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Cyclicity in Dirichlet type spaces on the Polydisc

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Abstract

In this thesis, we examine which functions are cyclic with respect to the shift operators in Dirichlet type spaces on the polydisc. That is, we investigate which functions have the property that the only closed subspace that contains the function, which is also invariant under the shift operators, is the entire Dirichlet type space itself.

In particular, we attempt to generalize methods used in the complete characterization of cyclic polynomials in two complex variables to higher dimensions. For example, we generalize a theorem from two variables up to arbitrary dimension, which relates non-vanishing Gaussian curvature of a certain part of the zero set of a function to non-cyclicity of the same function. However, whereas in two variables this theorem was almost always applicable, it turns out that in arbitrary dimension we are not as lucky. Essentially because in two dimensions, the relevant part of the zero set could only be a hypersurface or a finite set, but in higher dimensions there are far more possibilities.

Already in three variables we find a family of polynomials for which the previously mentioned theorem is not applicable, so in the second part of the thesis we attempt to understand the cyclicity properties of this special family. Interestingly enough, it turns out that even for the polynomials on which we could not apply the theorem, we still obtain the same bound on non-cyclicity.

Finally, for the special family of polynomials we develop a method for comparing these polynomials two polynomials in two variables. Using this method we manage to completely understand the cyclicity properties of three variable polynomials in this family whose zero set is either a finite set or a hypersurface, and for polynomials whose zero set is a curve, we show that the cyclicity properties are indeed better than for hypersurfaces, but worse than for finite sets.

Chapter 1

Introduction

In many areas of mathematics, one is interested in studying a set endowed with some sort of algebraic structure, together with an operation on that set. In that context, it is of great interest to characterize the *invariant subsets* of that operation, that is, the subsets that are contained in their own image under the operation. In general, understanding these subsets says a lot about the operation. For example, for linear operators on finite dimensional spaces, this corresponds to finding the eigenspaces of the operator. As is seen through the spectral theorem, this knowledge can be used to better understand how this operator acts on our space. Furthermore, through the spectral theorem of self-adjoint operators on Hilbert spaces, we see another example of how this knowledge is concretely used in order to better understand the operator in question.

In general, it is difficult to characterize all invariant subsets of an operator. However, one particular type of invariant subsets are the so called *cyclic subsets* corresponding to our operator. These are essentially created by taking out some fixed element of our set, and then creating a subset by simply adding all elements that can be obtained by applying our operator to this element. For example, the *orbits* of a group action on a set are exactly subsets of this kind.

In analysis, the structured set is often a Hilbert space of functions, and the operator is often some linear operator from the Hilbert space to itself. One of the most fundamental operators in this context is the *shift operator*, which is simply the operator of "multiplying by x", that is, the operation

$$S(f) : f(x) \to xf(x).$$

The name comes from the fact that this operation simply shifts the sequence of Taylor coefficients of a function. In this thesis, we are primarily interested in Hilbert spaces of holomorphic functions on the polydisc, that is the Cartesian product of unit discs. One of the most important such spaces is the *Hardy space*, which consists of holomorphic functions on the open unit disc, for which the Hardy norm

$$\|f\|_{H^2} = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^2 dt \right)^{1/2},$$

is finite.

Intuitively, this is the space of holomorphic functions on the unit disc, whose restriction to the boundary behaves well in the L^2-sense. The Hardy space is important, and has several applications, both in pure mathematics, but also in other fields, such as control theory, and scattering theory.

Although the above definition is the most intuitive one, it is sometimes easier to work with another, equivalent norm. Namely, the norm given by

$$\|f\|_{H^2} = \left(\sum_{k=0}^{\infty} |a_k|^2 \right)^{1/2},$$

where $\{a_k\}$ are the Taylor coefficients of the function f.

But what is known about the invariant subspaces of the shift operator acting on this space? Quite a lot actually! Arne Beurling showed that the *only* invariant subspaces of the shift operator on the Hardy space are the cyclic subspaces [11]. He also showed that a function f *generates* the whole Hardy space if and only if it is *outer*, which means that

$$f(z) = c \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log(g(e^{i\theta}))d\theta \right),$$

for some c on the unit circle, and some positive measurable function g for which $\log(g)$ is integrable on the circle.

That a function f generates the entire space means that the smallest invariant subspace which contains f is dense in H^2. That is

$$\overline{\operatorname{span}\{z^k f(z) : k \in \mathbb{N}\}} = H^2.$$

A function with the property that it generates the entire space is called a *cyclic function.*

Although the cyclic subspaces are interesting because they are the easiest example of an invariant subspace, they are also interesting on their own since these subspaces inherit a lot of properties from the generator. As an example, for the shift operator, we know that the entire subspace inherits the zeros of the generator. Similarly, it is of course interesting to understand which functions generate the whole space, since this says a lot about the space.

Another natural space which is closely related to the Hardy space is the *Dirichlet space.* The Dirichlet space is the subspace of the Hardy space which consists of all functions whose *Dirichlet integral* is finite, that is

$$\int_D |f'(z)|^2 dA,$$

where dA is the area measure on the unit disc.

However, the Dirichlet integral is not a norm, since for example all constants have Dirichlet integral equal to zero. But with the equivalent norm for the Hardy space in mind, we can endow the Dirichlet space with the norm given by

$$\|f\|_D = \left(\sum_{k=0}^{\infty} (k+1)|a_k|^2 \right)^{1/2},$$

where once again $\{a_k\}$ are the Taylor coefficients of our function.

With this norm, the Dirichlet space is a Hilbert space of holomorphic functions on the unit disk.

Similarly, for fixed $\alpha \in \mathbb{R}$ we can define the *Dirichlet type space with parameter* α as the space of holomorphic functions on the unit disc for which the norm

$$\left(\sum_{k=0}^{\infty} (k+1)^{\alpha}|a_k|^2 \right)^{1/2} < \infty.$$

Those who have studied partial differential equations might notice that these spaces are related to the Hardy space in a similar way as how the *Sobolev spaces* are related to L^p-spaces.

As before, it is of great interest to characterize the invariant subspaces of the Dirichlet type spaces. For the Dirichlet space, it is known that the only invariant subspaces are the cyclic ones, however this is not true for all Dirichlet type spaces. Furthermore, one wants to generalize these results to higher dimensions, but this turns out to be much more difficult than one might expect. For example, for the Hardy space on the bidisc, it is no longer true that all outer functions generate the whole space. Although, being outer is a necessary condition for a function to generate the entire space.

It is considered to be a very difficult problem to give a complete characterization of all invariant subspaces of the shift operator in all Dirichlet type spaces. But since all functions generate an invariant subspace, a first step towards such a characterization is to understand which functions generate the whole space and not. In order to solve this problem, we begin by thoroughly examining polynomials, since understanding which properties of a polynomial are relevant for cyclicity will surely help in understanding the phenomena which determine whether or not a general function generates the whole space.

Chapter 2

Dirichlet type spaces and Cyclicity

2.1 Dirichlet-type spaces

First off, we consider the n-dimensional polydisk

$$\mathbb{D}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| < 1, ..., |z_n| < 1\},$$

and the distinguished boundary of $\mathbb{D}^n$, given by

$$\mathbb{T}^n = \{(z_1, ..., z_n) \in \mathbb{C}^n : |z_1| = 1, ..., |z_n| = 1\}.$$

Note that $\mathbb{T}^n$ is not the topological boundary of the polydisk, but for many applications it is more useful. In this context this is mainly due to the fact that in several variables, Cauchy's integral formula translates to an integral over the distinguished boundary, rather than the topological boundary. See for example [4].

Next, we consider a family of Hilbert spaces of holomorphic function defined on the polydisk, namely the so called Dirichlet-type spaces. The Dirichlet-type space on $\mathbb{D}^n$ with parameter $\alpha \in (-\infty, \infty)$ consists of holomorphic functions $f : \mathbb{D}^n \to \mathbb{C}$ whose power series expansion

$$f(z_1, ..., z_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,...,k_n} z_1^{k_1} \cdots z_n^{k_n}$$

satisfies

$$\|f\|_\alpha^2 = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1 + 1)^\alpha \cdots (k_n + 1)^\alpha |a_{k_1,...,k_n}|^2 < \infty.$$

We denote by $\mathfrak{D}_\alpha$ the Dirichlet-type space with parameter α.

Since larger values of the parameter α requires faster decay of the Fourier coefficients in order to assure convergence, we have that $\alpha < \beta$ implies $\mathfrak{D}_\beta \subset \mathfrak{D}_\alpha$.

Note that the definition of the norm implies that any polynomial in $k_1, ..., k_n \in \mathbb{C}$ belongs to $\mathfrak{D}_\alpha$, since the series defining the norm reduces to a finite sum.

Furthermore, the subset of polynomials is a dense subspace of $\mathfrak{D}_\alpha$ for every α. This can be seen by noting that every $f \in \mathfrak{D}_\alpha$ can be approximated by polynomials. Since f is holomorphic we have that

$$f(z_1, ..., z_n) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,...,k_n} z_1^{k_1} \cdots z_n^{k_n}.$$

And since the series

$$\|f\|_\alpha^2 = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^\alpha \cdots (k_n+1)^\alpha |a_{k_1,...,k_n}|^2 < \infty,$$

is convergent, we have that for every $\epsilon > 0$, there exists an N such that

$$\sum_{k_1=N}^{\infty} \cdots \sum_{k_n=N}^{\infty} (k_1+1)^\alpha \cdots (k_n+1)^\alpha |a_{k_1,...,k_n}|^2 < \epsilon.$$

And so the polynomial $p(z_1, ..., z_n) = \sum_{k_1=0}^{N-1} \cdots \sum_{k_n=0}^{N-1} a_{k_1,...,k_n} z_1^{k_1} \cdots z_n^{k_n}$ satisfies

$$\|f-p\|_\alpha^2 = \sum_{k_1=N}^{\infty} \cdots \sum_{k_n=N}^{\infty} (k_1+1)^\alpha \cdots (k_n+1)^\alpha |a_{k_1,...,k_n}|^2 < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, this proves the statement.

Furthermore, for $f \in \mathfrak{D}_\alpha$ the following bound on point evaluation at $z \in \mathbb{D}^n$ holds

$$|f(z)| = \left| \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} a_{k_1,...,k_n} z_1^{k_1} \cdots z_n^{k_n} \right| \leq \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} |a_{k_1,...,k_n} z_1^{k_1} \cdots z_n^{k_n}|$$

$$= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{\alpha/2-\alpha/2} \cdots (k_n+1)^{\alpha/2-\alpha/2} |a_{k_1,...,k_n}||z_1|^{k_1} \cdots |z_n|^{k_n}$$

$$\leq \|f\|_\alpha \left(\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |z_1|^{k_1} \cdots |z_n|^{k_n} \right)^{1/2},$$

where we have used Cauchy Schwarz-inequality to obtain the last inequality.

For $z \in \mathbb{D}^n$, the above series converges, and so it follows that point evaluation is a bounded linear functional on $\mathfrak{D}_\alpha$.

This has several interesting consequences. In general, a Hilbert space in which point evaluation is continuous is called a *reproducing kernel space.* Since evaluating at a point z_0 is a bounded linear functional, it follows from Riesz representation theorem that there exists a function $g \in \mathfrak{D}_\alpha$ such that

$$\lambda(f) = f(z_0) = \langle f, g_{z_0} \rangle.$$

So we can construct a function $g(z, z_0) = g_{z_0}(z)$. This function g is called the *reproducing kernel* of our Hilbert space. For more on reproducing kernel Hilbert spaces, see for example [8].

Another interesting consequence of the inequality

$$|f(z)| \leq \|f\|_\alpha \left(\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |z_1|^{k_1} \cdots |z_n|^{k_n} \right)^{1/2},$$

is that it shows that norm convergence implies uniform convergence on compact subsets of $\mathbb{D}^n$. To see this, note that by the above inequality

$$|f(z)-f_n(z)| \leq \|f-f_n\|_\alpha \left(\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |z_1|^{k_1} \cdots |z_n|^{k_n} \right)^{1/2},$$

and so $\|f-f_n\|_\alpha \to 0$ implies that f_n converges pointwise to f. Furthermore, since the above series converges uniformly on compact subsets $S \subset \mathbb{D}^n$, this implies that $f_n \to f$ uniformly on compact subsets.

2.2 Shift operators, invariant subspaces, and cyclic functions

We now introduce a family of bounded linear operators on $\mathfrak{D}_\alpha$, called the shift operators. For $i = 1, 2, ..., n$, the shift operator S_i is defined by

$$S_i f(z_1, ...z_i, ..., z_n) = z_i f(z_1, ...z_i, ..., z_n).$$

These operators simply act by shifting the sequence of Fourier coefficients, hence the name. That this family of operators is linear is obvious, and by recalling the definition of the norm on $\mathfrak{D}_\alpha$, it is clear that the shift operators are bounded and that they map into $\mathfrak{D}_\alpha$.

We are interested in characterizing the invariant subspaces of these operators, that is, the subspaces A such that $S_i(A) \subset A$ for all S_i. For any $f \in \mathfrak{D}_\alpha$, we have that

$$[f] = \overline{\text{span}\{z_1^{k_1} \cdots z_n^{k_n} f : k_i \in \mathbb{N}\}},$$

is a closed subspace which is an invariant subspace for all shift operators S_i. As a first step in characterizing all invariant subspaces, we seek to determine for which $f \in \mathfrak{D}_\alpha$ we have that $[f] = \mathfrak{D}_\alpha$. Note that there exists functions whose span is dense in the entire space.

Example 1. The span of 1 is all (complex) polynomials, which is a dense subset of $\mathfrak{D}_\alpha$.

Furthermore, if f vanishes at some point in $\mathbb{D}^n$, then every function in $\text{span}\{z_1^{k_1} \cdots z_n^{k_n} f : k_i \in \mathbb{N}\}$ will inherit this zero, and since convergence in norm implies uniform convergence on compact subsets (in our case the point z_0 for which f vanishes), it follows that any function in $[f]$ will also inherit this zero, and so $[f] \neq \mathfrak{D}_\alpha$ since for example $1 \in \mathfrak{D}_\alpha$ but 1 will not vanish at any point.

To explicitly calculate $[f]$ and check if $[f] = \mathfrak{D}_\alpha$ is very difficult, so a perhaps easier way of characterizing cyclic functions is the following. Since $g \in [f] \Rightarrow [g] \subset [f]$, it follows that $1 \in [f]$ will imply that $[f] = \mathfrak{D}_\alpha$, which is the definition of f being cyclic. Of course, if $1 \notin [f]$ then $[f] \neq \mathfrak{D}_\alpha$, so f is cyclic if and only if 1 lies in $[f]$. This can equivalently be stated as that there exists a sequence of polynomials $(p_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \|p_n f - 1\|_\alpha = 0.$$

This characterization will be frequently used. Note however that since $g \in [f] \Rightarrow [g] \subset [f]$, it suffices to show that any cyclic function is contained in $[f]$ in order to show that f is cyclic.

2.3 Cauchy transforms, and linear functionals on $\mathfrak{D}_\alpha$

We will now investigate the relationship between cyclicity of a function f and the zero set of

$$\lim_{r\to 1^-} f(re^{iv_1}, ..., re^{iv_n}) \subset \mathbb{T}^n.$$

Note that for $\alpha \geq 0$, $\mathfrak{D}_\alpha$ is contained in H^2, and so the radial limits exist everywhere.

We denote by $Z(f)$ the set

$$Z(f) = \{(e^{iv_1}, ..., e^{iv_n}) \subset \mathbb{T}^n : \lim_{r\to 1^-} f(re^{iv_1}, ..., re^{iv_n}) = 0\}.$$

We showed earlier that no function which vanishes inside the polydisk can be cyclic, but it may or may not be possible for a function to vanish on the boundary and still be cyclic. We will show that if the zero set on the boundary is too large (in some sense), then $[f]$ cannot be the entire space. The idea of the proof is the following. If the zero set on the boundary is large, then it will support a measure with certain desired properties. These properties will allow us to construct a (non-trivial) bounded linear functional based on this measure that will annihilate every element of span$\{z_1^{k_1} \cdots z_n^{k_n} f : k_i \in \mathbb{N}\}$ as a consequence of the fact that the measure and f have disjoint supports on the boundary.

In order to do this, we must first clarify what we mean by the boundary set being "too large". However, this definition of "too large" is essentially constructed with the purpose of making our functional bounded, so instead, it is easier to begin by looking at the functional we want to create, and then the definition of "too large" will be made so that our arguments go through.

Henceforth, we will use multi-index notation, so z should be interpreted as $(z_1, ..., z_n)$, $k = (k_1, ..., k_n)$, the Fourier coefficients $\hat{f}(k) = \hat{f}(k_1, ..., k_n)$, $z^k = z_1^{k_1} \cdots z_n^{k_n}$, and $e^{it} = e^{i(t_1,...,t_n)} = (e^{it_1}, ..., e^{it_n})$ etc. However, sometimes these expressions will be written out for clarity.

Lemma 1. *For every $\alpha \in \mathbb{R}$, every $g \in \mathfrak{D}_{-\alpha}$ induces a bounded linear functional on $\mathfrak{D}_\alpha$ through the pairing*

$$\begin{aligned}(f, g) &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \hat{f}(k)\hat{g}(k) \\ &= \lim_{r\to 1^-} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(re^{it})g(re^{-it})dt_1 \cdots dt_n.\end{aligned}$$

for $f \in \mathfrak{D}_\alpha$. Here a_f and a_g are the Fourier coefficients of f and g respectively.

Proof. The linearity is obvious. It remains to show boundedness. We have that

$$\begin{aligned}
&\left|\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\hat{f}(k)\hat{g}(k)\right| \\
&\leq \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}((k_1+1)\cdots(k_n+1))^{\alpha/2-\alpha/2}|\hat{f}(k)||\hat{g}(k)| \\
&\leq \left(\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}((k_1+1)\cdots(k_n+1))^{\alpha}|\hat{f}(k)|^2\right)^{1/2} \\
&\quad\cdot\left(\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}((k_1+1)\cdots(k_n+1))^{-\alpha}|\hat{g}(k)|^2\right)^{1/2} < \infty,
\end{aligned}$$

by applying the Cauchy Schwarz inequality and the assumptions on the norm in $\mathfrak{D}_\alpha$ and $\mathfrak{D}_{-\alpha}$. □

Ultimately, we want to induce a linear map of this form from a measure defined on the boundary, but in order to do so we must first construct a holomorphic function (which lies in $\mathfrak{D}_{-\alpha}$) using this measure. We do this by means of the Cauchy transform.

Definition 1. *Given a positive Borel probability measure μ defined on $\mathbb{T}^n$, we construct a function g defined on $\mathbb{D}^n$ by*

$$g(z) = C[\mu](z) = \int_{\mathbb{T}^n}(1-e^{iv_1}z_1)^{-1}\cdots(1-e^{iv_n}z_n)^{-1}d\mu(v).$$

The function g is called the Cauchy transform of the measure μ.

First of all, we need to show that g is indeed holomorphic on $\mathfrak{D}_\alpha$. Since

$$(1-e^{iv_1}z_1)^{-1}\cdots(1-e^{iv_n}z_n)^{-1}$$

is holomorphic with respect to z, this will follow if we can differentiate under the integral sign. That this is permissible is a consequence of the dominated convergence theorem, and the fact that each of the derivatives of $(1-e^{iv_1}z_1)^{-1}\cdots(1-e^{iv_n}z_n)^{-1}$ with respect to z_i is uniformly bounded in v for every fixed $z \in \mathbb{D}^n$.

Furthermore, in order to allow us to construct a functional on $\mathfrak{D}_\alpha$ by means of the above pairing, we require g to lie in $\mathfrak{D}_{-\alpha}$, which means that we also require sufficient decay of the Fourier coefficients. Since g is closely related to μ, requirements on g will naturally translate to requirements on the measure μ.

Lemma 2. *For a Borel probability measure μ supported on $\mathbb{T}^n$, we have that $g(z) = C[\mu] = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \hat{\mu}(-k) z^k$, where $\hat{\mu}(k)$ denotes the Fourier coefficients of μ.*

Proof. By using the power series expansion of $(1 - e^{iv}z)^{-1}$, we obtain

$$\begin{aligned}
g(z) &= \int_{\mathbb{T}^n} (1 - e^{iv_1} z_1)^{-1} \cdots (1 - e^{iv_n} z_n)^{-1} d\mu(v) \\
&= \int_{\mathbb{T}^n} \left(\sum_{k_1=0}^{\infty} \left(e^{iv_1}\right)^{k_1} z_1^{k_1} \cdots \sum_{k_n=0}^{\infty} \left(e^{iv_n}\right)^{k_n} z_n^{k_n} \right) d\mu(v) \\
&= \int_{\mathbb{T}^n} \left(\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left(e^{iv_1 k_1}\right) z_1^{k_1} \left(e^{iv_n k_n}\right) \cdots z_n^{k_n} \right) d\mu(v) \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left(\int_{\mathbb{T}^n} \left(e^{iv_1 k_1}\right) \cdots \left(e^{iv_n k_n}\right) d\mu(v) \right) z^k \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \hat{\mu}(-k) z^k,
\end{aligned}$$

where changing orders of integration and summation is permissible because we have uniform convergence since the factors z^k decay rapidly. □

Furthermore, note that since μ is (by assumption) a real measure, we have that

$$\overline{\hat{\mu}(-k)} = \int_{\mathbb{T}^n} \left(e^{-iv\cdot k}\right) d\mu(v) = \hat{\mu}(k),$$

and so, the requirement that $g \in \mathfrak{D}_{-\alpha}$, i.e.

$$\|g\|_{-\alpha} = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |\hat{g}(k)|^2 < \infty$$

is equivalent to the statement

$$\begin{aligned}
&\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |\hat{\mu}(-k)|^2 \\
&= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |\hat{\mu}(k)|^2 < \infty.
\end{aligned}$$

Recall that our goal was to use a measure whose support is contained in $Z(f)$ to construct a functional which annihilates the entire span of f, thus showing that $[f] \neq \mathfrak{D}_\alpha$. We said that this was going to be possible if the set $Z(f)$ was "large enough", without actually clarifying what that means. By the above calculations we see that "large enough" for $Z(f)$ means that the set supports a measure for which the above norm is finite, i.e. it supports a measure for which

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^{-\alpha} \cdots (k_n+1)^{-\alpha} |\hat{\mu}(k)|^2 < \infty. \tag{1}$$

Now we could just use the existence of a measure with this property as our definition of $Z(f)$ being "large enough", but we can actually give another definition of size for a set which will imply the existence of such a measure. Ultimately, we want to find a connection between certain geometric properties of $Z(f)$ and existence of such measures, but in order to do this, we must first discuss the concept of capacity of a set.

2.4 Capacity of a set and measures of finite energy

Let $E \subset \mathbb{T}^n$ be a Borel set and μ be a probability measure supported on E. Let $K : [0, \infty)^n \to [0, \infty)$ be a continuous decreasing function. The potential of μ with respect to K is defined as

$$K\mu(x) = \int_{\mathbb{T}^n} K(\vec{x} - \vec{y}) d\mu(\vec{y}),$$

and the energy of μ is defined as

$$I_K[\mu] = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} K(\vec{x} - \vec{y}) d\mu(\vec{x}) d\mu(\vec{y}).$$

If E is a compact subset, we define the capacity of E with respect to K as

$$c_K(F) := 1/\inf\{I_K[\mu] : \mu \in P(E)\},$$

where $P(E)$ is the set of probability measures supported in E. If E supports no probability measure of finite energy, we say that it has *capacity zero*.

For a general Borel set S we define the capacity of S with respect to K as

$$\sup\{c_K(F) : F \subset S, F \text{ compact}\}.$$

Note that if $S_1 \subset S_2$, then $c_K(S_1) \leq c_K(S_2)$ since every probability measure μ_{S_1} on S_1 induces a probability measure on S_2 of the same energy by

$$\mu_{S_2}(E) = \mu_{S_1}(E \cap S_1).$$

The energy can be seen as a convolution with the kernel $K(\vec{x})$. It turns out that this energy can be connected to the Fourier coefficients of μ under certain circumstances, which means that finite energy (with respect to some suitable kernel) will imply that the Fourier coefficients of μ decays at a certain rate. Specifically, assuming that K is (2π-)periodic and that the Fourier series of K has good enough convergence to allow changing order of summation and integration, then

$$I[\mu] = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} K(x-y)d\mu(x)d\mu(y) \tag{2}$$

$$= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{K}(k) e^{ik\cdot x} e^{-ik\cdot y} \right) d\mu(x)d\mu(y) \tag{3}$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{K}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right) \tag{4}$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{K}(k) \overline{\hat{\mu}(k)} \hat{\mu}(k) \tag{5}$$

$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{K}(k) |\hat{\mu}(k)|^2, \tag{6}$$

where we have once again used that μ being a real measure implies that $\overline{\hat{\mu}(k)} = \hat{\mu}(-k)$.

By comparing (1) and (6), we see that if we can find a periodic kernel whose Fourier coefficients are proportional to $|k_1+1|^{-\alpha} \cdots |k_n+1|^{-\alpha}$, then having finite energy with respect to that kernel will imply that (1) holds.

Furthermore, it is known that in one variable, the Riesz potential with parameter $\alpha \in (0,1)$, $K_\alpha(x) = 1/|1-e^{ix}|^{1-\alpha}$ satisfies

$$c(|k|+1)^{-\alpha} \le \widehat{K_\alpha}(k) \le C(|k|+1)^{-\alpha}.$$

For some $0 < c < C < \infty$.

It follows that in n variables, we have that

$$h_\alpha(x_1, ..., x_n) = 1/|1-e^{ix_1}|^{1-\alpha} \cdots 1/|1-e^{ix_n}|^{1-\alpha}$$

has Fourier coefficients which satisfy

$$c(|k_1|+1)^{-\alpha} \cdots (|k_n|+1)^{-\alpha} \le \widehat{h_\alpha}(k) \le C(|k_1|+1)^{-\alpha} \cdots (|k_n|+1)^{-\alpha}, \tag{7}$$

since h_α admits separation of variables and so its Fourier coefficients is a product of its Fourier coefficients in each variables separately.

We obtain the notions of *Riesz capacity* and *Riesz energy* by using the Riesz potential in the definitions of energy of a measure and capacity of a set.

By comparing (7), (6), and (1), we see that $Z(f)$ having positive Riesz capacity, i.e. $Z(f)$ supports a measure with finite Riesz energy, will imply the existence of a functional that (hopefully) annihilates every function in span$\{z^k f : k \in \mathbb{N}^k\}$. We are now ready to prove our necessary condition for cyclicity of a function.

For more on capacities, and especially the connection to the Riesz potential, see for example [6] or [9].

2.5 A necessary condition for cyclicity

First, we motivated using the Riesz potential by comparing its Fourier coefficient with equation (6). However, for equations (3) – (4) to hold, we require "good convergence" on the Fourier series of the kernel. But since the Riesz potential has a singularity at the origin, this is not entirely obvious. However, this turns out to be manageable.

Lemma 3. *Let h_α be the Riesz potential with parameter $\alpha \in (0,1)$, we have that*

$$I_{h_\alpha}[\mu] = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1}-e^{iy_1}|^\alpha \cdots |e^{ix_n}-e^{iy_n}|^\alpha} d\mu(x)d\mu(y)$$
$$= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k)|\hat{\mu}(k)|^2.$$

Proof. We need to show that equations (2) – (6) hold, i.e. that the calculations

$$\begin{aligned}
I_{h_\alpha}[\mu] &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1}-e^{iy_1}|^\alpha \cdots |e^{ix_n}-e^{iy_n}|^\alpha} d\mu(x)d\mu(y)\\
&= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) e^{ik\cdot x} e^{-ik\cdot y} \right) d\mu(x)d\mu(y)\\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right)\\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) \overline{\hat{\mu}(k)} \hat{\mu}(k)\\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) |\hat{\mu}(k)|^2,
\end{aligned}$$

are valid. The only step that might fail is

$$\begin{aligned}
&\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) e^{ik\cdot x} e^{-ik\cdot y} \right) d\mu(x)d\mu(y)\\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right),
\end{aligned}$$

so that this holds is what we need to show.

For $r \in (0,1)$ we have that

$$\begin{aligned}
&\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1} - re^{iy_1}|^\alpha \cdots |e^{ix_n} - re^{iy_n}|^\alpha} d\mu(x) d\mu(y) \\
&= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) e^{ik\cdot x} r^{|k|} e^{-ik\cdot y} \right) d\mu(x) d\mu(y) \\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} r^{|k|} \widehat{h_\alpha}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right),
\end{aligned}$$

where changing the order of integration and summation is permissible because of the uniform convergence induced by the factor $r^{|k|}$.

We now let $r \to 1^-$ on both sides.

If

$$\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1} - e^{iy_1}|^\alpha \cdots |e^{ix_n} - e^{iy_n}|^\alpha} d\mu(x) d\mu(y)$$

is finite, then we can use that

$$\frac{1}{|e^{ix_1} - re^{iy_1}|^\alpha \cdots |e^{ix_n} - re^{iy_n}|^\alpha} \leq \frac{2^{n\alpha}}{|e^{ix_1} - e^{iy_1}|^\alpha \cdots |e^{ix_n} - e^{iy_n}|^\alpha}$$

for all $r \in (0,1)$, and apply the dominated convergence theorem in order to pass the limit inside the integral. We thus obtain

$$\begin{aligned}
&\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1} - e^{iy_1}|^\alpha \cdots |e^{ix_n} - e^{iy_n}|^\alpha} d\mu(x) d\mu(y) \\
&= \lim_{r\to 1^-} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{1}{|e^{ix_1} - re^{iy_1}|^\alpha \cdots |e^{ix_n} - re^{iy_n}|^\alpha} d\mu(x) d\mu(y) \\
&= \lim_{r\to 1^-} \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} r^{|k|} \widehat{h_\alpha}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right) \\
&= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) \left(\int_{\mathbb{T}^n} e^{ik\cdot x} d\mu(x) \right) \left(\int_{\mathbb{T}^n} e^{-ik\cdot y} d\mu(y) \right).
\end{aligned}$$

On the other hand, if it is infinite, we instead use Fatou's lemma to see that both sides are infinite. Note that applying Fatou's lemma is permissible since the integrand is non-negative.

This finishes the proof. □

The following proof is largely analogous to the proof of Theorem 5 in [1] in which they prove the same result for $n = 1$, and for $\alpha = 1$. The main differences lie in the fact that we use Riesz capacity instead of logarithmic capacity, and minor differences due to the change of dimension.

Theorem 1. *If $f \in \mathfrak{D}_\alpha$ for $\alpha \in (0,1)$, and $Z(f) \subset \mathbb{T}^n$ has positive Riesz capacity, then f is not cyclic.*

Proof. Since $Z(f)$ has positive Riesz capacity (by assumption), we know that it supports a measure with finite Riesz energy. However, in a later stage we will require a pointwise bound on $f(re^{iv_1}, ..., re^{iv_n})$ for $(e^{iv_1}, ..., e^{iv_n}) \in Z(f)$ in order to apply the dominated convergence theorem. So for that reason, we decompose $Z(f)$ in the following way:

Consider the sets

$$J_k = \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(re^{iv_1}, ..., re^{iv_n})| \leq k, 0 \leq r < 1\}.$$

Clearly $Z(f) = \cup_{n=1}^{\infty} J_k$. Furthermore, each J_k is a Borel set. This can be shown as follows:

We have that

$$\begin{aligned} J_k &= \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(re^{iv_1}, ..., re^{iv_n})| \leq k, 0 \leq r < 1\} \\ &= \bigcap_{q \in \mathbb{Q} \cap [0,1)} \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(qe^{iv_1}, ..., qe^{iv_n})| \leq k\}. \end{aligned}$$

Since this is a countable intersection, J_K is a Borel set if we can show that each of these sets is a Borel set. For a fixed q, we have that

$$\begin{aligned} &\{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(qe^{iv_1}, ..., qe^{iv_n})| \leq k\} \\ &= \bigcap_{n=1}^{\infty} \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(qe^{iv_1}, ..., qe^{iv_n})| < k + 1/n\}. \end{aligned}$$

Each of the sets in the above intersection is open since they are inverse images of open sets for a continuous function, and since

$$\bigcap_{n=1}^{\infty} \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(qe^{iv_1}, ..., qe^{iv_n})| < k + 1/n\}$$

is a countable intersection the statement follows.

Since $Z(f)$ has positive capacity, and since a countable union of Borel sets with capacity zero has capacity zero, it follows that for at least one integer, N, J_N must have positive capacity. It follows from the definition of capacity that there must be a compact subset $F \subset J_N$ that has positive Riesz capacity, since if all compact subsets of J_N had capacity zero, then J_N would have capacity zero. This implies that there exists a real valued probability measure μ whose support is contained in $F \subset J_N \subset Z(f)$, that has finite Riesz energy. That is,

$$\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{1}{|e^{ix_1} - e^{iy_1}|^{1-\alpha}} \cdots \frac{1}{|e^{ix_n} - e^{iy_n}|^{1-\alpha}} d\mu(x_1, ..., x_n) d\mu(y_1, ..., y_n) < \infty.$$

We now construct a function $g \in \mathfrak{D}_{-\alpha}$ by means of the Cauchy transform, namely

$$g(z) = C[\mu] = \int_{\mathbb{T}^n} (1 - e^{iv_1} z_1)^{-1} \cdots (1 - e^{iv_n} z_n)^{-1} d\mu(v).$$

By Lemma 2 we have that

$$g(z) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \hat{\mu}(-k) z^k. \tag{8}$$

Note that the convergence is uniform for $|z| < 1$.

Now consider the pairing of g and $pf \in \mathfrak{D}_\alpha$ from Lemma 1. We need to show (1), that $g \in \mathfrak{D}_{-\alpha}$ so that this pairing is indeed a functional, (2) that this functional is non-trivial, and (3) that this functional annihilates $[f]$.

We begin by proving (1). We need to show that

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (|k_1| + 1)^{-\alpha} \cdots (|k_n| + 1)^{-\alpha} |\hat{g}(k)|^2 < \infty.$$

From equation (8) and using that μ is a real measure, we have that

$$\hat{g}(k) = \hat{\mu}(-k) = \hat{\mu}(k),$$

for $k_1, ..., k_n \geq 0$, so we need to show that

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (|k_1| + 1)^{-\alpha} \cdots (|k_n| + 1)^{-\alpha} |\hat{\mu}(k)|^2 < \infty. \tag{9}$$

By applying Lemma 3 and using the assumption that μ has finite energy, we see that

$$\begin{aligned} I_{h_\alpha}[\mu] &= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{K}(k) |\hat{\mu}(k)|^2 \\ &= \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) |\hat{\mu}(k)|^2 < \infty. \end{aligned}$$

By using the bound on the Fourier coefficients of h_α from (7), we know that there exists a constant $c > 0$ such that

$$\begin{aligned} &c \cdot \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} (|k_1| + 1)^{-\alpha} \cdots (|k_n| + 1)^{-\alpha} |\hat{\mu}(k)|^2 \\ &\leq \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} \widehat{h_\alpha}(k) |\hat{\mu}(k)|^2 < \infty. \end{aligned}$$

Since all the terms in the sum are positive, this implies that

$$\sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (|k_1|+1)^{-\alpha} \cdots (|k_n|+1)^{-\alpha} |\hat{\mu}(k)|^2 < \infty.$$

It follows that $g = C[\mu] \in \mathfrak{D}_{-\alpha}$.

Since $g \in \mathfrak{D}_{-\alpha}$, it follows that the pairing from Lemma 1

$$\begin{aligned}(f, g) &= \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \hat{f}(k)\hat{g}(k) \\ &= \lim_{r \to 1^-} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} f(re^{it}) g(re^{-it}) dt_1 \cdots dt_n,\end{aligned}$$

Is a functional on $\mathfrak{D}_{-\alpha}$. Since $\hat{g}(k) = \hat{\mu}(k)$, for $k_1, ..., k_n \geq 0$, and $\hat{\mu}(k) > c(|k_1|+1)^{-\alpha} \cdots (|k_n|+1)^{-\alpha} > 0$ for some $c > 0$, it follows that the above pairing is not the zero functional.

We are now ready to prove that this pairing annihilates $\mathrm{span}\{z^k f : k \in \mathbb{N}^k\}$, i.e. it maps every function in $\mathfrak{D}_\alpha$ of the form $p(z_1, ..., z_n)f(z_1, ..., z_n)$, p a polynomial, to zero. This shows that $\mathrm{span}\{z^k f : k \in \mathbb{N}^k\}$ is not dense in $\mathfrak{D}_\alpha$, since if this was the case g would vanish on a dense subspace. So by continuity it vanishes on the entire space, which contradicts that it is not the zero functional.

Recall that $\mathrm{supp}(\mu) \subset F \subset J_N \subset Z(f)$, where

$$J_N = \{(e^{iv_1}, ..., e^{iv_n}) \in Z(f) : |f(re^{iv_1}, ..., re^{iv_n})| \leq N, 0 \leq r < 1\}.$$

For any polynomial p we have that

$$\begin{aligned}&(p \cdot f(z_1, ..., z_n), g(z_1, ..., z_n)) \\ &= \lim_{r \to 1^-} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} p \cdot f\left(re^{it}\right) g\left(re^{-it}\right) dt_1 \cdots dt_n \\ &= \lim_{r \to 1^-} \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} p \cdot f\left(re^{it}\right) \\ &\qquad \cdot \left(\int_{\mathbb{T}^n} \left((1 - e^{iv_1} re^{-it_1}) \cdots (1 - e^{iv_n} re^{-it_n}) \right)^{-1} d\mu(v) \right) dt_1 \cdots dt_n \\ &= \lim_{r \to 1^-} \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} \\ &\quad \left(\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{p \cdot f\left(re^{it}\right)}{(1 - e^{iv_1} re^{-it_1}) \cdots (1 - e^{iv_n} re^{-it_n})} dt_1 \cdots dt_n \right) d\mu(v).\end{aligned}$$

By applying Cauchy's integral formula to

$$\begin{aligned}
&\frac{p \cdot f\left(re^{it}\right)}{\left(1-e^{iv_1}re^{-it_1}\right)\cdots\left(1-e^{iv_n}re^{-it_n}\right)}dt_1\cdots dt_n \\
=&\frac{re^{it_1}\cdots re^{it_n}\cdot(p\cdot f)\left(re^{it}\right)}{\left(re^{it_1}-e^{iv_1}r^2\right)\cdots\left(re^{it_n}-e^{iv_n}r^2\right)}dt_1\cdots dt_n \\
=&\frac{i^n(p\cdot f)(z)}{\left(z_1-e^{iv_1}r^2\right)\cdots\left(z_n-e^{iv_n}r^2\right)}dz_1\cdots dz_n,
\end{aligned}$$

we see that the last equation equals

$$\lim_{r\to 1^-}\int_{\mathbb{T}^n}(p\cdot f)\left(r^2e^{iv}\right)d\mu(v) = \lim_{r\to 1^-}\int_F (p\cdot f)\left(r^2e^{iv}\right)d\mu(v).$$

Now finally, since $|f(re^{iv}))| \leq N$ for $e^{iv} \in F$, p is bounded since it is polynomial, F is bounded, and μ is a probability measure, the dominated convergence theorem yields

$$\begin{aligned}
&\lim_{r\to 1^-}\int_F (p\cdot f)\left(r^2e^{iv}\right)d\mu(v) \\
=&\int_F \lim_{r\to 1^-}(p\cdot f)\left(r^2e^{iv}\right)d\mu(v) = 0,
\end{aligned}$$

where the last equality holds since $\lim\limits_{r\to 1^-} f(re^{iv})$ vanishes on F.

Since $\operatorname{span}\{z^k f : k \in \mathbb{N}^k\}$ is annihilated by a non-trivial functional, it cannot be a dense subset of $\mathfrak{D}_\alpha$.

This finishes the proof.

□

Although this result seems to be fairly general, it is still unfortunately rather challenging to determine whether or not a given (or in this case implicitly given) set has finite Riesz capacity. It is therefore desirable to find some characteristic which will imply the existence of a measure of finite energy. By applying a generalization of van der Corput's lemma, which connects geometric properties of a set to the rate of decay of the Fourier coefficients of certain measures supported on the set, one can apply the above theorem in order to find a connection between certain geometric properties of $Z(f)$ and cyclicity.

2.6 Geometric conditions for non-cyclicity

Definition 2. *Let $S \subset \mathbb{T}^n$ be a smooth m-manifold. Let $\phi : I^m \to \mathbb{T}^n$ be a smooth parametrization, where $I \subset \mathbb{R}$ is an interval. We define the* type *of a point $\xi = \phi(x)$ as the smallest τ such that for all unit vectors $\eta \in \mathbb{R}^n$ there exists a multi-index $k \in \mathbb{N}^m$ with $|k| \leq \tau$ such that*

$$\left[\frac{d^k\phi}{dt^k} \cdot \eta\right]_{t=x} \neq 0.$$

We say that the m-manifold S has type τ is the maximum of the types of $\xi \in S$ is τ.

The following generalization of van der Corput's lemma from the theory of oscillatory integrals gives a connection between decay of the Fourier coefficients of absolutely continuous measures on S, and the type of S. See Theorem 2 on page 351 in [5] for the proof.

Theorem 2. *Let $S \subset \mathbb{T}^n$ is a locally smooth m-manifold of finite type $\tau \in \mathbb{N}$ and let σ be the measure on S induced by pulling back to the Lebesgue measure using the parametrization of S. If μ is a measure of the form $d\mu(x) = \phi(x)d\sigma(x), x \in S \subset \mathbb{T}^n$, where $\phi(x)$ is a non-negative smooth function with compact support (defined on S), then there exists a constant $C > 0$ such that*

$$|\hat{\mu}(k_1, ..., k_n)| \leq C(k_1^2 + \cdots + k_n^2)^{-1/2\tau}, \quad k_1, ..., k_n \in \mathbb{Z} \setminus \{0\}.$$

We will now apply the above estimate on the Fourier coefficients in order to show that certain geometric properties of $Z(f)$ implies existence of measures of finite Riesz energy, thus implying non-cyclicity.

Theorem 3. *Assume that $f \in \mathfrak{D}_\alpha$ is such that $Z(f) \subset \mathbb{T}^n$ contains a locally smooth m-manifold of finite type τ. Then f is not cyclic in $\mathfrak{D}_\alpha$ for any $\alpha > 1 - 2/n\tau$.*

Proof. Since $S \subset Z(f)$ we know that the Riesz capacity of S is less than or equal to the Riesz capacity of $Z(f)$. So if we can show that S supports a measure of finite Riesz energy for $\alpha > 1 - 1/\tau$, and thus show that S has positive Riesz capacity, then it follows that $Z(f)$ has positive Riesz capacity. By applying Theorem 1 the statement follows.

That S supports such a measure will be shown by using Theorem 2 together with Lemma 3, since these statements prove that all absolutely continuous measures have Fourier coefficients which decay quickly.

Let μ be any absolutely continuous measure on F. From Lemma 3 with $K = h_\alpha$ and the bound on the Fourier coefficients of the Riesz capacity from equation (7), we have that

$$I_{h_\alpha}[\mu] = \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \widehat{h_\alpha}(k)|\hat{\mu}(k)|^2 \tag{10}$$

$$\leq C \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \frac{|\hat{\mu}(k)|^2}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha}. \tag{11}$$

Since μ is a probability measure, $|\hat{\mu}(0,...,0)|^2 = 1$, and so, from Theorem 2 and the assumptions on μ, we have that

$$(11) \leq C \left(1 + \sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \frac{1}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha (k_1^2 + ... + k_n^2)^{1/\tau}} \right), \tag{12}$$

where $k_1, ..., k_n$ are not all equal to zero in the above series. The above expression is finite if and only if the series is finite. By using that the summand is even in each k_i, we have that

$$\sum_{k_n=-\infty}^{\infty} \cdots \sum_{k_1=-\infty}^{\infty} \frac{1}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha (k_1^2 + ... + k_n^2)^{1/\tau}} \tag{13}$$

$$\leq 2^n \sum_{k_n=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \frac{1}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha (k_1^2 + ... + k_n^2)^{1/\tau}}. \tag{14}$$

Note that we do not necessarily have equality since terms with at least one k_i equal to 0 are counted several times in the second expression. Furthermore, since the summand is symmetric in all arguments, we can bound the above expression by a series in which we only take the sum over ascending chains, and then multiply by the number of elements in the symmetric group of n elements. I.e., we only take one representative from each class of $(k_1, ...k_n)$, were two $n-$tuples are considered to be the same if they differ by a permutation. We thus obtain

$$\sum_{k_n=0}^{\infty} \cdots \sum_{k_1=0}^{\infty} \frac{1}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha (k_1^2 + ... + k_n^2)^{1/\tau}} \tag{15}$$

$$\leq C \sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_1=0}^{k_2} \frac{1}{(|k_n|+1)^\alpha \cdots (|k_1|+1)^\alpha (k_1^2 + ... + k_n^2)^{1/\tau}}, \tag{16}$$

where once again not all k_i are zero.

We are done if we can show that the above series is finite for $\alpha > 1 - 2/n\tau$. That this is true will (partly) be shown by induction on the dimension. The induction step is mainly used for dropping the part where $k_i = 0$, in order to get around the issue of having to check whether or not we know that a given k_i is zero. Namely, when using certain inequalities we would otherwise be forced to consider several cases, since some of the inequalities we want to use would not work since the denominator would become equal to zero. So the induction step is probably not a necessary part of the proof.

For $n = 1$ the above series becomes

$$\sum_{k_1=1}^{\infty} \frac{1}{(k_1+1)^{\alpha} k_1^{2/\tau}} < \sum_{k_1=1}^{\infty} \frac{1}{(k_1)^{\alpha+2/\tau}} < \infty,$$

since $\alpha + 2/\tau > 1 - 2/\tau + 2/\tau > 1$.

Now assume that

$$\sum_{k_{n-1}=0}^{\infty} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_1=0}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} (k_1^2 + ... + k_n^2)^{1/\tau}} < \infty,$$

where once again all k_i are not zero at the same time. We have that

$$\sum_{k_n=0}^{\infty} \sum_{k_{n-1}=0}^{k_n} \cdots \sum_{k_1=0}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} (k_1^2 + ... + k_n^2)^{1/\tau}} \tag{17}$$

$$= \sum_{k_n=1}^{\infty} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_1=1}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} (k_1^2 + ... + k_n^2)^{1/\tau}} \tag{18}$$

$$+ n \cdot \sum_{k_{n-1}=0}^{\infty} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_1=0}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} (k_1^2 + ... + k_n^2)^{1/\tau}}. \tag{19}$$

The equality is obtained by splitting the series up into two parts, one in which $k_i \geq 1$ for all i and one in which at least one, but not all $k_i = 0$. By symmetry all of the series of the second kind are the same, and by the induction hypothesis they are all finite since $\alpha > 1 - 2/n\tau > 1 - 2/(n-1)\tau$. It follows that we are done if we can show that

$$\sum_{k_n=1}^{\infty} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_1=1}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} (k_1^2 + ... + k_n^2)^{1/\tau}} < \infty.$$

That this is true can be seen in the following way.

$$\sum_{k_n=1}^{\infty} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_1=1}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha}(k_1^2+...+k_n^2)^{1/\tau}} \tag{20}$$

$$\leq C \sum_{k_n=1}^{\infty} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_1=1}^{k_2} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_1|+1)^{\alpha} k_n^{2/\tau}} \tag{21}$$

$$= C \sum_{k_n=1}^{\infty} k_n^{-2/\tau} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_2=1}^{k_3} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_2|+1)^{\alpha}} \sum_{k_1=1}^{k_2} \frac{1}{(|k_1|+1)^{\alpha}} \tag{22}$$

In order to proceed, we need the following inequality. For $(m-1)-m\alpha \leq 0$ we have that

$$\sum_{k=1}^{l}(k+1)^{(m-1)-m\alpha} \leq \int_1^l (x+1)^{(m-1)-m\alpha}dx \leq (l+1)^{m-m\alpha}. \tag{23}$$

And if $(m-1)-m\alpha > 0$

$$\sum_{k=1}^{l}(k+1)^{(m-1)-m\alpha} \leq \int_1^l ((x+1)+1)^{(m-1)-m\alpha}dx \tag{24}$$

$$\leq (l+2)^{m-m\alpha} \leq C(l+1)^{m-m\alpha}, \tag{25}$$

For some suitable constant C which is independent of l. That such a constant C exists is clear since

$$\frac{(l+2)^{m-m\alpha}}{(l+1)^{m-m\alpha}}$$

is bounded, for example by $2^{m-m\alpha}$.

So independently of the parity of the exponent there is an inequality of the form

$$\sum_{k=1}^{l}(k+1)^{(m-1)-m\alpha} \leq C(l+1)^{m-m\alpha}.$$

By applying this inductively to (22) we get that

$$(22) \leq C \sum_{k_n=1}^{\infty} k_n^{-2/\tau} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_2=1}^{k_3} \frac{(k_2+1)^{1-\alpha}}{(|k_n|+1)^{\alpha} \cdots (|k_2|+1)^{\alpha}} \tag{26}$$

$$= C \sum_{k_n=1}^{\infty} k_n^{-2/\tau} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_3=1}^{k_4} \frac{1}{(|k_n|+1)^{\alpha} \cdots (|k_3|+1)^{\alpha}} \sum_{k_2=1}^{k_3} (k_2+1)^{1-2\alpha} \tag{27}$$

$$\leq C \sum_{k_n=1}^{\infty} k_n^{-2/\tau} \sum_{k_{n-1}=1}^{k_n} \cdots \sum_{k_3=1}^{k_4} \frac{(k_3+1)^{2-3\alpha}}{(|k_n|+1)^{\alpha} \cdots (|k_4|+1)^{\alpha}} \tag{28}$$

$$\vdots \tag{29}$$

$$\leq C \sum_{k_n=1}^{\infty} k_n^{-2/\tau} (k_n+1)^{(n-1)-n\alpha} \leq C \sum_{k_n=1}^{\infty} (k_n+1)^{(n-1)-n\alpha-2/\tau}. \tag{30}$$

The last sum is finite if and only if $(n-1)-n\alpha-2/\tau < -1 \iff n-2/\tau < n\alpha \iff 1-2/\tau n < \alpha$, which is what we want to show. $\square$

The previous theorem generalizes Theorem 5.1 from [2] from dimension 2 to any dimension. The idea of using the generalization of van der Corput's lemma in connection to cyclicity was, as far as I know, first used in [2], and the above proof relies on the same fundamental idea, although the calculations become significantly longer.

In the "general case", the values of α for which we can determine non-cyclicity decreases when the dimensionen increases, which of course is not what we want. However, if we impose stronger assumptions on $Z(f)$, then we can use the following result from page 348 in [5] in order to determine non-cyclity of a function.

Theorem 4. *Let $S \subset \mathbb{T}^n$ is a smooth hypersurface of type 2, i.e. it can be parametrized using $n-1$ real parameters and it has non-vanishing Gaussian curvature, and let σ be the measure on S induced by pulling back to the Lebesgue measure using the parametrization of S. Then for every measure μ of the form $d\mu(x) = \phi(x)d\sigma(x), x \in S \subset \mathbb{T}^n$, where $\phi(x)$ is a non-negative smooth function with compact support (defined on S), there exists a constant $C > 0$ such that*

$$|\hat{\mu}(k_1, ..., k_n)|^2 \leq C(k_1^2 + ... + k_n^2)^{(1-n)/2}$$

We now use the above estimate in order to get a similar result as Theorem 3.

Theorem 5. *Assume that $f \in \mathfrak{D}_\alpha$ is such that $Z(f) \subset \mathbb{T}^n$ contains a locally smooth hyper-surface of type 2. Then f is not cyclic in $\mathfrak{D}_\alpha$ for any $\alpha > 1/n$.*

The proof of the above theorem is completely analogous to the proof of Theorem 3. Just replace the estimate on the Fourier coefficients from theorem 2 with the estimate from theorem 4 in all calculations.

The above result will later be applied in order to determine non-cyclicity of certain polynomials. In general, for functions whose zero set is fairly easy to understand, it is sometimes possible to apply the previous results in order to draw conclusions regarding cyclicity.

2.7 Cyclicity, factorizations, and slices

The proofs in this section are more or less completely analogous to the proofs of the corresponding statements for $n = 2$ from [2].

It is often easier to characterize properties of functions in lower dimensions. With this in mind, it is interesting to determine whether or not properties such as cyclicity is preserved by fixing some variables or by looking at individual factors in some factorization. The factorizations of functions $f \in \mathfrak{D}_\alpha$ that interest us is the ones in which both factors also lie in $\mathfrak{D}_\alpha$.

Definition 3. *We denote by $M(\mathfrak{D}_\alpha)$ the set of* multipliers *of $\mathfrak{D}_\alpha$. A function g on $\mathbb{D}^n$ lies in $M(\mathfrak{D}_\alpha)$ if for every $f \in \mathfrak{D}_\alpha$, we have that $gf \in \mathfrak{D}_\alpha$.*

Example 2. All polynomials lie in $M(\mathfrak{D}_\alpha)$. To see this, we argue as follows. Let

$$p(z) = \sum_{k\in\mathbb{N}^n} a_k z^k,$$

be a given polynomial. By the triangle inequality, we have that

$$\|pf\|_\alpha \leq \sum_{k\in\mathbb{N}^n} a_k \|z^k f\|_\alpha.$$

But since p is a polynomial, and thus has finite degree, this is a finite sum. So the expression on the right hand side is finite if and only if $\|z^k f\|_\alpha$ is finite for every fixed $k \in \mathbb{N}^n$. But by definition

$$\|z^k f\|_\alpha^2 = \sum_{i-k\in\mathbb{N}^n, i,k\geq 0} (i_1+1)^\alpha \cdots (i_n+1)^\alpha \hat{f}(i-k).$$

And this expression is finite since $(i_j+1) \leq (k_j+1)(i_j-k_j+1)$ for all i, and since f is by assumption an element of $\mathfrak{D}_\alpha$. Essentially, multiplying by z^k only shifts the coefficients in the series defining the norm. We do, however get a slight mismatch between the weights and the Fourier coefficients in the series, but this is only a minor issue since the mismatch is by a finite, and constant number of steps.

Note that since $1 \in \mathfrak{D}_\alpha$, we have that $M(\mathfrak{D}_\alpha) \subset \mathfrak{D}_\alpha$.

Definition 4. *Every function $g \in M(\mathfrak{D}_\alpha)$ induces a linear map from $\mathfrak{D}_\alpha$ to itself through multiplication by g. Namely, $M_g : \mathfrak{D}_\alpha \to \mathfrak{D}_\alpha$ is defined by $f \to gf$. Note that since the set of points $(f, gf) \subset \mathfrak{D}_\alpha \times \mathfrak{D}_\alpha$ is closed, the closed graph theorem implies that the operator M_g is bounded.*

We define the multiplier norm of a function $g \in M(\mathfrak{D}_\alpha)$ as the operator of the corresponding bounded operator M_g. We denote the operator norm by $\|g\|_M$, or simply $\|g\|$ if it is unambiguous from the context.

Note that for $g \in M(\mathfrak{D}_\alpha)$ we have that $g[f] \in [f]$ for all $f \in \mathfrak{D}_\alpha$. Since gf by assumption lies in $\mathfrak{D}_\alpha$, and so the tail of the series defining the norm must become arbitrarily small, it follows that

$$\lim_{n\to\infty} \|p_n f - gf\|_\alpha = 0,$$

if we let p_n be the partial sums of the power series expansion of g.

We can now give a result which connects cyclicity of a function with cyclicity of the factors in a given factorization.

Lemma 4. *Let $f \in \mathfrak{D}_\alpha$ and $g \in M(\mathfrak{D}_\alpha)$, then $gf \in \mathfrak{D}_\alpha$ is cyclic if and only if both f and g are cyclic.*

Proof. Suppose that both f and g are cyclic. Since f is cyclic there exists a sequence of polynomials $(p_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \|p_n f - 1\|_\alpha = 0.$$

This implies that

$$\lim_{n\to\infty} \|p_n gf - g\|_\alpha = \lim_{r\to\infty} \|M_g(p_n f - 1)\|_\alpha \leq \lim_{r\to\infty} \|M_g\|_M \|p_n f - 1\|_\alpha = 0,$$

and so $g \in [gf]$. This implies that $[g] \subset [gf]$, but since g was cyclic, this means that $\mathfrak{D}_\alpha \subset [gf]$, and so gf is cyclic.

Suppose next that gf is cyclic. Then there exists polynomials $(p_n)_{n=1}^\infty$ such that

$$\lim_{n\to\infty} \|p_n gf - 1\|_\alpha = 0.$$

Since $g \in M(\mathfrak{D}_\alpha)$, we have that $gf \in [f]$, and so $p_n gf \in [f]$. Since 1 clearly is a limit point of functions in $[f]$, we have that $1 \in [f]$, and so it follows that f is cyclic.

Now, let $(q_n)_{n=1}^\infty$ be polynomials for which

$$\lim_{n\to\infty} \|q_n - f\|_\alpha = 0,$$

then

$$\lim_{n\to\infty} \|q_n g - gf\|_\alpha \leq \lim_{n\to\infty} \|M_g\| \|q_n - f\|_\alpha = 0.$$

And so $gf \in [g] \Rightarrow [gf] \subset [g]$. Since gf is cyclic by assumption, this implies that $[g]$ is cyclic.

$\square$

This has several interesting implications. For example, if f is a polynomial in $\mathbb{C}^n$, then since all polynomials are multipliers, the above theorem implies that we only need to check whether or not the irreducible factors of f are cyclic or not in order to determine if f is cyclic or not.

Moving on, one would want to be able to determine whether or not a given function is cyclic or not by looking at different restrictions to lower dimensions. From here on we will denote by $\mathfrak{D}_\alpha^k$ the Dirichlet type space with parameter α defined on $\mathbb{D}^k$ for $k < n$, and we denote by $\|f\|_{k,\alpha}$ the norm on $\mathfrak{D}_\alpha^k$. Furthermore, a function on $\mathbb{D}^k$ given by fixing $n-k$ of the variables of f is called a *k-slice of f*. In general, cyclicity of a function is passed down to its k-slices. The following holds.

Theorem 6. *Let f be a cyclic function in $\mathfrak{D}_\alpha$. Then every k-slice of f is cyclic in $\mathfrak{D}_\alpha^k$.*

Proof. Assume without loss of generality that the $k-$slice of f, f_k, is given by fixing the last $n-k$ variables of f to $(a_{k+1}, ..., a_n) \in \mathbb{D}^{n-k}$, i.e. $f_k(z_1, ..., z_k) = f(z_1, z_2, ..., a_{k+1}, ..., a_n)$.

Recall that a function g is cyclic in $\mathfrak{D}_\alpha^k$ if and only if there exist a sequence of polynomials such that

$$\lim_{n\to\infty} \|p_n g - 1\|_{k,\alpha} = 0.$$

Now assume that we can find a bound of the form

$$\|g_k\|_{k,\alpha} \leq C\|g\|_\alpha, \tag{31}$$

where g_k is the function given by restricting the last $n-k$ variables of g to the point $(a_{k+1}, ..., a_n) \in \mathbb{D}^{n-k}$. Then, since f is cyclic in $\mathfrak{D}_\alpha$, there exists a sequence of polynomials $(q_n)_{n=0}^\infty$ such that

$$\lim_{n\to\infty} \|q_n f - 1\|_\alpha = 0,$$

the above bound will show that f_k is cyclic in $\mathfrak{D}_\alpha^k$ since

$$0 \leq \lim_{n\to\infty} \|q_{nk} f_k - 1\|_{k,\alpha} \leq \lim_{n\to\infty} C\|q_n f - 1\|_\alpha = 0.$$

It remains to show that a bound like (31) holds.

Since g is holomorphic, we have that

$$g(z_1, ...a_n) = \sum_{l_n=0}^\infty \cdots \sum_{l_1=0}^\infty \hat{g}(l) z_1^{l_1} \cdots z_k^{l_k} a_{k+1}^{l_{k+1}} \cdots a_n^{l_n} \tag{32}$$

$$= \sum_{l_1=0}^\infty \cdots \sum_{l_k=0}^\infty \left(\sum_{l_n=0}^\infty \cdots \sum_{l_{k+1}=0}^\infty \hat{g}(l) a_{k+1}^{l_{k+1}} \cdots a_n^{l_n} \right) z_1^{l_1} \cdots z_k^{l_k} \tag{33}$$

It follows that

$$\|g_k\|_{k,\alpha}^2 = \sum_{l_1=0}^{\infty} \cdots \sum_{l_k=0}^{\infty} (1+l_1)^\alpha \cdots (1+l_k)^\alpha \left| \sum_{l_n=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty} \hat{g}(l) a_{k+1}^{l_{k+1}} \cdots a_n^{l_n} \right|^2 .$$

By the Cauchy-Schwarz inequality, we see that

$$\left| \sum_{l_n=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty} \hat{g}(l) a_{k+1}^{l_{k+1}} \cdots a_n^{l_n} \right|^2 \tag{34}$$

$$= \left| \sum_{l_n=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty} (1+l_{k+1})^{\alpha/2-\alpha/2} \cdots (1+l_n)^{\alpha/2-\alpha/2} \hat{g}(l) a_{k+1}^{l_{k+1}} \cdots a_n^{l_n} \right|^2 \tag{35}$$

$$\leq \left(\sum_{l_n=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty} (1+l_{k+1})^\alpha \cdots (1+l_n)^\alpha |\hat{g}(l)|^2 \right) \tag{36}$$

$$\cdot \sum_{l_n=0}^{\infty} \cdots \sum_{l_{k+1}=0}^{\infty} \frac{|a_{k+1}|^{2l_{k+1}} \cdots |a_n|^{2l_n}}{(1+l_{k+1})^\alpha \cdots (1+l_n)^\alpha}. \tag{37}$$

Note that the second series converges for every $(a_{k+1}, ..., a_n) \in \mathbb{D}^{n-k}$, and furthermore that the series is independent of $(l_1, ..., l_k)$. By calling this series C, plugging this into the above expression for $\|g_k\|_{k,\alpha}^2$, recalling the definition of $\|g\|_\alpha$, and comparing with (31) finishes the proof.

□

Unfortunately, the converse statement does not hold, there exists functions all of whose k-slices are cyclic which are not cyclic themselves.

Example 3. The polynomial $1 - xy$ is cyclic in $\mathfrak{D}_\alpha$ if and only if $\alpha \leq 1/2$, but all of its slices are cyclic for all $\alpha \leq 1$. These statements will be proved in the next chapter.

However, if the function is *separable*, i.e. $f(z_1, ...z_n) = g(z_1, ..., z_k)h(z_{k+1}, ..., z_n)$, then f is cyclic if and only if g and h are cyclic. Note that if f admits such a factorization, then it follows from the definition of the norm on $\mathfrak{D}_\alpha$ that $\|f\|_\alpha = \|g\|_{k,\alpha}\|h\|_{n-k,\alpha}$. Furthermore, it is of course not necessary that f admits such a factorization in the first k variables only, since we can make any set of k variables the "first k variables" by simply relabeling them. We can now prove the following

Theorem 7. *Let $f(z_1, ..., z_n) = g(z_1, ..., z_k)h(z_{k+1}, ..., z_n)$ for $g \in \mathfrak{D}_\alpha^k$ and $h \in \mathfrak{D}_\alpha^{n-k}$. Then f is cyclic in $\mathfrak{D}_\alpha$ if and only if g is cyclic in $\mathfrak{D}_\alpha^k$ and h is cyclic in $\mathfrak{D}_\alpha^{n-k}$.*

Proof. Note that g is a constant multiple of a k-slice of f and that h is a constant multiple of an $n-k$-slice of f, so it follows from Theorem 4 that g and h are cyclic if f is cyclic.

Assume now that both g and h are cyclic in their respective spaces. Let (p_n) and (q_n) be sequences of polynomials such that

$$\lim_{n\to\infty} \|p_n g - 1\|_{k,\alpha} = \lim_{n\to\infty} \|q_n h - 1\|_{n-k,\alpha} = 0.$$

Since $p_n gh - h = (p_n(z_1, ..., z_k)g(z_1, ..., z_k) - 1)h(z_{k+1}, ..., z_n)$ is separable, we have that

$$\|p_n f - h\|_\alpha = \|p_n g - 1\|_{k,\alpha}\|h\|_{n-k,\alpha},$$

which tends to 0 as n goes to infinity. It follows that h lies in $[f] \subset \mathfrak{D}_\alpha$, and so $[h] \subset [f]$, But since h only depends on the last $n-k$ variables, we have that

$$\|q_n h - 1\|_\alpha = \|q_n h - 1\|_{n-k,\alpha},$$

and the right hand side tends to zero as n goes to infinity. It follows that h is cyclic in $\mathfrak{D}_\alpha$, and so $[h] \subset [f]$ implies that f is cyclic.

This finishes the proof.

□

The above statement shows that under certain circumstances, it is possible to draw conclusions regarding cyclicity of a function by examining each factor in a given factorization separately. For example, since it is known that any polynomial which is non-vanishing in $\mathbb{D}$ is cyclic in $\mathfrak{D}^1_\alpha$ for $\alpha \leq 1$, it follows that any polynomial of the form $p_1(z_1)\cdots p_n(z_n)$ is cyclic in $\mathfrak{D}_\alpha$, where each p_i is non-vanishing in the unit disc.

As a final remark, note that since a function f is cyclic if and only if there exists a sequence of polynomials p_n for which

$$\lim_{n\to\infty} \|p_n f - 1\|_\alpha = 0,$$

it is clear that good behavior of the actual inverse of f (that is $1/f$) will imply the existence of such a sequence, which in turn will imply cyclicity. More precisely, the following holds.

Lemma 5. *Given a multiplier $f \in M$ which is nowhere zero in $\mathbb{D}^n$, then f is cyclic if $1/f \in \mathfrak{D}_\alpha$.*

Proof. Since f is nowhere vanishing on $\mathbb{D}^n$, $1/f$ is analytic on $\mathbb{D}^n$. Furthermore, since multiplication by f is a bounded operator, we have that

$$\|p_n f - 1\|_\alpha \leq ||f||_{M(\mathfrak{D}_\alpha)}\|p_n - 1/f\|_\alpha.$$

And since $1/f \in \mathfrak{D}_\alpha$ and polynomials are dense in $\mathfrak{D}_\alpha$, the sequence p_n can be chosen to make the right hand side tend to zero.

This finishes the proof. $\square$

The above proof is completely analogous to the proof of the corresponding statement for $n = 2$, which is given in [3].

So far, we have drawn several conclusions regarding cyclicity. However, many of the conditions for cyclicity or non-cyclicity that we have are hard to check. For example, we are still unable to easily use the above machinery in order to determine if something as simple as the polynomial $1 - z_1 z_2 \in \mathfrak{D}_\alpha$ is cyclic or not. We will now restrict our attention to polynomials and, characterize certain properties of polynomials that will imply cyclicity or non-cyclicity.

Chapter 3

Cyclic Polynomials

3.1 Cyclic Polynomials in one Variable

The main goal of this section is to characterize the cyclic polynomials in $\mathfrak{D}^1_\alpha$. We know that a cyclic polynomial cannot vanish in $\mathbb{D}$. But it turns out that this is our only requirement. We will show that a polynomial on $\mathbb{D}$ is cyclic in $\mathfrak{D}^1_\alpha$ if and only if it has no zeros on $\mathbb{D}$. By the fundamental theorem of algebra and Lemma 4, it suffices to prove our claim for polynomials of the form $z-\zeta$. We will first prove that any polynomial which does not vanish on $\overline{\mathbb{D}}$ is cyclic. This will be done by showing that $z-\zeta$ is cyclic if $|\zeta|>1$.

Lemma 6. *If $f = z-\zeta$ for $|\zeta|>1$, then f is cyclic in $\mathfrak{D}^1_\alpha$.*

Proof. Since $|\zeta| > 1$, $1/(z-\zeta)$ has a convergent power series expansion everywhere on an open disc which contains $\overline{\mathbb{D}}$. It follows that the successive partial sums of the power series of $1/(z-\zeta)$, denote it by (p_n), converges uniformly on $\overline{\mathbb{D}}$, and thus on $\mathbb{D}$. It follows that

$$\|p_n f - 1\|_\alpha = \|p_n f - f/f\|_\alpha \le C\|p_n - 1/f\|_\alpha = C\left(\sum_{k=n+1}^{\infty} \frac{(k+1)^\alpha}{|\zeta^k|^2}\right)^{1/2},$$

where C is the multiplier norm of f, and the series on the right hand side clearly tends to zero as $n\to\infty$. Since $1\in[f]$, this finishes the proof. □

The above proof for $|\zeta|>1$ was essentially carried out by explicit calculation. This is however not possible for $|\zeta|=1$, since the last sum will not converge. So in order to prove the corresponding statement for $|\zeta|=1$, we use the following proof due to Brown and Shields.

Lemma 7. *If $f = z - \zeta$ with $|\zeta| = 1$, then f is cyclic in $\mathfrak{D}^1_\alpha$ for all $\alpha \leq 1$.*

Proof. Assume without loss of generality that $f = z - 1$. If not, we can factor out ζ and just rename z.

We will argue by contradiction. Assume that $\text{span}\{z^k f : k \in \mathbb{N}\}$ is not dense in $\mathfrak{D}_\alpha$. We will reach a contradiction to a corollary of the Hahn-Banach theorem by showing that the only element of the dual space of $\mathfrak{D}_\alpha$ which annihilates $\text{span}\{z^k f : k \in \mathbb{N}\}$ is the zero functional.

By the Riesz representation theorem, every element of the dual space of $\mathfrak{D}_\alpha$ can be represented as taking the inner product with some element of $\mathfrak{D}_\alpha$. That is, there exists a $g \in \mathfrak{D}_\alpha$ such that $f \to \langle g, f\rangle_\alpha$ is the desired functional. Assume that this $g = \sum_{k=0}^{\infty} a_k z^k$ is such that it annihilates all of $\text{span}\{z^k f : k \in \mathbb{N}\}$. Then in particular

$$\langle g, z^k(1-z)\rangle_\alpha = a_k(k+1)^\alpha - a_{k+1}(k+2)^\alpha = 0.$$

By considering these equalities for all $k \in \mathbb{N}$, we see that

$$a_k(k+1)^\alpha = a_0 \iff a_k = a_0/(k+1)^\alpha$$

for all $k \in \mathbb{N}$. And so

$$g(z) = \sum_{k=0}^{\infty} \frac{a_0}{(k+1)^\alpha} z^k.$$

Recall however, that $g(z) \in \mathfrak{D}_\alpha$, and so

$$\|g\|_\alpha^2 = \sum_{k=0}^{\infty} \frac{|a_0|^2}{(k+1)^\alpha} < \infty.$$

But since $\alpha \leq 1$, this is true if and only if $a_0 = 0$. But this implies that all coefficients in the power series expansion of g are equal to 0. It follows that the zero functional is the only functional which annihilates all of $\text{span}\{z^k f : k \in \mathbb{N}\}$, a contradiction.

□

Putting these statements together yields the main result of this section.

Theorem 8. *A polynomial $f \in \mathbb{C}[z]$ is cyclic in $\mathfrak{D}^1_\alpha$ for $\alpha \leq 1$ if and only if f has no zeros in $\mathbb{D}$.*

Proof. By the fundamental theorem of algebra, $f(z) = C(z-\zeta_1)\cdots(z-\zeta_n)$. If f has no zeros in $\mathbb{D}$, then $|\zeta_i| \geq 1$ for all $i = 1, ..., n$. It follows from the previous two lemmas that each factor of f is cyclic. And since all polynomials are multipliers, it follows from lemma 4 that f is also cyclic.

If however f has zeros in $\mathbb{D}$, then f is clearly not cyclic.

This finishes the proof.

□

This has some interesting implications. In general, a holomorphic function defined on $\mathbb{D}^k$ can be extended to a holomorphic function on $\mathbb{D}^n$ for $n \geq k$, by simply regarding it as constant in the last $n-k$ variables. For such a function, $\|f\|_\alpha^k = \|f\|_\alpha^n$, and so, if there exist polynomials (p_n) such that $fp_n \to 1$ in $\mathfrak{D}_\alpha^k$, then the same polynomials will make $fp_n \to 1$ in $\mathfrak{D}_\alpha^n$ for all $n > k$. Since any polynomial in one variable only, which does not vanish on $\mathbb{D}$ is cyclic in $\mathfrak{D}_\alpha^1$ for $\alpha \leq 1$, and thus is cyclic in $\mathfrak{D}_\alpha^n$ for any $n \geq 1$, it follows from Lemma 4 that any product of polynomials in one variable only, is cyclic in $\mathbb{D}_\alpha$.

We will now move one to study polynomials in more than one variable.

3.2 Cyclicity of polynomials with finite zero set on $\mathbb{T}^n$

By Lemma 4 and the fact that all polynomials are multipliers, it follows that we only need to prove our results for irreducible polynomials.

The main goal of this section is to prove that any polynomial with only finitely many zeros on $\mathbb{T}^n$ is cyclic in $\mathfrak{D}_\alpha$ for $\alpha \leq 1$. In order to do this we will use the following inequality due to Łojasiewicz.

Theorem 9 (Łojasiewicz's inequality)**.** *Let f be a non-zero real analytic function on an open set $U \subset \mathbb{R}^n$. Assume the zero set of f in U, $Z(f)$, is non-empty. Let E be a compact subset of U. Then there are constants $C > 0$ and $q \in \mathbb{N}$, depending on E, such that*

$$|f(x)| \geq C \cdot \operatorname{dist}(x, Z(f))^q$$

for every $x \in E$.

The proof can be found in [10].

When talking about polynomials, we will denote the zero set of f by $Z(f)$. Note that since any polynomial on $\mathbb{D}^n$ is continuous on $\overline{\mathbb{D}}^n$, we have that $Z(f) \cap \mathbb{T}^n$ equals our previous definition of $Z(f)$, namely the points of $\mathbb{T}^n$ for which the radial limit of f as we approach them equals zero.

The idea of our proof is the following. Given a polynomial f with only finitely many zeros on the distinguished boundary, we will use Łojasiewicz's inequality in order to create a polynomial p, which is known to be cyclic, such that p/f is sufficiently smooth on $\mathbb{T}^n$. The smoothness will imply rapid decay of the Fourier coefficients, thus showing that p/f lies in the appropriate Dirichlet type space. From there we can use that $p \in f\mathfrak{D}_\alpha$ along with the cyclicity of p in order to conclude that f is cyclic.

To make the proof more transparent, we will put the existence of such a polynomial in a separate lemma.

Lemma 8. *Let $f \in \mathbb{C}[z_1, ..., z_n]$ be an irreducible polynomial with no zeros in $\mathbb{D}^n$, and only finitely many zeros on $\mathbb{T}^n$. Then, for any positive integer k there exist polynomials $p_i \in \mathbb{C}[z_i]$ for $i = 1, ..., n$ with zeros only on $\mathbb{T}$ such that the function Q defined by*

$$Q(z_1, ..., z_n) = \frac{p_1(z_1) \cdots p_n(z_n)}{f(z_1, ..., z_n)}$$

is k times continuously differentiable on $\mathbb{T}^n$.

Since the numerator is a product of one variable polynomials, it follows from Theorem 7 and the fact that one variable polynomials which do not vanish in $\mathbb{D}$ are cyclic, that the numerator is cyclic in $\mathfrak{D}_\alpha$.

Proof. Denote by $r(x_1, ..., x_n)$ the function $|f(e^{ix_1}, ..., e^{ix_n})|^2$. Since f is a polynomial (and in particular analytic), and since e^{iv} is analytic, it follows that r is a real-analytic function defined on all of $\mathbb{R}^n$. Furthermore, r clearly inherits all of its zeros from the zeros of f on $\mathbb{T}^n$, and so, on every bounded subset of $\mathbb{R}^n$, r will have finitely many zeros. Now, denote by E the compact set $[0, 2\pi]^n$. By Łojasiewicz's inequality there is a constant $C > 0$ and a positive integer q such that

$$r(x) \geq C \cdot \operatorname{dist}(x, Z(r) \cap E)^q$$

for all $x \in E$. Since $Z(r) \cap E$ is a finite set, there is a constant $c > 0$ such that

$$\operatorname{dist}(x, Z(r) \cap E)^2 \geq c \prod_{y \in Z(r) \cap E} |x - y|^2$$

for all $x \in E$. For example setting $1/c$ to be the length of the diagonal of E squared, times the number of zero will be sufficient.

By putting this together, we see that for $x \in E$ we have that

$$r(x) \geq C \left(\prod_{y \in Z(r) \cap E} |x - y|^2 \right)^{q/2},$$

and so

$$\frac{\left(\prod_{y \in Z(r) \cap E} |x - y|^2\right)^{q/2}}{r(x)} = \frac{\left(\prod_{y \in Z(r) \cap E} |x - y|^2\right)^{q/2}}{|f(e^{ix_1}, ..., e^{ix_n})|^2}$$

is bounded.

However, in order to attain a similar inequality as the one above but with the numerator on the correct form, we proceed as follows.

We have that

$$|x - y|^2 = |x_1 - y_1|^2 + ... + |x_n - y_n|^2 \geq |e^{ix_1} - e^{iy_1}|^2 + ... + |e^{ix_n} - e^{iy_n}|^2.$$

By inductively applying that $a^2+b^2 \geq 2ab$ on the left hand side, we see that

$$\begin{aligned}
&|e^{ix_1}-e^{iy_1}|^2+...+|e^{ix_n}-e^{iy_n}|^2\\
\geq& 2|e^{ix_1}-e^{iy_1}|\left(|e^{ix_2}-e^{iy_2}|^2+...+|e^{ix_n}-e^{iy_n}|^2\right)^{1/2}\\
&\vdots\\
\geq& 2|e^{ix_1}-e^{iy_1}|2^{1/2}|e^{ix_2}-e^{iy_2}|^{1/2}\cdots\left(|e^{ix_{n-1}}-e^{iy_{n-1}}|^2+|e^{ix_n}-e^{iy_n}|^2\right)^{1/2^{n-2}}\\
\geq& 2|e^{ix_1}-e^{iy_1}|\cdots 2^{1/2^{n-2}}|e^{ix_{n-1}}-e^{iy_{n-1}}|^{1/2^{n-2}}|e^{ix_n}-e^{iy_n}|^{1/2^{n-2}}.
\end{aligned}$$

By comparing this to the above quotient, it follows that

$$\frac{\prod_{y\in Z(r)\cap E}|e^{ix_1}-e^{iy_1}|^{q/2}|e^{ix_2}-e^{iy_2}|^{q/2^2}\cdots|e^{ix_n}-e^{iy_n}|^{q/2^{n-1}}}{|f(e^{ix_1},...,e^{ix_n})|^2}$$

is bounded for all $x \in E$. By replacing e^{ix_i} with z_1 and e^{iy_i} by ζ_i we get that

$$\frac{\prod_{y\in Z(r)\cap E}|z_1-\zeta_1|^{q/2}|z_2-\zeta_2|^{q/2^2}\cdots|z_n-\zeta_n|^{q/2^{n-1}}}{|f(z_1,...,z_n)|^2}$$

is bounded for all $z \in \mathbb{T}^n$.

The numerator of the above expression, call it Q, is on the correct form, and has some regularity in the sense that it is continuous, but we promised more. However, if we exponentiate Q sufficiently many times, we can achieve the desired regularity. Since both numerator and denominator are polynomials, the question of differentiability is purely about how many times we can differentiate until nothing in the numerator kills the singularities from the zeros of f on $\mathbb{T}^n$. Since the polynomial Q is separable, and since the factor in each variable is of the form $(z_i-\zeta_i)^n$, the partial derivatives in each variable will be $n(z_i-\zeta_i)^{n-1}$, and therefore will inherit all zeros, although the zeros will be of a lower degree. So for some large integer N, we will achieve the desired regularity of the expression Q^N/f. Of course, we consider the continuous extension of these expressions, so the function obtained by exponentiating the numerator will be extended to all of $\mathbb{T}^n$ by setting it equal to 0 on the zeros of f. Furthermore, note that after exponentiating, we still have a numerator of the form $p_1(z_1)\cdots p_n(z_n)$, where all the zeros of p_i lies on $\mathbb{T}$.

This finishes the proof.

$\square$

With the above lemma at hand, the proof of the desired result is straightforward.

Theorem 10. *Let $f \in \mathbb{C}[z_1, ..., z_n]$ be a polynomial with no zeros in $\mathbb{D}^n$ and only finitely many zeros on $\mathbb{T}^n$. Then f is cyclic in $\mathfrak{D}_\alpha$ for $\alpha \leq 1$.*

Proof. By the previous lemma, there exist one variable polynomials $p_i \in \mathbb{C}[z_i]$ which vanish only on $\mathbb{T}$, such that

$$Q(z_1, ..., z_n) = \frac{p_1(z_1) \cdots p_n(z_n)}{f(z_1, ..., z_n)}$$

is n times continuously differentiable on $\mathbb{T}^n$. By integrating by parts, we see that the Fourier coefficients of Q satisfies

$$\sum_{k \in \mathbb{Z}^n} |\widehat{Q}(k)|^2 (1 + k_1)^n \cdots (1 + k_n)^n < \infty,$$

and so $f \in \mathfrak{D}_\alpha$ for $\alpha \leq n$. It follows that $p_1(z_1) \cdots p_n(z_n) \in f\mathfrak{D}_\alpha$ for $\alpha \leq n$. Since f is a polynomial, and thus a multiplier, we have that $f\mathfrak{D}_\alpha \subset \mathfrak{D}_\alpha$, and so there exists a g in $\mathfrak{D}_\alpha$ such that $p_1(z_1) \cdots p_n(z_n) = fg$. But since g is analytic and $fg \in \mathfrak{D}_\alpha$ (and so the tail of the series that defines the norm of fg tends to zero), we have that

$$\lim_{n \to \infty} \|p_1(z_1) \cdots p_n(z_n) - fq_n\|_\alpha = 0,$$

where q_n is the partial sums of the power series expansion of g. It follows that $p_1(z_1) \cdots p_n(z_n) \in [f]$. Since each p_i is a polynomial in one variable which does not vanish on $\mathbb{D}$, it follows that each p_i lies in $\mathfrak{D}^1_\alpha$ for $\alpha \leq 1$, and so, by Theorem 7, we have that $p_1(z_1) \cdots p_n(z_n)$ is cyclic in $\mathfrak{D}_\alpha$. Since $[f]$ contains a cyclic function, it follows that f is cyclic.

□

This generalizes Theorem 3.1 from [2], which is the corresponding statement for polynomials in $\mathbb{C}[z_1, z_2]$. The main idea of the proof, namely to use Łojasiewicz's inequality in order to compare our polynomial with a product of polynomials in one variable, originates from the same article in order to prove the corresponding theorem for $n = 2$.

An interesting consequence of the above theorem is connected to the Riesz capacity of certain sets. Recall from Theorem 1, that if $Z(f) \cap \mathbb{T}^n$ has positive Riesz capacity with parameter α, then f is *not* cyclic in $\mathfrak{D}_\alpha$.

Example 4. Consider the polynomials

$$f_k(z_1, ..., z_k) = k - \sum_{i=1}^{k} z_k.$$

Since $Z(f_k) \cap \mathbb{T}^k$ consists of only one point, namely $(1, ..., 1)$, it follows from the previous theorem that f_k is cyclic in $\mathfrak{D}_\alpha^k$ for *all* $\alpha \leq 1$. And so, f_k is cyclic in $\mathfrak{D}_\alpha^n$ for all $\alpha \leq 1$ and for all $n \geq k$.

However, for $f_k \in \mathfrak{D}_\alpha^n$ with $n \geq k$, we have that

$$Z(f_k) \cap \mathbb{T}^n = \{1\} \times \cdots \times \{1\} \times \mathbb{T}^{n-k}.$$

But since f_k is cyclic in $\mathfrak{D}_\alpha^n$, theorem 1 implies that $\mathbb{T}^n = \{1\} \times \cdots \times \{1\} \times \mathbb{T}^{n-k}$ cannot have positive Riesz capacity for any parameter $\alpha \leq 1$.

3.3 Some notes on zeros on $\overline{\mathbb{D}}^n \setminus \mathbb{T}^n$

Let f be a cyclic function in $\mathfrak{D}_\alpha$. Since f is cyclic, we know that f has no zeros inside $\mathbb{D}^n$. But now, assume that f has a zero on $\overline{\mathbb{D}}^n \setminus \mathbb{T}^n$. Assume without loss of generality that this means that there exist a point (p, q) with $p = (p_1, ..., p_k) \in \mathbb{D}^k$ and $q \in \mathbb{T}^{n-k}$ for some $1 \leq k \leq n-1$, such that $f(p, q) = 0$. We now create a sequence of functions defined on $\mathbb{D}$ by

$$f_n(z) = f\left(p_1, ..., p_{i-1}, z, p_{i+1}, ...p_k, \left((1 - \frac{1}{n}\right) q\right).$$

Since f is non-vanishing in $\mathbb{D}^n$, this means that every $f_n(z)$ is non-vanishing in $\mathbb{D}$. However, as n tends to infinity, the limit will have a zero for $z = p_i$. But since f_n is a sequence of holomorphic functions in one variable, Hurwitz's theorem yields that the limit function is either non-zero, or it is constantly equal to zero. But since we know the limit to have one zero, it follows that it must be zero everywhere. We now have that $f(p_1, ..., z, ...p_k, q) = 0$ for all $z \in \mathbb{D}$. Note that $1 \leq i \leq k$ was arbitrary. By pushing z out to the boundary, we see that f is zero on $\mathbb{D}^{k-1} \times \mathbb{T} \times \mathbb{T}^{n-k}$, and so, it follows by induction that f will vanish everywhere on $\mathbb{T}^k \times \{q\}$. As a special case, we see that f will have zeros on $\mathbb{T}^n$ if it has zeros on the topological boundary of $\mathbb{D}^n$. It is worth noting that the above arguments do not explicitly use that f is cyclic, only that f is holomorphic and that f is non-zero everywhere on $\mathbb{D}^n$.

An important consequence of this is that the zero set cannot leave $\mathbb{D}^n$ without passing through $\mathbb{T}^n$. This can be used to characterize which polynomials are candidates for being cyclic.

Let $p(z)$ be any polynomials for which $p(0) \neq 0$ (if this was the case, then $p(z)$ is clearly not cyclic). Now define $r > 0$ as

$$r = |\inf\{d \in \mathbb{R} : p(dz) \text{ has no zeros in } \overline{\mathbb{D}}^n\}|.$$

From this definition, it follows that $p(rz)$ is a polynomial with no zeros in $\mathbb{D}^n$, so it is a candidate for being cyclic. Furthermore, it will have zeros on $\overline{\mathbb{D}}^n$, and thus on $\mathbb{T}^n$, and so, it is not obviously cyclic for all α. All interesting polynomials can therefore be obtained in the way described above. However, it is still rather challenging to use this "characterization" in order to draw any conclusions about cyclicity.

However, regarding cyclicity, we can assume without loss of generality that $p(r, r, ..., r) = 0$, since we know that $p(rz)$ has zeros on $\mathbb{T}^n$, and since rotations do not affect cyclicity. After rescaling again, this implies that the polynomials which we need to understand are polynomials of the form

$$1 - \sum_{k \in \mathbb{N}^n} c_k z^k,$$

where $\sum c_k = 1$. By considering real and imaginary parts, this means that $\sum Re(c_k) = 1$ and $\sum Im(c_k) = 0$. Note however, that not every polynomial of the above form whose coefficients satisfy the previous criteria is a candidate for being cyclic, since there are possibilities for such a polynomial to have zeros inside the polydisc. But if we assume that c_k are positive real numbers, then every polynomial of the above form will be a candidate for being cyclic.

3.4 Parametrizations and the type of $Z(f) \cap \mathbb{T}^n$ for $n \geq 3$

In what follows, we will refer to submanifolds of codimension 2 as *curves*, and we will refer to hypersurfaces simply as surfaces.

Recall that if $Z(f) \cap \mathbb{T}^n$ contains a subset of type 2, then f can not be cyclic in $\mathfrak{D}_\alpha$ for any $\alpha > 1 - 1/n$. With this in mind, it is of great interest to find certain conditions under which the existence of such a subset is guaranteed. Furthermore, we will introduce a method through which we can "force" certain curves to be of type 2 locally. This will essentially be carried out by applying Möbius transformations, m, on each variable separately. The new curve will be a curve of type 2 on which $f \circ m$ vanishes. This implies that $f \circ m$ is not cyclic, which in turn can be used to show that f is not cyclic.

The above idea was originally used in [2] in order to characterize the cyclic polynomials in two variables. However, in higher dimensions several new issues can arise.

This section will have two main parts. In the first one, we work with functions whose zero set locally looks like a curve, and in the second, we work with functions whose zero set locally looks like a surface. For concrete functions, it is sometimes possible to find a concrete parametrization of the zero set, in which case that is probably easier. But for completeness, we will give certain criteria for when such a parametrization is possible.

Given a function f which extends continuously to $\mathbb{T}^n$, for example a polynomial, we can parametrize f on $\mathbb{T}^n$ by $f(e^{iv_1}, ..., e^{iv_n})$. This gives us a function from $\mathbb{R}^n$ to $\mathbb{C}$. By splitting f up into real and imaginary parts, that is by noting that $f(e^{iv_1}, ...e^{iv_n}) = u(v_1, ..., v_n) + iv(v_1, ..., v_n)$, we see that the zeros of f on $\mathbb{T}^n$ are exactly the zeros of $(u(v_1, ..., v_n), v(v_1, ..., v_n)) \subset \mathbb{R}^2$. So assuming that there is some point $p \in Z(f) \cap \mathbb{T}^n$, for which some pair of (v_i, v_j) satisfy

$$\begin{vmatrix} u'_{v_i}(p) & v'_{v_i}(p) \\ u'_{v_j}(p) & v'_{v_j}(p) \end{vmatrix} \neq 0, \tag{38}$$

then we can apply the implicit function theorem in order to parametrize $Z(f) \cap \mathbb{T}^n$ locally around p. The above condition essentially means that there is some point on $Z(f) \cap \mathbb{T}^n$ where the partial derivatives of f span $\mathbb{C}$ in the sense that every point in $\mathbb{C}$ can be written as a linear combination of the partial derivatives with real coefficients. Note however that the above criteria is not always satisfied.

Example 5. The polynomial

$$f(z_1, ...z_n) = n - \sum_{k=1}^{n} z_i$$

has only one zero on $\mathbb{T}^n$, namely for $z_1 = ... = z_n = 1$. But the partial derivatives at that point clearly do not span $\mathbb{C}$ since they are all parallel due to symmetry.

But moving on, assuming there is such a point, and furthermore, assume without loss of generality that the last two variables satisfy the above criteria, then we can apply the implicit function theorem in order to parametrize $Z(f) \cap \mathbb{T}^n = Z(f(e^{iv_1}, ..., e^{iv_n}))$ around p. The parametrization will be given by $(v_1, ..., v_{n-2}, h(v_1, .., v_{n-2}), g(v_1, .., v_{n-2}))$. We will often denote $(v_1, ..., v_{n-2}) \in [0, 2\pi]^{n-2}$ by v.

Note that the function $f(e^{iv_1}, ..., e^{ih(v_1,...,v_{n-2})}, e^{ig(v_1,...,v_{n-2})})$ is constantly equal to zero everywhere in an open neighborhood, and so all partial derivatives vanish there. This means that

$$f'_j + f'_{n-1}h'_j + f'_n g'_j = 0,$$

for all j.

But by the above assumption, $f'_{n-1}h'_j + f'_n g'_j \neq 0$ at the point p, and so, this implies that *all* partial derivatives must vanish at the point p. In particular, a necessary condition for the above assumption to hold is that all partial derivatives are non-vanishing at the point p. However, this is not sufficient since it might be the case that they do all align, and so no pair will span $\mathbb{C}$.

We now ask ourselves under which conditions we can guarantee that the piece of $Z(f) \cap \mathbb{T}^n$ that is parametrized above is indeed of type 2. Assume that the point $p = (p', h(p'), g(p'))$ is not of type 2, then this would imply that there is some $\eta \in S^n$ for which

$$\eta_i + h'_i(p')\eta_{n-1} + g'_i(p')\eta_n = 0, \tag{39}$$

and

$$h''_{i,j}(p')\eta_{n-1} + g''(p')_{i,j}\eta_n = 0, \tag{40}$$

for *all* $1 \leq i, j \leq n-2$.

This is only possible if $\{(h''_{i,j}(p'), g''_{i,j}(p')\}$ does not span $\mathbb{R}^2$, or if they do span $\mathbb{R}^2$, but the *unique* solution to

$$h''_{i,j}(p')\eta_{n-1} + g''_{i,j}\eta_n$$

and

$$h''_{i',j'}(p')\eta_{n-1} + g''_{i',j'}\eta_n = 0,$$

is also a solution to the corresponding equations for all i and j, and furthermore, the unique solution η given by

$$\eta_i = -(h'_i(p')\eta_{n-1} + g'_i(p')\eta_n),$$

has to lie on S^n.

Example 6. Consider the polynomial

$$2 - xy - xz.$$

The zero set of this polynomial on the torus is parametrized by $(t, -t, -t)$. This zero set does not have type 2 at any point, since for example $\eta = (2, 1, 1)/\sqrt{6} \in S^3$ satisfies all of the above equations.

It is not entirely clear how big a constraint equations (39) and (40) are. The above example shows that these equations might have solutions, but the zero set of the above polynomial is entirely flat. Recall that we can parametrize around *any* point, which means that we only have problems if equations (39) and (40) fail around every point. If this is to happen, then it will truly imply serious constraints on how the zero set may look.

If we are in any of the unfortunate situations above, then we might still be able to force the curve to be of type 2 by applying Möbius transformations to the parameters. Consider the Möbius transformations

$$m_a(z) = \frac{a - z}{1 - \bar{a}z},$$

where $a \in \mathbb{T}$. Note that $m_a(z)$ is its own inverse, that is $(m_a \circ m_a)(z) = z$, and so if $(e^{iv_1}, ..., e^{if(v)}, e^{ig(v)})$ parametrizes $Z(f) \cap \mathbb{T}^n$ around p, then

$$\left(\arg(e^{im_{a_1}(v_1)}), ..., \arg(e^{im_{a_{n-2}}(v_{n-2})}), h(v), g(v)\right)$$

parametrizes $Z(f \circ e^{iv} \circ m)$ around $m(p)$, where

$$m(v) = (m_{a_1}(v_1), ..., m_{a_{n-2}}(v_{n-2}), v_{n-1}, v_n).$$

Now assume without loss of generality that $p = 0$, and denote $\arg(e^{im_{a_i}(z)})$ by $\phi_{a_i}(z)$. That this assumption is indeed without loss of generality is clear, since if $p \neq 0$, then we can simply move it to 0 through a change of variables, and this will not affect the cyclicity of f. Through direct calculations, one can show that $\phi'_{a_i}(0) > 0$ and $\phi''_{a_i}(0) \neq 0$ for any $a_i \in \mathbb{T}$ with $\mathrm{Im}(a) \neq 0$, see for example Section 2.3 in [2] for some further notes on this method.

We now ask ourselves under which circumstances this new curve has type 2.

If this curve does not have type 2, then there exists some $\eta \in S^n$ for which

$$\phi'_{a_i}(0)\eta_i + h'_i(0)\eta_{n-1} + g'_i(0)\eta_n = 0, \tag{41}$$

and

$$\phi''_{a_i}(0)\eta_i + h''_{i,i}(0)\eta_{n-1} + g''_{i,i}(0)\eta_n = 0, \tag{42}$$

and

$$h''_{i,j}(0)\eta_{n-1} + g''_{i,j}(0)\eta_n = 0, \tag{43}$$

for all $1 \leq i, j \leq n-2$.

The main difference between equations (41)$-$(43) and the equations (39) and (40) that were used for determining whether or not our initial parametrization was of type 2 is the second family of equations, that is equation (42).

We point out that we do not have to apply Möbius transformations on all parameters. If there is one parameter, v_i, on which we do not apply a Möbius transformation, then this will correspond to $\phi''_{a_i}(0)\eta_i = 0$ in the above equations.

We now have two major cases, either all pairs $\{(h''_{i,j}(p'), g''_{i,j}(p')\} = 0$, or at least one of them is non-zero.

If all of them equal zero, then equations (42) will force $\eta_i = 0$ for all $1 \leq i \leq n-2$. It follows that this curve will not be of type 2 if and only if there exists $(\eta_{n-1}, \eta_n) \in S^2$ such that

$$h'_i(0)\eta_{n-1} + g'_i(0)\eta_n = 0,$$

for all $1 \leq i \leq n-2$.

If not all pairs $\{(h''_{i,j}(p'), g''_{i,j}(p')\} = 0$, then for at least one pair, either $h''_{i,j}$ or $g''_{i,j} \neq 0$. Assume without loss of generality that $g''_{i,j} \neq 0$. If $i = j$, then we set the corresponding m_{a_i} to just be the identity map, and so $\phi''_{a_i} = 0$. In any case, this will force

$$\eta_n = \frac{-h''_{i,j}(0)}{g''_{i,j}(0)}\eta_{n-1}.$$

Note that $\eta_{n-1} \neq 0$, since this would force *all* $\eta_i = 0$.

By plugging this into the remaining equations and comparing expressions given by the first and second family of equations, we get that

$$\eta_l = \frac{1}{\phi'_{a_l}(0)}\left(\frac{g'_l(0)h''_{i,j}(0)}{g''_{i,j}(0)} - h'_l(0)\right)\eta_{n-1} = \frac{1}{\phi''_{a_l}(0)}\left(\frac{g''_{l,l}(0)h''_{i,j}(0)}{g''_{i,j}(0)} - h''_{l,l}(0)\right)\eta_{n-1},$$

for all $1 \leq l \leq n-2$.

We have two possibilities. Either

$$\left(\frac{g_l'(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_l'(0)\right) = \left(\frac{g_{l,l}''(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_{l,l}''(0)\right) = 0$$

for *all* $1 \leq l \leq n-2$, and so $\eta_l = 0$ for all l.

Furthermore, this would imply that

$$h_i'(0)/g_i'(0) = h_{i,j}''(0)/g_{i,j}''(0) = c,$$

for *all* i, j. Once again, note that $g_{i,j}''(0) \neq 0$.

The second alternative is that either

$$\left(\frac{g_{l,l}''(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_{l,l}''(0)\right) \neq 0,$$

or

$$\left(\frac{g_l'(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_l'(0)\right) \neq 0.$$

for some $1 \leq l \leq n-2$. Assume without loss of generality that the second possibility occurs. This would imply that

$$\frac{\phi_{a_i}''(0)}{\phi_{a_i}'(0)} = \frac{\frac{g_{l,l}''(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_{l,l}''(0)}{\frac{g_l'(0)h_{i,j}''(0)}{g_{i,j}''(0)} - h_l'(0)}.$$

However, the left hand side is *not* constant with respect to a_i, whilst the right hand side is. It follows that we can always choose a_i in such a way as to make this scenario impossible.

In conclusion, given $f \in \mathfrak{D}_\alpha$ which extends analytically to $\mathbb{T}^n$, then if there is some point $p = (e^{iv_1}, ..., e^{iv_n}) \in Z(f) \cap \mathbb{T}^n$, such that for some pair of (v_i, v_j) we have that

$$\begin{vmatrix} u_{v_i}'(p) & v_{v_i}'(p) \\ u_{v_j}'(p) & v_{v_j}'(p) \end{vmatrix} \neq 0,$$

where $u(v_1, ..., v_n) + iv(v_1, ..., v_n) = f(e^{iv_1}, ..., e^{iv_n})$. Then some neighborhood of $Z(f) \cap \mathbb{T}^n$ which contains p can be parametrized by

$$(v_1, ..., h(v_1, ..., v_{n-2}), g(v_1, ..., v_{n-2})).$$

Either this set already is of type 2 at the point p. If not, then as long as there is *no* $(\eta_{n-1}, \eta_n) \in S^1$ for which

$$h''_{i,j}(p)\eta_{n-1} + g''_{i,j}(p)\eta_n = 0$$

and

$$h'_i(p)\eta_{n-1} + g'_i(p)\eta_n = 0$$

for *all* $1 \le i, j \le n-2$, then we can apply Möbius transformations in order to make the curve to be of type 2 at the point p.

Once again, even though we have altered the set by applying Möbius transformations, this is still okay. Since if the new set has type 2, then the function $f \circ m$ will not be cyclic for $\alpha > 1 - 1/n$. But as we will see in Section 3.7, composition with Möbius transformations turns out to be bounded operators between Dirichlet type spaces, so the fact that $f \circ m$ is non-cyclic will imply that f is non-cyclic.

In some ways the above characterizations are unsatisfactory. We have very harsh constraint on what type of curve we cannot force to be of type 2 locally, but in general, it is still rather difficult to verify all of the above criteria, although it might get a little bit easier if one assumes that all partial derivatives of order 1 are non-zero, that is, we have we have a non-singular curve in $\mathbb{T}^n$

Another big problem is that the assumption on the differential does not always hold, which means that we can not necessarily make the above parametrization at all.

Example 7. For example, there is no such point p in $Z(f) \cap \mathbb{T}^n$ for either of our standard polynomials

$$f(z_1, ..., z_n) = n - \sum_{i=1}^{n} z_i \text{ and } g(z_1, ..., z_n) = 1 - \prod_{i=1}^{n} z_i.$$

For the first polynomial, this makes sense because the intersection consists of just a single point. For the second polynomial however, the problem is related to the fact that the zero set is actually much nicer, namely we can parametrize it through $(v_1, ..., v_{n-1}, f(v_1, ..., v_{n-1}))$.

In fact for all polynomials which are nice in the same way, the above methods for parameterizing and transforming $Z(f) \cap \mathbb{T}^n$ locally will yield much more concrete results. That is, we get better control over functions whose zero set is a surface locally instead of a curve. We give a condition which guarantees this to happen, provide an example of this, and then move on with the calculations for the surfaces.

Let $f \in \mathfrak{D}_\alpha$ be a function which admits an analytic continuation to $\overline{\mathbb{D}^n}$, and assume furthermore that $f(z) = u(z) + iv(z)$ is such that $u(z) = 0$ if and only if $v(z) = 0$ for all $z \in \mathbb{T}^n$. It follows that $(v_1, ..., v_n) \in [-\pi, \pi]^n$ is a zero of $f \circ e^{iv}$ if and only if it is a zero of $(u \circ e^{iv}, v \circ e^{iv}) \subset \mathbb{R}^2$, $u \circ e^{iv}$, or $v \circ e^{iv}$. Assume now that there is some point $p \in [-\pi, \pi]^n$ such that *either* $u \circ e^{ip} = 0$, or $v \circ e^{ip} = 0$. So if *any* partial derivative of either u or v is non-zero at the point p, then we can apply the implicit function theorem, which yields that $(v_1, ..., g(v_1, ..., v_{n-1}))$ parametrizes the zero set of $u \circ e^{iv}$ or $v \circ e^{iv}$, and so it parametrizes the zero set of $f \circ e^{iv}$ locally around the point p.

Example 8. The polynomial $1 - xyz$ has this property, since $\cos(v_1 + v_2 + v_3) = 1$ if and only if $\sin(v_1 + v_2 + v_3) = 0$.

We now ask ourselves if the subset of $Z(f) \circ \mathbb{T}^n$ which is parametrized by $(v_1, ..., g(v_1, ..., v_{n-1}))$ has type 2. If it does not have type 2, then there exist some $\eta \in S^n$ such that

$$\eta_i + g_i'(p)\eta_n = 0 \tag{44}$$

and

$$g_{i,j}''(p)\eta_n = 0 \tag{45}$$

for all $1 \leq i, j \leq n - 1$.

If not all $g_{i,j}''(p) = 0$, then the only possibility is that $\eta_n = 0$, and so the first family of equations forces η_i to be 0 for all i, which is impossible. This implies that the surface has type 2 if and only if there is some $1 \leq i, j \leq n-1$ such that $g_{i,j}''(p) \neq 0$.

If however all partial derivatives of order 2 are equal to zero, then under certain circumstances, we can still force the surface to be of type 2 by applying Möbius transformations, m_{a_i} with $a \in \mathbb{T}$ and $Im(a_i) \neq 0$, on all parameters. We can once again assume without loss of generality that $p = 0$, otherwise we rotate it there. We get that the the surface $(\arg(e^{m_{a_1}(v_1)}), ..., \arg(e^{m_{a_{n-1}}(v_{n-1})}), f(v_1, ..., v_{n-1}))$ parametrizes the zero set of $Z(f \circ e^{iv} \circ m)$ for $m(v) = (m_{a_1}(v_1), ..., v_n)$. We denote $\arg(e^{m_{a_i}(v_i)})$ by $\phi_i(v_i)$ and recall that $\phi_{a_i}'(0) > 0$, and $\phi_{a_i}''(0) \neq 0$. It follows that this surface does not have type 2 if and only if there is some $\eta \in S^n$ such that

$$\phi_{a_i}'(0)\eta_i + g_i'(0)\eta_n = 0 \tag{46}$$

and

$$\phi_{a_i}''(0)\eta_i + g_{i,i}''(0)\eta_n = 0 \tag{47}$$

and

$$g''_{i,j}(0)\eta_n = 0 \tag{48}$$

for all $1 \le i, j \le n-1$.

Since we assumed that the first surface was not of type 2, $g''_{i,j} = 0$ for all i, j, and so the second family of equations implies that $\eta_i = 0$ for all i. It follows that this surface does not have type 2 if and only if $g'_i = 0$ for *all* i. But if we know that there exists some point where not all derivatives vanish, then we can force the surface to be of type 2 locally by the above methods.

Note that if all partial derivatives of g vanish, then every partial derivative, f'_i, of f has to vanish for $1 \le i \le n-1$. This is because

$$f(e^{iv_1}, ..., e^{iv_{n-1}}, e^{ig(v_1,...,v_{n-1})}) = 0$$

everywhere in some open neighborhood. In particular, all partial derivatives must vanish, and so

$$f'_i + f'_n g'_i = 0$$

for all i. But since g'_i equals zero for all $1 \le i \le n-1$, so does f'_i. This implies that this situation occurs if and only if all but one of the partial derivatives of f vanish at the point p. Furthermore, note that since $f'_n(p)$ is assumed to be non-zero, we know that f'_n is non-zero everywhere in an open neighborhood containing p. Since $(v_1, ..., g(v_1, ..., v_{n-1}))$ parametrizes $Z(f) \cap \mathbb{T}^n$ in another open neighborhood around p, we see that in the intersection of this neighborhood and the neighborhood in which $f'_n \neq 0$, either we have that $f'_i = 0$ for *all* $1 \le i \le n-1$ everywhere in this intersection, or there is some other point p' in this neighborhood for which $f(p') = 0$, around which we can apply the above methods to construct a surface of type 2.

Example 9. Once again, consider the polynomial

$$f(z_1, ..., z_n) = 1 - \prod_{i=1}^{n} z_i.$$

This polynomial satisfies the above criteria, since a point $v \in [-\pi, \pi]^n$ is a zero of $f \circ e^{iv}$ if and only if $v = (v_1, ..., v_{n-1}, -(v_1 + ... + v_{n-1})) = (v_1, ..., f(v_1, .., v_{n-1}))$. Although all partial derivatives of f of order 2 are equal to 0 everywhere, *no* first order derivative is zero anywhere, and so we can force this curve to be of type 2 locally by the above methods.

3.5 Some notes on Lifts

A frequently discussed example so far has been the polynomial

$$p(z_1, ..., z_n) = 1 - \prod_{i=1}^{n} z_i.$$

The discussion in the previous section implies that this polynomial is not cyclic in for any $\alpha > 1 - 1/n$. However, we have not yet concluded anything regarding when it for sure is cyclic. The goal of this section will be to determine cyclicity for certain functions by comparing their norms with the norm of a related one-variable function.

Given $f \in \mathfrak{D}_\alpha^1$, we define its *lift* corresponding to $k = (k_1, ..., k_n)$ with $k_i \in \mathbb{N} \setminus \{0\}$, $L_k : \mathfrak{D}_\alpha^1 \to \mathfrak{D}_\beta$ by

$$L_k(f)(z_1, ..., z_n) = f(z_1^{k_1} \cdots z_n^{k_n}).$$

We require $k_i \geq 1$ for all $1 \leq i \leq n$ in order to assure that the lift is truly a function in $n-$variables. Otherwise we could just consider the corresponding lift to a lower-dimensional space, and then expand up in dimension through our previous methods.

The norm of the lift can be bounded by the norm of f in the following way, which, as a consequence, lets us relate β to α.

Theorem 11. *For $f \in \mathfrak{D}_\alpha^1$, and $L_k(f) \in \mathfrak{D}_\beta$ for $\beta \geq 0$, we have that*

$$\|L_k(f)\|_\beta^2 \leq C\|f\|_{n\beta}^2,$$

for some $C > 0$. In particular, if $f \in \mathfrak{D}_\alpha$ then $L_k(f) \in \mathfrak{D}_{\alpha/n}$.

Proof. Since f is analytic in $\mathbb{D}$, we have that

$$f(z) = \sum_{l=0}^{\infty} a_l z^l,$$

and so

$$L_k(f)(z_1, ..., z_n) = \sum_{m_n=0}^{\infty} \cdots \sum_{m_1=0}^{\infty} b_m z_1^{m_1} \cdots z_n^{m_n} = \sum_{l=0}^{\infty} a_l (z_1^{k_1} \cdots z_n^{k_n})^l.$$

It follows that $b_m = a_l$ if $m = (m_1, ..., m_n) = (k_1 l, ..., k_n l)$ for some $l \in \mathbb{N}$, and zero if there is no such l.

From this we get that

$$
\begin{aligned}
\|L_k(f)\|_\beta^2 &= \sum_{m_n=0}^{\infty} \cdots \sum_{m_1=0}^{\infty} |b_m|^2 (m_1+1)^\beta \cdots (m_n+1)^\beta \\
&= \sum_{l=0}^{\infty} |a_l|^2 (1+k_1 l)^\beta \cdots (1+k_n l)^\beta \\
&\leq \sum_{l=0}^{\infty} |a_l|^2 (1+(k_1\cdots k_n) l)^\beta \cdots (1+(k_1\cdots k_n) l)^\beta \\
&\leq \sum_{l=0}^{\infty} |a_l|^2 ((k_1\cdots k_n)+(k_1\cdots k_n) l)^\beta \cdots ((k_1\cdots k_n)+(k_1\cdots k_n) l)^\beta \\
&= C\sum_{l=0}^{\infty} |a_l|^2 (1+l)^\beta \cdots (1+l)^\beta = C\sum_{l=0}^{\infty} |a_l|^2 (1+l)^{n\beta} = C\|f\|_{n\beta}^2,
\end{aligned}
$$

where $C = k_1 \cdots k_n$. This finishes the proof. $\square$

Note that if $k_1 = ... = k_n = 1$, then the above inequalities are in fact equalities, i.e. $\|L_{(1,..,1)}(f)\|_\alpha = \|f\|_{n\alpha}$.

The above result can be used to find certain parameters α for which our standard polynomials are cyclic. Let

$$f_k(z_1, ..., z_n) = 1 - \prod_{i=1}^{k} z_k,$$

for $k \leq n$. This function is cyclic in $\mathfrak{D}_\alpha$ if and only if $1 - \prod_{i=1}^{k} z_k$ is cyclic in $\mathfrak{D}_\alpha^k$. Since this is simply the lift of the function $1 - z$ corresponding to $k = (1, ..., 1)$, that is

$$1 - \prod_{i=1}^{k} z_k = L_{(1,...,1)}(1-z)(z_1, ..., z_k),$$

and $1-z$ is a one variable polynomial with no zeros in $\mathbb{D}$, and hence is cyclic for all $\alpha \leq 1$, the above theorem yields that $f_k(z_1, ..., z_n) \in \mathfrak{D}_\alpha$ is cyclic for all $\alpha \leq 1/k$, since if p_n is a sequence of one variable polynomials for which

$$\lim_{n\to\infty} \|p_n(1-z) - 1\|_\alpha = 0,$$

then $L_{(1,..,1)}(p_n)$ will provide the corresponding sequence of polynomials for $f_k(z_1, .., z_n)$.

In particular

$$f(z_1, ..., z_n) = 1 - \prod_{i=1}^{n} z_i$$

is cyclic for all $\alpha \leq 1/n$.

3.6 An application of the Brown and Shields recursion

We have previously used a method, originally due to Brown and Shields, in order to characterize the parameters α for which one variable polynomials are cyclic. In this section, the same idea will be applied in order to give an alternative proof for the statement that

$$f(z_1, ..., z_n) = 1 - \prod_{i=1}^{n} z_i$$

is cyclic for $1 \leq \alpha \leq 1/n$, and non-cyclic if $1/n < \alpha \leq 1$.

As a consequence of the Hahn Banach theorem, $f(z)$ is cyclic if and only if the only element in the dual space of $\mathfrak{D}_\alpha$ that annihilates $p(z)f(z)$ for all polynomials p is the zero functional. Since $\mathfrak{D}_\alpha$ is a Hilbert space, every bounded linear functional can be represented as $f \to \langle g, f \rangle$ for some $g \in \mathfrak{D}_\alpha$. Henceforth we will write $\tilde{k} = (k, ..., k)$.

Assume now that $g(z) = \sum_{k \in \mathbb{N}^n} \hat{g}(k) z^k$ annihilates pf for every polynomial p. Then in particular $\langle g, z^{\tilde{k}+n} f \rangle = 0$ for $n \in \mathbb{N}^n$, and so

$$\langle g, z^n - z^{\tilde{k}+n} \rangle = \langle g, (z^n - z^{\tilde{1}+n}) + (z^{\tilde{1}+n} + z^{\tilde{2}+n}) - z^{\tilde{2}+n} + ... + z^{\tilde{k}+n} \rangle = \sum_{i=1}^{k} \langle g, z^{\tilde{i}+n} f \rangle = 0.$$

By plugging this into the definition of the inner product, the above equality yields that

$$\hat{g}(n) \prod_{i=1}^{n} (1 + n_i)^\alpha = \hat{g}(n + \tilde{k}) \prod_{i=1}^{n} (1 + n_i + k)^\alpha.$$

Any $g \in \mathfrak{D}_\alpha$ which satisfies the above recursion will annihilate pf for all polynomials p, because it will annihilate monomials, and by linearity of the inner product.

It remains to investigate for which parameters α there exists a non-zero function whose Fourier coefficients satisfy the above recursion will have finite norm.

Note that every $n \in \mathbb{N}^n$ can be written as $n' + \tilde{k}$. for some $k \in \mathbb{N}$ and for n' with at least one component equal to zero. This will be used in order to represent the norm on $\mathfrak{D}_\alpha$ in a way which allows us to use the above recursion. When investigating finiteness of the sum over $\mathbb{N}^n$, we only need to consider the part of the sum whose indexes are of the form $n + \tilde{k}$ for

$n = (n', 0)$ with $n' \in \mathbb{N}^{n-1}$. This is due to symmetry and the fact that we are free to choose $\hat{g}(n', 0)$. That is, if there is no choice of $\hat{g}(n', 0)$ which makes the sum over $(n', 0) + \tilde{k}$ converge, then the entire series will not converge. And if there is come choice that makes the partial sum converge, then we can set $\hat{g}(0, n_1, ..., n_{n-1}) = ... = \hat{g}(n_1, ..., n_{n-1}, 0)$ and thus make the entire sum converge.

Now consider the series given by the norm on $\mathfrak{D}_\alpha$

$$\sum_{n' \in \mathbb{N}^{n-1}} |\hat{g}(n', 0)|^2 \prod_{i=1}^{n-1} (1 + n_i)^{2\alpha} \sum_{k \in \mathbb{N}} \frac{1}{\prod_{j=1}^{n} (1 + n_j + k)^\alpha}.$$

For *every* fixed value of $(n', 0)$, the series

$$\sum_{k \in \mathbb{N}} \frac{1}{\prod_{j=1}^{n} (1 + n_j + k)^\alpha}$$

will diverge if $n\alpha \leq 1$. So the only possibility for $\hat{g}(n', 0)$ to save the convergence is if they are all equal to zero. But this implies that g is the zero functional. Since this is the only g for which the series converges, and thus the only element in the dual space of $\mathfrak{D}_\alpha$ which annihilates all pf, it follows that f is cyclic. If however $\alpha > 1/n$, then the series

$$\sum_{k \in \mathbb{N}} \frac{1}{\prod_{j=1}^{n} (1 + n_j + k)^\alpha}$$

converges for every $(n', 0)$. Furthermore, the series is bounded for every $(n', 0)$, and so, by choosing $\hat{g}(n', 0)$ small enough, for example by setting it equal to 0 everywhere except for at the origin, then we get a non-trivial functional which annihilates all pf, and so f is *not* cyclic.

The above method can be used in order to get slightly more general results about cyclicity of certain polynomials. For example, consider polynomials of the form $1 - p(z)$, where $p(z) = cz^n$ for $n \in \mathbb{N}^n$. We will always assume that $c = 1$, since if $c < 1$ the polynomial will always be cyclic. And if $c > 1$, then the polynomial will never be cyclic. Furthermore, we assume that all variables are represented in p. Otherwise we just consider the polynomial as a polynomial of fewer variables, determine cyclicity there, and then finally we use that cyclicity is passed on upwards in dimension. This means that $n_i \geq 1$ for all i.

Again, assume that there is a $g \in \mathfrak{D}_\alpha$ such that $\langle g, q(1 - p) \rangle = 0$ for all polynomials q. Then by the same methods as above, we see that this g annihilates all $q(1 - p)$ if and only if

$$\hat{g}(k) \prod_{i=1}^{n} (1 + k_i)^\alpha = \hat{g}(k + ln) \prod_{i=1}^{n} (1 + ln_i + k_i)^\alpha,$$

for every $l \in \mathbb{N}$. We now want to determine if such a g can indeed lie in $\mathfrak{D}_\alpha$. Denote by S the smallest subset of $\mathbb{N}^n$ such that for every $m \in \mathbb{N}^n$, there exists $l \in \mathbb{N}$ for which $m - ln \in S$. Furthermore, note that for every $m \in \mathbb{N}^n$, the pair $s \in S$ and $l \in \mathbb{N}$ is unique. That we lose this property is one of many reasons for why this argument is difficult to generalize to $p(z)$ consisting of more than one term.

Using the above recursion formula we now see that

$$\|g\|_\alpha^2 = \sum_{s \in S} \sum_{l \in \mathbb{N}} |\hat{g}(s+ln)|^2 (1+s_i+ln_i)^\alpha = \sum_{s \in S} \prod_{i=1}^{n} (1+s_i)^{2\alpha} |\hat{g}(s)|^2 \sum_{l \in \mathbb{N}} \frac{1}{(1+s_i+ln_i)^\alpha}.$$

Just as before, we note that

$$\sum_{l \in \mathbb{N}} \frac{1}{(1+s_i+ln_i)^\alpha}$$

diverges for all $\alpha \leq 1/n$, and so $g \in \mathfrak{D}_\alpha$ if and only if all of its Fourier coefficients are equal to zero, which in turn will imply that f is cyclic. If however the series converges, as is the case for $\alpha > 1/n$, then by choosing $\hat{g}(s)$ small enough, it follows that there are non-trivial functional which annihilate all $q(1-p)$, and therefore f cannot possibly be cyclic.

3.7 An equivalent norm and Möbius transformations

Recall that the norm on $\mathfrak{D}_\alpha$ is given by

$$\|f\|_\alpha^2 = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} (k_1+1)^\alpha \cdots (k_n+1)^\alpha |a_{k_1,\dots,k_n}|^2 < \infty.$$

If one splits the above series up into parts, one in which no variables are zero, one in which exactly one variable is zero, one in which exactly two are equal to zero etc, then by manipulating power series and applying Parseval's identity, it can be shown that the above norm is equivalent to the norm given by

$$|f(0,0)|^2 + \sum_{i=1}^{n} \int_{\mathbb{D}} |f_i'(0,...,z_i,...,0)|^2 (1-|z_i|^2)^{1-\alpha} dA(z_i)$$

$$+ \sum_{1\le i\le j}^{n} \int_{\mathbb{D}^2} |\partial_i \partial_j f(0,...,z_i,...,z_j,...0)|^2 (1-|z_i|^2)^{1-\alpha}(1-|z_j|^2)^{1-\alpha} dA(z_i)dA(z_j)$$

$$+ \int_{\mathbb{D}^n} |\partial_1 \cdots \partial_n f(z_1,...,z_n)|^2 (1-|z_1|^2)^{1-\alpha} \cdots (1-|z_n|^2)^{1-\alpha} dA(z_1)\cdots dA(z_n),$$

where $A(z)$ is the normalized Lebesgue measure.

See for example [7] for a thorough discussion on how to endow the Dirichlet type spaces with norms, and on the equivalence of different such norms.

We will mainly focus in the last term of the sum. The point of introducing this equivalent norm is to show that composing with Möbius transforms is a bounded operator from $\mathfrak{D}_\alpha$ to itself. This is easily seen by plugging $f \circ m_a$ into the above norm, make a change of variables, and comparing with the norm of f. Boundedness will follow since the Jacobian, $|m_a'(z)|^2$, is bounded and since there exist $c > 0$ such that $(1-|z|^2) \le c(1-|m_a(z)|^2)$, for all $z \in \mathbb{D}$. We will denote the operator norm of the operation of composing with m_a simply by $\|m_a\|$.

We can now give a partial result, which in itself is mainly interesting since it shows a lot of the machinery that we have built up in action.

Lemma 9. *Given $f \in \mathfrak{D}_\alpha$ which extends continuously to $\overline{\mathbb{D}}$, which has the property that $Z(f) \cap \mathbb{T}^n$ contains a hyper-surface S, and that for some point $p \in S$, we have that $f_i'(p) \neq 0$ and $f_j'(p) \neq 0$. Then f is not cyclic in $\mathfrak{D}_\alpha$ for any $\alpha > 1/n$.*

Proof. We have two possibilities. Either S has type 2, or it does not have type 2. If S has type 2, then the statement follows immediately from Theorem 5.

Let S be parametrized by $(v_1, ..., v_{n-1}, g(v_1, ..., v_{n-1}))$. Assume now that $p \in S$ does not have type 2 and assume without loss of generality that $p = 0$. From the discussion in the section about *Parametrizations and the type of* $Z(f) \cap \mathbb{T}^n$, we can apply Möbius transformations to the parameters in the parametrization in order to obtain a new curve that is of type 2 at the point $p = 0$. This new curve will parametrize $Z(f) \cap \mathbb{T}^n$ for the function $f \circ m$, where $m(v) = (m_{a_1}(v_1), ..., m_{a_{n-1}}(v_{n-1}), v_n)$. By applying Theorem 5, we can conclude that $f \circ m$ is *not* cyclic for any $\alpha > 1/n$. This in turn implies that f is not cyclic, since assume that f was cyclic, then there exists a sequence of polynomials p_n for which

$$\lim_{n\to\infty} \|p_n f - 1\|_\alpha = 0.$$

But this is not possible since this would imply that

$$\|(p_n \circ m) \cdot (f \circ m) - 1\|_\alpha \leq \|m\| \|p_n f - 1\|_\alpha,$$

where the right hand side tends to zero as $n \to \infty$. But by approximating $p_n \circ m$ by polynomials q_n, this implies that

$$\lim_{n\to\infty} \|q_n f \circ m - 1\|_\alpha = 0,$$

and so $f \circ m$ would be cyclic, a contradiction. $\square$

The above result gives a fairly good condition for determining whether or not a function whose zero set is a surface or non-cyclic, but on its own it is still not entirely satisfactory. Mainly because it requires $Z(f) \cap \mathbb{T}^n$ to have real dimension $n - 1$, whilst the generic case is probably for the zero set to have as low dimension as possible, since we only allow it to hit the torus, but not enter the polydisc. Furthermore, the assumption that two partial derivatives are non-vanishing at some point is necessary for the proof to hold. The case, in which $Z(f) \cap \mathbb{T}^n$ has real dimension $n - 2$ is seemingly more difficult, but we can still say the following.

Lemma 10. *Given $f \in \mathfrak{D}_\alpha$ which extends continuously to $\overline{\mathbb{D}^n}$, which has the property that $Z(f) \cap \mathbb{T}^n$ contains a point p, such that*

$$(v_1, ..., h(v_1, ..., v_{n-2}), g(v_1, ..., v_{n-2}))$$

parametrizes $Z(f) \cap \mathbb{T}^n$ locally around p. and furthermore that no $(x, y) \in S^1$ has the property that

$$h_i'(0)x + g_i'(0)y = h_{i,j}''(0)x + g_{i,j}''(0)y = 0,$$

for all $1 \leq i, j \leq n - 2$. Then f is not cyclic in $\mathfrak{D}_\alpha$ for $\alpha > 1 - 1/n$.

The proof is analogous to the previous one, with the minor difference that one has to use Theorem 3 instead of Theorem 5, and that the requirements for forcing the set to be of type 2 changes.

Note that the criteria for non-vanishing derivatives concerns the function f in Lemma 9, but that the corresponding requirement for Lemma 10 regards the derivatives of the functions in the parametrization. In both cases, the requirements originates from the requirements for being able to force the parametrized set to be locally of type 2. In the first case, we saw that this was possible if and only if not all other derivatives of f vanished at the point. In the second case, this is not possible.

In general, one would want to find similar criteria for subsets of any real dimension. We can provide several polynomials whose cyclicity is still hard to determine.

Example 10. Once again, consider the polynomial

$$p(x, y, z) = 2 - xy - xz.$$

This polynomial does *not* satisfy the criteria of Lemma 10, and so we do not know anything regarding whether or not it is cyclic.

Furthermore, consider any polynomial in $\mathbb{C}[z_1, ..., z_n]$ of the form

$$p(z) = a - \sum_{i=1}^{l} c_i z^{b_i}, \tag{49}$$

where $b_i \in \mathbb{N}^n$ and $\sum c_i = a$ where c_i are positive real numbers.

A polynomial of the above form will not have any zeros in $\mathbb{D}^n$, and so is a candidate for being cyclic. Furthermore, since $Z(p) \cap \mathbb{T}^n$ is given by e^{iv} for which $p(e^{iv_1}, ..., e^{iv_n}) = 0$, we see that solving the equation

$$a - \sum_{i=1}^{l} c_i \cos(b_{i,1}v_1 + ... + b_{i,n}v_n) + i \sum_{i=1}^{l} c_i \sin(b_{i,1}v_1 + ... + b_{i,n}v_n) = 0,$$

will give a parametrization of the zero set. Note that for this particular kind of polynomials, the real part is zero if and only if *all* $\cos(b_{i,1}v_1 + ... + b_{i,n}v_n) = 1$, which means that *all* equations

$$b_{i,1}v_1 + ... + b_{i,n}v_n = 0,$$

must hold for $1 \leq i \leq l$. Furthermore, if the above equations hold, then the imaginary part will automatically be zero, and so the solutions to the above system of equations yields the entire zero set. Since the above system of equations can be arbitrarily big, we can not in general expect to parametrize the set with either $n-1$ or $n-2$ parameters. In fact, since we are free to choose $b_{i,j}$, we can create polynomials whose zero set has arbitrarily many parameters (between 0 and n of course). Furthermore, we can create polynomials whose zero sets are parametrized by

$$(v_1, ..., v_{n-2}, g(v_1, ..., v_{n-2}), h(v_1, ..., v_{n-2}))$$

where g and h are linear functions. Since all second derivatives of linear functions vanish, and since we can construct polynomials for which h and g are parallel, we see that the requirements on the parametrization in the previous lemma are *not* always automatically satisfied. However, it is still not clear if the fact that we can not force the set to be of type 2 by applying Möbius transformations means that the polynomial is cyclic, or if it is just that our current machinery is not powerful enough.

To round of this section, we will give examples which shows that the requirements we have for being able to force the zero set to be of type 2 are in fact necessary. That is, there are functions whose zero sets do not meet this criteria, and so we can not use the above machinery in order to determine non-cyclicity.

Example 11. Consider the following parametrization with $n-2$ parameters.

$$-v_n = -v_{n-1} = \sum_{i=1}^{n-2} v_i.$$

The corresponding equations are $b_1 = (1, ..., 1, 0, 1)$ and $b_2 = (1, ..., 1, 0)$, and so, the corresponding polynomials are

$$p(z) = a - c_1 z_1^1 \cdots z_{n-2}^1 z_n^1 - c_2 z_1^1 \cdots z_{n-1}^1,$$

where $a = c_1 + c_2$.

The above polynomial has a zero set whose intersection with $\mathbb{T}^n$ is parametrized with $n-2$ parameters. However, it is *not* possible to apply Möbius transformations in order to force it to be of type 2 locally.

Next, we will consider polynomials of the above form whose zero set is a hypersurface, but which do not satisfy the above criteria for determining non-cyclicity. Assume that

$$(v_1, ..., v_{n-1}, g(v_1, ..., v_{n-1}))$$

parametrizes the zero set of a polynomial on the distinguished boundary. If we are not able to apply the above results in order to determine non-cyclicity, then this means that $g_i' = 0$ for *all* $1 \leq i \leq n-1$ and *all* $v \in [-\pi, \pi]^{n-1}$. Since g is a linear function, this means that g is constant with respect to all variables, that is, it is constant. But for a polynomial of the above form, this is only possible if it does not depend on any of the variables $z_1, ..., z_{n-1}$. This means that any polynomial of the form (49) which depends on all variables, and for which $Z(p) \cap \mathbb{T}^n$ is a hypersurface, is non-cyclic for all $\alpha \geq 1/n$.

In general, we are only interested in polynomials which depend on all variables, since otherwise we can examine the same polynomial in a lower dimensional space and use that cyclicity is preserved under embedding into Dirichlet type spaces of more variables. It is now natural to ask if the above result holds for all polynomials which depend on all variables, and for which $Z(f) \cap \overline{\mathbb{D}}^n = Z(f) \cap \mathbb{T}^n \neq \emptyset$. That is, if $Z(f) \cap \mathbb{T}^n$ contains a hypersurface, will this imply that the polynomial is non-cyclic for $\alpha > 1/n$?

The above problems do not occur in lower dimensions. For $n = 2$, we have that either the zero set is finite, which we can handle in general, or it can be parametrized with $n-1 = 1$ parameters, which is also a fairly nice situation.

3.8 Polynomials in $\mathbb{C}[z_1, z_2, z_3]$

The previous section indicates that cyclicity for polynomials is more difficult in higher dimensions than in 2 dimensions. In 2 dimensions, we always have that the zero set is finite, or a hypersurface, both of which are situations that can be handled even in higher dimensions. The problem in higher dimensions is mainly related to the fact that there are far more possibilities for the dimension of the zero set, namely the zero set can be an embedded manifold of every dimension less than or equal to n, not only a finite set and a hypersurface, and all of these different possibilities seem to behave differently. The goal of this section is to consider the special kind of polynomials constructed in the previous section, and determine cyclicity for them. In general, these polynomials will either have a finite zero set, a one-dimensional zero set, or a two dimensional zero set. If it is finite, then we now that they are all cyclic. If it is two dimensional, then we know that they are never cyclic for $\alpha > 1/3$. But what happens if it is one-dimensional?

One tool which we will use in order to determine this is by comparing our polynomials to other polynomials whose cyclicity properties are already known. The following proof is similar to the final arguments in the proof of Theorem 10. A proof for a slightly more general statement for the Dirichlet space in one dimension was originally given in [1].

Lemma 11. *Let f and $g \in \mathfrak{D}_\alpha$ be polynomials such that $|f(z)| > |g(z)|$ for $z \in \mathbb{D}^n$, and where g is cyclic in $\mathfrak{D}_\alpha$ for $\alpha \leq 1$. Then f is cyclic in $\mathfrak{D}_\alpha$.*

Proof. Since $|f| > |g|$, we have that g/f is bounded and continuous in $\overline{\mathbb{D}}^3$. In particular, this implies that g^2/f is continuously differentiable, and so, by integrating by parts and applying Parsevals equality, we see that $g^2/f \in \mathfrak{D}_\alpha$ for all $\alpha \leq 1$. This means that there exists $h \in \mathfrak{D}_\alpha$ such that $g^2 = fh$. Denote by p_n the partial sums of the Fourier series of h. We get that

$$\|g - fp_n\| \leq \|g - fh\| + \|fh - fp_n\| \leq c\|h - h_n\|,$$

where c is the multiplier norm of f (recall that f is a polynomial and that polynomials are multipliers). The right hand side tends to zero has $n \to \infty$ since $h \in \mathfrak{D}_\alpha$, and so $g^2 \in [f]$. Since $g^2 = g \cdot g$, and g is cyclic, it follows that g^2 is cyclic. This implies that $[f]$ contains a cyclic function, and so f is cyclic. $\square$

In the previous section, we discovered a large class of polynomials for which our previous methods did not apply. In what follows, we will try to thoroughly understand these specific polynomials.

The polynomials we are interested in are polynomials of the form

$$p(z) = a - \sum_{i=1}^{k} c_i z^{b_i} \tag{50}$$

where $b_i \in \mathbb{N}^3$ and c_i are positive real numbers whose sum equals a. These polynomials have no zeros inside the tridisc, and e^{iv} is a zero on $\mathbb{T}^3$ if and only if (v_1, v_2, v_3) is a solution to

$$b_{i,1}v_1 + b_{i,2}v_2 + b_{i,3}v_3 = 0$$

for all i. In this context, we are primarily interested in the situation where $Z(f) \cap \mathbb{T}^3$ is parametrized by one parameter, that is polynomials on the above form for which $Z(f) \cap \mathbb{T}^3 = (e^{it}, e^{ibt}, e^{ict})$, and where b and c are rational numbers (as a consequence of $b_i \in \mathbb{N}^3$). Furthermore, we can assume without loss of generality that both b and c are negative rational numbers. If they were both positive, then (v_1, bv_2, cv_3) cannot possibly be a solution to the above system of equations. If only one of them is non-negative, then we can get the desired situation by simply switching parameter.

Why is this of interest? What we will now try to accomplish is to compare polynomials in three variables whose zero set is a line, with polynomials in two variables, and then use the Lemma 11 to determine cyclicity. In the general situation this is hard do achieve directly, so what we do is the following. Every line which is the zero set of a polynomial in three variables is also the zero set of a polynomial of the form

$$p(x, y, z) = 2 - x^a y^b - x^c z^d \tag{51}$$

Assume that the zero set is parametrized by $(v_1, -bv_1/a, -dv_1/c)$, with $a, b, c, d \in \mathbb{N}$. Then the above polynomial will have the same zero set, and by the previous discussion the only lines that are of relevance are the lines of that form.

This can also be seen through basic linear algebra. When solving the system of equations which determine the zero set of a polynomial of the form (50), we end up with a system of linear equations with integer coefficients. Every term $kx^a y^b z^c$ contribute with an equation

$$ax + by + cz = 0$$

to the system of linear equations.

If we assume that the zero set is a curve, we know that we will have a one-parameter solution to the system of equations, and so we can use Gauss elimination with the first two equations (corresponding to the first two terms) and clear all other equations. But the system of equations we end up with then will have a zero set coming from a polynomial with only two terms.

Example 12. Consider the polynomial

$$3 - 2xyz - x^2yz/2 - x^3y^2z^2/2.$$

The zero set of this polynomial on $\mathbb{T}^3$ will be parametrized by the solution to the system of equations

$$x + y + z = 2x + y + z = 3x + 2y + 2z = 0.$$

After Gauss elimination, we get that

$$x = y + z = 0,$$

which is the corresponding system of equations for the polynomial

$$1 - x/2 - yz/2.$$

We give one more example which is slightly more complicated.

Example 13. Consider the polynomial

$$3 - x^2yz - xy^2z - x^3y^3z^2.$$

The corresponding system of equations is given by

$$2x + y + z = x + 2y + z = 3x + 3y + 2z = 0,$$

which after Gauss elimination becomes

$$3y + z = -x + z = 0,$$

which is *not* a system of equations coming from one of our polynomials, since we have a minus sign.

However, the zero set is parametrized by $(t, -3t, t)$, and by setting $v = -3t$, we get a parametrization of the form $(-v/3, v - v/3)$. But this zero set is clearly the zero set of the system of equations

$$y = -x/3, \quad y = -z/3 \iff x + 3y = z + 3y = 0,$$

which is the system of equations from the polynomial

$$2 - xy^3 - zy^3.$$

Since every polynomial in our class has the same zero set as a polynomial on the above form, it suffices to understand cyclicity of these polynomials (the details of how this comparison will be carried out will be made clear later).

Now, these polynomials are of a particularly easy form. What we will do is to generalize the *lift* which was used earlier. Generalizing it in such a way to cover all polynomials is probably not possible, due to technical reasons which will become apparent later. But for the lifts associated with polynomials of the above form, a lot can be said.

Definition 5. *For $k = (a, b, 0)$ and $l = (c, 0, d)$, with $a, b, c, d \in \mathbb{N}$, we introduce the* 2-3 lift $L_{k,l}$ *defined by $L_{k,l}(f(x, y))(z) = f(z^k, z^l)$, where x, y and $z \in \mathbb{D}$.*

For these lifts, we can say the following

Lemma 12. *Let f be a holomorphic function on the bidisc, and $L_{k,l}$ be a 2-3 lift. Then*

$$c\|f\|_{3\alpha/2} \leq \|L_{k,l}(f)\|_\alpha \leq C\|f\|_{2\alpha},$$

if $\alpha \geq 0$, and

$$C\|f\|_{3\alpha/2} \geq \|L_{k,l}(f)\|_\alpha \geq c\|f\|_{2\alpha},$$

if $\alpha < 0$.

In this section, we are primarily interested in positive α, however, the result for negative α will be applied later.

Proof. We know that

$$f(x, y) = \sum_{m,n\in\mathbb{N}} \hat{f}(k, l)x^m y^n,$$

and so

$$L_{k,l}(f)(z) = \sum_{p\in\mathbb{N}^3} A_p z^p = \sum_{m,n\in\mathbb{N}} \hat{f}(k, l)z^{mk+nl} = \sum_{m,n\in\mathbb{N}} \hat{f}(k, l)z^{(ma+nc,bm,cn)}.$$

It follows that $A_p = \hat{f}(m, n)$ if $p = mk + ln$ for $m, n \in \mathbb{N}$, and 0 if there are no such m, n. This is one of the reasons for why we need to restrict ourselves to particular k and l. For arbitrary choices, there might not be a one to one correspondence.

For $\alpha \geq 0$, we have that

$$\begin{aligned}
\|L_{k,l}(f)\|_\alpha^2 &= \sum_{p\in\mathbb{N}^3} |A_p|^2(1 + p_1)^\alpha(1 + p_2)^\alpha(1 + p_3)^\alpha \\
&= \sum_{m,n\in\mathbb{N}} |\hat{f}(m, n)|^2(1 + ma + nc)^\alpha(1 + mb)^\alpha(1 + nc)^\alpha \\
&\leq \sum_{m,n\in\mathbb{N}} |\hat{f}(m, n)|^2(1 + ma)^\alpha(1 + nc)^\alpha(1 + mb)^\alpha(1 + nc)^\alpha \\
&\leq C \sum_{m,n\in\mathbb{N}} |\hat{f}(m, n)|^2(1 + m)^{2\alpha}(1 + n)^{2\alpha} = C\|f\|_{2\alpha}^2.
\end{aligned}$$

By using that $(1+ma+nc) = ((1+2ma)+(1+2nc))/2 \geq (1+2ma)^{1/2}(1+2nc)^{1/2}$ in the above calculations instead of replacing it by $(1+ma)^{\alpha}(1+nc)^{\alpha}$, we get that

$$c\|f\|_{3\alpha/2}^2 \leq \|L_{k,l}(f)\|_{\alpha}^2.$$

Finally, for negative α the above inequalities are just reversed.

This finishes the proof.

$\square$

This result has several interesting implications for our specific polynomials. In particular, we can say the following.

Lemma 13. *Every polynomial in* $\mathbb{C}[z_1, z_2, z_3]$ *of the form*

$$p(z_1, z_2, z_3) = 2 - z_1^a z_2^b - z_1^c z_3^d$$

is cyclic in $\mathfrak{D}_{\alpha}^3$ *for all* $\alpha \leq 1/2$.

Proof. A polynomial of the above form is simply a *2-3 lift* of the polynomial $2 - x - y$. This polynomial has only one zero on $\mathbb{T}^2$, and therefore is cyclic for all $\alpha \leq 1$. This means that there exists a sequence of polynomials q_n such that

$$\lim_{n\to\infty} \|p_n(2 - x - y) - 1\|_1 = 0.$$

By applying the previous lemma, this implies that

$$\lim_{n\to\infty} \|L_{l,k}(q_n)p - 1\|_{1/2} = 0,$$

and so p is cyclic for all $\alpha \leq 1/2$. $\square$

The motivation for studying this particular kind of polynomials is because every line which is the zero set of *any* polynomial of the form

$$a - \sum_{i=1}^{n} c_i z^{b_i},$$

for $c_i > 0$ such that $\sum c_i = a$, is also the zero set of one of the easier polynomials of the form (51). This can be used to compare every polynomial with one of the easier polynomials.

For any polynomial $q(z_1, z_2, z_3)$, with no zeros in $\mathbb{D}^3$, and whose zero set on $\mathbb{T}^3$ is a curve, there is a polynomial of the form

$$p(z_1, z_2, z_3) = 2 - z_1^a z_2^b - z_1^c z_3^d,$$

such that $Z(q) \cap \mathbb{T}^3 = Z(p) \cap \mathbb{T}^3$. And furthermore there exists $c > 0$ and $d \in \mathbb{N}$ such that

$$|q(z)| > c|p(z)|^d,$$

everywhere in $\mathbb{D}^3$.

This can be seen by splitting $\mathbb{D}^3$ up into parts, one neighborhood u close to $Z(p) \cap \mathbb{T}^3$ in which p is smaller than ϵ with $1 > \epsilon > 0$, and the complement of this neighborhood. First, choose d large enough so that the inequality $|q(z)| > |p(z)|^d$ holds everywhere in this neighborhood u. Next, choose $0 < c < 1$ such that $|q(z)| > c|p(z)|^d$ holds everywhere in $\mathbb{D}^3$. This is possible since we have a continuous function on a compact set, since once again, the polynomials are continuous on $\overline{\mathbb{D}}^3$, which is compact.

We are now ready to prove the main results of this section.

Lemma 14. *Every polynomial in $\mathbb{C}[z_1, z_2, z_3]$ of the form*

$$p(z) = a - \sum_{i=1}^{n} c_i z^{b_i},$$

for $c_i > 0$ such that $\sum c_i = a$, for which $Z(f) \cap \mathbb{T}^3$ is a curve, is cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \leq 1/2$.

Proof. Let $p(z)$ be a polynomial with the above properties. Then there exists a polynomial $q(z)$ of the form

$$q(z) = 2 - z_1^a z_2^b - z_1^c z_3^d,$$

such that $Z(q) \cap \mathbb{T}^3 = Z(p) \cap \mathbb{T}^3$. By the previous lemma $q(z)$ is cyclic for all $\alpha \leq 1/2$. It follows that $cq(z)^d$ is cyclic for all $\alpha \leq 1/2$. Since c and d can be chosen such that

$$|p(z)| > c|q(z)|^d$$

for all $z \in \mathbb{D}^3$, the first lemma of this section implies that $p(z)$ is cyclic for all $\alpha \leq 1/2$.

This finishes the proof.

□

We can actually use the same technique as above in order to determine cyclicity for polynomials, $q(z_1, z_2, z_3)$, of the above form whose zero set is a hyperplane. In fact, it is actually easier, since in this case, the zero set will be of the form $(e^{iv_1}, e^{iv_2}, e^{i(-av_1/c - bv_2/c)})$, where a, b and c are non-negative integers (well, c is even positive). It is easily seen that the polynomial

$$p(z_1, z_2, z_3) = 1 - z_1^a z_2^b z_3^c,$$

has the same zero-set. But this polynomial is simply a standard lift of the polynomial $1 - z$, which we know to be cyclic for all $\alpha \leq 1$. It follows that $p(z)$ is cyclic for all $\alpha \leq 1/3$. By the same arguments as above, we can find $c > 0$ and $q \in \mathbb{N}$ such that $|q(z)| > |p(z)|$ for all $z \in \mathbb{D}^3$, which in turn implies that q is cyclic for all $\alpha \leq 1/3$. We state this as a theorem.

Theorem 12. *Every polynomial in $\mathbb{C}[z_1, z_2, z_3]$ of the form*

$$p(z) = a - \sum_{i=1}^{n} c_i z^{b_i},$$

for $c_i > 0$ such that $\sum c_i = a$, for which $Z(f) \cap \mathbb{T}^3$ is a hyper-surface, is cyclic in $\mathfrak{D}_\alpha$ for all $\alpha \leq 1/3$.

In conclusion, we now know the following about polynomials of the form

$$p(z) = a - \sum_{i=1}^{n} c_i z^{b_i},$$

for which all variables are represented.

(1) If $Z(p) \cap \mathbb{T}^3$ is a finite set, then $p(z)$ is cyclic for all $\alpha \leq 1$.

(2) If $Z(p) \cap \mathbb{T}^3$ is a curve, then $p(z)$ is cyclic for all $\alpha \leq 1/2$.

(3) If $Z(p) \cap \mathbb{T}^3$ is a hyper-surface, then $p(z)$ is cyclic if and only if $\alpha \leq 1/3$.

We end this section by discussing the remaining problems in order to obtain a full characterization of the cyclic polynomials in 3 variables.

First of, even for the class of polynomials which we have primarily worked with so far, we do not yet know anything about when we can say for sure that a polynomial for which $Z(p) \cap \mathbb{T}^3$ is *not* cyclic. It is worth emphasizing that the previous results regarding non-cyclicity of functions whose zero set has real dimension $n - 2$ is *never* applicable for these polynomials. Essentially since we have only one equation which must hold in order to conclude that we do not have type 2, and we have one degree of freedom for points in S^1. However, if one tries to generalize this to higher dimensions, then that result might be useful, since we will then have a lot of equations which must hold in order for the curve not to be of type 2, but we still only have one degree of freedom.

Second, it is still not clear if these methods are possible to generalize to all polynomials that are candidates for being cyclic. If it turns out that the zero set of the general polynomial is a union between straight lines, points, and hyper-planes, then it should be possible to find a polynomial which is a product of polynomials which we understand well, whose zero set is the same as that of the general polynomial. In this case, we can use the above results in order to determine cyclicity for each factor, and then finally use the above methods to compare our well understood polynomial with the general one. There is however, a possibility that the zero set is not a union of points, straight lines, and hyperplanes, in which case the characterization will be much harder.

3.9 Integration currents and $Z(f) \cap \mathbb{T}^3$

The goal of this section is to use find conditions under which polynomials for which $Z(f) \cap \mathbb{T}^3$ is a curve, can be determined to be non-cyclic. The idea is the following.

Recall from Section 2.5, that if there exist a measure μ, whose support is contained in $Z(f) \cap \mathbb{T}^3$, for which $C[\mu] \subset \mathfrak{D}_{-\alpha}$, then f is *not* cyclic in $\mathfrak{D}_\alpha$. As usual, $C[\mu]$ denotes the Cauchy transform of the measure μ. Also, note that if $C[\mu] \in \mathfrak{D}_{-\alpha}$, then $C[\mu] \in \mathfrak{D}_{-\beta}$ for all $\beta > \alpha$, and so f will not be cyclic in $\mathfrak{D}_\beta$ for any $\beta > \alpha$.

We will mainly restrict our attention to a certain type of measure supported on $Z(f) \cap \mathbb{T}^3$, and attempt to conclude for which α we have that $C[\mu] \in \mathfrak{D}_{-\alpha}$ for these specific measures.

Given $Z(f) \cap \mathbb{T}^3$ which can be parametrized by $(e^{iv_1}, e^{ibv_1}, e^{icv_1})$, where b and c are negative rational numbers, we consider the probability measure μ that has constant density on $Z(f) \cap \mathbb{T}^3$ with respect to the pullback of the Lebesgue measure, and whose density is zero everywhere else. Note that since b and c are rational numbers, we only need to go a finite number of periods (of v_1) until we have covered the entire zero set. In particular, this is important since this curve will not be space filling or anything unpleasant. Furthermore, for the sake of making the calculations easier, we note that the zero set can be parametrized by $(e^{ias}, e^{ibs}, e^{ics})$ where $a, b, c \in \mathbb{Z}$ and where $s \in [0, 2\pi)$.

Example 14. Not all polynomials in our family have a zero set of this form, since it may be the case that a, b, c or d is zero. For example, the zero set of

$$p(x, y, z) = 2 - x - yz$$

is given by $(0, v, -v)$.

However, since this zero set is truly two dimensional, one might expect this polynomial to behave like a polynomial in two variables. Although, nothing has been proved yet.

But moving on, for the above parametrization of the zero set, the Cauchy transform of μ is given by

$$C[\mu](z_1, z_2, z_3) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{(1 - e^{ias} z_1)(1 - e^{ibs} z_2)(1 - e^{ics} z_3)} ds.$$

We illustrate this method through an example. The general case for our special polynomials is in fact shown in more or less exactly the same way.

Example 15. Consider the polynomial

$$p(z_1, z_2, z_3) = 2 - z_1 z_2 - z_1 z_3.$$

The zero set of this polynomial is parametrized by $(s, -s, -s)$, and so, the corresponding Cauchy transform is given by

$$C[\mu](z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{(1-e^{is}z_1)(1-e^{-is}z_2)(1-e^{-is}z_3)}ds.$$

By replacing the factors in the integral by their power series expansions and changing orders of integration and summation, we get that the Fourier coefficients of $C[\mu]$ are given by

$$\widehat{C}(k,l,m)z_1^k z_2^l z_3^m = \frac{1}{2\pi}\int_0^{2\pi} e^{iks}z_1^k e^{-ils}z_2^l e^{-ims}z_3^m ds.$$

By orthonormality, we have that the above expression equals $z_1^k z_2^l z_3^m$ if $k = l + m$ and zero otherwise. It follows that

$$C[\mu](z_1, z_2, z_3) = \sum_{l,m\in\mathbb{N}} (z_1 z_2)^l (z_1 z_3)^m = \frac{1}{1 - z_1 z_2}\frac{1}{1 - z_1 z_3}.$$

If the above function lies in $\mathfrak{D}_{-\alpha}$, then $C[\mu]$ will induce a bounded linear functional on $\mathfrak{D}_\alpha$ which annihilates the entire span of $p(z)$, and so $p(z)$ is not cyclic in $\mathfrak{D}_\alpha$.

We have that $C[\mu]$ lies in $\mathfrak{D}_\alpha$ if and only if

$$\|C[\mu]\|_\alpha^2 = \sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(1+l+m)^\alpha(1+m)^\alpha(1+l)^\alpha < \infty.$$

The above series is clearly divergent if α is non-negative, so from here on we assume that $\alpha < 0$.

By symmetry, the above series is finite if and only if

$$\sum_{l=0}^{\infty}(1+l)^\alpha \sum_{m=0}^{l}(1+l+m)^\alpha(1+m)^\alpha < \infty.$$

Since $m \leq l$, we have that

$$\sum_{m=0}^{l}(1+l+m)^\alpha(1+m)^\alpha \geq (1+2l)^\alpha \sum_{m=0}^{l}(1+m)^\alpha \geq c(1+2l)^\alpha(1+l)^{\alpha+1}.$$

By plugging this into the above series, we see that

$$\sum_{l=0}^{\infty}(1+l)^\alpha \sum_{m=0}^{l}(1+l+m)^\alpha(1+m)^\alpha \geq c\sum_{l=0}^{\infty}(1+l)^\alpha(1+2l)^\alpha(1+l)^{\alpha+1},$$

and the last expression is finite if and only if $3\alpha + 1 < -1$, which implies that $\alpha < -2/3$.

In fact, by using that $(1 + l + m)^\alpha < (1 + l)^\alpha$ instead, we get that

$$\sum_{l=0}^{\infty}(1+l)^{2\alpha}\sum_{m=0}^{l}(1+m)^\alpha \leq C\sum_{l=0}^{\infty}(1+l)^{3\alpha+1},$$

which is finite if and only if $\alpha < -2/3$.

It follows that $C[\mu] \in \mathfrak{D}_\alpha$ if (and only if) $\alpha \leq -2/3$, and so $2 - z_1z_2 - z_1z_3$ is *not* cyclic for any $\alpha > 2/3$.

We know from earlier results that this polynomial is cyclic if $\alpha \leq 1/2$, however, we do not yet know what happens for $\alpha \in (1/2, 2/3]$.

Independently, the above example shows that there is indeed harder for a polynomial whose zero set is a curve to be cyclic than it is for a polynomial whose zero set is just a finite set.

But how does this method work for other polynomials whose zero set can be parametrized as $(as, -bs, -cs)$ where $a, b, c \in \mathbb{N}$? Well, actually exactly the same. The Cauchy transform of the corresponding measure has previously been expressed, and by once again using the power series expansion av each factor, and then change orders of integration and summation, we get that $C[\mu] \in \mathfrak{D}_\alpha$ if and only if

$$\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}(1+\frac{b}{a}l+\frac{c}{a}m)^\alpha(1+m)^\alpha(1+l)^\alpha < \infty.$$

Clearly, the same results regarding convergence will hold, we can in fact use the same approximations. We state this as a theorem.

Theorem 13. *Every polynomial in* $\mathbb{C}[z_1, z_2, z_3]$ *of the form*

$$p(z) = a - \sum_{i=1}^{n} c_i z^{b_i},$$

for $c_i > 0$ *such that* $\sum c_i = a$*, for which* $Z(f) \cap \mathbb{T}^3$ *is a 3 dimensional curve, is non-cyclic in* $\mathfrak{D}_\alpha$ *for all* $\alpha > 2/3$*.*

As previously stated, we have now shown cyclicity for all $\alpha \leq 1/2$, and shown non-cyclicity for all $\alpha > 2/3$ for this special family of polynomials. But what happens in between?

Chapter 4

Summary and some open problems

In this thesis, we have primarily attempted to generalize results regarding cyclicity in Dirichlet type spaces in one and two variable up to higher dimensions. By using a method originally developed in [1] , we started by proving that a function with a large zero set on the distinguished boundary in the sense of having positive Riesz capacity, cannot be cyclic in certain Dirichlet type spaces. Although this result is very nice and quite general, it is unfortunately very difficult to check whether or not a set has positive Riesz capacity or not. To remedy this, we use known results from the theory of oscillatory integrals in order to show that a sufficient condition for a set to have positive Riesz capacity is for it to contain a point with non-vanishing Gaussian curvature. This idea was originally developed in [2] with the purpose of determining cyclicity for polynomials in two complex variables. However, whereas in two dimensions, every submanifold of the zero set on the distinguished boundary will have dimensions one or zero, there are far more possibilities in higher dimensions. When generalizing this result to arbitrary dimension, we get two different results, one weak result which works for submanifolds of any dimension, and one strong result which only works for hypersurfaces. But it is especially when trying to apply these results to concrete functions that things start to behave differently from the situation in two variables. In two variables, every polynomial whose zero set was a curve could be compared to a function whose zero set has a point of non-vanishing Gaussian curvature, but this turns out to be false in higher dimensions. The strong result which is related to functions whose zero set is a hypersurface is usually applicable, all we require is that the zero set contains a point in which two partial derivatives do not vanish. However, for polynomials whose zero set is a curve, we can not apply the above theorem for any polynomial whose zero set is in some sense flat. Furthermore, this condition turns out not to be empty, in fact, we can construct a rather large family of polynomials for which this condition does not hold.

In order to understand cyclicity of this particular family of polynomials for which the previously established methods do not work, we are forced to develop new methods. We begin by generalizing a result from [2] regarding cyclicity for finite zero sets from two variables to any dimension. By using the same proof idea, we show that every polynomial whose zero set consists of finitely many points is cyclic for all α. After that, we notice that all polynomials in this family whose zero set is either a curve or a hypersurface can be compared to other, slightly simpler polynomials in this family. In order to understand these polynomials, we generalize the concept of a lift developed in [3], and through this method it becomes possible to compare these polynomials to polynomials in lower dimension. In the situation where the zero set is a hypersurface, this method together with the methods using curvature gives us sharp results regarding cyclicity. But for the curves, the results we obtain are probably not sharp. Nevertheless, they are still very interesting, because the results we obtain show us that the cyclicity properties are truly different for polynomials whose zero sets have different dimensions! Finally, for the polynomials in our family whose zero set is a three dimensional flat curve, we explicitly construct bounded linear functionals which annihilate the entire generated set of the polynomial. We do this by applying Cauchy transforms of induced Lebesgue measures on the zero set, thus proving non-cyclicity for certain parameter values.

Several questions still remain unanswered. The most obvious question is whether or not the bound on cyclicity for our special polynomials whose zero set is a curve, is in fact sharp. Another thing which is interesting to understand is how big a constriction the requirements for applying the method of non-vanishing Gaussian curvature really are. The requirements clearly impose severe restrictions on how the curve may look, more precisely, it demands that the second derivative is parallel to the first derivative in all points. When showing non-cyclicity for the polynomials with flat zero set, we never actually used that they were polynomials, just that the zero set could be parametrized by a straight line with rational coefficients. If it turns out that the only zero sets of polynomials which do not satisfy the curvature conditions are the flat zero set, then this will imply that we already have sufficient tools for completely understanding the situation. This is probably very optimistic, but it will for sure help our understanding of the problem to understand in which situations our current methods are applicable.

Chapter 5

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