2\sqrt{a}$$

$$\leftrightarrow (\sqrt{a}-1)^2 \geq 0$$

We have thereby shown that $f(x) \geq 1 + \sqrt{a}x$.

We start by setting $a = \frac{1}{2}$ giving us $f(x) \geq 1 + ax$ for $a = \frac{1}{2^{\frac{1}{2^k}}}$ for all $k \geq 1$ which also converges to 1 meaning that the only possibility is $f(x) = 1 + x$ for all $x \geq 0$.

This means that $\sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + ...}}} = 3$.

5.4.2 Ramanujan II

$$\sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + 4\sqrt{9 + ...}}}}$$

This radical is not quite as popular as the one previously demonstrated but does often appear as an extra challenge when tackling it.

We shall denest this infinitely nested radical through the tools that we have obtained through the demonstrated examples and later go on to viewing Ramanujan's solution. Convergence can be verified in accordance with Theorem 10.

Method I

We start by expanding our assumed solution in order to obtain the radical.

$$4 = \sqrt{16} = \sqrt{6 + 2 \cdot 5} = \sqrt{6 + 2\sqrt{25}}$$

$$= \sqrt{6 + 2\sqrt{7 + 3 \cdot 6}} = \sqrt{6 + 2\sqrt{7 + 3\sqrt{36}}}$$

$$= \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + ... + n\sqrt{(n + 5) + (n + 1)(n + 4)}}}}$$

To prove that the radical converges to exactly 4, we need to find an upper and lower bound.

Again we consider

$$u_1 = \sqrt{6 + 2}$$

$$u_2 = \sqrt{6 + 2\sqrt{7 + 3}}$$

$$u_3 = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + ...}}}$$

and see that each part of the radical is smaller than the finite radical that is equal to 4 meaning the upper bound is 4. We now need to find a lower bound by showing, given a small arbitrary $\epsilon < 4$, there exists an integer N, where $n > N$ which gives

$$u_n = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + ... + n\sqrt{(n + 5)}}}} > 4 - \epsilon.$$

We set $4-\epsilon=4r$, meaning that $0<r=1-\frac{\epsilon}{4}<1$.

We will show that $u_n>4r=r\sqrt{6+2\sqrt{7+...+n\sqrt{(n+5)+(n+1)(n+4)}}}$

We rewrite the inequality as

$$\sqrt{6+2\sqrt{7+3\sqrt{8+...+n\sqrt{(n+5)}}}}>$$
$$\sqrt{6r^2+2\sqrt{7r^4+...+n\sqrt{r^{2^n}[(n+5)+(n+1)(n+4)]}}}.$$

We know from the definition of r that $1>r^{2^i}$, $i=1,2,3...$ It follows that there exists an integer N such that

$$n+5>r^{2^n}[(n+5)+(n+1)(n+4)]=r^{2^n}(n+3)^2$$

when $n\geq N$, because $r^{2^n}(n+3)^2\to 0$ when $n\to\infty$. This shows that the nested radical has both an upper and a lower bound that is equal to 4, meaning that the nested radical must be equal to 4 and the proof is done.

Method II / General formula

We begin by finding the general radical.

$$f(x)=\sqrt{(x+4)+x\sqrt{((x+4)+1)+(x+1)\sqrt{((x+4)+2)+...}}}$$

for all $x>0$, we get the equation

$$f(x)=\sqrt{(x+4)+xf(x+1)}$$

The following inequalities derive from the fact that $x>0$:

$$f(x)\leq\sqrt{(x+4+x)f(x+1)}\leq\sqrt{(x+4)+x}\sqrt{\sqrt{(x+4+1+x+1)f(x+2)}}\leq...$$

Therefore

$$f(x)\leq\prod_{k=1}^{\infty}((x+3+k)+(x+k-1))^{\frac{1}{2^k}}.$$

Thus, using a similar reasoning as in the previous radical,

$$\prod_{k=1}^{\infty}((x+3+k)+(x+k-1))^{\frac{1}{2^k}}\leq\prod_{k=1}^{\infty}(8kx)^{\frac{1}{2^k}}=8x\prod_{k=1}^{\infty}k^{\frac{1}{2^k}}<8x\prod_{k=1}^{\infty}2^{\frac{k-1}{2^k}}\leq 16x.$$

Meaning that for any $x\geq 0$ we have $f(x+1)<16(x+1)$ leading to,

$$f(x)=\sqrt{(x+4)+xf(x+1)}<\sqrt{(x+4)+16x(x+1)}\leq 5x+2.$$

We introduce some $a>0$ while $x\geq 0$.

Assume that $f(x) < 2 + (1+a)x$, which implies the inequality

$$f(x+1) < 2 + (1+a)(x+1) = (a+1)x + (a+3).$$

Using the above, we get

$$f(x) = \sqrt{(x+4) + xf(x+1)} < \sqrt{(a+1)x^2 + (a+4)x + 4} < 2 + \left(1 + \frac{a}{2}\right)x$$

Starting with our initial $a = 5$, we use the inequality $f(x) \leq ax+2$ recursively for $a = 3, 2, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$... which is a sequence with the limit 1. This leaves us with $f(x) \leq x + 2$ for all $x \geq 0$.

Now that we have found an upper bound, we will be looking for a lower bound.

Since $f(x) \leq f(x+1)$, then $f(x) \geq \sqrt{(x+4) + xf(x)}$ due to $f(x) = \sqrt{(x+4) + xf(x+1)}$. This leads to $f(x) \geq 2 + \frac{x}{2}$ due to

$$f(x) \geq \sqrt{(x+4) + xf(x)}$$
$$\leftrightarrow f^2(x) \geq (x+4) + xf(x)$$
$$\leftrightarrow f^2(x) - xf(x) - (x+4) \geq 0$$
$$\leftrightarrow f(x) \geq \frac{x}{2} + \frac{\sqrt{x^2+4x+16}}{2}$$

And since $x \geq 0$, then $f(x) \geq \frac{x}{2} + 2$ is also true.

If $a > 0$ satisfies the following inequality $f(x) \geq 2 + ax$ for all $x \geq 0$ then $f(x) \geq 2 + \sqrt{a}x$. This inequality is true due to

$$f(x) = \sqrt{(x+4)f(x)} \geq \sqrt{(x+4)x(2+a(x+1)} = \sqrt{x+4+2x+ax^2+ax}.$$

$$3x + ax + ax^2 + 4 \geq ax^2 + 4\sqrt{a}x + 4$$
$$\leftrightarrow 3 + a \geq 4\sqrt{a}$$
$$\leftrightarrow (\sqrt{a} - 2)^2 \geq 1$$

This comes to show that $f(x) \geq 2 + \sqrt{a}x$.

We start with $a = \frac{1}{2}$, giving us $f(x) \geq 2 + ax$ for $a = \frac{1}{2^{\frac{1}{2^k}}}$ for all $k \geq 1$ which also converges to 1 meaning that the only possibility is $f(x) = 2 + x$ for all $x \geq 0$.

This means that $\sqrt{2 + 6\sqrt{3 + 7\sqrt{4 + ...}}} = 4$.

We will now view how Ramanujan approached this radical in order to see the difference between the three solutions as well as acknowledge that nested radicals can be solved using different methods.

Theorem 13. Ramanujan's solution (in [3] page 108.)

$$f(x) = x + n + a = \sqrt{ax + (n+a)^2 + x\sqrt{a(x+n) + (n+a)^2 + (x+n)\sqrt{\dots}}}$$

Where $x > 0, a \geq 0$ *and* $n > 0$.

In the viewed radical we substitute $a = 1$, $n = 1$ and $x = 2$ and get

$$2 + 1 + 1 = \sqrt{6 + 2\sqrt{7 + 3\sqrt{8 + \dots}}}$$

A proof will not be shown due to flaws in the initial proof, although it may be viewed in the given reference.

6 Conclusion

This paper has given us insight on how broad the subject of nested radicals really is. Solving nested radicals does not merely revolve around getting an answer but understanding the methods one is using and whether the radical converges to begin with. The topic has not been researched very much and most theorems available revolve around finitely nested radicals, some of which are not complete. This paper gives an overview of what can be studied within the field.

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