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The inability of Spekkens' epistemic view of quantum states to reproduce the solution to the mean king's problem

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The inability of Spekkens' epistemic view of quantum states to reproduce the solution to the mean king's problem

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For the significant part of the past century, the familiar Copenhagen interpretation, an inherently probabilistic stance, has been the dominant vantage point for research on quantum mechanics. Lately, however, new experimental results once again fuel debate and contending interpretations make their claim. Most notably, perhaps, being the pilot wave formulation, an inherently deterministic viewpoint. A recurring, more specific, front of this debate is whether quantum mechanics behaves in a manner described by an *ontic* or an *epistemic* theory. Spekkens, to the defence of the latter, outlines in his paper [1] a toy theory explicitly rooted in a belief that there exists fundamental limits to an observers knowledge of a particle's state. The argument the author, very reasonably, makes for epistemology is that an extension of the pile of experiments covered and explained by it, while failing for the contenders, continuously adds to its plausibility. In this paper, however, we present a failure of the toy theory to replicate the solution to the mean king's problem as presented by Vaidman, Aharonov and Albert [2]. Strictly speaking, this is not evidence against all of epistemology, rather an argument only against this toy theory which sets a seemingly arbitrary restriction on its foundation. A restriction Spekkens refers to as the *knowledge balance principle.*

I. The mean king's problem

A scholar is challenged by the king. The king will measure the observable spin-value of either σ_x, σ_y or σ_z on a particle, but the scholar doesn't get to know which of these the king will make. While lacking this seemingly crucial bit of information, the scholar is nonetheless challenged to ascertain the value the king obtained on his measurement, with unit probability. The rules are such that before the king's measurement, the scholar gets to prepare the particle in any state of her choosing. After the king's measurement, she herself gets to do one final measurement on the particle.

As pointed out in the original paper [2], one solution is, with the aid of an external particle alongside the king's, to prepare the entangled state $\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$. After the king's measurment, the scholar performs her measurement on the particle w.r.t an observable A, which has nondegenerate eigenvectors

$$|\phi_1\rangle = \frac{\sqrt{2}}{2}|00\rangle + \frac{1}{2}(e^{-i\frac{\pi}{4}}|01\rangle + e^{i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_2\rangle = \frac{\sqrt{2}}{2}|00\rangle - \frac{1}{2}(e^{-i\frac{\pi}{4}}|01\rangle + e^{i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_3\rangle = \frac{\sqrt{2}}{2}|11\rangle + \frac{1}{2}(e^{i\frac{\pi}{4}}|01\rangle + e^{-i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_4\rangle = \frac{\sqrt{2}}{2}|11\rangle - \frac{1}{2}(e^{i\frac{\pi}{4}}|01\rangle + e^{-i\frac{\pi}{4}}|10\rangle), \quad (1)$$

with corresponding eigenvalues a_1, a_2, a_3 and a_4. Now, since A is an observable, she is certain to get one, and only one, of these values upon measurement. The scholar can now, depending on the outcome, infer the precise result from the king's measurement in all three cases in accordance with Table 1.

The value of A	σ_x	σ_y	σ_z
a_1	$\uparrow$	$\uparrow$	$\uparrow$
a_2	$\downarrow$	$\downarrow$	$\uparrow$
a_3	$\uparrow$	$\downarrow$	$\downarrow$
a_4	$\downarrow$	$\uparrow$	$\downarrow$

Table 1. Infered σ-values from outcomes of A.

As an example, this implies that obtaining a_1 from the measurement of A would ascertain $\uparrow$ in all three cases of the king's measurement. This procedure solves the challenge.

Preliminary quantum theory and basic algebra, alongside details of this scheme, is outlined in Appendix I and II, respectively.

II. Aspects of the toy theory

A. Ontology and epistemology

Seemingly against common intuition, quantum behaviour has since discovery proved to be a serious challenge for theorists seeking a satisfactory interpretation of it. During the heated debates of the 1920's, a major turn of the tides occurred when Born first suggested that physics should be understood by means of probability distributions, rather than of casual cause. This new notion of *indeterminism* was met with very vocal resistance, perhaps most by Einstein. He refused the implication of randomness to events, with no underlying cause. Nonetheless, an indeterministic theory emerged as victorious and remained dominant for the rest of the century. However, recent new experiments have once again sparked debate and renewed interest in *deterministic* theories.

Deterministic theories are rooted in ontological ideology. Notable examples being the many-worlds interpretation and various branches of hidden variable theories. The latter of which the pilot wave formulation is a variety of. It was introduced 1927 by Broglie. Although dismissed at first, recent findings has garnered new experimental support. Deterministic theories actively try to avoid notions such as the wave-particle duality, wave collapses and paradoxes associated with the thought experiment of Shroedinger's cat. Attributes typically inherent in probabilistic theories.

Indeterministic, and thus probabilistic, theories belong to the school of epistemology. Notable examples are the consistent histories interpretation and the Copenhagen interpretation. The latter being the base to most quantum research for the majority of the past century.

An ontic state, in contrast to an epistemic one, is a state of determined reality. If an observer knows the ontic state of a particle, then she has factual information about its position and momentum, and its past and future can therefore be calculated and predetermined. This notion should be familiar from classical mechanics.

In the toy theory provided by Spekkens, epistemic states are defined in terms of ontic states. It is simply a matter of introducing an ambiguity in what precise ontic state a particle resides in, at all times. The observer can never have sufficient information to deduce the ontic state of a particle, and has to settle for the fact that the particle is in one of several possible ontic states. This state of knowledge is called an epistemic state.

B. The knowledge balance principle

Spekkens' toy theory is moulded by epistemic ideology and thus requires a limit to the amount of knowledge an observer can have of a particle, at any given time. Epistemology in itself does not, however, define this limit. The hypothesis by Spekkens, in this particular toy theory, is referred to as the knowledge balance principle. In plain text, it explicitly states that

If one has maximal knowledge, then for every system, at every time, the amount of knowledge one possesses about the ontic state of the system at that time must equal the amount of knowledge one lacks.

From this hypothesis, a symmetry between knowledge and ignorance follows. There can be no imbalance between these quantities when the observer has maximal knowledge of a given system. This, as we will see shortly, has profound impact on the mathematical aspects of the toy theory.

C. Elementary systems

To quantitatively understand the notion of an elementary epistemic state, we begin by introducing the concept of a canonical set of yes-no questions. This is the set of yes-no questions about a system that is sufficient to identify its true ontic state. The questions should also be formulated in such a manner that the set contains a minimal amount of elements.

Elementary systems in the toy theory can, as a consequence of the knowledge balance principle, reside in a possibility of four different ontic states. No more, no less. Thus, an observer currently in the state of *least* amount of knowledge about a particular system would describe its state as $1 \vee 2 \vee 3 \vee 4$, where $\vee$ is notation for the logical *or*.

The most efficient canonical set of questions for these kind of systems is one in which the questions divides the set of possible ontic states by two, with each question. Since there are four possible ontic states, the number of questions in the canonical set is two. For instance, one example of a pair of questions is "Is it in the set $\{1, 4\}$, or not?" and "Is it in the set $\{1, 3\}$, or not?".

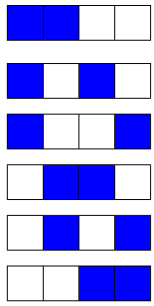
Now, and this is key for the toy theory, application of the knowledge balance principle on the canonical set of questions infers that we are only allowed to obtain the answer to one of the two questions. The principle clearly states that while an observer is in a state of maximal knowledge, her knowledge about the system must be equal to the amount of ignorance.

As a concrete example, consider an observer in a state of least amount of knowledge about a system, given by $1 \vee 2 \vee 3 \vee 4$. Further, assume she asks "Is it in the set $\{1, 2\}$, or not?" regarding the ontic state of the system, and is returned with the value *true*. In that case, she has sufficient information to reject the ontic states 3 and 4. Thus, from this point onward, she would describe the epistemic state of the system as $1 \vee 2$, and she now has as much knowledge about the true ontic state of the system as she has ignorance. According to the knowledge balance principle, this is as much as she is allowed to have. Hence, this epistemic state, i.e $1 \vee 2$, is an example of one of maximal knowledge in the toy theory.

Although the conclusion of the preceding paragraph indeed stands, it does not tell the whole story. Physically, there are no barriers limiting the observer from proceeding to ask another question about the system, now described as $1 \vee 2$. The observer might conduct a subsequent measurement as to whether the system is in the set $\{1, 3\}$, or not. By coupling the result with the previously obtained information, one might believe to be in a position to deduce the true ontic state of the system. But this is not the case, as each measurement imposes a disturbance on the system, effectively nullifying the results from previous measurements, unless it is the same measurement. We discuss this further in SEC. II E.

There are $\binom{4}{2} = 6$ different elementary epistemic states of maximal knowledge. Visually, they are represented as

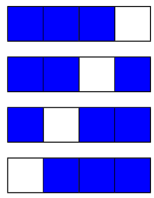
$1 \vee 2 \leftrightarrow$
$1 \vee 3 \leftrightarrow$
$1 \vee 4 \leftrightarrow$
$2 \vee 3 \leftrightarrow$
$2 \vee 4 \leftrightarrow$
$3 \vee 4 \leftrightarrow$



(2)

There are, of course, other forms of elementary systems. States of non-maximal knowledge are given by

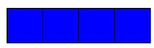
$1 \vee 2 \vee 3 \leftrightarrow$
$1 \vee 2 \vee 4 \leftrightarrow$
$1 \vee 3 \vee 4 \leftrightarrow$
$2 \vee 3 \vee 4 \leftrightarrow$



(3)

And lastly, the state of least amount of knowledge is given by

$1 \vee 2 \vee 3 \vee 4 \leftrightarrow$

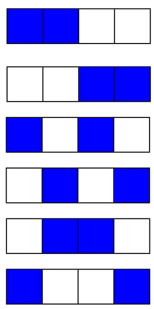


(4)

D. Analogies with quantum mechanics

Physically measurable quantities are in mechanics called observables. Generally speaking, quantities of interest are most often position and momentum, but not always. In quantum mechanics, a quantity of significant importance is the intrinsic spin of particles. In fact, Spekkens directly defines a 1-to-1 correlation between the elementary states of maximal knowledge and the eigenvectors of three orthogonal spin-directions. Specifically, we have the following correlations

$|0\rangle \leftrightarrow$
$|1\rangle \leftrightarrow$
$|+\rangle \leftrightarrow$
$|-\rangle \leftrightarrow$
$|+i\rangle \leftrightarrow$
$|-i\rangle \leftrightarrow$



(5)

where $|0\rangle$ and $|1\rangle$ corresponds, respectively, to the spin-$\uparrow$ and spin-$\downarrow$ eigenvectors of the σ_z-operator. Similarly, $|+\rangle$ and $|-\rangle$ corresponds to those of σ_x. And lastly, $|+i\rangle$ and $|-i\rangle$ to the eigenvectors of σ_y.

Using these correlations and having the knowledge balance principle as a staple for the toy theory, Spekkens is able to reproduce a wide range of quantum phenomenon within his framework. Examples include non-commutativity of measurements, teleportation, entanglement, the impossibility of a universal state inverter, no-cloning and many others.

E. Measurements

A measurement on an elementary system can be defined as a physical action which distinguishes the state in question to one, out of two possible, epistemic states. There are only three such possible measurements, and we represent them in terms of the different epistemic states they have as possible outcomes. These are

$$\{1 \vee 3,\ 2 \vee 4\}, \tag{6}$$

$$\{2 \vee 3,\ 1 \vee 4\}, \tag{7}$$

$$\{1 \vee 2,\ 3 \vee 4\}. \tag{8}$$

For starters, by combining this with the correlations of the previous subsection, we note that each one of these simply correspond to a spin measurement in a different direction. Its clear that (6), (7) and (8) measures spin in the x, y and z-direction, respectively.

To find the probability of a given outcome, we need only take note of the amount of ontic states the outcome in question, and the state being measured upon, has. As some concrete examples, assume first we measure $\{1 \vee 2,\ 3 \vee 4\}$ on the state $1 \vee 2$. The probability distribution in this case is (1,0), i.e the first outcome is certain to occur. Now, assume we conduct the same measurement on the state $2 \vee 3$. This will instead yield the distribution $(\frac{1}{2}, \frac{1}{2})$. Both outcomes thus has the same probability to occur.

In the toy theory, we only consider reproducible measurements. Reproducible in the sense that if repeated on a system, it must yield the same result. In order to satisfy this axiom, it is assumed that the state being measured upon always transforms into the outcome state of the measurement. Clearly, this ensures that repeated measurements yield the same results. From this, it follows that spin measurements in different directions do not commute.

F. Bipartite systems

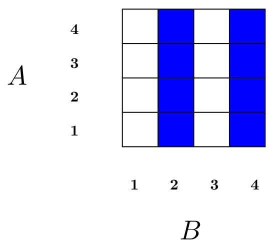
Combined systems in the toy theory are defined in a natural manner. The epistemic state of the composite system AB is simply defined to be the possible combinations of ontic states from system A and B, respectively. As an example, consider A to be in the state $1 \vee 2 \vee 3 \vee 4$ and B to be in $2 \vee 4$. The composite system AB is then said to be in

$$(1 \vee 2 \vee 3 \vee 4) \cdot (2 \vee 4) = \tag{9}$$

$$= (1 \cdot 2) \vee (1 \cdot 4) \vee (2 \cdot 2) \vee (2 \cdot 4) \vee (3 \cdot 2) \vee (3 \cdot 4) \vee (4 \cdot 2) \vee (4 \cdot 4), \tag{10}$$

where $\cdot$, as per Spekkens' convention, represents logical *and*. To clarify, since the true ontic state of A is in $\{1, 2, 3, 4\}$ and B in $\{2, 4\}$, the true ontic state of AB is given by an element of $\{1, 2, 3, 4\} \times \{2, 4\}$.

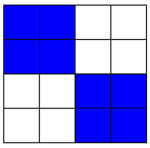
Fortunately, bipartite systems also have simple visual manifestations. The example just given is graphically represented by



(11)

As seen, the rows of this figure represent the different ontic states of A, while columns represent those of B. Marked intersections are possible true ontic states for AB.

Expression (9) and (10) are logically equivalent, but it is of fundamental importance to point out that not all composite systems can be written in the form given by (9). More specifically, not all systems AB can be decomposed in the form A·B. The states that can are called *product states*. Clearly, if both states of system A and B individually are described by an arbitrary choice amongst those given by (2), (3) or (4), then AB will be a product state. An example of a composite state which is not a product state is



(12)

This can only be expressed in the form given by (10), which explicitly is

$$(1\cdot 3)\vee(1\cdot 4)\vee(2\cdot 3)\vee(2\cdot 4)\vee(3\cdot 1)\vee(3\cdot 2)\vee(4\cdot 1)\vee(4\cdot 2). \quad (13)$$

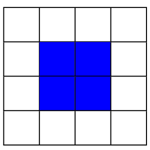
States that are not product states are called *entangled states*. This ordeal is analogous to a notion in vector algebraic quantum mechanics, where a bipartite system $|\psi\rangle_{AB}$ is said to be a product state if it can be expressed as

$$|\psi\rangle_{AB}=\sum_i c_i\,|i\rangle_A\otimes\sum_j c_j\,|j\rangle_B\,, \quad (14)$$

where $\{|i\rangle_A\}$ is a base for H_A, the Hilbert space of A, and $\{|j\rangle_B\}$ is a base for H_B. If $|\psi\rangle_{AB}$ cannot be decomposed as in (14), it is said to be an entangled state.

G. Bipartate systems of particular significance

A combined system AB is said to be in an *uncorrelated* state of maximal knowledge if as much as possible is known about the true ontic state of its subsystems, individually. Therefore, a bipartite system AB is in an uncorrelated state of maximal knowledge if A and B both are represented by an arbitrary choice among the elementary states of maximal knowledge given by (2). An example of such a state is



(15)

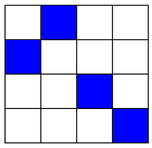
which in logical notation is described as

$$(2\cdot 2)\vee(2\cdot 3)\vee(3\cdot 2)\vee(3\cdot 3). \quad (16)$$

These states are uncorrelated in the sense that further information (although disallowed by the knowledge balance principle) about system A (or B) does not imply the same about system B (or A).

From this discussion, and the fact that the true ontic state of an elementary system can at most be pinpointed to two different possibilities, we deduce that the true ontic state of bipartite systems can at most be pinpointed to four different possibilities.

A second type of state of particular importance are the *perfectly correlated* states of maximal knowledge, also called maximally entangled states. An example of such a state is



(17)

which in logical notation is described as

$$(1\cdot 4)\vee(2\cdot 3)\vee(3\cdot 1)\vee(4\cdot 2). \quad (18)$$

We can understand the perfect correlation (or maximal entanglement) by analysing (18). Clearly, we see that the different ontic states of subsystem A are bound to a unique ontic state of B, and thus vice-versa. This is not true with e.g. the states described by (13) or (16).

In perfectly correlated states of maximal knowledge, we do not know as much as possible about the subsystems individually, but instead we know everything about the relation between them.

H. Mathematical formalism

For the purposes of this paper, the most important states are uncorrelated and perfectly correlated states of maximal knowledge, and there are many such states. It would be quite inefficient and clunky to explicitly write out all of them. To simplify our work, we will in this section introduce an underlying mathematical structure for bipartite states. Using this, we will be able to surgically identify all states of interest. This new structure will also aid in defining, amongst other things, measurements for composite systems.

Bipartite quantum states in the toy theory can be represented by elements of a certain set, hereby denoted as $\mathrm{B}_{4\times 4}\{0,1\}$. This is the set of 4×4 boxes containing a single binary digit, in each box. The graphical representation of these elements are, for the sake of consistency, such that the boxes containing 0's are shown empty, while those containing 1's are fully marked with blue color. For $a\in \mathrm{B}_{4\times 4}\{0,1\}$, the binary digits in each of its boxes are naturally referred to as a_{ij}, where i signifies the row and j the column, in an orientation consistent with (11).

Following this new definition, we say that an element $b \in \mathrm{B}_{4\times4}\{0,1\}$ is an uncorrelated state of maximal knowledge if it has precisely four non-zero elements b_{ij}, b_{kl}, b_{mn}, b_{op} which together satisfies

$$i = k,\ j = n,\ l = p,\ m = o,\ j \neq\ l \text{ and } n \neq p. \quad (19)$$

The state described by (15), which has non-zero elements b_{22}, b_{23}, b_{32} and b_{33}, is a clear example of this. Further, and this might not be obvious at first glance, we see from condition (19) that all other uncorrelated states of maximal knowledge can be obtained e.g. from the state (15) either through a permutation of its rows, a permutation of its columns, or any combination of these operations.

Analogously, we have that $c \in \mathrm{B}_{4\times4}\{0,1\}$ is a perfectly correlated state of maximal knowledge if it has precisely four non-zero elements c_{ij}, c_{kl}, c_{mn}, c_{op} which together satisfies

$$i \neq k \neq m \neq o \text{ and } j \neq l \neq n \neq p. \quad (20)$$

The state described by (17) with non-zero elements c_{14}, c_{23}, c_{31} and c_{42}, clearly satisfies these conditions. All other maximally entangled states can be obtained from this state via the same permutations described above.

I. Orthogonality

Using this new formalism, we can define orthogonality. We begin by defining the *size*, $|z|$, of a bipartite element $z \in \mathrm{B}_{4\times4}\{0,1\}$. It is given by

$$|z| \equiv \sum_{i,j} z_{ij}, \quad (21)$$

which simply corresponds to the amount of non-zero digits of z. Using this, we can now define a measure f of distance between $x, y \in \mathrm{B}_{4\times4}\{0,1\}$. This is a function

$$f : \mathrm{B}_{4\times4}\{0,1\} \times \mathrm{B}_{4\times4}\{0,1\} \to \{0, \tfrac{1}{n}, \tfrac{2}{n}, \tfrac{3}{n}, ..., \tfrac{n-1}{n}, 1\},$$

defined as

$$f(x,y) \equiv \tfrac{1}{n} \textstyle\sum_{i,j} x_{ij} y_{ij}, \quad (22)$$

where $n = \max(|x|, |y|)$.

From the definition of f, we see that

$$f(x,y) = 0 \Rightarrow x_{ij} y_{ij} \neq 1 \ \forall\ i,j \in \{1,2,3,4\}. \quad (23)$$

In other words, implying that x and y have no non-zero digits in common, and thus are orthogonal in this sense. Further, we see that $f(x,y) = 1 \Rightarrow x = y$.

If x and y are any combination of uncorrelated and perfectly correlated states of maximal knowledge, then $n = 4$ and the image of f becomes $\{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$.

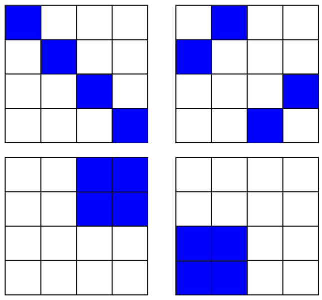
J. Observables

Physically measurable quantities are in mechanics called observables, as previously mentioned. In the vector algebraic language of quantum mechanics, observables are represented by hermitian elements of $\mathbb{C}^{n\times n}$. According to the spectral theorem, hermitian matrices can be decomposed as a linear sum in terms of its eigenvectors, weighted by their corresponding eigenvalues. See Appendix I for further details.
The observable A, used in SEC. I by the scholar, lives in $\mathbb{C}^{4\times4}$ and can in this manner be rewritten in the form

$$A = \sum_{i=1}^{4} a_i \left|\phi_i\right\rangle \left\langle\phi_i\right|. \quad (24)$$

Similar to (24), there is an analogous manner to describe observables in the toy theory, in terms of their eigenstates. For starters, just as with the eigenvectors of A, the eigenstates of observables in the toy theory must be pairwise orthogonal.

Further, the eigenvectors of hermitian matrices constitute a base for the space in which they live. Since the eigenvectors of A live in $\mathbb{C}^4$, there are therefore, as expected, precisely four of them. While the notion of a base is not as clear in the toy theory, the maximum amount of eigenstates a bipartite observable can have is also four. This occurs when the eigenstates are uncorrelated or perfectly correlated states of maximal knowledge, or a combination of both. It can not, however, simply be any combination, since the condition of pairwise orthogonality applies. We call a measurement of a bipartite observable, with four eigenstates, a *maximally informative measurement*. An example of a set of four eigenstates, which together compose an observable, is



(25)

A sometimes more convenient manner to illustrate observables, similar to the left-hand side of (24), is to incorporate the eigenstates into a single 4×4 figure. In order to distinguish the eigenstates from one other, we mark the non-zero digits of each eigenstate differently. We can use Greek letters for this purpose. Following this, the observable given by the eigenstates in (25) can graphically be represented as

ϕ_1	ϕ_2	ϕ_3	ϕ_3
ϕ_2	ϕ_1	ϕ_3	ϕ_3
ϕ_4	ϕ_4	ϕ_1	ϕ_2
ϕ_4	ϕ_4	ϕ_2	ϕ_1

(26)

K. Outcome probabilities for maximally informative measurements

In general, epistemic theories give statistical, and not definite, predictions on measurement outcomes. There are a few exceptions to this, such as repeated measurements. But in general, this is not the situation at hand. In vector algebraic language, an observer can find the probability of an outcome that results in $|\phi_i\rangle$, when conducting a measurement on a system in the state $|\alpha\rangle$, by calculating $|\langle\phi_i|\alpha\rangle|^2$. One can repeat this calculation for the different eigenvectors of an observable to find the probability for each outcome.

Similarly, we can find the probability for each outcome of a maximally informative measurement in the toy theory simply by calculating f, once for each eigenstate, with the state with being measured upon as the other argument of the function. More concretely, if we want to find the probability distribution $P(\Phi, \alpha)$ for a measurement of an observable Φ, with eigenstates ϕ_i, on a system in the state α, we calculate

$$P(\Phi, \alpha) = (f(\phi_1, \alpha), f(\phi_2, \alpha), f(\phi_3, \alpha), f(\phi_4, \alpha)). \quad (27)$$

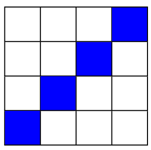
As an example, assume we'd like to find out the outcome probabilities of the observable given by (26) when measuring on the state given by (12). The probability distribution would in this manner be $(\frac{1}{2}, \frac{1}{2}, 0, 0)$. If the same observable were to be measured on the state given by (15), we would instead have the distribution $(\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{4})$.

III. Proof of inability

A. The king's action

The main strategy of the proof will be to show that there exist no observable capable of reproducing the algebraic solution provided by Vaidman, Aharonov and Albert. We will do this by systematically deducing, by a case-by-case basis, that there exist no eigenstate in the toy theory capable of ascertaining three orthogonal spin-outcomes, as the eigenvectors of A in SEC. I evidently are capable of.

The challenge begins by the scholar being handed the particle which the king will conduct a measurement on. By convention, we denote the system of this particle A. She is now allowed to prepare it in any state of her choosing. With the aid of an external particle, the system of which we denote as B, she prepares the entangled state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. In the toy theory, this state, explicitly according to Spekkens, corresponds to the perfectly correlated, or maximally entangled, state



(28)

which in logical terms is described as

$$(1 \cdot 1) \vee (2 \cdot 2) \vee (3 \cdot 3) \vee (4 \cdot 4). \tag{29}$$

The king subsequently retakes control and conducts a spin measurement, on system A, in a direction (x, y or z) of his choosing. He keeps the decision, and the result of the measurement, hidden from the scholar.

It is now important to deconstruct and understand the king's action since it does not leave the state (29) unaltered. To see this, assume that he chooses to conduct a spin measurement, on system A, in e.g. the x-direction. As discussed in SEC. II E, this corresponds to the measurement given by (6). The measurement has the possible outcomes $1 \vee 3$ and $2 \vee 4$. Assume that $1 \vee 3$ (or spin-$\uparrow$, or just $\uparrow$ for shorts) is obtained. As mentioned in the same subsection, the original state of A will then transform to this state. And, as explained in SEC. II G, since (29) is perfectly correlated, this transformation will have consequences on system B.

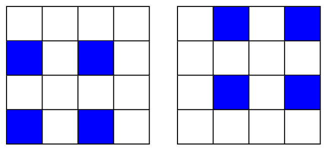
To see how, bear in mind that A is now known to be in the ontic state 1 or 3, which means that we must reject the possibilities of 2 and 4. We must also reject the ontic states of B correlated with these. Duo to the relations of (29), the ontic states of B which must be rejected happens to be the same, 2 and 4. Thus the composite system AB, after the king measures $\uparrow$ in the x-direction, would now be

$$\mathrm{A} \cdot \mathrm{B} \;=\; (1 \vee 3) \cdot (1 \vee 3). \tag{30}$$

If instead the king would have obtained the result $2 \vee 4$ ($\downarrow$), in the x-direction, the state AB would instead transform into

$$\mathrm{A} \cdot \mathrm{B} \;=\; (2 \vee 4) \cdot (2 \vee 4). \tag{31}$$

These new states can, respectively, be graphically represented as



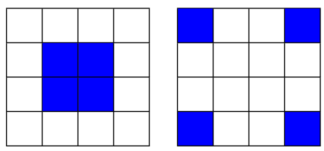
(32)

Analogously, the transformed state AB, for a result of either $\uparrow$ or $\downarrow$ in the y-direction, will respectively be

$$\mathrm{A} \cdot \mathrm{B} \;=\; (2 \vee 3) \cdot (2 \vee 3), \tag{33}$$

$$\mathrm{A} \cdot \mathrm{B} \;=\; (1 \vee 4) \cdot (1 \vee 4). \tag{34}$$

These are graphically represented as



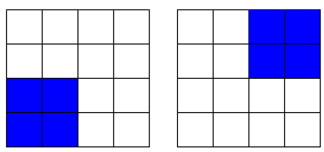
(35)

Lastly, we have the case of a spin measurement in the z-direction. The results $\uparrow$ or $\downarrow$ will respectively alter the state AB into

$$\mathrm{A} \cdot \mathrm{B} \;=\; (1 \vee 2) \cdot (1 \vee 2), \tag{36}$$

$$\mathrm{A} \cdot \mathrm{B} \;=\; (3 \vee 4) \cdot (3 \vee 4), \tag{37}$$

which can be graphically represented as



(38)

B. Eigenstate analysis

In the previous subsection, we were able to quantitatively understand how the prepared state (29) transformed during different actions of the king. It is also important to understand that even though we (acting as the scholar) were able to outline all the different transformations, we do not know which one actually occurred since the king keeps that information to himself. Nonetheless, the knowledge we actually did obtain about the transformed states will be crucial in our attempt to create an observable with eigenstates capable of distinguishing them and thus pinpoint the king's actual result with unit probability.

We now commence our pursuit of a potent observable. To maintain a close analogue to the observable A of SEC. I, our initial aim will be an observable with four eigenstates. The other cases are left to the subsequent subsection. From the discussion in SEC. II J, we know that an observable with four eigenstates is only possible if the eigenstates are uncorrelated or perfectly correlated states of maximal knowledge, or some suitable combination of both.

The outcomes of A in SEC. I are, individually, capable of ascertaining precisely one outcome of each spin measurement alongside the three orthogonal directions. As an example, the outcome a_1 (corresponding to the eigenvector $|\phi_1\rangle$), guarantees that the king measured ($\uparrow$, $\uparrow$, $\uparrow$) in (x, y, z). The state $|\phi_1\rangle$ accomplishes this by having zero probability of being the outcome if the state being measured upon is any of those consistent with the opposite results, in this case being ($\downarrow$, $\downarrow$, $\downarrow$). More specifically, this is done by ensuring that the eigenvector is orthogonal to these states.

The probability, in the toy theory, of an eigenstate ϕ_i to be the outcome upon measurement on a system in the state α, as seen in SEC. II K, is given by $f(\phi_i, \alpha)$. Thus, in order for an eigenstate to guarantee e.g. ($\uparrow$, $\uparrow$, $\uparrow$), this product must be zero with the states consistent with the outcomes ($\downarrow$, $\downarrow$, $\downarrow$). By the definition of f given in SEC. II I, we see that this is only possible if this hypothetical eigenstate, just as the eigenvector $|\phi_1\rangle$ of A, is orthogonal to the these states, i.e does not have any non-zero digits in common with them.

The states consistent to the result ($\downarrow$, $\downarrow$, $\downarrow$) are given by (31), (34) and (37), respectively. It is clear from their graphical representations where the non-zero digits reside.

If we forbid the union of their non-zero digits, the remaining allowed boxes to construct our eigenstate can be found. These boxes are marked as ✓ below

$$\begin{array}{|c|c|c|c|}\hline & & & \\ \hline \checkmark & \checkmark & & \\ \hline \checkmark & & \checkmark & \\ \hline & \checkmark & \checkmark & \\ \hline \end{array} \tag{39}$$

But by considering the structural conditions for uncorrelated and perfectly correlated states of maximal knowledge, given respectively by (19) and (20), we see that such states are not possible in the space illustrated in (39).

To see this, we can analyse both conditions separately. We start with the simplest one, given by (20) for perfectly correlated states. This is simply a condition requiring that the four non-zero digits each reside in a unique row and column. Compare with (17) and (28). Now, since (39) explicitly forbids access to one of its rows and columns, we deduce that no perfectly correlated state is possible in the available space.

We move on to condition (19) for uncorrelated states of maximal knowledge. This condition imposes a requirement of rectangular symmetry for the non-zero digits. Compare with (15), (32), (35) and (38), all of which are states of the desired type. This can further be reinforced by, as discussed in SEC. II H, the fact that that states of this type can be obtained from (15) by either a permutation of its rows, its columns or any combination of these permutations. From this, we see that a rectangular symmetry, in the available space of (39), is not possible either.

We have thus deduced that there exists neither an uncorrelated or perfectly correlated state of maximal knowledge capable of rejecting ($\downarrow$, $\downarrow$, $\downarrow$), and hence possibly guaranteeing ($\uparrow$, $\uparrow$, $\uparrow$). Since we have now shown that an eigenstate corresponding to atleast one of the eigenvectors of A is not possible, we can also conclude that a 1-to-1 correlation of the scheme is not possible. However, one might argue that there exist a different set of combinations we might be able to create suitable eigenstates for.

Each spin measurement has two outcomes and there are three such measurements. Therefore, we have in total 2^3 different combinations which we can attempt to reject, including the one we have already. These eight combinations are

$$(\downarrow, \downarrow, \downarrow), (\downarrow, \uparrow, \uparrow), (\uparrow, \uparrow, \downarrow), (\uparrow, \downarrow, \uparrow), \tag{40}$$
$$(\uparrow, \uparrow, \uparrow), (\uparrow, \downarrow, \downarrow), (\downarrow, \downarrow, \uparrow), (\downarrow, \uparrow, \downarrow). \tag{41}$$

The first combination of (40) is the one we have already dealt with. However, visually lining up the allowed boxes, for each of these first four combinations, yields valuable insight. Respectively, these are

✓	✓		
✓		✓	
	✓	✓	

✓		✓	
✓			✓
		✓	✓

✓	✓		
✓			✓
	✓		✓

	✓	✓	
	✓		✓
		✓	✓

(42)

As seen, these combinations give rise to some notable patterns of allowed boxes. In particular, they all forbid access to one of their rows and columns, and we can therefore immediately deduce that no perfectly correlated state can possibly be constructed. By further review, we also see, as in the case for (39), that a rectangular symmetry is not possible in any of them either.

We now turn our attention to the combinations given by (41). These, respectively, have the allowed boxes

✓	✓	✓	✓
			✓
			✓
			✓

	✓		
	✓		
✓	✓	✓	✓
	✓		

		✓	
✓	✓	✓	✓
		✓	
		✓	

✓			
✓			
✓			
✓	✓	✓	✓

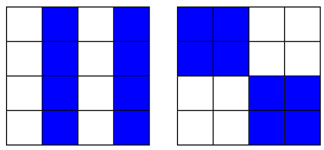
(43)

For them, a much more obvious pattern emerge. The allowed boxes constitute a full row, and a full column. However, non-zero digits of perfectly correlated systems need to reside in a unique row and column, thereby making it impossible for one to be constructed in any of the spaces in (43). Following the same line of argument, rectangular symmetries, at bare minimum, require access to space in two different rows and columns, which we do not have.

We can thus conclude that no combination, whatsoever, of spin-outcomes is able to be guaranteed using eigenstates of maximal knowledge, which in turn implies that all maximally informative measurements fails for this purpose. Close, but no cigar.

C. Non-maximally informative measurements

The scheme proposed by Vaidman, Aharonov and Albert has been shown to not be possible, in the toy theory, using observables consisting of eigenstates of maximal knowledge. It remains to show likewise for eigenstates of non-maximal knowledge. There are two, non-trivial, types. Examples of both of these, which we already have encountered, are



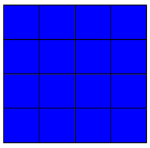
(44)

a. b.

The first type, given by (44a), are called mixed uncorrelated states. States of this type are product states. The second type, given by (44b), are called mixed correlated states (or entangled states). These, as stated in SEC. II F, are not product states.

Although the structural conditions for these are not as straightforward, they all have one thing in common. The amount of non-zero digits in each valid state is strictly eight [ ]. But the spaces illustrated in (42) and (43) was independent of the kind of eigenstate we planned to utilize, and they all allow for less than eight non-zero digits. We can therefore conclude that observables consisting of these types of eigenstates also fails the scheme, since an overlap with forbidden space inevitably occurs.

Lastly, we have the remaining case of the trivial observable consisting of only one eigenstate. This is the least-informative measurement and its sole eigenstate is given by



(45)

This least-informative state is the bipartite counterpart to (4). It is unable to distinguish any state from another and a measurement of this kind is therefore futile.

This concludes the proof of the inability of Spekkens' toy theory to solve the mean king's problem, as done by Vaidman, Aharonov and Albert.

IV. Conclusion

In the prelude of this paper, we reiterated Spekkens' claim about theories gaining traction as phenomena consistently are explained by it. This is at the very heart of scientific methodology. And many quantum phenomenon do find, mathematically speaking, an explanation in this epistemic toy theory. But others do not. In his paper, Spekkens explicitly highlights various other phenomena not covered, and suggest that these provide insight on how to proceed with this research programme.

Analysing the eigenvectors of A in SEC. I might provide clues on why this scheme failed in the toy theory. All four eigenvectors consist of very specific linear weights of the standard basis for $\mathbb{C}^4$. In vector quantum mechanics, one is in general allowed to construct arbitrary linear weights. The sole condition is that the given vector is of unit length. Nonetheless, this provides for uncountably many valid states, even for elementary systems. By contrast, in the toy theory, we are not in control over the weights. A system is strictly defined in terms of its ontic states and they are assumed to be of equal weight, at all times. Elementary systems can only be described by eleven states in total, and bipartite systems not even an order of magnitude more than that [3]. This imposes a severe limit to the variety of this theory.

The solution to the problem provided by Vaidman, Aharonov and Albert, is closely tied to the concept of *mutually unbiased bases* (MUBs). These are two sets of orthonormal bases $\{e_1,\ e_2,\ \dots\ ,\ e_d\}$, $\{f_1,\ f_2,\ \dots\ ,\ f_d\}$ s.t $\langle e_i,\ f_j\rangle = \frac{1}{d}$, where $\langle\cdot,\cdot\rangle$ is the inner product of the space and d its dimension.

Spekkens defines a similar concept, *mutually unbiased partitionings* (MUPs), for the toy theory. These all involve states of maximal knowledge, of which use we have already proven ineffective, possibly due to the reason explained above.

Appendix I

A. Preliminary quantum theory

Quantum mechanics is the theory of dynamics of nature at low scale and low energies of atomic and subatomic particles. As particles, in the classical limit, are governed by Newton's laws, quantum particles evolve according to the Schrödinger equation, which for one dimensional systems is

$$\Bigg[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x} + V(x,t)\Bigg]\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t), \tag{46}$$

where $\hbar$ is the Planck constant and $V(x,t)$ the potential of which the particle, described by $\Psi(x,t)$, is subject to. The bracketed terms are often, for shorts, denoted as $H(x,t)$. This is the time-dependent Hamiltonian operator and corresponds to the total energy of the system, in most cases. The equation can be separated if we make the ansatz $\Psi(x,t) = \psi(x)f(t)$ and $V(x,t) = V(x)$. The resulting equations are

$$\Bigg[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial^2 x} + V(x)\Bigg]\psi(x) = E\psi(x), \tag{47}$$

$$\frac{d}{dt}f(t) = -\frac{iE}{\hbar}f(t), \tag{48}$$

where E is interpreted as the total energy of the solution $\psi(x)f(t)$. The former is called the time-independent Schrödinger equation. The latter can be integrated to yield the normalized solution

$$f(t) = e^{-iEt/\hbar}. \tag{49}$$

Evolution in time of $\Psi(x,t)$ is determined by this factor. Note that $f(0) = 1$ implies that $\psi(x)$ fully describes the system at $t = 0$.

The space spanned by the set of solutions $\{\psi_1 f_1,\ \psi_2 f_2,\ \ldots\ \}$ is called *Hilbert space*, and is denoted $\mathcal{H}$. Mathematicians more commonly know it as $L_2(a,b)$. It may be finite or infinite [4], but we will only consider the finite case. The elements of this set are mutually orthonormal w.r.t the inner product of the space, as follows

$$\langle \psi_k f_k,\ \psi_l f_l\rangle = f_k^* f_l \int_a^b \psi_k^* \psi_l \ dx = \delta_{kl}, \tag{50}$$

where * denotes complex conjugation.

In the Copenhagen interpretation, which is an inherently epistemic view, one can only make probabilistic claims on a particle's whereabouts. The probability distribution for elementary systems is given by $\psi^*\psi = |\psi|^2$. The normalisation condition for probabilities is implicitly stated by (50) for the case $k = l$. This ensures that they add to unity in the interval in which the system is contained, in this case $[a, b]$.

The general solution to a linear differential equation is a linear sum of solutions. Hence, if $\{\psi_1 f_1,\ \psi_2 f_2,\ \ldots\ ,\ \psi_n f_n\}$ is a set of solutions, then

$$\Psi = \sum_{i=1}^{n} \lambda_i \psi_i f_i, \quad \lambda_i \in \mathbb{C}, \tag{51}$$

is also a solution. It is a valid *physical* solution if the normalization condition is fulfilled. For this, we require

$$\begin{aligned} 1 = \langle \Psi, \Psi\rangle &= \int_a^b \Big(\sum_{i=1}^{n}\lambda_i\psi_i f_i\Big)^* \Big(\sum_{j=1}^{n}\lambda_j\psi_j f_j\Big)\ dx = \\ &= |\lambda_1|^2 \int_a^b |\psi_1|^2\ dx + |\lambda_2|^2\int_a^b |\psi_2|^2\ dx + \ldots = \\ &= |\lambda_1|^2 + |\lambda_2|^2 + \ldots + |\lambda_n|^2, \end{aligned} \tag{52}$$

where we in the second line utilized relation (50). By this result, we see that that the explicit condition on the constants is

$$\sum_{i=1}^{n} |\lambda_i|^2 = 1. \tag{53}$$

Further, we interpret (51) as $|\lambda_i|^2$ being the probability of Ψ to reside in $\psi_i f_i$, upon a suitable measurement. Therefore, it is natural to require that these probabilities add to unity. By once again using (50), one can find $|\lambda_i|^2$ through Ψ. We have that

$$\langle \psi_i f_i, \Psi\rangle = \int_a^b (\psi_i f_i)^* \Big(\sum_{j=1}^{n}\lambda_j\psi_j f_j\Big)\ dx = \lambda_i, \tag{54}$$

from which it follows that

$$|\langle \psi_i f_i, \Psi\rangle|^2 = |\lambda_i|^2. \tag{55}$$

The set of elementary solutions are *complete* in the sense that any general solution in the space can be described uniquely as a linear combination of them.

B. Transition to linear algebra

The inherent linearity of the Schrödinger equation makes linear algebra a suitable language, for practical purposes. As we have seen in (51), and further discussed in the same subsection, the general solutions can be described uniquely by a set of constants corresponding to the weight of each elementary solution. In $\mathbb{C}^n$, these elementary solutions correspond to the standard basis and the general solution can thus be uniquely described as

$$\Psi = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ ... \\ \lambda_n \end{pmatrix}. \tag{56}$$

Furthermore, this complex vector space is equipped with the standard inner product. For $x,\ y\ \in \mathbb{C}^n$, we have

$$\langle x, y \rangle = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}^* \begin{pmatrix} y_1 \\ y_2 \\ ... \\ y_n \end{pmatrix} = \sum_{i=1}^{n} {x_i}^* y_i. \tag{57}$$

The property of mutual orthonormality of elementary solutions, as given by (50), implicitly follows from (56) and (57). Explicitly, we can reiterate this as

$$\langle \psi_k f_k,\ \psi_l f_l \rangle = f_k^* f_l \sum_{i=1}^{n} \delta_{ik}\delta_{il} = \delta_{kl}. \tag{58}$$

The normalization requirement for general solutions manifests itself analogously as

$$\langle \Psi, \Psi \rangle = \sum_{i=1}^{n} \lambda_i^* \lambda_i = \sum_{i=1}^{n} |\lambda_i|^2 = 1. \tag{59}$$

Lastly, (55) follows from

$$\langle \psi_i f_i, \Psi \rangle = \sum_{j=1}^{n} \delta_i \lambda_j = \lambda_i,$$

which implies that

$$|\langle \psi_i f_i, \Psi \rangle|^2 = |\lambda_i|^2. \tag{60}$$

C. Bra-ket notation

For further efficiency in many situations, Dirac's bra-ket notation is often adopted to be the standard of describing quantum states. For $\alpha\ \in\ \mathbb{C}^n$, we denote

$$\langle \alpha | = \begin{pmatrix} \alpha_1 & \alpha_2 & ... & \alpha_n \end{pmatrix}^* \quad \text{and} \quad | \alpha \rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ ... \\ \alpha_n \end{pmatrix}. \tag{61}$$

The notation for the conjugated row vector $\langle \alpha |$ is what we call a *bra*, and the notation for the column vector $|\alpha\rangle$ is called a *ket*. It is clear that

$$\langle \alpha | = | \alpha \rangle^\dagger , \tag{62}$$

where † denotes the combination of vector transposition and complex conjugation. The inner product of $\mathbb{C}^n$, shown in (57), is re-denoted as

$$\langle x, y \rangle \to \langle x | y \rangle . \tag{63}$$

When a bra and a ket joins together for an inner product, as in (63), we say that they form a *braket*.

As a further note on the inner product, we have, in this notation, that the square of the absolute value of it, i.e

$$| \langle x | y \rangle |^2 \tag{64}$$

is interpreted to be the probability of $|y\rangle$ to be found in $|x\rangle$, or vica-versa, upon a suitable measurement, given that both states are initially normalized.

Furthermore, the time-independent Schrödinger equation, given by (47), can in this notation be rewritten as

$$H |\psi\rangle = E |\psi\rangle , \tag{65}$$

where the time-independent Hamiltonian H is represented by a suitable square matrix. It is clear from this result that our initial problem of solving a linear differential equation has, in the language of linear algebra, translated into an eigenvector and eigenvalue problem.

D. Hermitian matrices and observables

An element $A \in \mathbb{C}^{n\times n}$ is called hermitian (or self-adjoint) if

$$A^\dagger = A. \tag{66}$$

It is but a simple exercise to show that the eigenvalues of hermitian matrices are real, and that the corresponding eigenvectors are orthogonal.

Now, given (66), we see that hermitian matrices are also normal matrices since clearly $A^\dagger A = AA^\dagger$. And since A is normal, we can apply the Spectral theorem [5]. The theorem, in this case, states that there exists an orthonormal basis for $\mathbb{C}^n$ consisting of the eigenvectors of A. Further, A is diagonal in this basis and can be written as a linear combination of pairwise orthogonal projections, in the following manner

$$A = \sum_{i=1}^{n} a_i \left|\phi_i\right\rangle \left\langle\phi_i\right|, \tag{67}$$

where $\{\left|\phi_1\right\rangle, \left|\phi_2\right\rangle, \ \dots\ , \left|\phi_n\right\rangle\}$ is the set of orthonormal eigenvectors of A, $\{a_1,\ a_2,\ \dots\ ,\ a_n\}$ their corresponding eigenvalues and $\left|\cdot\right\rangle\left\langle\cdot\right|$ notation for the vector outer product.

Physically measurable quantities (such as position and momentum) are called observables by physicists. In the properties of hermitian matrices stated above, we see some examples of why observables, in the language of mathematics, are necessarily represented by them. We e.g. have that the eigenvalues are real, which is what the outcome of physical measurements are as well.

The Hamiltonian H, as discussed previously, is the observable corresponding to energy. Following equation (65), we can express the time-independent Hamiltonian as

$$H = \sum_{i=1}^{n} E_i \left|\psi_i\right\rangle \left\langle\psi_i\right|, \tag{68}$$

where E_i is the (energy) eigenvalue of the eigenvector $\left|\psi_i\right\rangle$. To prove this, we calculate

$$H\left|\psi_i\right\rangle = \textstyle\sum_{j=1}^{n} E_j \left|\psi_j\right\rangle \left\langle\psi_j|\psi_i\right\rangle = \sum_{j=1}^{n} E_j \left|\psi_j\right\rangle \delta_{ji} = E_i \left|\psi_i\right\rangle. \tag{69}$$

E. Composite quantum systems

If an arbitrary system A is described by the state $\left|\alpha\right\rangle$, and another system B by $\left|\beta\right\rangle$, we say that the composite system AB is described by $\left|\alpha\right\rangle \otimes \left|\beta\right\rangle$, where $\otimes$ is the tensor product. For convenience, one often denotes $\left|\alpha\right\rangle \otimes \left|\beta\right\rangle = \left|\alpha\beta\right\rangle$.

The tensor product has several interesting properties. First of, it is both associative and bilinear, i.e

$$a\left|v\right\rangle \otimes \left|w\right\rangle = \left|v\right\rangle \otimes a\left|w\right\rangle = a(\left|v\right\rangle \otimes \left|w\right\rangle),\ a \in \mathbb{C}. \tag{70}$$

$$\begin{aligned} \left|uv\right\rangle + \left|wv\right\rangle &= (\left|u\right\rangle + \left|w\right\rangle) \otimes \left|v\right\rangle, \\ \left|uv\right\rangle + \left|uw\right\rangle &= \left|u\right\rangle \otimes (\left|v\right\rangle + \left|w\right\rangle). \end{aligned} \tag{71}$$

Secondly, if $u, u' \in U$ and $v, v' \in V$, then the inner product for elements of the composite system UV is given by

$$\left\langle uv|u'v'\right\rangle = \left\langle u|u'\right\rangle \left\langle v|v'\right\rangle, \tag{72}$$

i.e, a simple multiplication between the inner products of the different spaces. Now, assume that the set $\{\left|e_1\right\rangle, \left|e_2\right\rangle, \ \dots\ , \left|e_n\right\rangle\}$ is an orthonormal basis for U, and $\{\left|f_1\right\rangle, \left|f_2\right\rangle, \ \dots\ , \left|f_m\right\rangle\}$ an orthonormal basis for V. It follows, from (72), that the set $\{\left|e_1 f_1\right\rangle, \ \dots\ , \left|e_1 f_m\right\rangle, \ \dots\ , \left|e_n f_1\right\rangle, \ \dots\ , \left|e_n f_m\right\rangle\}$ is an orthonormal basis for UV, as confirmed by

$$\left\langle e_i f_j|e_k f_l\right\rangle = \left\langle e_i|e_k\right\rangle \left\langle f_j|f_l\right\rangle = \delta_{ik}\delta_{jl}. \tag{73}$$

Another important property of the tensor product is that if $X : U \to U$ and $Y : V \to V$, then

$$(X \otimes Y)(\left|u\right\rangle \otimes \left|v\right\rangle) = X\left|u\right\rangle \otimes Y\left|v\right\rangle, \tag{74}$$

from which we deduce that $X \otimes Y : UV \to UV$.

In the context of quantum mechanics, U and V are substitutes for the Hillbert spaces of the individual systems in question. Thus, UV is the Hilbert space of the composite system.

F. Projective measurements

The wave collapse hypothesis, shared by many epistemic views of quantum mechanics, is an attempt to explain the experimental phenomenon of identical results following repeated measurements. In mathematical terms, this translates to a projection of the state in question onto the state corresponding to the outcome of the measurement.

As an example, say that we have a system in the state $|\alpha\rangle$ and that we intend to conduct a measurement of an arbitrary quantity on it. As described in APP. I D, measurable quantities are necessarily represented by hermitian matrices. The normalized eigenvectors of a hermitian matrix are an orthonormal basis for the space in which they live, and since $|\alpha\rangle$ lives in the same space, it can be expressed as a linear combination of them. Assuming the eigenvectors in question are $|\phi_1\rangle\,,\ |\phi_2\rangle\,,\ ...\,,|\phi_n\rangle$, we can write

$$|\alpha\rangle = \alpha_1 |\phi_1\rangle + \alpha_2 |\phi_2\rangle + ... + \alpha_n |\phi_n\rangle\,, \tag{75}$$

where $|\alpha_i|^2$ is interpreted to be the probability of the measurement to yield the outcome that corresponds to the eigenvector $|\phi_i\rangle$.

Now, assume that we did obtain the outcome $|\phi_i\rangle$. In order to satisfy the wave collapse hypothesis, we act on $|\alpha\rangle$ with the projector $|\phi_i\rangle\langle\phi_i|$, followed by a normalisation. This process yields

$$|\phi_i\rangle\langle\phi_i|\alpha\rangle = \alpha_i |\phi_i\rangle \to |\phi_i\rangle\,. \tag{76}$$

The clear result is that our initial state is fully transformed into the outcome state, an operation which will guarantee the outcome upon repeated measurement since the probability for all others is now zero.

We can extend this procedure for composite systems. Assume we have two Hilbert spaces, U and V. Let $|\alpha\rangle \in U$, and let the state be defined as in (75). Further, let $|v\rangle \in V$ be some normalized state. The composite state is then described by $|\alpha\rangle \otimes |v\rangle$. Although the system is now composite, we can still conduct a separate measurement on $|\alpha\rangle$, if we wish. Assume we do. In this case, the appropriate projector will instead be $|\phi_i\rangle\langle\phi_i| \otimes \mathbb{1}_V$, where $\mathbb{1}_V$ is the identity on V. Acting with this operator on our state, and using property (74), we get

$$\begin{aligned}(|\phi_i\rangle\langle\phi_i| \otimes \mathbb{1}_V)(|\alpha\rangle \otimes |v\rangle) &= |\phi_i\rangle\langle\phi_i|\alpha\rangle \otimes \mathbb{1}_V |v\rangle = \\ &= \alpha_i |\phi_i\rangle \otimes |v\rangle \to |\phi_i\rangle \otimes |v\rangle\,.\end{aligned} \tag{77}$$

These update rules imply that measurements of several observables commute iff they have the same eigenvectors. More generally, if A and B represent observables, measurements commute iff $[A, B] = 0$.

G. Spin

A quantity of utmost interest in quantum mechanics is the *spin* of a system, which is a form of intrinsic angular momentum. Usually, one is interested in its directions alongside three orthogonal axes, usually denoted as x, y and z. In each, the spin can point in two different directions, up ($\uparrow$) or down ($\downarrow$). For elementary systems, the hermitian matrices corresponding to each of these is denoted σ_x, σ_y and σ_z, where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{78}$$

The standard basis for $\mathbb{C}^2$ is also the eigenvectors of σ_z. By convention, we denote

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{79}$$

where $|0\rangle$ corresponds to spin-$\uparrow$ while $|1\rangle$ to spin-$\downarrow$ on the z-axis. Also by convention, we express the eigenvectors of the other two matrices by linear combinations of these. Doing so, we have that spin-$\uparrow$ and spin-$\downarrow$ on the x-axis are respectively given by

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|0\rangle - |1\rangle}{\sqrt{2}}. \tag{80}$$

Analogously, we have that spin-$\uparrow$ and spin-$\downarrow$ on the y-axis are given by

$$\frac{|0\rangle + i\,|1\rangle}{\sqrt{2}} \quad \text{and} \quad \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}}. \tag{81}$$

For convenience, as in the general case described in APP. I E, we denote $|i\rangle \otimes |j\rangle = |ij\rangle\,,\ i, j \in \{0, 1\}$. Therefore, e.g.

$$|0\rangle \otimes \frac{|0\rangle - i\,|1\rangle}{\sqrt{2}} = \frac{|00\rangle - i\,|01\rangle}{\sqrt{2}}. \tag{82}$$

Appendix II

A. The scheme

In this subsection, and in the following, we expand upon the details omitted in the article by Vaidman, Aharonov and Albert [2]. Familiarity with the prerequisites provided by APP. I is sufficient for an adequate understanding of the process.

The king presents his challenge to the scholar. As per the rules she is, before the king's measurement, allowed to prepare the particle in question in an initial state of her choosing. We denote the system of this particle A. Now, alongside an external particle, of which system we denote B, she prepares the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_{\mathrm{B}} \otimes |0\rangle_{\mathrm{A}} + |1\rangle_{\mathrm{B}} \otimes |1\rangle_{\mathrm{A}}) = \\ = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle). \quad (83)$$

From now on we shall stick to the authors convention that the first argument of the tensor product is information on system B, while the second is of system A, and thereby scraping the need for subnotations.

As the kings retakes control, he conducts a single spin measurement on system A alongside one of the three orthogonal axes denoted as x, y and z. However, he keeps the nature of his measurement, and its result, secret.

The scholar, now unaware of the state of the system since it has been altered by the king's measurement, is again handed control. By the rules, she is now allowed to do one last measurement of her choosing. Thereafter, she is to present to the king the result of his own measurement. The authors propose that she conducts a measurement w.r.t the observable A, with eigenvectors

$$|\phi_1\rangle = \frac{\sqrt{2}}{2}|00\rangle + \frac{1}{2}(e^{-i\frac{\pi}{4}}|01\rangle + e^{i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_2\rangle = \frac{\sqrt{2}}{2}|00\rangle - \frac{1}{2}(e^{-i\frac{\pi}{4}}|01\rangle + e^{i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_3\rangle = \frac{\sqrt{2}}{2}|11\rangle + \frac{1}{2}(e^{i\frac{\pi}{4}}|01\rangle + e^{-i\frac{\pi}{4}}|10\rangle),$$

$$|\phi_4\rangle = \frac{\sqrt{2}}{2}|11\rangle - \frac{1}{2}(e^{i\frac{\pi}{4}}|01\rangle + e^{-i\frac{\pi}{4}}|10\rangle),$$

and with corresponding eigenvalues a_1, a_2, a_3 and a_4. By doing this, the claim is that the results of Table 1, from SEC. I, follows. In order to see this, we need to get into the details of the probability function given by

$$p(C = c_n, i) = \frac{|\langle\phi_i| P_{C=c_n}|\psi\rangle|^2}{\sum_j |\langle\phi_i| P_{C=c_j}|\psi\rangle|^2}, \quad (84)$$

where $C \in \{x, y, z\}$, $c_n \in \{\uparrow, \downarrow\}$ and $i \in \{1, 2, 3, 4\}$.

As an example, we interpret $p(x = \uparrow,\ 3)$ to be the probability of the king having measured spin-$\uparrow$ in the x-direction, given that the scholar obtained a_3 in her subsequent measurement. We can understand this clearly by analysing the probability function, term-by-term.

The initial state is, as previously stated, $|\psi\rangle$. Measurements alter the state being measured upon, and the king's is no exception. $P_{C=c_n}$ therefore represents a projector that alters the initial state in different ways, depending on the king's choice of measurement axis C, and the result c_n. Thus, $P_{C=c_n}|\psi\rangle$ is the altered state handed back to the scholar. Usually, a projection must be followed by a normalisation but since the denominator of the probability function does that for us, we do not have to worry about that here.

The scholar now performs her measurement on the altered state $P_{C=c_n}|\psi\rangle$. Although this altered state is unknown to her, she can still find the probability of each outcome a_i by calculating $p(C = c_n, i)$. Usually, it would be sufficient to simply calculate $|\langle\phi_i| P_{C=c_n}|\psi\rangle|^2$, but since we did not normalize the system after the projection, we need to go one step further. In any case, since she does not know the nature of the altered state, she has to calculate the product for each possible case and from that see if there are any conclusions to be made. This is indeed the case, which we will see in the next subsection. We will find that certain outcomes of her measurement are not possible given certain results of the king's measurement, in a way that allows her to deduce, with probability one, the actual result from the king's action, for every possible case.

B. The detailed procedure

In order to establish quantitative conclusions, the scholar needs to analyse every possible situation in this experiment, case-by-case. The observable she is conducting a measurement with has four outcomes. She is, by definition, guaranteed to obtain one, and only one of these. The king, on the other hand, is free to choose between three different measurements. He can measure spin along the x-, y- or z-axis. Each of these has two different outcomes, $\uparrow$ and $\downarrow$. These six different cases of the king's measurement alters the initial state $|\psi\rangle$ in different ways. The scholar, while analysing the probabilities for each one of her outcomes, therefore has to take each of these six in account.

The six possible altered states she may receive from the king are $P_{C=c_n}|\psi\rangle$, for $C \in \{x, y, z\}$ and $c_n \in \{\uparrow, \downarrow\}$. In each case, the projector is given by

$$P_{z=\uparrow} = \mathbb{1} \otimes |0\rangle \langle 0| , \tag{85}$$

$$P_{z=\downarrow} = \mathbb{1} \otimes |1\rangle \langle 1| , \tag{86}$$

$$P_{x=\uparrow} = \mathbb{1} \otimes \frac{1}{2}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|), \tag{87}$$

$$P_{x=\downarrow} = \mathbb{1} \otimes \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|), \tag{88}$$

$$P_{y=\uparrow} = \mathbb{1} \otimes \frac{1}{2}(|0\rangle + i\,|1\rangle)(\langle 0| - i\,\langle 1|), \tag{89}$$

$$P_{y=\downarrow} = \mathbb{1} \otimes \frac{1}{2}(|0\rangle - i\,|1\rangle)(\langle 0| + i\,\langle 1|). \tag{90}$$

It follows e.g. that the altered state received by the scholar, given that the king measured spin-$\uparrow$ alongside the z-axis, is

$$P_{z=\uparrow}|\psi\rangle = (\mathbb{1} \otimes |0\rangle \langle 0|)(\frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) =$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \langle 0|0\rangle + |1\rangle \otimes |0\rangle \langle 0|1\rangle =$$

$$= \frac{1}{\sqrt{2}}|0\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle . \tag{91}$$

Similarly, we can calculate the altered states for the remaining cases. These are

$$P_{z=\downarrow}|\psi\rangle = \frac{1}{\sqrt{2}}|11\rangle , \tag{92}$$

$$P_{x=\uparrow}|\psi\rangle = \frac{1}{2\sqrt{2}}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \tag{93}$$

$$P_{x=\downarrow}|\psi\rangle = \frac{1}{2\sqrt{2}}(|00\rangle - |01\rangle - |10\rangle + |11\rangle), \tag{94}$$

$$P_{y=\uparrow}|\psi\rangle = \frac{1}{2\sqrt{2}}(|00\rangle + i\,|01\rangle - i\,|10\rangle + |11\rangle), \tag{95}$$

$$P_{y=\downarrow}|\psi\rangle = \frac{1}{2\sqrt{2}}(|00\rangle - i\,|01\rangle + i\,|10\rangle + |11\rangle). \tag{96}$$

For each of the four possible outcomes of her measurement, the scholar can now proceed to calculate the product $|\langle\phi_i| P_{C=c_n}|\psi\rangle|^2$. Let us begin by, say, $i = 1$. We obtain

$$|\langle\phi_1| P_{z=\uparrow}|\psi\rangle|^2 = |\langle\phi_1| P_{y=\uparrow}|\psi\rangle|^2 =$$

$$= |\langle\phi_1| P_{z=\uparrow}|\psi\rangle|^2 = \frac{1}{4}, \tag{97}$$

while on the other hand

$$|\langle\phi_1| P_{z=\downarrow}|\psi\rangle|^2 = |\langle\phi_1| P_{y=\downarrow}|\psi\rangle|^2 =$$

$$= |\langle\phi_1| P_{z=\downarrow}|\psi\rangle|^2 = 0. \tag{98}$$

The former is not yet normalized and can therefore not be interpreted as a probability. Inserting the results from (97) and (98) into the probability function (84), which normalizes them for her, yields

$$p(x = \uparrow,\ 1) = p(y = \uparrow,\ 1) = p(z = \uparrow,\ 1) = 1, \tag{99}$$

$$p(x = \downarrow,\ 1) = p(y = \downarrow,\ 1) = p(z = \downarrow,\ 1) = 0. \tag{100}$$

The conclusion here is that is that the outcome a_1 for the scholar is not even possible in the cases where the king obtained spin-$\downarrow$ alongside any direction. Instead, she can rest assured that the king measured spin-$\uparrow$.

Analogously, she can repeat this procedure for the cases $i = 2, 3, 4$. By doing so, it can be shown that

$$p(x = \downarrow,\ 2) = p(x = \uparrow,\ 3) = p(x = \downarrow,\ 4) = 1, \tag{101}$$

$$p(y = \downarrow,\ 2) = p(y = \downarrow,\ 3) = p(y = \uparrow,\ 4) = 1, \tag{102}$$

$$p(z = \uparrow,\ 2) = p(z = \downarrow,\ 3) = p(z = \downarrow,\ 4) = 1. \tag{103}$$

In a straightforward manner, it can be shown that for the remaining cases, which are precisely those not included in (99)-(103), the probabilities are all zero. As expected, these results are consistent with Table 1.

As a last comment, note that while the scholar, with this scheme, is able to infer, with probability one, the result of the king's measurement in the three different measurement cases, she is never able to deduce the actual direction of said measurement. Fortunately for her though, that was not part of the challenge.

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