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Asymptotic zero distributions of iterated derivatives of certain meromorphic functions

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Abstract

We study the asymptotic zero distributions of the sequences $\left\{\frac{\mathrm{d}^{\lfloor \alpha n \rfloor}}{\mathrm{d}z^{\lfloor \alpha n \rfloor}}(R(z)^n)\right\}$ and $\left\{\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(R(z)e^{T(z)}\right)\right\}$, where $R(z)$ is a rational function, $T(z)$ is a polynomial, and $\alpha > 0$ is a real number. Additionally, some numerical results on the above zero distributions are presented along with related conjectures.

Acknowledgments

I would like to thank my supervisor Rikard Bögvad for his insight, patience, advice and support, and for promoting my interest in zero distributions of sequences of polynomials.

Contents

1 Introduction

In 1816, the future French banker, mathematician and social reformer Olinde Rodrigues discovered what is now called *Rodrigues' formula* as part of his doctoral dissertation in mathematics. His formula states that the n:th *Legendre polynomial* $P_n(z)$ is given explicitly by

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left[\left(z^2 - 1 \right)^n \right]. \tag{1}$$

(Recall that Legendre polynomials appear in many areas of physics and mathematics.) In this thesis, a derivation of Rodrigues' formula is presented in section 2, which provides an elementary introduction to the classical orthogonal polynomials. Some results from a recent paper by Rikard Bögvad, Boris Shapiro and this author (see [7]) are derived and discussed in section 3, in which the asymptotic zero distribution of $\frac{\mathrm{d}^{\lfloor \alpha n \rfloor}}{\mathrm{d}z^{\lfloor \alpha n \rfloor}}(R(z)^n)$, for rational functions R and real numbers $\alpha > 0$, is investigated. Furthermore, the results of some numerical studies inspired by this paper are presented in section 3.4, which will hopefully motivate further research. Finally, a generalization of the main result in a recent paper by Rikard Bögvad and this author (see [6]) to a larger class of meromorphic functions is proven in section 4.

2 Legendre polynomials and Rodrigues' formula

In this section, we explain a connection between the Legendre polynomials and the classical theory of electromagnetism. Furthermore, we derive an explicit formula for their calculation, known as *Rodrigues' formula*, and explain why they are called orthogonal.

2.1 Electrostatic potentials

Consider two point charges q_1 and q_2, each at rest and a distance r apart. Coulomb's law, first announced by the French physicist Charles Augustin de Coulomb on experimental grounds in 1784, states that the electrostatic force $\vec{\mathbf{F}}$ acting on q_2 from q_1 is given by

$$\vec{\mathbf{F}} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\vec{\mathbf{r}}, \tag{2}$$

where ε_0 is the electric constant (called vacuum permittivity), and $\vec{\mathbf{r}}$ is a unit vector pointing toward (away from) q_1 if the two charges have different (the same) signs. (By Newton's third law, the force acting on q_1 from q_2 equals $-\vec{\mathbf{F}}$.)

Now, assume that $q_1 > 0$ is placed at the origin, and that $q_2 > 0$ is placed infinitely far away from the origin, so that it does not feel any electric force from q_1. The energy W required to bring q_2 to a distance r away from q_1 along some path C is given by

$$W = -\int_C \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{s}} = -\int_\infty^r (|\vec{\mathbf{F}}|\cos\theta)\,\mathrm{d}s, \tag{3}$$

where $\vec{\mathbf{F}}$ is the electric force from q_1 acting on q_2, and θ is the angle between the force vector and the displacement vector $\mathrm{d}\vec{\mathbf{s}}$[19]. (The minus sign here is due to the fact that $\vec{\mathbf{F}}$ is equal in magnitude but opposite in direction to the external force.) Since (time-invariant) electric fields are conservative (see [30]), any path C in (3) with prescribed start and end points can be chosen. Thus, we can choose the path to be inward along the radial direction, $\cos\theta = -1$, and $\mathrm{d}s = -\mathrm{d}r$. Since $\vec{\mathbf{F}}$ is given by equation (2), it follows that the energy required is then given by

$$W = \frac{-q_1 q_2}{4\pi\varepsilon_0}\int_\infty^r \frac{1}{r^2}\,\mathrm{d}r = \frac{q_1 q_2}{4\pi\varepsilon_0 r}. \tag{4}$$

The last expression in (4) is known as the *electric potential energy* of the two point charges q_1 and q_2.

In general, if we consider a point charge q_0 in some electric field $\vec{\mathbf{E}}$ generated by one or more charges, the force acting on q_0 is equal to $q_0\vec{\mathbf{E}}$. In this case, the electric potential energy PE_E associated with q_0 and $\vec{\mathbf{E}}$ is given by $PE_E = q_0 V(r)$, where $V(r)$ is a function that depends only on the other charges and their distribution in space. Thus, the function $V(r)$ represents the electric potential energy per unit charge, and is known as the *electric potential* or *potential* (associated with q_0 and $\vec{\mathbf{E}}$). Here, q_0 is the charge located at the position where the potential is being determined. If we consider the special case with two point charges in equation (4) (and label $q_1 = q'$ and $q_2 = q_0$), the equation $PE_E = q_0 V(r)$ reduces to

$$V(r) = \frac{q'}{4\pi\varepsilon_0 r}, \tag{5}$$

which is the electric potential of a point charge q' at the origin. This potential is known as the *Coulomb potential* in $\mathbb{R}^3$. In general, the Coulomb potential in $\mathbb{R}^n$ at a distance r from the point charge q' at the origin is given by

$$V(r) = K_n q' \frac{\mathrm{d}^{n-2}}{\mathrm{d}r^{n-2}} \log r, \qquad \text{if } n \geq 2, \tag{6}$$

where K_n is a real number (see [9, 10]). Thus, in the plane, the Coulomb potential is logarithmic, which is related to the origins of potential theory (i.e. the study of harmonic functions; see [11]) in 19th century physics, and in particular to the logarithmic potentials we will see in sections 3 and 4.

2.2 Legendre polynomials

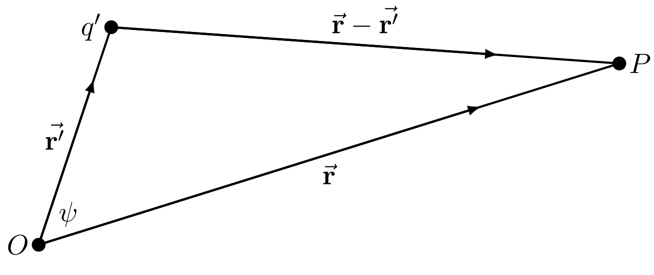

Figure 1: A potential situation giving rise to Legendre polynomials.

In this and the next sections, we will closely follow the exposition in *The Legendre Polynomials* (see [18]). Consider the situation in Figure 1, where the charge q' has been moved away from the origin O to some new position $\vec{\mathbf{r}'}$. The potential at the point P with position $\vec{\mathbf{r}}$ is then, by equation (5), given by

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{|\vec{\mathbf{r}} - \vec{\mathbf{r}'}|}\right) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{(r^2 - 2\vec{\mathbf{r}} \cdot \vec{\mathbf{r}'} + r'^2)^{1/2}}\right). \tag{7}$$

If $r > r'$ and ψ denotes the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{r}'}$, it follows that

$$V(\vec{\mathbf{r}}) = \frac{q'}{4\pi\varepsilon_0 r}\left(\frac{1}{\{1 - 2(r'/r)\cos\psi + (r'/r)\}^{1/2}}\right). \tag{8}$$

By letting $z := \cos\psi$ and $t := r'/r$, equation (8) becomes

$$V(\vec{\mathbf{r}}) = \frac{q'}{4\pi\varepsilon_0 r}(1 - 2zt + t^2)^{-1/2}. \tag{9}$$

For a given value of z, the second factor in (9) can be expanded in terms of powers of t:

$$g(z,t) := (1 - 2zt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)t^n \tag{10}$$

The coefficients $P_n(z)$ in (10) are polynomials of order n, known as *Legendre polynomials*, whose generating function is $g(z,t)$. These polynomials can also be shown to be solutions to Legendre's differential equation

$$(1 - z^2)\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - 2z\frac{\mathrm{d}y}{\mathrm{d}z} + n(n+1)y = 0, \tag{11}$$

which, for instance, arises when solving Laplace's equation in spherical coordinates[8]. (Here, any fixed $n \in \mathbb{N}$ gives rise to a solution $y(z) = P_n(z)$.) They also arise as the eigenfunctions in the solution of the three-dimensional time-independent Schrödinger equation[20] and in many other areas of physics.

It should be noted here that the general solution of (11) is

$$y_n(z) = AP_n(z) + BQ_n(z),$$

where A and B are constants, and $Q_n(z)$ are *Legendre functions of the second kind*, whose generating function is

$$\sum_{n=0}^{\infty} Q_n(z)t^n = (1 - 2tz + t^2)^{-1/2}\log\left(\frac{z - t + \sqrt{1 - 2tz + t^2}}{\sqrt{z^2 - 1}}\right).$$

In general, the functions $Q_n(z)$ are not polynomials, but they still admit various elegant representations (e.g. in terms of integrals of Legendre polynomials) when restrictions on $z \in \mathbb{C}$ are used (see [25]).

Recall that the general binomial theorem (first discovered by Newton around 1665 and rigorously proven by Gauss in 1812) states that

$$(1+y)^\alpha = 1 + \alpha y + \frac{\alpha(\alpha-1)}{2!} y^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} y^3 + \cdots \tag{12}$$

for real numbers α and y with $|y| < 1$. By letting $y = -2zt + t^2$ and $\alpha = -1/2$, equation (12) becomes

$$\begin{aligned}(1-2zt+t^2)^{-1/2} &= 1 + (-1/2)(-2zt+t^2) + \frac{(-1/2)(-3/2)}{2!}(-2zt+t^2)^2 \\ &\quad + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-2zt+t^2)^3 + \cdots\end{aligned} \tag{13}$$

By comparing the coefficients of t^n in equations (10) and (13), the first four Legendre polynomials are evidently $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = (3z^2-1)/2$, and $P_3(z) = (5z^3-3z)/2$. In general,

$$\sum_{n=0}^{\infty} P_n(z) t^n = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2n-1}{2}\right)}{n!}(-2zt+t^2)^n,$$

or,

$$\sum_{n=0}^{\infty} P_n(z) t^n = \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}(2zt-t^2)^n. \tag{14}$$

By writing the coefficient of $(2zt-t^2)^n$ in equation (14) more succinctly and using the binomial theorem again, we see that

$$\begin{aligned}\sum_{n=0}^{\infty} P_n(z) t^n &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} t^n (2z-t)^n \\ &= \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2} t^n \sum_{r=0}^{\infty} \frac{n!}{(n-r)!r!}(2z)^{n-r}\cdot(-t)^r \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}(-1)^r \frac{(2n)!}{2^{n+r}n!(n-r)!r!} z^{n-r}t^{n+r} \\ &= \sum_{n=0}^{\infty}\sum_{r=0}^{[n/2]}(-1)^r \frac{(2n-2r)!}{2^n(n-r)!(n-2r)!r!} z^{n-2r}t^n,\end{aligned} \tag{15}$$

where in the last step, we have exchanged the outer summation variable n for $n+r$, and used the fact that the terms for which $n = 0, 1, \ldots, r-1$ or

$2r > n$ are zero (due to the $(n-r)!$ and $(n-2r)!$ factorials, respectively). Consequently, by identifying the coefficients of t^n in (15), we see that

$$P_n(z) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n (n-r)!(n-2r)!r!} \, z^{n-2r},$$

or equivalently,

$$P_n(z) = \sum_{r=0}^{[n/2]} (-1)^r \frac{1}{2^n r!(n-r)!} \left(\frac{(2n-2r)!}{((2n-2r)-n)!} \, z^{(2n-2r)-n} \right). \qquad (16)$$

Now, since

$$\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}z^n} \left(z^\rho\right) = \begin{cases} \dfrac{\rho!}{(\rho-n)!} z^{\rho-n}, & \text{if } \rho \geq n, \\ 0, & \text{if } \rho < n, \end{cases}$$

it follows that

$$\frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}z^n} \left(z^{2n-2r}\right) = \begin{cases} \dfrac{(2n-2r)!}{((2n-2r)-n)!} z^{(2n-2r)-n}, & \text{if } r \leq n/2, \\ 0, & \text{if } r > n/2. \end{cases} \qquad (17)$$

By using (17) in (16), and changing the upper summation limit to n (which can be done because all terms greater than $r = [n/2]$ vanish due to (17)), we see that

$$P_n(z) = \sum_{r=0}^{n} (-1)^r \frac{1}{2^n r!(n-r)!} \frac{\mathrm{d}^{\mathrm{n}}}{\mathrm{d}z^n} \left(z^{2n-2r}\right),$$

or

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \left(z^2\right)^{n-r} (-1)^r. \qquad (18)$$

Since the sum in (18) is the binomial expansion of $\left(z^2-1\right)^n$, it finally follows that

$$P_n(z) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left[\left(z^2-1\right)^n\right]. \qquad (19)$$

Equation (19) is known as Rodrigues' formula for the Legendre polynomials, named after Olinde Rodrigues who first introduced it in 1816. Since $\left(z^2-1\right)^n$ is a polynomial of degree $2n$, it follows directly from Rodrigues' formula that $P_n(z)$ is a polynomial of degree n.

2.3 Orthogonality of Legendre polynomials

Consider the integral

$$I_{m,n} = \int_{-1}^{1} P_m(z) P_n(z) \, \mathrm{d}z,$$

where $P_m(z)$ and $P_n(z)$ are two distinct Legendre polynomials. We can, without loss of generality, assume that $m < n$. By using equation (19) and integrating by parts, it follows that

$$\begin{aligned} I_{m,n} &= \frac{1}{2^n n!} \int_{-1}^{1} P_m(z) \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left[\left(z^2-1\right)^n \right] \mathrm{d}z \\ &= \frac{1}{2^n n!} \left(\left[P_m(z) \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\left(z^2-1\right)^n \right] \right]_{z=-1}^{1} - \int_{-1}^{1} P_m'(z) \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\left(z^2-1\right)^n \right] \mathrm{d}z \right) \\ &= \frac{(-1)}{2^n n!} \int_{-1}^{1} P_m'(z) \frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}} \left[\left(z^2-1\right)^n \right] \mathrm{d}z. \end{aligned} \tag{20}$$

Consequently, we see from (20) that n integrations by part yield that

$$I_{m,n} = \frac{(-1)^n}{2^n n!} \int_{-1}^{1} \left(z^2-1\right)^n \frac{\mathrm{d}^n}{\mathrm{d}z^n} P_m(z) \, \mathrm{d}z. \tag{21}$$

Since $\deg P_m(z) = m < n$, the factor $\frac{\mathrm{d}^n}{\mathrm{d}z^n} P_m(z)$ vanishes in (21), and it follows that

$$I_{m,n} = \int_{-1}^{1} P_m(z) P_n(z) \, \mathrm{d}z = 0, \quad m \neq n. \tag{22}$$

Equation (22) is known as the orthogonality relation of the Legendre polynomials. As a consequence of (22), the Legendre polynomials $\{P_n(z)\}$ are said to form an orthogonal sequence on the interval $-1 \leq z \leq 1$ with weight function 1.

In section 2.6, we explore sequences of other *orthogonal polynomials* that satisfy relations similar to (22).

2.4 Other representations of Legendre polynomials

We return briefly to equation (10) from section 2.2. By differentiating both sides of the equation with respect to t, it is seen that

$$\frac{z-t}{\sqrt{1-2zt+t^2}} = (1-2zt+t^2) \sum_{n=0}^{\infty} n P_n(z) t^{n-1}. \tag{23}$$

Clearly, from (10), equation (23) can be written as

$$(z-t)\sum_{n=0}^{\infty} P_n(z)t^n = (1-2zt+t^2)\sum_{n=0}^{\infty} nP_n(z)t^{n-1}. \tag{24}$$

By combining the series with some algebra, it is seen that equation (24) is equivalent to

$$zP_0(z)-P_1(z)+\sum_{n=1}^{\infty}\left[(2n+1)zP_n(z)-(n+1)P_{n+1}(z)-nP_{n-1}(z)\right]t^n = 0. \tag{25}$$

Since each coefficient of t^n in (25) (for each fixed $n = 0, 1, 2, \dots$) must be equal to 0, it follows that

$$(n+1)P_{n+1}(z) = (2n+1)zP_n(z) - nP_{n-1}(z), \tag{26}$$

where $P_0(z) = 1$ and $P_1(z) = z$. The recurrence relation (26) is known as *Bonnet's recursion formula.*

In addition to Rodrigues' formula, expressions for $P_n(z)$ include

$$\begin{aligned} P_n(z) &= \frac{1}{2\pi i}\oint (1-2tz+t^2)^{-1/2}t^{-n-1}\,\mathrm{d}t \\ &= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (z-1)^{n-k}(z+1)^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\left(\frac{1-z}{2}\right)^k \\ &= 2^n\cdot\sum_{k=0}^{n}(-1)^k z^k\binom{n}{k}\binom{\frac{n+k-1}{2}}{n}, \end{aligned}$$

where the simple closed contour in the integral encloses the origin and is traversed counterclockwise (see [28]).

2.5 Zeros of Legendre polynomials

An important application of the zeros of Legendre polynomials dates back to Gauss (see [1, 27]). Specifically, Gauss considered the problem of obtaining the best numerical estimate of the Riemann integral of some function $f(z)$ on some interval L by optimally picking the points $z_1, \dots, z_n \in L$ at which to evaluate $f(z)$. If L is the closed interval $[-1, 1]$, and if $f(z)$ has no singularities in L, it turns out, by the *fundamental theorem of Gaussian* *quadrature*, that z_k can 'optimally' be chosen as the k:th root of the n:th Legendre polynomial $P_n(z)$. Specifically, the approximation in this case is given by

$$\int_{-1}^{1} f(z)\,\mathrm{d}z \approx \sum_{k=1}^{n} w_k f(z_k), \tag{27}$$

where

$$w_k := \frac{2}{\left(1 - z_k^2\right)\left[P_n'(z_k)\right]^2}$$

is the weight used for the approximation. Here, 'optimally' means that equation (27) holds exactly if $f(z)$ is a polynomial of degree at most $2n-1$. For other functions $f(z)$, the error in the approximation depends on n and (informally) on how well $f(z)$ can be approximated by polynomials on L.

Formula (27) analogously (and optimally in the same sense) holds for other domains of integration when the weights w_k and points z_k depend on other orthogonal polynomials, which we will mention in the next section.

Also note that since the polynomial $(z^2 - 1)^n$ has n zeros in each of the points $z = -1$ and $z = 1$, it follows from Rodrigues' formula (equation (19) on page 10) and the Gauss-Lucas theorem (Theorem A.1 on page 60) that all n zeros of $P_n(z)$ (which can be shown to be simple) reside in the open interval $(-1, 1)$. Finally, we plot some Legendre polynomials (Figure 2) and their zero distribution when $n = 500$ (Figure 3).

2.6 Classical orthogonal polynomials

Before we close this section, we note that the Legendre polynomials are part of the theory of *classical orthogonal polynomials*, which satisfy analogues of many of the above equations. For instance, all classical orthogonal polynomials are polynomial solutions $y_n(z) = P_n(z)$ to the differential equation

$$Qy_n'' + Ly_n + \lambda_n y_n = 0, \tag{28}$$

where $Q = Q(z)$ and $L = L(z)$ are polynomials and $\lambda_n \in \mathbb{R}$ are constants that depend on the orthogonal polynomial in question (compare this to equation (11)). These polynomials are said to be orthogonal because they satisfy orthogonality relations of the form

$$\int_{a}^{b} W(z) P_m(z) P_n(z)\,\mathrm{d}z = 0, \quad m \neq n, \tag{29}$$

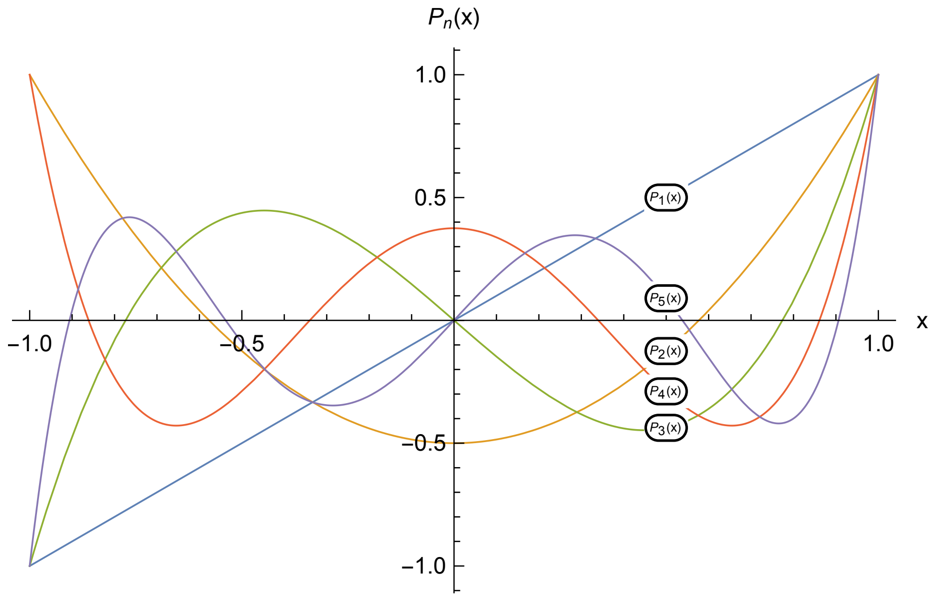

Figure 2: Five plots of Legendre polynomials of a real variable z.

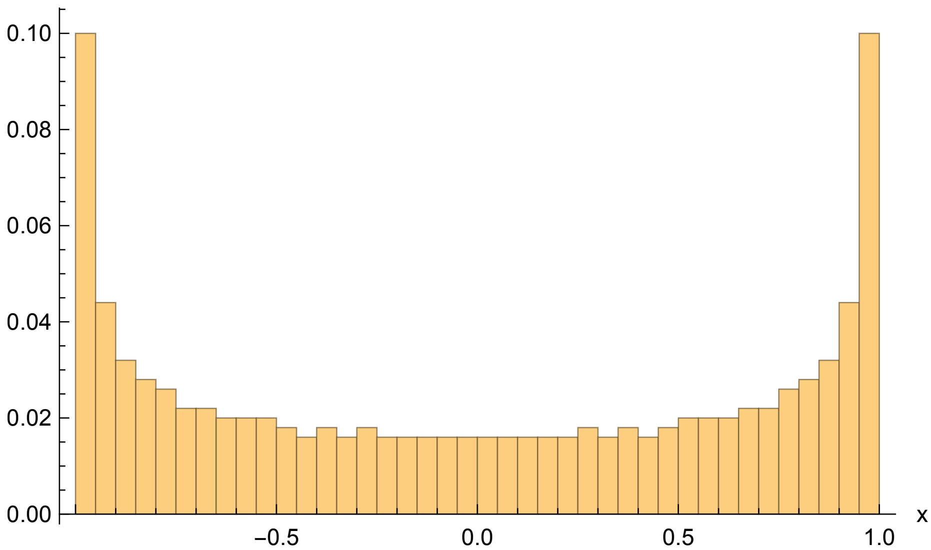

Figure 3: A histogram of the frequency of zeros of $P_{500}(z)$ on $(-1, 1)$.

where $W : \mathbb{R} \to \mathbb{R}^+$ is a weight function, and a, b are the endpoints of the *interval of orthogonality*.

If certain restrictions on Q and L are given, e.g. if Q is a quadratic polynomial with distinct real zeros, L is a linear polynomial whose zero lies between the zeros of Q, and the leading coefficients of Q and L have the same sign, then $\deg P_n = n$, and the weight function is given by

$$W(z) = \frac{e^{\int (L(z)/Q(z))\,\mathrm{d}z}}{Q(z)}. \tag{30}$$

Furthermore, we can also define the classical orthogonal polynomials by Rodrigues' formula in the following form:

$$P_n(z) = \frac{1}{e_n W(z)} \frac{\mathrm{d}^n}{\mathrm{d}z^n} \left(W(z) \left[Q(z) \right]^n \right). \tag{31}$$

Here, the numbers e_n depend on the class of polynomials in question.

(As we shall see, equation (31) and the exponential weight (30) bear some resemblance to (and thus provide some motivation for) the expression in Remark 3.2 on page 22, and $f^{(n)}$ in section 4.)

We list some of the properties of the classical orthogonal polynomials in the table below[26, 29] (where the first column of the first table follows directly from comparison of equations (11), (19), (28), and (31)).

Name	Legendre	Hermite	Laguerre	Chebyshev	Chebyshev (second kind)
Interval	$[-1, 1]$	$(-\infty, \infty)$	$[0, \infty)$	$[-1, 1]$	$[-1, 1]$
$W(z)$	1	e^{-z^2}	e^{-z}	$(1 - z^2)^{-1/2}$	$\sqrt{1 - z^2}$
$Q(z)$	$1 - z^2$	1	z	$1 - z^2$	$1 - z^2$
$L(z)$	$-2z$	$-2z$	$1 - z$	$-z$	$-3z$
e_n	$(-2)^n n!$	$(-1)^n$	$n!$	$(-2)^n \frac{\Gamma(n+1/2)}{\sqrt{\pi}}$	$2(-2)^n \frac{\Gamma(n+3/2)}{(n+1)\sqrt{\pi}}$
λ_n	$n(n + 1)$	$2n$	n	n^2	$n(n + 2)$

Name	Associated Laguerre, $L_n^{(\alpha)}$	Gegenbauer, $C_n^{(\alpha)}$	Jacobi, $P_n^{(\alpha,\beta)}$
Interval	$[0, \infty)$	$[-1, 1]$	$[-1, 1]$
$W(z)$	$z^\alpha e^{-z}$	$(1-z^2)^{\alpha-1/2}$	$(1-z)^\alpha(1+z)^\beta$
$Q(z)$	z	$1-z^2$	$1-z^2$
$L(z)$	$\alpha+1-z$	$-(2\alpha+1)z$	$\beta-\alpha-(\alpha+\beta+2)z$
e_n	$n!$	$\frac{(-2)^n\, n!\,\Gamma(2\alpha)\,\Gamma(n+1/2+\alpha)}{\Gamma(n+2\alpha)\Gamma(\alpha+1/2)}$	$(-2)^n n!$
λ_n	n	$n(n+2\alpha)$	$n(n+1+\alpha+\beta)$

3 Zeros of certain classes of polynomials with Rodrigues-like formulas

In this section, we present and elaborate on some of the results from the manuscript *Around Rodrigues' formula* by Rikard Bögvad, Boris Shapiro and this author (see [7]).

3.1 Numerical motivation

A question that arises in the study of the zeros of Legendre polynomials is as follows: *What happens if we replace the polynomial $z^2 - 1$ in Rodrigues' formula (see (19)) by a more general polynomial $P(z)$ of degree $d \geq 2$?* Since the coefficient $(2^n n!)^{-1}$ is immaterial to the zero distribution, we can write the resulting polynomials of interest simply as $\mathcal{P}_n(z) := (P^n)^{(n)}$, which are of degree $n(d-1)$. By the Gauss-Lucas theorem, the zeros of $\mathcal{P}_n$ are contained in the convex hull of the zeros of $P(z)$; see Figure 4.

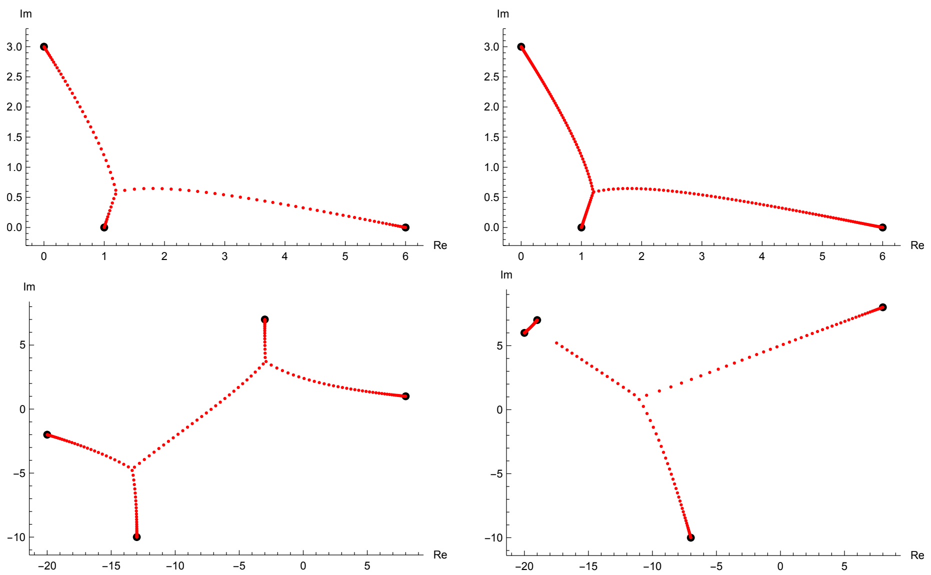

Figure 4: The zeros of $(P^n)^{(n)}$ (small dots), where $n = 50$ (top left), $n = 100$ (top right), $n = 50$ (bottom left), and $n = 50$ (bottom right). In each case, the large dots are the simple zeros of P.

As Figure 4 indicates, the zeros of $\mathcal{P}_n$ appear to asymptotically be supported by forests (i.e. cycle-free, undirected graphs; see definitions A.5-A.13 on pages 61-62 for some elementary terminology from graph theory) in the complex plane. Furthermore, the trees bear some resemblance to the concept of mother bodies introduced by D. Zidarov and later developed by B. Gustafsson (see [31, 12]), and to Steiner trees.

To strive for generality, we can further assume that the order of the derivative is $m = \lfloor \alpha n \rfloor$, where $\alpha \geq 0$ is a real number, and that the repeated differentiation acts on a rational function rather than on a polynomial. In other words, if P and Q are polynomials, we consider $\mathcal{R}_n(z) := ((P/Q)^n)^{(\lfloor \alpha n \rfloor)}$. Examples of the zeros of $\mathcal{R}_n$ are shown below in figures 5 and 6.

3.2 A generalized Legendre differential equation

It is natural to wonder if the rational functions $\mathcal{R}_n(z)$ from the previous section satisfy some differential equation analogous to (28). As it turns out, they do.

Proposition 3.1. *The rational function* $\mathcal{R}_{m,n}(z) := ((P/Q)^n)^{(m)}$ *satisfies the linear homogeneous differential equation*

$$\sum_{j=0}^{d}\sum_{k=0}^{j} \frac{(m+d-1)!\,(m+d+(n-1)j-2nk)}{(m+d-j)!\,(j-k)!\,k!} P^{(k)} Q^{(j-k)} y^{(d-j)} = 0 \quad (32)$$

of order $d := \deg P + \deg Q$.

Proof. Consider the first-order differential equation

$$PQw' + n(PQ' - P'Q)w = 0, \quad (33)$$

Clearly, if $R = P/Q$, then $w = R^n$ satisfies (33). By differentiating (33) $\ell \geq d-1$ times (or $\ell > d-1$ times if $d = 0$) and using Leibniz's rule for the derivative of a product, we get

$$\sum_{i=0}^{\ell} \frac{\ell!}{i!\,(\ell-i)!} S^{(i)} w^{(\ell+1-i)} + n \cdot \sum_{i=0}^{\ell} \frac{\ell!}{i!\,(\ell-i)!} T^{(i)} w^{(\ell-i)} = 0, \quad (34)$$

where $S := PQ$ and $T := PQ' - P'Q$. In the first sum, remove the first term and replace i by $r+1$ in the remaining sum. In the second sum, replace i by r and remove the last term. By combining the two resulting sums and simplifying, equation (34) becomes

$$Sw^{(\ell+1)} + nT^{(\ell)}w + \sum_{r=0}^{\ell-1} \frac{\ell!}{(\ell-r)!\,r!} \left(\frac{\ell-r}{r+1} S^{(r+1)} + nT^{(r)} \right) w^{(\ell-r)} = 0. \quad (35)$$

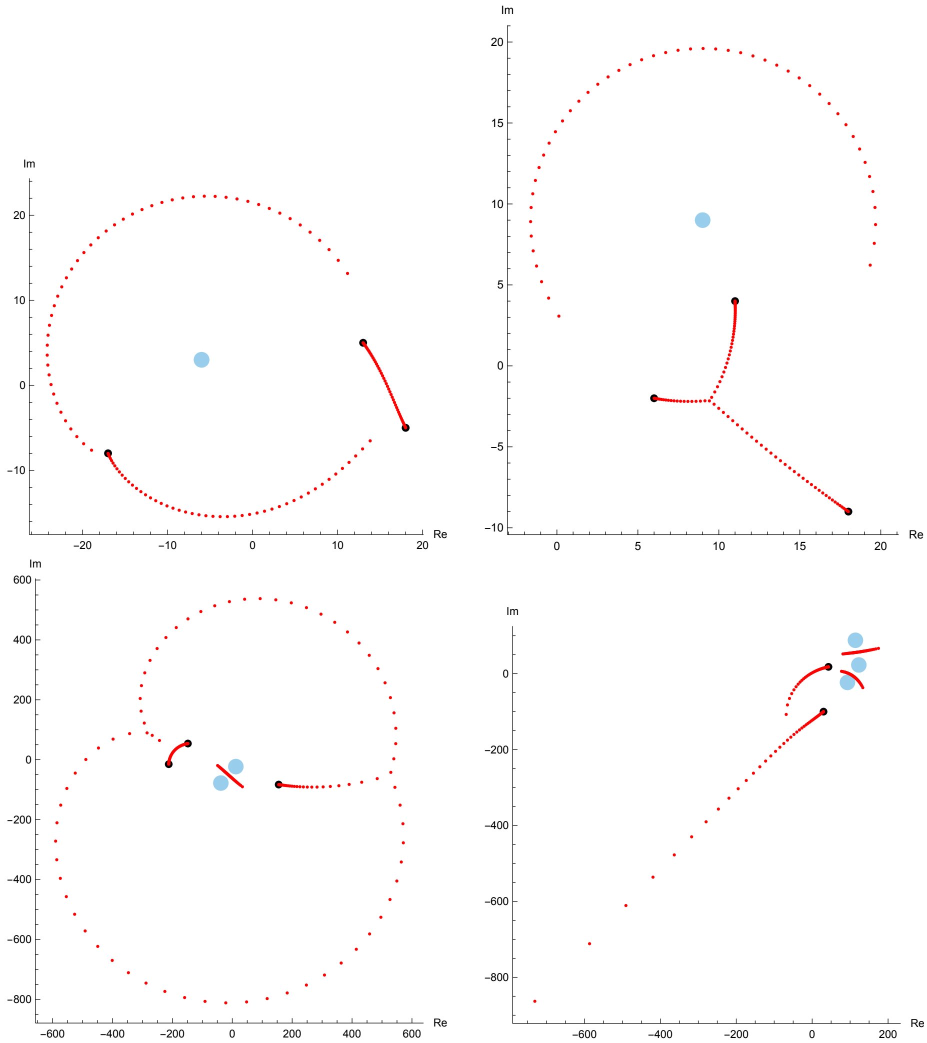

Figure 5: The zeros of $((P/Q)^n)^{(n)}$ (small dots), where $n = 50$. In each case, the mid-sized dots are the zeros of P, and the large dots are the zeros of Q. Note that cycles seem to occur in some asymptotic supports of the zeros. Furthermore, when $\deg Q \geq \deg P$, it is frequently (or always?) the case that the zeros appear to be unbounded in n.

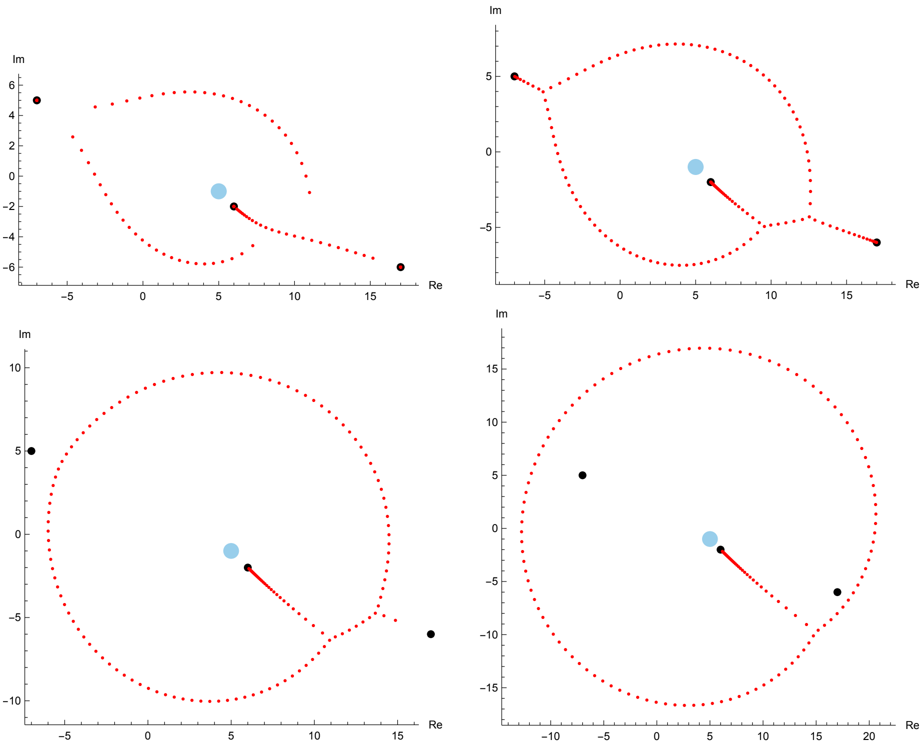

Figure 6: The zeros of $((P/Q)^n)^{(\lfloor \alpha n \rfloor)}$ (small dots), where $n = 50$, and $\alpha = 1/2$ (top left), $\alpha = 1$ (top right), $\alpha = 3/2$ (bottom left), and $\alpha = 2$ (bottom right). In each case, the mid-sized dots are the zeros of P, and the large dots are the zeros of Q, all of which are fixed.

By changing the upper limit of summation in (35) from $\ell-1$ to ℓ, the term $nT^{(\ell)}w$ is encompassed by the sum. Since S and T are degree d and $d-1$ polynomials, respectively, and $\ell \geq d-1$, we can change the upper limit of summation further to $d-1$, since higher terms vanish. That is, we obtain the equation

$$Sw^{(\ell+1)} + \sum_{r=0}^{d-1} \frac{\ell!}{(\ell-r)!\,r!}\left(\frac{\ell-r}{r+1}S^{(r+1)} + nT^{(r)}\right) w^{(\ell-r)} = 0,$$

or equivalently, by replacing r by $j-1$, separating the sums, and dividing by $\ell! \neq 0$ to simplify the calculations,

$$\sum_{j=0}^{d} \frac{1}{(\ell-j)!\,j!} S^{(j)} w^{(\ell+1-j)} + n \cdot \sum_{j=1}^{d} \frac{T^{(j-1)}}{(\ell+1-j)!\,(j-1)!} w^{(\ell+1-j)} = 0. \quad (36)$$

Expanding $S^{(j)} = (PQ)^{(j)}$ and $T^{(j-1)} = (PQ' - P'Q)^{(j-1)}$ in (36) with Leibniz's rule yields, after some calculations, that

$$\begin{aligned} &\sum_{j=0}^{d}\sum_{k=0}^{j} \frac{\ell+1-j}{(\ell+1-j)!\,(j-k)!\,k!} P^{(k)} Q^{(j-k)} w^{(\ell+1-j)} \\ &+ n \cdot \sum_{j=0}^{d}\sum_{k=0}^{j} \frac{j-2k}{(\ell+1-j)!\,(j-k)!\,k!} P^{(k)} Q^{(j-k)} w^{(\ell+1-j)} = 0. \end{aligned} \quad (37)$$

By combining the two double sums in (37), reversing the order of summation with respect to the outer sum, and letting $m := \ell + 1 - d$, the equation is transformed into

$$\sum_{j=0}^{d}\sum_{k=0}^{d-j} \frac{m+j+(d-j)n-2nk}{(m+j)!\,(d-j-k)!\,k!} P^{(k)} Q^{(d-j-k)} w^{(m+j)} = 0,$$

for all $\ell = m+d-1 \geq d-1 \iff m \geq 0$. Thus, $y = w^{(m)} = (R^n)^{(m)}$ satisfies the differential equation

$$\sum_{j=0}^{d}\sum_{k=0}^{d-j} \frac{m+j+(d-j)n-2nk}{(m+j)!\,(d-j-k)!\,k!} P^{(k)} Q^{(d-j-k)} y^{(j)} = 0.$$

By reversing the order of summation again, and multiplying by $(m+d-1)!$ (i.e. $\ell!$, which we previously divided by to simplify our calculations), the proposition follows. □

Corollary 3.1. *If $P(z) \not\equiv 0$ is a polynomial of degree d and $0 \leq m < nd$, then the polynomial $(P^n)^{(m)}$ satisfies the linear homogeneous differential equation*

$$\sum_{k=0}^{d} \frac{(m+d-1)!\,[m-nd-(k-d)(n+1)]}{(m+d-k)!\,k!} P^{(k)} y^{(d-k)} = 0 \quad (38)$$

of order d.

Remark 3.1. *A differential equation similar to (38), for the special case $m = n$, was first discovered in [15].*

Remark 3.2. *Similarly to (33), the differential equation $PQw' + n(PQ' - PQT' - P'Q)w = 0$ is satisfied by $w = \big((P/Q)e^T\big)^n$, so it is reasonable to derive a differential equation analogous to (32) for $\big[\big((P/Q)e^T\big)^n\big]^{(m)}$. We leave the details to the reader.*

3.3 The asymptotic zero-counting measure of $((P/Q)^n)^{(\lfloor \alpha n \rfloor)}$, and a formal algebraic equation for its asymptotic Cauchy transform

Before we can proceed, we need a few tools from potential theory. If $\tilde{P}$ is a monic polynomial of degree $d \geq 1$ with zero-counting measure μ (that is, a probability measure that assigns mass $1/d$ to each zero of $\tilde{P}$, accounting for multiplicity), then the logarithmic potential $\mathcal{L}_\mu(z)$ of μ is given by

$$\mathcal{L}_\mu(z) := \int \log|z - \zeta| \,\mathrm{d}\mu(\zeta) = \frac{1}{d} \log\left|\tilde{P}(z)\right|. \tag{39}$$

Similarly, the Cauchy transform of μ (defined as above) is given by

$$\mathcal{C}_\mu(z) := \int \frac{\mathrm{d}\mu(\zeta)}{z - \zeta} = \frac{\tilde{P}'(z)}{d\tilde{P}(z)}. \tag{40}$$

The logarithmic potential and the Cauchy transform can be used to reconstruct the measure μ from either $\mathcal{L}_\mu(z)$ or $\mathcal{C}_\mu(z)$ by the formula

$$\mu = \frac{1}{2\pi} \cdot \Delta\mathcal{L}_\mu = \frac{1}{\pi} \cdot \frac{\partial \mathcal{C}_\mu}{\partial \overline{z}}, \tag{41}$$

where $\Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2$ is the Laplace operator and $\partial/\partial\overline{z} = (\partial/\partial x + i\partial/\partial y)/2$ (see [2]). Here, the derivatives are defined in the distributional sense (see [23]).

Now, since $\mathcal{R}_{m,n}(z) = ((P/Q)^n)^{(m)}$ is a rational function $\mathcal{P}/\mathcal{Q}$ (where we assume $\gcd(\mathcal{P}, \mathcal{Q}) = 1$), we can let $P_{m,n}(z)$ denote the monic version of the polynomial $\mathcal{P}$. Furthermore, we let $\mu_{m,n}$ be the zero-counting measure of $P_{m,n}(z)$. Since the zeros of $P_{m,n}(z)$ are the zeros of $\mathcal{R}_{m,n}(z)$, we also refer to $\mu_{m,n}$ as the zero-counting measure of $\mathcal{R}_{m,n}(z)$ from now on.

Furthermore, let α be a positive number, and as above, let $\mathcal{C}_\mu(z)$ be the Cauchy transform of μ, where $\mu := \lim_{n\to\infty} \mu_{\lfloor \alpha n \rfloor, n}$. Then the following algorithm can be used to transform (32) into an algebraic equation for $\mathcal{C}_\mu(z)$ (which is valid under certain hypotheses on the convergence which we will not detail here; see [5]):

Step 1: Replace m by αn, and divide both sides by y.

Step 2: Replace $y^{(d-j)}/y$ by $(n(d-\alpha)\mathcal{C}_\mu)^{d-j}$ in the resulting equation, and divide both sides by n^d.

Step 3: Let $n \to \infty$.

By carrying out the three steps above (where we leave the technical details of establishing a weak limit for $\mu_{\lfloor \alpha n \rfloor, n}$, and a pointwise limit for $C_{\mu_{\lfloor \alpha n \rfloor, n}}(z)$ to [7]), we arrive at the following algebraic equation for $\mathcal{C}_\mu(z)$ (which exists a.e.):

$$\sum_{j=0}^{d}\sum_{k=0}^{j} \frac{\alpha^{j-1}(\alpha+j-2k)(d-\alpha)^{d-j}}{(j-k)!\,k!} P^{(k)}Q^{(j-k)}\mathcal{C}_\mu^{d-j} = 0. \tag{42}$$

If we divide both sides of equation (42) by α^{d-1} and substitute $W := \frac{d-\alpha}{\alpha}\mathcal{C}_\mu$, it can be written more simply as

$$\sum_{j=0}^{d}\sum_{k=0}^{j} \frac{\alpha+j-2k}{(j-k)!\,k!} P^{(k)}Q^{(j-k)}W^{d-j} = 0, \tag{43}$$

or, if we write the $1+2+\cdots+(d+1)$ terms of (43) on $d+1$ rows and sum along diagonals,

$$\sum_{j=0}^{d}\sum_{k=0}^{d-j} \frac{\alpha+j-k}{j!\,k!} P^{(k)}Q^{(j)}W^{d-j-k} = 0. \tag{44}$$

Note that the Taylor expansion of $P(z+u)Q(z+u)$ about $u=0$ is

$$\sum_{j=0}^{d}\sum_{k=0}^{j} \frac{P^{(k)}(z)Q^{(j-k)}(z)}{(j-k)!\,k!}u^j.$$

Similarly,

$$u \cdot \frac{\partial}{\partial u}\left(P(z+u)Q(z+u)\right) = \sum_{j=0}^{d}\sum_{k=0}^{j} j\frac{P^{(k)}(z)Q^{(j-k)}(z)}{(j-k)!\,k!}u^j,$$

and

$$u \cdot Q(z+u) \cdot \frac{\partial}{\partial u}P(z+u) = \sum_{j=0}^{d}\sum_{k=0}^{j} k\frac{P^{(k)}(z)Q^{(j-k)}(z)}{(j-k)!\,k!}u^j.$$

Consequently, by letting $W = 1/u$ in equation (43), we see that it can (if we assume that u, $W \neq 0$) be written as $\alpha\, P(z+u)Q(z+u) + u \cdot \frac{\partial}{\partial u}(P(z+u)Q(z+u)) - 2u \cdot Q(z+u) \cdot \frac{\partial}{\partial u} P(z+u) = 0$, or equivalently,

$$\alpha\, P(z+W^{-1})Q(z+W^{-1})+W^{-1}\left[P(z+W^{-1})Q'(z+W^{-1}) - P'(z+W^{-1})Q(z+W^{-1})\right] = 0. \tag{45}$$

Clearly, if F denotes the left-hand side of equation (45), and $z = b$ is any zero of either P or Q of multiplicity $m \geq 2$, then $W = (b-z)^{-1}$ solves $F = 0$ (and thus also equation (43)). More generally, by differentiating (45) with respect to z or W repeatedly, we see that the same solution satisfies

$$\frac{\partial^{k+\ell} F}{\partial z^k \partial W^\ell} = 0, \qquad 0 \leq k + \ell \leq j - 2,\ \ j, k, \ell \in \mathbb{N}_0, \tag{46}$$

whenever $m = j \geq 2$. Outside of these solution curves $W = (b-z)^{-1}$, it follows from (45) that the scaled Cauchy transform W satisfies

$$W = \frac{P'(z+W^{-1})Q(z+W^{-1}) - P(z+W^{-1})Q'(z+W^{-1})}{\alpha\, P(z+W^{-1})Q(z+W^{-1})} \tag{47}$$

almost everywhere (except along $W = (b-z)^{-1}$). Equation (47) can be written more compactly as

$$\alpha W = \frac{\mathrm{d} \log R(z+W^{-1})}{\mathrm{d}z}, \tag{48}$$

where $R := P/Q$. For the original Cauchy transform $\mathcal{C}_\mu(z)$, this implies the following proposition.

Proposition 3.2. *Let μ be the asymptotic zero-counting measure of $(R^n)^{(\lfloor \alpha n \rfloor)}$, where $R = P/Q$. Furthermore, assume that hypotheses i)-iii) of Proposition 3 in [5] are satisfied. Then its Cauchy transform $\mathcal{C}_\mu(z)$ satisfies*

$$\sum_{j=0}^{d} \sum_{k=0}^{j} \frac{\alpha^{j-1}(\alpha + j - 2k)(d-\alpha)^{d-j}}{(j-k)!\, k!} P^{(k)} Q^{(j-k)} \mathcal{C}_\mu^{d-j} = 0. \tag{49}$$

If $z = b$ is a zero of either P or Q of multiplicity at least 2, then (49) has a "trivial" solution $C_\mu = \alpha/((d-\alpha)(b-z))$ for each such b. Other solutions of (49) satisfy

$$(d-\alpha)\mathcal{C}_\mu = \frac{\mathrm{d} \log R\left(z + \frac{\alpha}{d-\alpha}\mathcal{C}_\mu^{-1}\right)}{\mathrm{d}z}. \tag{50}$$

Remark 3.3. *We conjecture that the assumptions in [5] are unnecessary for the validity of Proposition 3.2.*

3.4 A numerical study of zeros as motivation for further research

3.4.1 Two-dimensional regions of zeros

As previously, let P be a polynomial of degree $d \geq 2$. As an example of a not yet understood property of the zero distribution of $(P^n)^{(m)}$, it is interesting to consider the union $U_{P,n}$ of all $nd(nd+1)/2$ zeros of P^n, $(P^n)'$, $(P^n)''$, $\ldots$, $(P^n)^{(nd-1)}$. For generic P, this union appears to asymptotically form a two-dimensional concave region in the convex hull of $Z(P)$; see Figure 7.

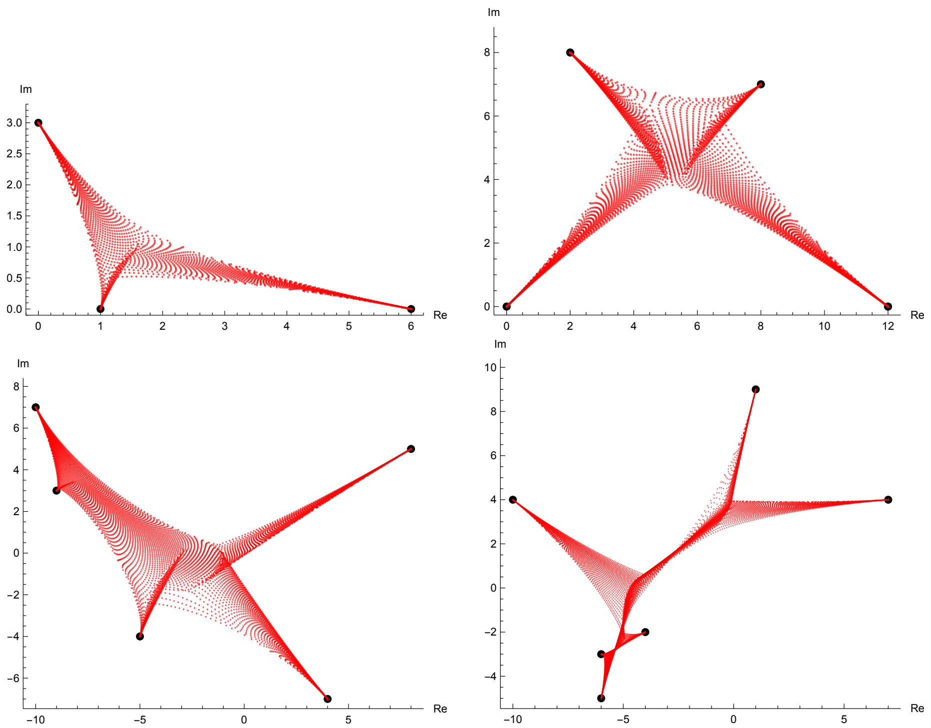

Figure 7: The union of all zeros of $(P^n)^{(m)}$, for $m = 0, 1, \ldots, (\deg P)n - 1$ (small dots). Here, $n = 30$, and in each case, the mid-sized dots are the zeros of P.

If we consider the rational function $R = P/Q$ instead, where $p := \deg P$ and

$q := \deg Q$, then $(R^n)^{(m)}$ can be decomposed into polar parts as

$$\begin{aligned}(R^n)^{(m)} = H_{m,n}(z) &+ \frac{C_{z_1,1}}{(z-z_1)^{m+1}} + \frac{C_{z_1,2}}{(z-z_1)^{m+2}} + \cdots + \frac{C_{z_1,n}}{(z-z_1)^{m+n}} \\ &+ \frac{C_{z_2,1}}{(z-z_2)^{m+1}} + \frac{C_{z_2,2}}{(z-z_2)^{m+2}} + \cdots + \frac{C_{z_q,n}}{(z-z_q)^{m+n}},\end{aligned} \tag{51}$$

where $z_1, \ldots, z_q$ are the zeros of Q, the $C_{z_i,j}$ are complex numbers, and $H_m(z)$ is a polynomial of degree $\max\{(p-q)n-m,\, 0\}$. In particular, $H_{m,n}(z)$ is nonzero if and only if $p \geq q$ and $0 \leq m \leq (p-q)n$. Thus, when $H_{m,n}(z)$ is nonzero, we can write the terms in the right-hand side of (51) as a fraction with a common denominator, such that the polynomial $H_{m,n}(z)[(z-z_1)(z-z_2)\cdots(z-z_q)]^{m+n}$ dominates the degree of the polynomial in its numerator. Hence, assuming that $Z(H_{m,n}) \cap Z(Q) = \emptyset$, $(R^n)^{(m)}$ has $(p-q)n-m+q(m+n) = pn+(q-1)m$ zeros in this case.

Consequently, it makes sense to consider the polynomials $(P^n)^{(m)}$ (and their zeros) previously discussed as a special subcase of $(P/Q)^{(m)}$, where $p \geq q$ and $0 \leq m \leq (p-q)n$. Corresponding generalizations of the shapes in Figure 7 for this class of rational functions are shown below in Figure 8.

The zeros of $H_{m,n}(z)$, for various P and Q, are shown below in Figure 9.

3.4.2 The unit m-star

Motivated by the previous balance between exponents and orders of the derivative, we consider the polynomial

$$P_{m,n}(z) := \left(\sum_{k=0}^{n} z^{mk}\right)^{(n)} = \sum_{k=\min\left(\lfloor\frac{n+m}{m}\rfloor, \lceil\frac{n}{m}\rceil\right)}^{n} \frac{(mk)!}{(mk-n)!} z^{mk-n} \underset{\text{if } z^m \neq 1}{=} \left(\frac{z^{m(n+1)}-1}{z^m-1}\right)^{(n)}, \tag{52}$$

defined for all $n \geq 0$ and $m \geq 1$. Clearly, $P_{m,n}$ is a degree $(m-1)n$ polynomial, so we illustrate its zero distribution for $n = 100$ and a few $m \geq 2$ below in Figure 10.

Finally, linear combinations of such polynomials $P_{m,n}$ give rise to other star-like zero distributions, as seen in Figure 11. Variations in the orders of the derivative and in the upper limits of summation give rise to other interesting structures.

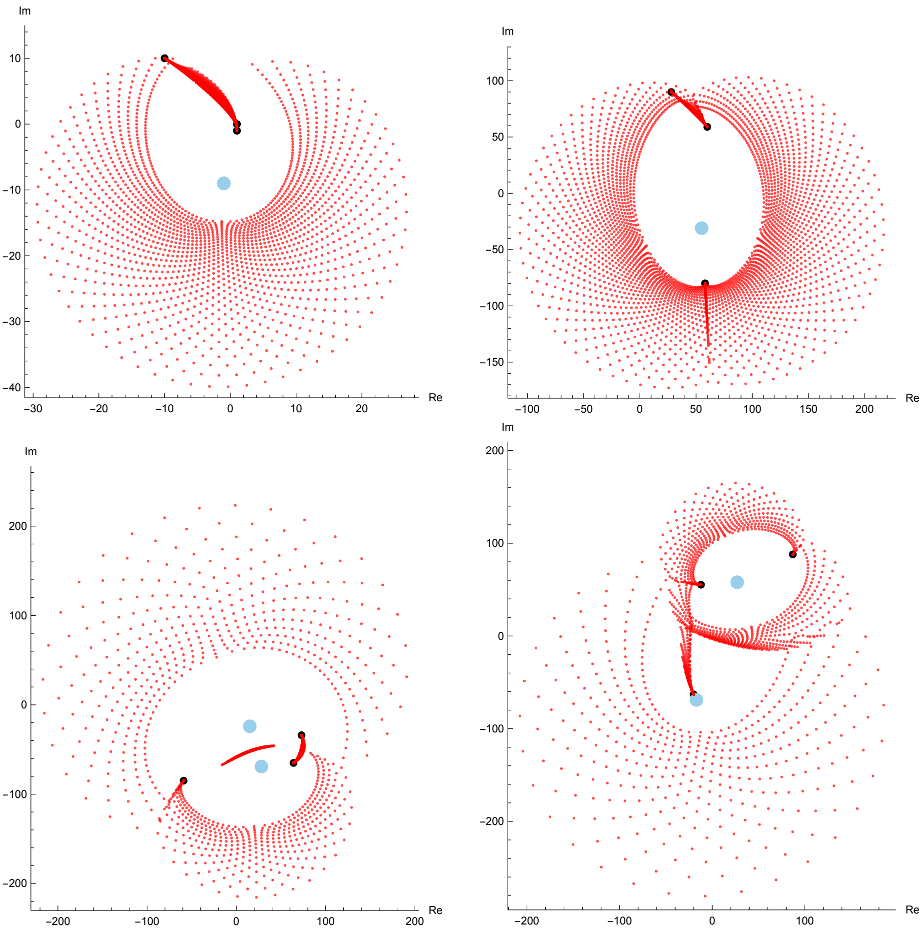

Figure 8: The union of all zeros of $((P/Q)^n)^{(m)}$, for $m = 0, 1, \ldots, (\deg P - \deg Q)n$ (small dots). Here, $n = 30$, and in each case, the mid-sized dots are the zeros of P, and the large dots are the zeros of Q.

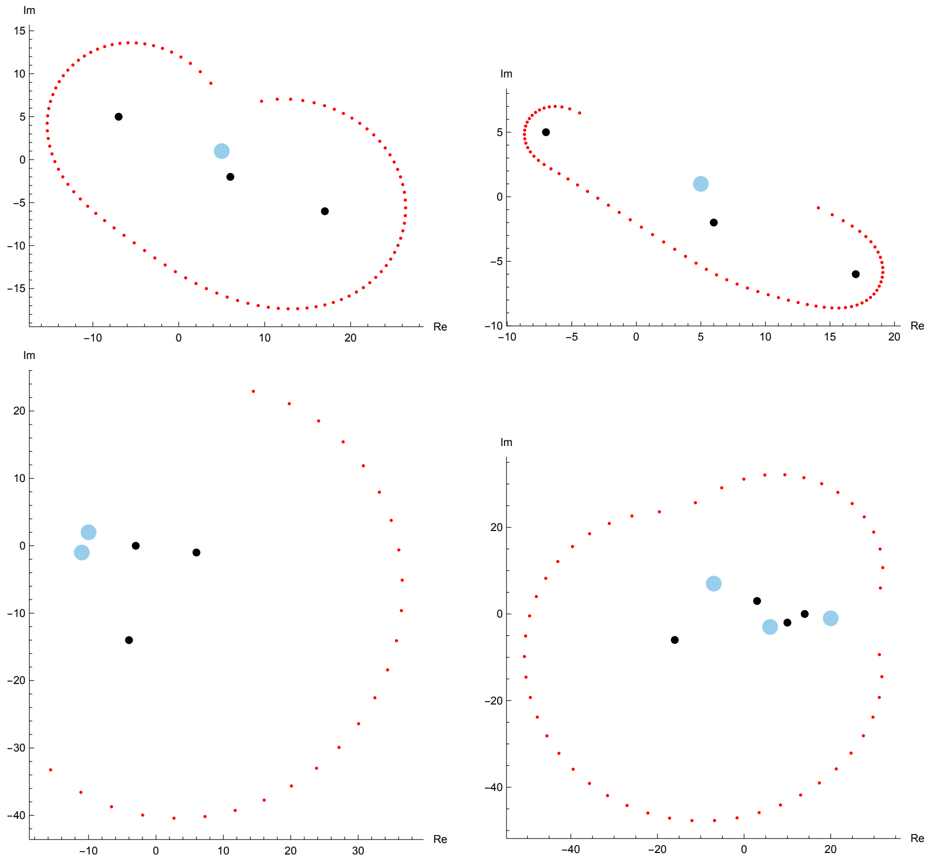

Figure 9: The zeros of $H_{m,50}(z)$ (small dots), where $m = 0$ (top left), $m = 25$ (top right), $m = 25$ (bottom left), and $m = 0$ (bottom right). In each case, the mid-sized dots are the zeros of P, and the large dots are the zeros of Q.

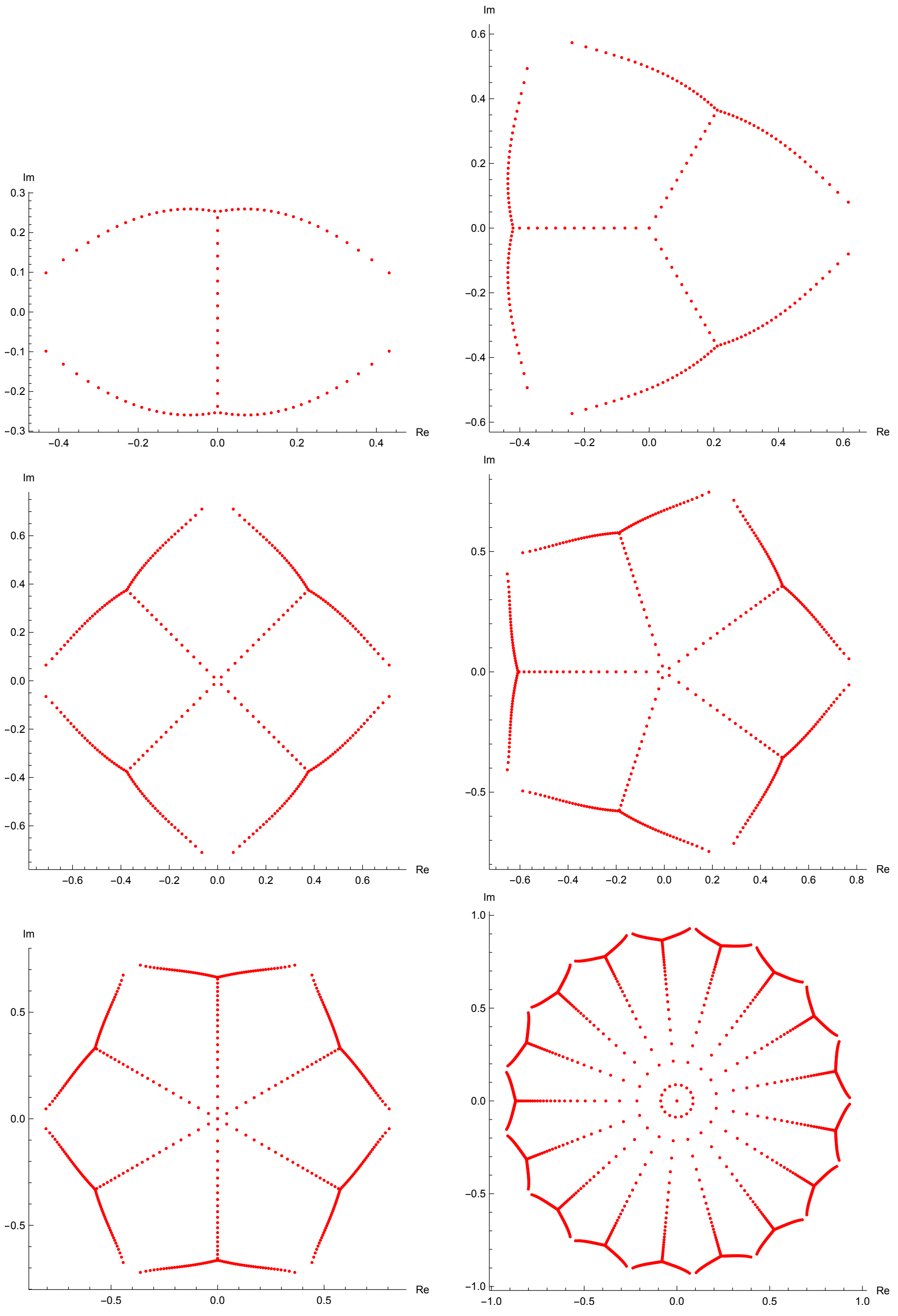

Figure 10: The zeros of $P_{m,n}$ for $n = 100$, and $m = 2$ (top left), $m = 3$ (top right), $m = 4$ (middle left), $m = 5$ (middle right), $m = 6$ (bottom left), and $m = 17$ (bottom right).

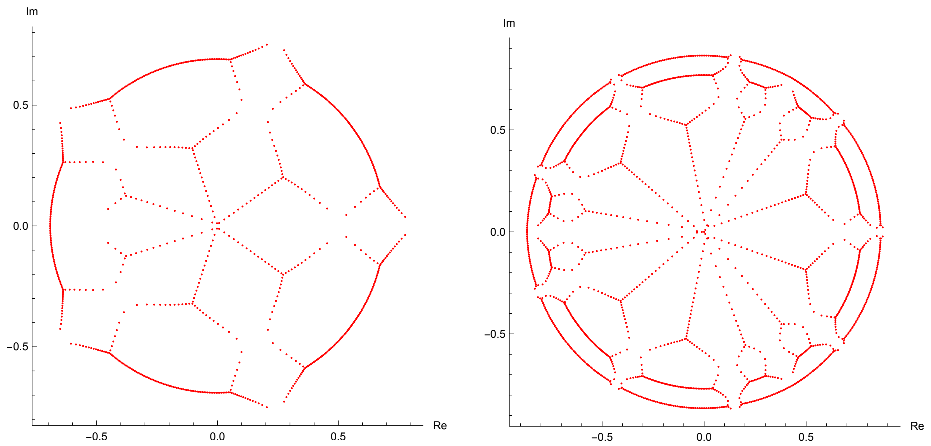

Figure 11: The zeros of $\left(\sum_{k=0}^{n} z^{2k}\right)^{(n)} + \left(\sum_{k=0}^{n} z^{5k}\right)^{(n)}$ (left), and $\left(\sum_{k=0}^{n} z^{3k}\right)^{(n)} + \left(\sum_{k=0}^{n} z^{6k}\right)^{(n)} + \left(\sum_{k=0}^{n} z^{9k}\right)^{(n)}$ (right). Here, $n = 200$.

3.4.3 Zeros separated by fixed angles

In preparation for the next section, we provide a conjecture on the behavior of the zeros of $(e^T)^{(n)}$, where T is a polynomial of degree 3.

Conjecture 3.1. *Let $\beta \in \mathbb{R}_+$, and assume that the polynomial $T = (z - \beta\alpha_1)(z - \beta\alpha_2)(z - \beta\alpha_3)$ has three distinct zeros that are not the vertices of an equilateral triangle. Then the zeros of $(e^T)^{(n)}$ asymptotically lie on*
(i) two orthogonal lines, if n is held constant and $\beta \to \infty$, or
(ii) three line segments that emerge from a point with $2\pi/3$ angles between adjacent pairs, if β is held constant and $n \to \infty$.

Parts (i) and (ii) of Conjecture 3.1 are illustrated below in figures 12 and 13, respectively. Furthermore, we conjecture that the zeros of $(e^T)^{(n)}$ asymptotically lie on a finite number of lines with interesting relations between their angles for more general polynomials $T = (z - \beta\alpha_1) \cdots (z - \beta\alpha_d)$.

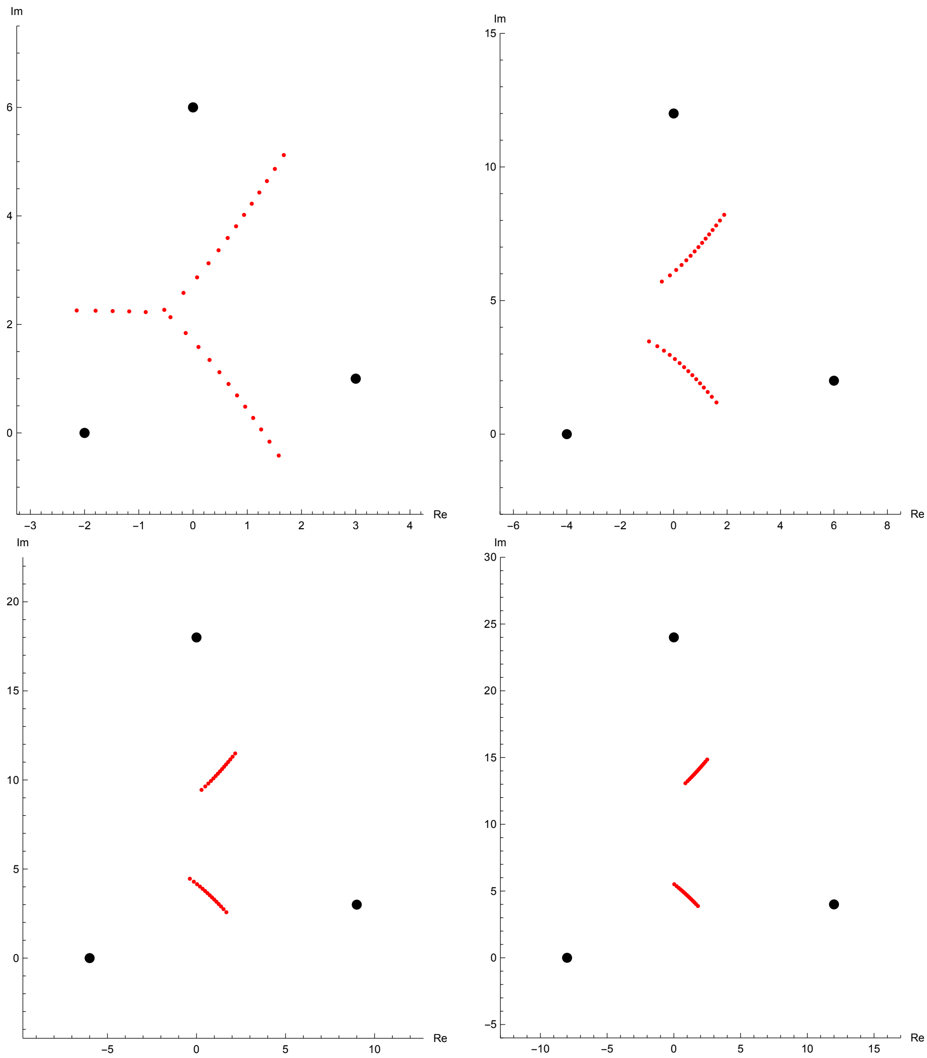

Figure 12: The zeros of $T = (z + 2\beta)(z - 6i\beta)(z - (3 + i)\beta)$ (shown by big dots), and the zeros of $\left(e^T\right)^{(n)}$ (shown by small dots). Here, $n = 15$, and $\beta = 1$ (top left), $\beta = 2$ (top right), $\beta = 3$ (bottom left), and $\beta = 4$ (bottom right).

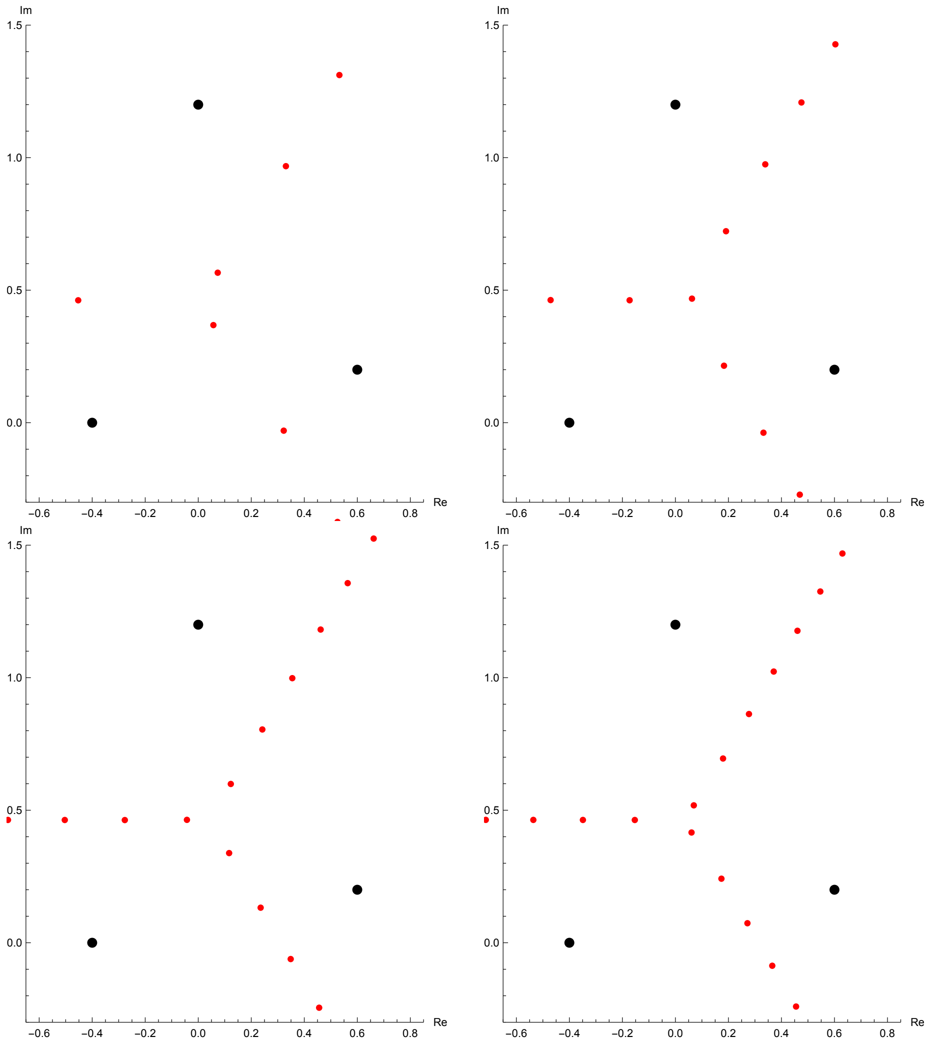

Figure 13: The zeros of $T = (z + 2/5)(z - 6i/5)(z - 3/5 - i/5)$ (shown by big dots), and zeros of $\left(e^T\right)^{(n)}$ (shown by small dots). Here, $n = 10$ (top left), $n = 20$ (top right), $n = 30$ (bottom left), and $n = 40$ (bottom right).

4 Zeros on Voronoi diagrams

4.1 Iterated derivatives of a class of meromorphic functions

We take a detour into related territory. Consider a meromorphic function f with set of poles S. Pólya proved in 1922 that the zeros of the iterated derivatives f', f'', $f^{(3)}, \dots$ of such a function asymptotically accumulate along the boundaries of the Voronoi diagram associated with S. This classical but little-known result is called Pólya's Shire theorem (see Theorem 4.2 on page 35). In a recent paper by Rikard Bögvad and this author (see [6]), a measure-theoretic refinement of Pólya's Shire theorem was given for the special case that $f = P/Q$, where P and Q are polynomials with $\gcd(P, Q) = 1$ and $P \not\equiv 0$.

In this section, we generalize the main result of the aforementioned paper (see Theorem 1 of [6]) to the situation that $f = (P/Q)e^T$, where P and Q are defined as previously, and T is a polynomial. These functions are of particular interest if Q has at least two distinct zeros. Under these conditions, it follows from Hadamard's theorem (see Theorem 4.4 on page 39) that the class of such functions is equivalent to the class of meromorphic functions that are quotients of two entire functions of finite order, each with a finite number of zeros. For convenience, we denote $p := \deg P$, $q := \deg Q$ and $t := \deg T$ throughout this section, and additionally set $P = \sum_{k=0}^{p} b_k z^k$, $Q = \sum_{k=0}^{q} c_k z^k$ and $T = \sum_{k=0}^{t} d_k z^k$.

The main result of this section is stated below in Theorem 4.1. The terminology of this theorem is gradually explained during its proof in sections 2 and 3.

Theorem 4.1. *Let $f := (P/Q)e^T$, where P, Q and T are polynomials with $\gcd(P, Q) = 1$, $P \not\equiv 0$, $\deg Q \geq 2$ and $\deg T \geq 1$. Furthermore, assume that all zeros $z_1, \dots, z_q$ of Q are distinct. Then*
(i) the zero-counting measures μ_n of the sequence $\left\{f^{(n)}\right\}_{n=1}^{\infty}$ converge to a measure μ_S with mass $(q-1)/(q-1+t)$.
(ii) The logarithmic potential $\mathcal{L}_{\mu_n}(z)$ of μ_n diverges as $n \to \infty$.
(iii) The shifted logarithmic potential $\widetilde{\mathcal{L}}_{\mu_n}(z) := \mathcal{L}_{\mu_n}(z) - \frac{\log n!}{n(q+t-1)+p}$ of μ_n converges in L^1_{loc} to the shifted logarithmic potential $\Psi(z)$ of μ_S, where

$$\Psi(z) = \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} + \log|Q| - \log\left(|c_q|\,|d_t|\,t\right)\right). \quad (53)$$

(iv) The measure μ_S is given by $\frac{1}{2\pi}\Delta\Psi(z)$.

Note that if $t = 0$, it follows from Theorem 1 of [6] that the logarithmic potential of the asymptotic zero-counting measure μ is given by $\mathcal{L}_\mu(z) =$

$\frac{1}{q-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\}+\log|Q|\right)$. Thus, there are strong similarities with Theorem 4.1 above. An illustration of Theorem 4.1 is given in Figure 14.

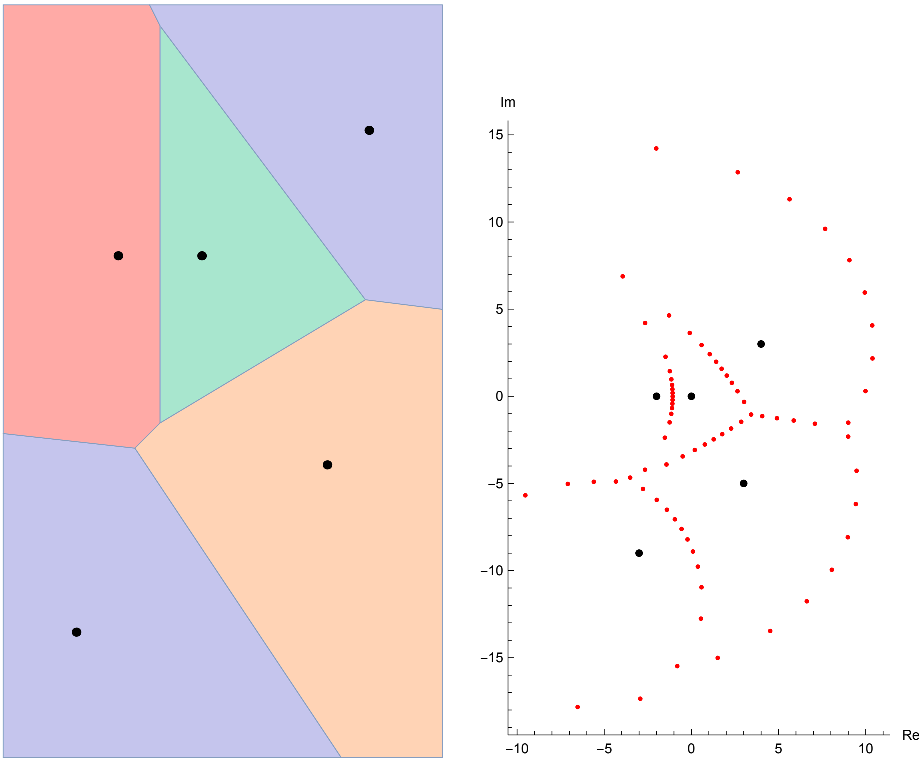

Figure 14: The Voronoi diagram generated by the zeros of the polynomial $Q = z(z+2)(z-4-3i)(z-3+5i)(z+3+9i)$ (left), and the 75 zeros of $\left((1/Q)e^T\right)^{(15)}$ (small dots), where $T = z+1$ (right).

Before we begin with the proof of Theorem 4.1, it is for historical reasons worth noting that Pólya himself investigated functions similar to f. For example, he proved that all entire functions that can be approximated uniformly on compact subsets of the plane by real polynomials with only real zeros can be written on the form $g(z) = w(z)\,e^{az^2+bz+c}$. Here, $a \leq 0$, b and c are real numbers, and w is a canonical product of genus at most one, with only real zeros. Due to this result, the class of such functions $g(z)$ is known as the *Laguerre-Pólya class*, denoted LP. Since LP is closed under differentiation, all derivatives of a function in LP have only real zeros. Furthermore, Pólya proved that functions of the form $h(z) = He^T$, where H and T are real polynomials, belong to the class LP, assuming that $h^{(n)}$ has only real zeros for all $n \geq 0$. McLeod generalized the latter result to the case that H is an entire function with the growth constraint $\log M(\rho, H) = o(\rho^{\deg T - 1})$, $\rho \to \infty$ where $M(\rho, H) = \max_{|z| \le \rho} |H(z)|$ (see [3]).

4.2 Voronoi diagrams

Consider a set of q distinct points $S = \{z_1, \ldots, z_q\} \subset \mathbb{C}$. The Voronoi diagram associated with S, denoted $\mathrm{Vor}_{\mathrm{S}}$, is a partitioning of $\mathbb{C}$ into q distinct cells $V_1, \ldots, V_q$, where any interior point α_i in V_i is closest to z_i of all points in S. The boundary between two adjacent cells V_i and V_j consists of a segment on the line $|z - z_i| = |z - z_j|$.

Based on the aforementioned definition of Voronoi diagrams, it is natural to stratify the complex plane using the function

$$\Phi(z) := \min_{i=1,\ldots,q} \{|z - z_i|\}. \tag{54}$$

Thus, the (closed) cell V_i that contains the point $z_i \in S$ is equal to the set

$$V_i = \{z : \Phi(z) = |z - z_i|\}.$$

Similarly, the boundary V_{ij} between two cells V_i and V_j is given by

$$V_{ij} = \{z : \Phi(z) = |z - z_i| = |z - z_j|\}.$$

Together, these boundaries form the $1-$ *skeleton* of $\mathrm{Vor}_{\mathrm{S}}$, which we denote $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}}$ (where B means boundary). Finally, the vertices of $\mathrm{Vor}_{\mathrm{S}}$ are points z such that at least three distances $|z - z_i|$, $i = 1, \ldots, q$, coincide with $\Phi(z)$.

4.3 Some theorems of Pólya and Hadamard

The following result is due to Pólya (see [21]), which is required in the next section where equation (53) in Theorem 4.1 is derived.

Theorem 4.2 (Pólya's Shire theorem, version 1)**.** *Let $F(z)$ be a meromorphic function with set of poles S, and consider the sequence*

$$\left|\frac{F'(z)}{1!}\right|, \left|\frac{F''(z)}{2!}\right|^{\frac{1}{2}}, \ldots, \left|\frac{F^{(n)}(z)}{n!}\right|^{\frac{1}{n}}, \ldots \tag{55}$$

of nonnegative functions. Then
(i) The sequence (55) converges uniformly on $D(R, \delta)$, where $D(R, \delta)$ is the the disk $|z| \le R$ with all points closer than $\delta > 0$ to a pole of $F(z)$ removed.
(ii) The sequence (55) converges pointwise a.e. in the open set $\mathbb{C} \backslash \left(\mathrm{Vor}_S^B \cup S\right)$ to $\max\left\{\frac{1}{|z-\zeta|}, \zeta \in S\right\}$.

Proof. We reproduce the proof of Pólya. To prove (i), consider the region $D\left(R+\frac{\delta}{2},\frac{\delta}{2}\right)$. In this region, $F(z)$ is regular (i.e. holomorphic and single-valued in the sense of monodromy) and has a well-defined maximum value M. Note that a circle of radius $\delta/2$ can be drawn around any point $z \in D(R,\delta)$ such that it is contained in $D\left(R+\frac{\delta}{2},\frac{\delta}{2}\right)$. Thus, according to a classical inequality in function theory,

$$\left|\frac{F^{(n)}(z)}{n!}\right| \leq \frac{M}{\left(\frac{\delta}{2}\right)^n},$$

or equivalently,

$$\left|\frac{F^{(n)}(z)}{n!}\right|^{\frac{1}{n}} \leq \frac{2M^{\frac{1}{n}}}{\delta}$$

for any point $z \in D(R,\delta)$, from which (i) follows.

To prove (ii), consider a pole $a \in S$ of order $q+1$, and decompose $F(z)$ into polar parts as

$$F(z) = \frac{A_0}{z-a}+\frac{1!\,A_1}{(z-a)^2}+\frac{2!\,A_2}{(z-a)^3}+\cdots+\frac{q!\,A_q}{(z-a)^{q+1}}+\Lambda(z), \quad (A_q \neq 0,\ q \geq 0),$$

where $\Lambda(z)$ is a function that is regular at a. By differentiating $F(z)$ n times, we see that

$$F^{(n)}(z) = (-1)^n\left(\frac{n!\,A_0}{(z-a)^{n+1}}+\frac{(n+1)!\,A_1}{(z-a)^{n+2}}+\cdots+\frac{(n+q)!\,A_q}{(z-a)^{n+q+1}}\right)+\Lambda^{(n)}(z),$$

so it follows that

$$\begin{aligned}\frac{F^{(n)}(z)}{n!} &= (-1)^n\frac{(n+1)(n+2)\cdots(n+q)\,A_q}{(z-a)^{n+q+1}}\left(1+\frac{(z-a)^{q+1}\,(a-z)^n\,\Lambda^{(n)}(z)}{(n+1)\cdots(n+q)\,n!\,A_q}+\right.\\ &\quad\left.+\sum_{k=0}^{q-1}\frac{A_k\,(z-a)^{q-k}}{A_q\,(n+k+1)\cdots(n+q)}\right)\\ &= (-1)^n\frac{(n+1)(n+2)\cdots(n+q)\,A_q}{(z-a)^{n+q+1}}(1+\varphi_n(z)). \qquad (56)\end{aligned}$$

Next, let $\varrho(z)$ be the radius of convergence of the power series of $\Lambda(z)$ about the point z. Recall that $\varrho(z)$ is a continuous function of z. (Consider a point z' that satisfies $|z'-z| < \varrho(z)$, i.e. z' lies inside the circle of convergence about z. By considering the largest and smallest circles with center z' that touch the circle of convergence about z, we obtain that

$$\varrho(z) - |z'-z| \leq \varrho(z') \leq \varrho(z) + |z'-z|).$$

Next, consider the function $\frac{|z-a|}{\varrho(z)}$, which is continuous at every point where $\varrho(z) > 0$. Thus, $\frac{|z-a|}{\varrho(z)}$ is also continuous in the Voronoi cell $V_i(a)$ that contains a; furthermore, $\frac{|z-a|}{\varrho(z)} < 1$ everywhere in $V_i(a)$. Now, let U denote a compact subset of $V_i(a)$, and let $\varrho_0 := \min_{z\in U}\{\varrho(z)\}$ and $\alpha := \max_{z\in U}\left\{\frac{|z-a|}{\varrho(z)}\right\}$. Thus,

$$0 < \varrho_0 \leq \varrho(z), \quad \frac{|z-a|}{\varrho(z)} \leq \alpha < 1, \quad \forall z \in U. \tag{57}$$

Choose a fixed β such that

$$\alpha < \beta < 1, \tag{58}$$

and consider disks of radius $\beta\varrho(z)$ centered around every point $z \in U$. We denote the set covered by all of these disks as U^*. A point $z \in U^*$ lies at a distance $\geq (1-\beta)\varrho_0$ from the nearest singularity of $\Lambda(z)$. The function $\Lambda(z)$ is therefore regular everywhere in U^*, and there exists a number M such that for every $z \in U^*$,

$$|\Lambda(z)| \leq M.$$

But for every point $z \in U$, there is a disk of radius $\beta\varrho(z)$ that lies entirely in U^*. Therefore, for every $z \in U$,

$$\frac{|\Lambda^{(n)}(z)|}{n!} \leq \frac{M}{(\beta\varrho(z))^n},$$

which together with (57) yields that

$$\left|\frac{(z-a)^n\,\Lambda^{(n)}(z)}{n!}\right| \leq \left(\frac{\alpha}{\beta}\right)^n M. \tag{59}$$

Thus from (56), (58) and (59), it follows that the function $\varphi_n(z)$ described by the first of these equations satisfies

$$\lim_{n\to\infty}\varphi_n(z) = 0 \tag{60}$$

everywhere in U.

Finally, if we let $P_a(z) := (n+1)(n+2)\cdots(n+q)A_q$, it follows from (60) that $\lim_{n\to\infty}|P_a(z)|^{1/n} = 1$ and $\lim_{n\to\infty}|1+\varphi_n(z)|^{1/n} = 1$. Consequently, we see from (56) that

$$\left|\frac{F^{(n)}(z)}{n!}\right|^{\frac{1}{n}} = \frac{|P_a(z)|^{1/n}}{|z-a|^{(n+q+1)/n}}|1+\varphi_n(z)|^{1/n} \to \frac{1}{|z-a|}$$

as $n \to \infty$. This completes the proof of (ii). □

An alternative formulation of Pólya's Shire theorem in the context of Nevanlinna theory is given below. We refer the interested reader to [13] for a proof.

Theorem 4.3 (Pólya's Shire theorem, version 2)**.** *Suppose that $f(z)$ is meromorphic in $|z - z_0| < R$, where $0 < R \leq \infty$, and has at least two distinct poles there. Let r be the radius of the largest circle with centre z_0 containing no pole of $f(z)$ in its interior. Then*
(i) if the circle $|z - z_0| = r$ contains at least two distinct poles of f(z), then for every positive δ, the equation $f^{(n)}(z) = 0$ has roots in $|z - z_0| < \delta$, when n is sufficiently large.
(ii) if the circle $|z - z_0| = r$ contains only one pole of $f(z)$, then if δ is sufficiently small, $f^{(n)}(z) \to \infty$ as $n \to \infty$ uniformly in $|z - z_0| \leq \delta$.

Corollary 4.1. *For all sufficiently large n, $f^{(n)}(z)$ has zeros in every disk in which $f(z)$ is meromorphic and has at least two distinct poles.*

We remind the reader than an entire function $f(z)$ is said to be of finite order if there exists a positive number A such that, as $|z| = r \to \infty$,

$$f(z) = O(e^{rA}).$$

The lower bound ρ of A for which this is true is called the *order* of the function (see [24]).

Furthermore, define a function

$$E(u,s) = \begin{cases} 1-u, & \text{if } s = 0, \\ (1-u)e^{u+\frac{u^2}{2}+\cdots+\frac{u^s}{s}}, & \text{if } s = 1, 2, \ldots . \end{cases}$$

The functions $E(u,0)$, $E(u,1)$, $\ldots$ are sometimes called *primary factors.* It can be shown that if $f(z)$ is an entire function of finite order with zeros $z_1, z_2, \ldots$, then there exists a natural number s such that the product

$$\prod_{k=1}^{\infty} E(z/z_k, s) \tag{61}$$

converges for all $z \in \mathbb{C}$. In particular, if s is chosen as the smallest integer for which the series

$$\sum_{k:z_k \in Z(f(z))} (r/|z_k|)^{s+1}$$

converges for any $r \in \mathbb{R}$ (which also necessitates that (61) converges), then the product in (61) is called the *canonical product* formed with the zeros of $f(z)$. Here, s is called its *genus.* In this notation, the following important decomposition of entire functions can be given, which we present without proof.

Theorem 4.4 (Hadamard's factorization theorem)**.** *If $f(z)$ is an entire function of order ρ with zeros $z_1, z_2, \dots$ ($f(0) \neq 0$), then*

$$f(z) = e^{Q(z)} P(z),$$

where $P(z)$ is the canonical product formed with the zeros of $f(z)$, and $Q(z)$ is a polynomial with $\deg Q \leq \rho$.

4.4 Zero-counting measures and logarithmic potentials

We return to the function $f = (P/Q)e^T$, defined as in Theorem 4.1. By Pólya's Shire theorem (see Theorem 4.2 on page 35), the zeros of the iterated derivatives f', f'', f''', ... of f tend to accumulate along $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}}$, where $S = \{z_1, \dots, z_q\}$ is the set of zeros of Q.

Simple computations show that

$$\begin{aligned} f &= \frac{P_0}{Q} e^T, \\ f' &= \frac{P_1}{Q^2} e^T, \\ f'' &= \frac{P_2}{Q^3} e^T, \\ &\vdots \\ f^{(n)} &= \frac{P_n}{Q^{n+1}} e^T, \qquad (62) \\ &\vdots \end{aligned}$$

where $P_0 := P$, $P_1 = QP' + P(QT' - Q'), P_2, \dots, P_n, \dots$ are polynomials. Clearly, the zeros of $f^{(k)}$ are the zeros of the polynomial P_k, so it is of interest to investigate the structure of P_k. Since $\left(f^{(n-1)}\right)' = f^{(n)}$, by using equation (62), we see that

$$\left(\frac{P_{n-1}}{Q^n} e^T\right)' = \frac{P_n}{Q^{n+1}} e^T,$$

or equivalently,

$$\frac{P'_{n-1} Q^n - nQ^{n-1} Q' P_{n-1}}{Q^{2n}} e^T + \frac{P_{n-1}}{Q^n} T' e^T = \frac{P_n}{Q^{n+1}} e^T. \qquad (63)$$

Since e^T is nonzero in $\mathbb{C}$ for any polynomial T, dividing both sides of equation (63) by e^T and simplifying yields the recurrence relation

$$P_n = (QT' - nQ')P_{n-1} + QP'_{n-1}, \; n \geq 1, \qquad (64)$$

where $P_0 := P$.

To proceed, we make use of the assumptions that $t = \deg T \geq 1$ and $P \not\equiv 0$ in Theorem 4.1. In this situation, we see from equation (64) that the $QT'P_{n-1}$ term dominates the degree of P_n. Thus, it follows for P_k that

$$
\begin{aligned}
\deg P_0 &= \deg P = p, \\
\deg P_1 &= \deg QT'P_0 = q + t - 1 + p, \\
\deg P_2 &= \deg QT'P_1 = q + t - 1 + (q + t - 1 + p) = 2(q + t - 1) + p, \\
&\vdots \\
\deg P_n &= \deg QT'P_{n-1} = n(q + t - 1) + p. \qquad (65) \\
&\vdots
\end{aligned}
$$

In addition to the degree of P_n, we will soon make use of the coefficient A_n of the highest-power term of P_n. To determine it explicitly, let $\alpha_1, \ldots, \alpha_{\deg P_n}$ be the zeros of P_n, and let $P_n = A_n \prod_{k=1}^{\deg P_n}(z-\alpha_k)$. Furthermore, we remind the reader that $P = \sum_{k=0}^{p} b_k z^k$, $Q = \sum_{k=0}^{q} c_k z^k$ and $T = \sum_{k=0}^{t} d_k z^k$. Now note that $A_0 = b_p = (c_q\, d_t\, t)^0\, b_p$. Since A_n depends only on the $QT'P_{n-1}$ term in (64) when $t \geq 1$, $A_n = c_q\, d_t\, t\, A_{n-1}$, $n \geq 1$, and thus,

$$A_n = (c_q\, d_t\, t)^n \cdot b_p, \quad n \geq 0, \quad t \geq 1. \qquad (66)$$

Now, according to Theorem 4.2, $\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}$ converges pointwise a.e. in any open Voronoi cell to $\max\left\{\frac{1}{|z-z_i|},\ i = 1, \ldots, q\right\}$. To make use of this, we see from equation (62) that

$$
\begin{aligned}
\log\left(\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}\right) &= \frac{1}{n}\log\left|\frac{P_n}{n!\,Q^{n+1}}e^T\right| \\
&= \frac{\log|P_n|}{n} + \frac{\log\left|e^T\right|}{n} - \frac{\log|n!|}{n} - \frac{(n+1)\log|Q|}{n} \qquad (67) \\
&= \frac{\log|A_n|}{n} + \frac{\deg P_n}{n} \cdot \frac{\log\left|\prod_{k=1}^{\deg P_n}(z-\alpha_k)\right|}{\deg P_n} + \\
&\quad + \frac{\log\left|e^T\right|}{n} - \frac{\log|n!|}{n} - \frac{(n+1)\log|Q|}{n}.
\end{aligned}
$$

By equation (39), the term $\frac{\log\left|\prod_{k=1}^{\deg P_n}(z-\alpha_k)\right|}{\deg P_n}$ is the logarithmic potential $\mathcal{L}_{\mu_n}(z)$ of the zero-counting measure μ_n of P_n/A_n. We note from equation

(66) that $\frac{\log|A_n|}{n} = \log(|c_q|\,|d_t|\,t) + \frac{1}{n}\log|b_p|$ for $n \geq 1$. By passing to the limit in n in equation (67) and making use of (65), we see that

$$\lim_{n\to\infty} \frac{\log|A_n|}{n} = \log(|c_q|\,|d_t|\,t), \tag{68}$$

$$\lim_{n\to\infty} \frac{\deg P_n}{n} = q + t - 1, \tag{69}$$

$$\lim_{n\to\infty} \frac{\log\left|e^T\right|}{n} = 0, \tag{70}$$

$$\lim_{n\to\infty} -\frac{\log n!}{n} = -\infty, \tag{71}$$

and

$$\lim_{n\to\infty} -\frac{(n+1)\log|Q|}{n} = -\log|Q|. \tag{72}$$

Thus, since the left-hand side of (67) converges to

$$\log\left(\max_{i=1,\ldots,q}\left\{\frac{1}{|z-z_i|}\right\}\right) = \max_{i=1,\ldots,q}\left\{\log|z-z_i|^{-1}\right\} \tag{73}$$

inside open Voronoi cells, which is finite outside of S, it follows that $\lim_{n\to\infty}\mathcal{L}_{\mu_n}(z) = \infty$, i.e. the logarithmic potential of the asymptotic zero-counting measure $\lim_{n\to\infty}\mu_n$ does not exist in $\mathbb{C}\setminus\left(\mathrm{Vor}_S^B \cup S\right)$. This proves part (ii) of Theorem 4.1. In the same region, however, by using Stirling's formula ($\log n! = n\log n - n + O(\log n)$), it follows from (68)-(73) that

$$\mathcal{L}_{\mu_n}(z) \approx (q+t-1)^{-1}\cdot\log\left(\frac{n|Q|}{e\,t\,|c_q|\,|d_t|\,\Phi}\right)$$

for sufficiently large n, where Φ is defined in equation (54) on page 35.

Although no logarithmic potential of the asymptotic zero-counting measure μ_S exists, we can define

$$\widetilde{\mathcal{L}}_{\mu_n}(z) = \mathcal{L}_{\mu_n}(z) - \frac{\log n!}{n(q+t-1)+p},$$

which we refer to as a *shifted logarithmic potential* (of the measure μ_n). Then equation (67) can be rewritten as

$$\widetilde{\mathcal{L}}_{\mu_n}(z) = \frac{n}{n(q+t-1)+p}\left(\log\left(\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}\right) + \frac{(n+1)\log|Q|}{n} - \frac{\log|A_n|}{n} - \frac{\log|e^T|}{n}\right),$$

(74)

or, as we will find use for later, by using the expression for $f^{(n)}$ in (62),

$$\begin{aligned}\widetilde{\mathcal{L}}_{\mu_n}(z) &= \frac{1}{n(q+t-1)+p}\left(\log\left|\frac{P_n}{A_n}\right| - \log n!\right)\\ &= \frac{1}{n(q+t-1)+p}\left(\sum_{k=1}^{\deg P_n}\log|z-\alpha_k| - \log n!\right).\end{aligned} \tag{75}$$

By letting $n \to \infty$ in (74), we obtain the equation

$$\Psi(z) := \lim_{n\to\infty}\widetilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} + \log|Q| - \log\left(|c_q|\,|d_t|\,t\right)\right), \tag{76}$$

where the right-hand side has converged pointwise to a *continuous subharmonic function* defined in the whole complex plane, as we will see in Lemma 4.1 in the next section. Although the existence of the asymptotic zero-counting measure $\mu_S := \lim_{n\to\infty}\mu_n$ will not be proven until section 4.8, we will (conditioned on its existence) refer to $\Psi(z)$ as the shifted logarithmic potential of μ_S. Furthermore, note that (by part (ii) of Theorem 4.2), this convergence is pointwise in any open Voronoi cell V_i^o.

In addition to pointwise convergence, the aforementioned convergence can be shown to be uniform on compact subsets of V_i^o. To see this, let W denote a compact subset of V_i^o that does not contain a zero of Q. It follows (from Theorem 4.2 again) that there is a number N_W such that for all $n > N_W$, no zeros of P_n are contained in W. Hence for $n > N_W$, both $S_n(z) := \frac{\log\left|\prod_{k=1}^{\deg P_n}(z-\alpha_k)\right|}{n}$ and $\lim_{n\to\infty}(\deg P_n/n)\mathcal{L}_{\mu_n}(z)$ are continuous and harmonic functions on W, and consequently, so is $\widetilde{\mathcal{L}}_{\mu_n}(z)$.

By Theorem 4.2, the term $\log\left(\left|\frac{f^{(n)}}{n!}\right|^{\frac{1}{n}}\right)$ converges uniformly to $\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\}$ on W. Since the remaining terms in the right-hand side of (74) converge uniformly on W in accordance with equations (72), (68) and (70), respectively (e.g. due to the continuity of Q and absence of zeros of Q in W), it follows that $\widetilde{\mathcal{L}}_{\mu_n}(z)$ converges uniformly to $\Psi(z)$ on W.

This result can be extended to all compact subsets of V_i^o. Assume that $\tilde{W} \subset V_i^o$ is a compact set. We can further assume that $\tilde{W}$ contains a zero z_i of Q in its interior, since if the result holds for this situation, then it can easily be extended for the more general set $\tilde{W}$. Since z_i lies in the interior of $\tilde{W}$, we can assume that the open disk $D(z_i,\rho) \subset \tilde{W}$. Let $W = \tilde{W} \setminus D(z_i,\rho)$

and use the previous result to get a number N such that $n \geq N$ implies that $-\epsilon < S_n(z) - \lim_{n\to\infty}(\deg P_n/n)\mathcal{L}_{\mu_n}(z) < \epsilon$ in W. Since W contains the boundary of $D(z_i, \rho)$ and both $S_n(z)$ and $\lim_{n\to\infty}(\deg P_n/n)\mathcal{L}_{\mu_n}(z)$ are harmonic in $D(z_i, \rho)$, it follows by the maximum principle that $-\epsilon < S_n(z) - \lim_{n\to\infty}(\deg P_n/n)\mathcal{L}_{\mu_n}(z) < \epsilon$ in $\tilde{W}$.

We summarize the aforementioned results about convergence in Proposition 4.1 below.

Proposition 4.1. *Let* $\mathcal{L}_{\mu_n}(z) = \frac{\log|P_n| - \log|A_n|}{\deg P_n}$ *be the logarithmic potential of the zero-counting measure* μ_n *of* P_n/A_n*. Furthermore, let* $\widetilde{\mathcal{L}}_{\mu_n}(z) = \mathcal{L}_{\mu_n}(z) - \frac{\log n!}{n(q+t-1)+p}$*. Then for any* z *in the interior of the Voronoi cell* V_i^o*, we have pointwise convergence*

$$\lim_{n\to\infty} \widetilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} + \log|Q| - \log\left(|c_q|\,|d_t|\,t\right)\right) =: \Psi(z). \tag{77}$$

The convergence is uniform on compact subsets of V_i^o*.*

It is intuitive to use equation (41) to reconstruct what will be shown to be the asymptotic zero-counting measure μ_S from $\Psi(z)$ as

$$\mu_S = \frac{1}{2\pi(q+t-1)}\left(\Delta \max_{i=1,\dots,q}\left\{\log|z-z_i|^{-1}\right\} + \Delta\log|Q|\right). \tag{78}$$

Of course, the same measure μ_S is obtained by first multiplying both sides of equation (74) by $(2\pi)^{-1}$, taking the Laplacian, using (41), and letting $n \to \infty$. Thus, if μ_S exists, it must be given by (78), which is the same asymptotic zero-counting measure, up to multiplication by a constant, as in the rational case discussed in [6].

4.5 Harmonic, subharmonic and superharmonic functions

To describe some additional properties of $\Psi(z)$, and to prove that the asymptotic zero-counting measure of P_n/A_n given by equation (78) exists, we need a few more definitions from potential theory (see [17]).

Definition 4.1. *Let* D *be a domain (a connected open set) in* $\mathbb{C}$*, and let* $g(z)$ *be a real-valued function of* $z = x + iy \in D$*. Then* $g(z)$ *is said to be harmonic in* D *if all of its second-order partial derivatives are continuous in* D *and if, at each point of* D*,* g *satisfies Laplace's equation*

$$\Delta g := \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 0. \tag{79}$$

Clearly, $\Psi(z)$ is harmonic in the interior of each cell V_i, and is an example of a *piecewise harmonic* function.

Definition 4.2. *Let U be an open subset of $\mathbb{C}$. A function g is called subharmonic if g is upper semi-continuous and satisfies the local submean inequality. Namely, for any $w \in U$, there exists a $\rho > 0$ such that*

$$g(w) \leq \frac{1}{2\pi} \int_0^{2\pi} g(w + \rho e^{i\tau}) \, \mathrm{d}\tau \quad (0 \leq r \leq \rho). \tag{80}$$

The integral (80) is the Lebesgue integral and is well-defined for upper semi-continuous functions. Finally, a function h is called *superharmonic* if $-h$ is subharmonic.

Lemma 4.1. *The function $\Psi(z)$, defined in $\mathbb{C}$, is a continuous subharmonic function, and is harmonic in the interior of any cell V_i.*

Proof. In the Voronoi cell V_i, we have

$$(q + t - 1)\Psi(z) = \log |z - z_i|^{-1} + \log |Q| - \log (|c_q| \, |d_t| \, t), \tag{81}$$

hence $\Psi(z)$ is continuous in $\mathbb{C}$, and harmonic in the interior of V_i. Also, $\Psi(z)$ is the maximum of a finite number of harmonic functions, and is thus subharmonic (see [14, 22]). □

Since $\Psi(z)$ is subharmonic, $\Delta\Psi(z) = 4\frac{\partial^2 \Psi(z)}{\partial \bar{z} \partial z}$ is a positive measure with support on $\mathrm{Vor}_{\mathrm{S}}^{\mathrm{B}}$.

Proposition 4.2. *Define a measure with support on the line $l_{ij} : \; |z - z_i| = |z - z_j|$ as*

$$\delta_{ij} = \frac{1}{4(q + t - 1)} \frac{|z_i - z_j|}{|(z - z_i)(z - z_j)|} \, \mathrm{d}s.$$

where $\mathrm{d}s$ *is Euclidean length measure in the complex plane. Then*

1. $\frac{\partial^2 \Psi}{\partial \bar{z} \partial z}$ *is the sum of all δ_{ij}, each restricted to V_{ij}.*
2. $\mu_S := \frac{2}{\pi} \frac{\partial^2 \Psi}{\partial \bar{z} \partial z}$ *has mass $(q - 1)/(q + t - 1)$.*

Remark 4.1. *The connection between the probability measures μ_n and the measure μ_S with mass $(q-1)/(q+t-1)$ may appear surprising, as it implies that a mass of $t/(q + t - 1)$ disappears as $n \to \infty$. Intuitively, however; consider the situation in Figure 14, where $q = 5$, $t = 1$, and $\deg P_n = 75$. In this case, $t/(q + t - 1) = 1/5$, which is approximately the fraction of the zeros of P_n that reside on "bubbles" whose structure does not appear to* *converge on the Voronoi diagram of $Z(Q)$ (compare this to Figure 1 in [6], where $t = 0$). Numerical experiments indicate that these "bubbles" expand toward ∞ asymptotically, so we conjecture that this structure is responsible for the mass discrepancy.*

We will later, in Corollary 4.2, also see that $\Psi(z)$ is the logarithmic potential of μ_S. For the proof of Proposition 4.2 in section 4.7, we need to recall some general facts about piecewise harmonic functions, which we present in section 4.6. The material in these next two sections is almost identical to that in the previous paper of Rikard Bögvad and this author (see [6]), with some minor adjustments for our present situation.

4.6 The Laplacian of a piecewise harmonic function

Assume that Ψ is a continuous and subharmonic function, defined on $\mathbb{C}$, harmonic and equal to H_i in the open sets V_i, $i = 1, \ldots, q$, that form a disjoint cover of $\mathbb{C}$ a.e. Furthermore, assume that each H_i is defined and harmonic in an open set containing V_i. Let $A_i := \partial H_i / \partial z$. Assume finally that $\partial V_i = \cup_j V_{ij}$ is a finite union of C^1 curves, where $V_{ij} \subset \partial V_i \cap \partial V_j$ are closed as sets. We let $V_{ij} = V_{ji}$, and also define $V_{ij} = \emptyset$ if V_i and V_j have no common curve segment as a border between them. Orient V_{ij} so that $V_{\min(i,j)}$ is on the right of V_{ij}.

Lemma 4.2. *As distributions, acting on a test function $f \in C_c(\mathbb{C})$,*

1.
$$\left\langle \frac{\partial \Psi}{\partial z}, f \right\rangle = \sum_i \int_{V_i} A_i(z) f(z) \, \mathrm{d}x \wedge \mathrm{d}y.$$

2.
$$\left\langle \frac{\partial^2 \Psi}{\partial \bar{z} \partial z}, f \right\rangle = \sum_{i<j} \frac{1}{2i} \int_{V_{ij}} (A_i(z) - A_j(z)) f(z) \, \mathrm{d}z.$$

Proof. In an interior point $z \in V_i$, Ψ has a pointwise derivative $A_i = \partial H_i / \partial z$, and hence Ψ has a pointwise derivative a.e. (with respect to Lebesgue measure). This derivative is a L^1_{loc}-function, and so defines a distribution. For a general function, the pointwise derivative does not necessarily coincide with the distributional derivative; however for a function that is continuous and piecewise differentiable, this is true (see e.g. Prop. 2 in [4]). This proves (1).

Hence

$$\left\langle \frac{\partial^2 \Psi}{\partial \bar{z} \partial z}, f \right\rangle = -\left\langle \frac{\partial \Psi}{\partial z}, \frac{\partial f(z)}{\partial \bar{z}} \right\rangle = -\sum_i \int_{V_i} A_i(z) \frac{\partial f(z)}{\partial \bar{z}} \, \mathrm{d}x \wedge \mathrm{d}y.$$

But, since A_i is holomorphic,

$$A_i(z) \frac{\partial f(z)}{\partial \bar{z}} = \frac{\partial (A_i(z) f(z))}{\partial \bar{z}}, \ z \in V_i,$$

so the last sum equals, using Stokes' theorem ([16, 1.2.2]),

$$-\frac{1}{2i} \sum_i \int_{\partial V_i} A_i(z) f(z) \, \mathrm{d}z. \tag{82}$$

The curve integrals in this sum use an orientation of ∂V_i, such that V_i is to the left of ∂V_i. Rewriting (82) as a sum of curve integrals on V_{ij}, noting that each V_{ij} occurs in both ∂V_i and ∂V_j with opposite orientation gives (2). □

4.7 Proof of Proposition 4.2

Let $\Psi(z)$ be as in equation (77). If we let $\widetilde{\Psi}(z)$ be the logarithmic potential of the zero-counting measure $\widetilde{\mu}_S$ used in the situation of Proposition 2.2 of [6] (i.e. where $t = 0$), this gives the relation

$$\Delta \Psi(z) = \frac{q-1}{q+t-1} \Delta \widetilde{\Psi}(z). \tag{83}$$

Now define,

$$A_i(z) = \frac{\partial \widetilde{\Psi}}{\partial z} = (2q-2)^{-1} \left(-(z-z_i)^{-1} + \sum_{j=1}^{q} (z-z_j)^{-1} \right), \ z \in V_i. \tag{84}$$

Consequently, by (84),

$$A_i(z) - A_j(z) = \frac{z_j - z_i}{(2q-2)(z-z_i)(z-z_j)}.$$

V_{ij} is a segment of the orthogonal bisector of the line segment between z_1 and z_2, that is given by the equation $z = (1/2)(z_i + z_j) - ti(z_j - z_i), \ t \in \mathbb{R}$ (with the orientation that V_i is to the right of V_{ij}). Assume that $z \in V_{ij}$. Then $z - z_i = (z_j - z_i)(1/2 - ti)$ as well as $z - z_j = (z_j - z_i)(-1/2 - ti)$, and since $\mathrm{d}z = -i(z_j - z_i) \, \mathrm{d}t$,

$$(A_i(z) - A_j(z)) \, \mathrm{d}z = \frac{i \, \mathrm{d}t}{(2q-2)(1/4 + t^2)}. \tag{85}$$

The Pythagorean theorem applied to the triangle with vertices z, z_i, and $(1/2)(z_i + z_j)$ gives that $|z - z_i|^2 = |z_j - z_i|^2((1/4) + t^2)$. Length measure along V_{ij} is given by $\mathrm{d}s = |z_j - z_i|\,\mathrm{d}t$. Inserting these expressions for t and $\mathrm{d}t$ in (85), and applying Lemma 4.2 (2), gives part (1) of the proposition.

For the second part of the proof, let C_ρ be the circle $|z| = \rho$, and D_ρ be the disk $|z| \leq \rho$. Choose ρ so large that all bounded cells in the Voronoi diagram are contained in the interior of D_ρ, and let χ_ρ be the characteristic function of D_ρ. By (82) in the proof of Lemma 4.2,

$$\widetilde{\mu}_S(D_\rho) = \frac{2}{\pi}\left\langle \frac{\partial^2 \widetilde{\Psi}}{\partial \bar{z} \partial z}, \chi_\rho \right\rangle = -\frac{1}{\pi i} \sum_i \int_{\partial V_i \cap D_\rho} A_i(z)\,\mathrm{d}z.$$

The contribution of the boundaries of the bounded cells to the sum is zero, and the contribution of the unbounded cells $V_i,\ i = 1, \ldots, q$ equals the curve integral of $A_i(z) = \frac{\partial \widetilde{\Psi}}{\partial z}$ on $V_i \cap C_\rho$ (both statements by Cauchy's theorem), so that

$$\widetilde{\mu}_S(D_\rho) = \frac{1}{\pi i} \sum_{i=1}^{q} \int_{\partial V_i \cap C_\rho} A_i(z)\,\mathrm{d}z.$$

Note that the induced orientation on C_ρ is positive. It remains only to exploit that $A_i(z) \sim 1/2z$ at infinity. Let $\frac{\partial \widetilde{\Psi}}{\partial z} = (1/2z) + B(z)$. By (84),

$$B(z) = \frac{1}{2q-2} \sum_{j \neq i} \frac{z_j}{z(z - z_j)}, \qquad \text{for } z \in V_i,$$

so that it, by trivial estimates, satisfies $\lim_{\rho\to\infty} \int_{C_\rho} B(z)\,\mathrm{d}z = 0$. Hence

$$\lim_{\rho\to\infty} \widetilde{\mu}_S(D_\rho) = \frac{1}{2\pi i} \int_{C_\rho} \frac{\mathrm{d}z}{z} = 1.$$

Finally, by (83),

$$\lim_{\rho\to\infty} \mu_S(D_\rho) = \frac{q-1}{q+t-1}.$$

4.8 Proof of Theorem 4.1

Uniform convergence a.e. as in Proposition 4.1 does not by itself imply convergence of the logarithmic potentials in L^1_{loc}, though it tells us that there is only one possible limit, since a function in L^1_{loc} is determined by its behavior a.e. We will prove the L^1_{loc} convergence directly, with the main difficulty being the unboundedness of the zeros of P_n as $n \to \infty$. To deal with this problem, we give rough bounds of the growth of the zeros of P_n in Lemma 4.4 below.

4.8.1 Growth of zeros

Consider a fixed meromorphic function $f(z) := (P/Q)e^T$ as in Theorem 4.1. Lemma 4.3 below shows that if the statement of the theorem holds for $f(z)$, it also holds for $\widehat{f}(z) := f(\tau z + a)$, $\tau \in \mathbb{R}_+$, $a \in \mathbb{C}$, i.e. the theorem is invariant under scaling and translation. For convenience, let $\widehat{\mu}_n$ be the zero-counting measure of $\widehat{f}^{(n)}$ (or, technically, of the polynomial $\prod_k (z - \widehat{\alpha}_k)$, where the product is taken over all zeros $\widehat{\alpha}_1, \widehat{\alpha}_2, \ldots$ of $\widehat{f}^{(n)}$), and let $\widetilde{\mathcal{L}}_{\widehat{\mu}_n}(z)$ be its shifted logarithmic potential.

Lemma 4.3. *Assume that $\widetilde{\mathcal{L}}_{\mu_n}(z) \to \Psi(z)$ in L^1_{loc}, where $\Psi(z)$ is the shifted logarithmic potential given by (77) of the asymptotic zero-counting measure $\lim_{n\to\infty} \mu_n$. Then $\widetilde{\mathcal{L}}_{\widehat{\mu}_n}(z) \to \widehat{\Psi}(z)$ in L^1_{loc}, where $\widehat{\Psi}(z)$ is the shifted logarithmic potential of $\lim_{n\to\infty} \widehat{\mu}_n$.*

Proof. We see from (62) that

$$\widehat{f}^{(n)}(z) = \frac{\widehat{P}_n(z)}{(Q(\tau z + a))^{n+1}} e^{T(\tau z + a)}, \tag{86}$$

for some polynomial $\widehat{P}_n(z) := \widehat{A}_n \prod_{k=1}^{n(q+t-1)+p} (z - \widehat{\alpha}_k)$, where $\widehat{A}_n$ is a complex number. Similarly,

$$\begin{aligned} \widehat{f}^{(n)}(z) &= (f(\tau z + a))^{(n)} = \tau^n f^{(n)}(\tau z + a) \\ &= \tau^n \left(\frac{P_n(\tau z + a)}{(Q(\tau z + a))^{n+1}} e^{T(\tau z + a)} \right). \end{aligned} \tag{87}$$

By comparing equations (86) and (87), we see that

$$\widehat{P}_n(z) = \tau^n P_n(\tau z + a). \tag{88}$$

Consequently, by using the definitions of $\widehat{P}_n(z)$ and $P_n(z)$ in (88), it follows that

$$\widehat{A}_n \prod_{k=1}^{n(q+t-1)+p} (z - \widehat{\alpha}_k) = \tau^n A_n \prod_{k=1}^{n(q+t-1)+p} (\tau z + a - \alpha_k),$$

or equivalently,

$$\widehat{A}_n \prod_{k=1}^{n(q+t-1)+p} (z - \widehat{\alpha}_k) = \tau^{n(q+t)+p} A_n \prod_{k=1}^{n(q+t-1)+p} \left(z - \frac{\alpha_k - a}{\tau} \right). \tag{89}$$

Hence, it follows from (89) that

$$\widehat{A}_n = \tau^{n(q+t)+p} A_n. \tag{90}$$

Thus, by using (88) and (90) in (75),

$$\begin{aligned}
\widetilde{\mathcal{L}}_{\widehat{\mu}_n}(z) &= \frac{1}{n(q+t-1)+p}\left(\log\left|\frac{\widehat{P}_n(z)}{\widehat{A}_n}\right| - \log n!\right) \\
&= \frac{1}{n(q+t-1)+p}\left(\log\left|\frac{P_n(\tau z+a)}{\tau^{n(q+t-1)+p}A_n}\right| - \log n!\right) \\
&= \frac{1}{n(q+t-1)+p}\left(\log\left|\frac{P_n(\tau z+a)}{A_n}\right| - \log n!\right) - \log\tau \\
&= \widetilde{\mathcal{L}}_{\mu_n}(\tau z+a) - \log\tau.
\end{aligned} \tag{91}$$

As a result of (91) and the assumption of the lemma, $\widetilde{\mathcal{L}}_{\widehat{\mu}_n}(z) \to \Psi(\tau z+a) - \log\tau$ in L^1_{loc}.

To see that $\widehat{\Psi}(z) := \Psi(\tau z+a) - \log\tau$ is the correct shifted logarithmic potential of $\lim_{n\to\infty}\widehat{\mu}_n$ (rather than some other L^1_{loc}-function), we also need to prove that it satisfies equation (53). To do this, define $\widehat{c}_k$ and $\widehat{d}_k$ as the coefficients of z^k in $Q(\tau z+a)$ and $T(\tau z+a)$, respectively. Then, by using the definitions of $Q(z)$ and $T(z)$, we see that

$$Q(\tau z+a) = \sum_{k=0}^{q}\widehat{c}_k z^k = \sum_{k=0}^{q} c_k(\tau z+a)^k,$$

so

$$\widehat{c}_q = \tau^q c_q, \tag{92}$$

and similarly,

$$\widehat{d}_t = \tau^t d_t. \tag{93}$$

Furthermore, for each zero z_i of $Q(z)$, $\widehat{z}_i := (z_i - a)/\tau$ is a zero of $Q(\tau z+a)$. Consequently, by using this bijective correspondence between z_i and $\widehat{z}_i$,

$$\max_{i=1,\dots,q}\left\{\log|z-\widehat{z}_i|^{-1}\right\} = \max_{i=1,\dots,q}\left\{\log\left|z+\frac{a-z_i}{\tau}\right|^{-1}\right\} = \max_{i=1,\dots,q}\left\{\log|\tau z+a-z_i|^{-1}\right\} + \log\tau. \tag{94}$$

Finally, by using (92), (93), and (94) in the right-hand side of (53) for $\widehat{f}(z)$, we see that

$$\begin{aligned}
&\frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|z-\widehat{z}_i|^{-1}\right\} + \log|Q(\tau z+a)| - \log\left(|\widehat{c}_q||\widehat{d}_t|t\right)\right) \\
&= \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|\tau z+a-z_i|^{-1}\right\} + \log\tau + \log|Q(\tau z+a)| - \log\left(\tau^{q+t}|c_q||d_t|t\right)\right) \\
&= \frac{1}{q+t-1}\left(\max_{i=1,\dots,q}\left\{\log|\tau z+a-z_i|^{-1}\right\} + \log|Q(\tau z+a)| - \log\left(|c_q||d_t|t\right)\right) - \log\tau \\
&= \Psi(\tau z+a) - \log\tau = \widehat{\Psi}(z).
\end{aligned}$$

□

Next, let $D_\rho(b)$ denote the open disk with center b and radius ρ. By choosing b as one of the poles of $f(z)$, and by letting ρ be sufficiently small, it follows from Theorem 4.2 that $D_\rho(b)$ contains no zeros of $f^{(n)}(z)$ for all large enough n. More precisely, we may after a scaling and translation assume that the following holds due to Lemma 4.3:

(*) The closed disk $\bar{D}_2(0)$ contains exactly one pole $z_i = 0$ (so that $Q(0) = 0$).

It follows from (*), by Proposition 4.1, that there is a number N such that $z \in \bar{D}_1(0) \subset V_i^o \implies P_n(z) \neq 0$, if $n \geq N$. Equivalently, if $n \geq N$ and $P_n(z) = 0$, then $|z| > 1$.

Before we give bounds for the growth of the zeros of P_n, we define some additional notation for convenience. For $K \subset \mathbb{C}$, let

$$|z_{K,n}| := \prod_{z \in K : P_n(z) = 0} |z|,$$

(zeros taken with multiplicities; note that if there are no zeros of $P_n(z)$ in K, then $|z_{K,n}| = 1$). Let $\mathcal{D}_\rho := D_\rho(0) = \{z : |z| < \rho\}$, for $\rho > 0$, and let $m_n := \deg P_n = n(q + t - 1) + p$.

Lemma 4.4. *Assume (*). Then there are real numbers C_1, C_2, C_3, and N such that*

$$\lim_{n\to\infty} \frac{1}{m_n} \log\left|\frac{P_n(0)}{n!}\right| = \frac{\log|Q'(0)|}{q+t-1} =: C_1, \tag{95}$$

$$\lim_{n\to\infty} \frac{1}{m_n} \log\left(\frac{|z_{\mathcal{D}_\rho,n}|}{n!}\right) = -\infty, \tag{96}$$

$$C_2 \leq \frac{1}{m_n} \log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right) \leq C_3, \qquad \textit{for all } n \geq N. \tag{97}$$

Proof. Since $Q(0) = 0$ by (*), it follows from the assumptions in Theorem 4.1 that $P(0) \neq 0$, and $Q'(0) \neq 0$. Consequently, the recurrence relation (64) yields that

$$P_n(0) = -nQ'(0)P_{n-1}(0), \ \forall n \geq 1. \tag{98}$$

Because $P_0(0) = P(0)$ by definition, the solution of (98) is

$$P_n(0) = n!(-Q'(0))^n P(0), \ \forall n \geq 0. \tag{99}$$

Thus, it follows from (99) that

$$\frac{1}{m_n} \log\left|\frac{P_n(0)}{n!}\right| = \frac{n\log|Q'(0)|}{n(q+t-1)+p} + \frac{\log|P(0)|}{n(q+t-1)+p},$$

which proves equation (95) when $n \to \infty$.

Next, we consider the situation in which $0 < \rho \leq 1$. Due to Proposition 4.1, there exists a number N such that $|z_{\mathcal{D}_\rho,n}| = 1$ for all $n \geq N$. Consequently, equation (96) follows in this case from a trivial calculation involving (71).

Since $|P_n(0)| = |A_n||z_{\mathcal{D}_\rho,n}||z_{\mathcal{D}_\rho^c,n}|$, we obtain the equation

$$\frac{1}{m_n}\log\left|\frac{P_n(0)}{n!}\right| = \frac{1}{m_n}\Big(\log|A_n| + \log|z_{\mathcal{D}_\rho,n}| + \log|z_{\mathcal{D}_\rho^c,n}| - \log n!\Big). \quad (100)$$

Because $\lim_{n\to\infty}(1/m_n)\log|A_n| = (\log|c_q|d_t||t)/(q+t-1)$ by (68), $(1/m_n)\log|z_{\mathcal{D}_\rho,n}| = 0$ for all $n \geq N$, and the fact that the left-hand side (and thus also the right-hand side) of equation (100) converges due to the limit in (95), it follows that

$$\lim_{n\to\infty}\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right) = \frac{\log|Q'(0)| - \log(|c_q||d_t|t)}{q+t-1} =: C.$$

Hence, for any fixed $\epsilon > 0$, we can choose $C_2 = C - \epsilon$ and $C_3 = C + \epsilon$. Consequently, there exists a number $N = N(\epsilon)$ such that (97) follows in this case.

We proceed with the case $\rho > 1$. In this situation, we see from (100) that

$$\lim_{n\to\infty}\frac{1}{m_n}\Big(\log|z_{\mathcal{D}_\rho,n}| + \log|z_{\mathcal{D}_\rho^c,n}| - \log n!\Big) = C. \quad (101)$$

Assume that $\lim_{n\to\infty}(1/m_n)\log(|z_{\mathcal{D}_\rho^c,n}|/n!) = \infty$, for some subsequence of n. In order for (101) to be valid, we must have that $\lim_{n\to\infty}(1/m_n)\log|z_{\mathcal{D}_\rho,n}| = -\infty$ over the same subsequence. Furthermore, note that there exists a number N' such that all the zeros of R_n in $\mathcal{D}_\rho$ are contained in the annulus $\{z : 1 \leq |z| < \rho\}$ for all $n \geq N'$. Hence, $\rho > 1 \implies |z_{\mathcal{D}_\rho,n}| \geq 1 \implies (1/m_n)\log|z_{\mathcal{D}_\rho,n}| \geq 0$ for all large enough n, resulting in a contradiction. Thus, there exists a number C_3 such that

$$\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right) \leq C_3, \quad (102)$$

for all $n \geq N'$.

Next, assume that $\lim_{n\to\infty}(1/m_n)\log(|z_{\mathcal{D}_\rho^c,n}|/n!) = -\infty$ for some subsequence of n. Then by (101), it follows that $\lim_{n\to\infty}(1/m_n)\log|z_{\mathcal{D}_\rho,n}| = \infty$ over the same subsequence. Since the number of zeros of R_n in $\mathcal{D}_\rho$ is at most $m_n = n(q+t-1)+p$ for any fixed n, it follows that $|z_{\mathcal{D}_\rho,n}| < \rho^{m_n}$, or equivalently, $(1/m_n)\log|z_{\mathcal{D}_\rho,n}| < (1/m_n)\log(\rho^{m_n}) = \log\rho$, for all $n \geq 1$. This is another contradiction. Consequently, there exist numbers C_2 and N'' such that

$$C_2 \leq \frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right), \tag{103}$$

for all $n \geq N''$. Thus, by choosing $N = \max\{N', N''\}$, equation (97) follows from (102) and (103) in this case.

By subtracting $(1/m_n)\log n!$ from both sides of equation (100) and rearranging its terms, it is seen that

$$\frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}_\rho,n}|}{n!}\right) = \frac{1}{m_n}\left(\log\left|\frac{P_n(0)}{n!}\right| - \log|A_n| - \log\left(\frac{|z_{\mathcal{D}_\rho^c,n}|}{n!}\right) - \log n!\right). \tag{104}$$

Since the first three terms in the right-hand side of equation (104) are bounded (due to (95), (68), and (97), respectively), while its fourth term is unbounded, equation (96) follows in this case as well. □

An immediate consequence of Lemma 4.4 (under its assumptions) and equation (100) is that there exist real numbers C_4, C_5, and N such that $C_4 \leq (1/m_n)\log|z_{\mathcal{D}_\rho,n}| \leq C_5$ for all $n \geq N$.

4.8.2 L^1_{loc} convergence of the logarithmic potentials

Recall that we have previously proven (ii) of Theorem 4.1 in section 4.4. Note that if we prove (iii) of the theorem, then the whole theorem follows, since parts (i) and (iv) are immediate consequences of (iii).

To proceed, fix a number $0 < \epsilon < 1$. Recall that $\mathcal{D}_\rho$ is the disk of fixed radius $\rho > 0$ centered at the origin, and let $U \subset \mathcal{D}_\rho$ be the set of points on $\mathcal{D}_\rho$ that are at least a distance ϵ away from $\mathrm{Vor}^{\mathrm{B}}_{\mathrm{S}}$. To prove that the convergence of $\widetilde{\mathcal{L}}_{\mu_n}(z)$ to $\Psi(z)$ is L^1_{loc}, we must show that, for arbitrary ρ,

$$I_1 := \int_{\mathcal{D}_\rho}\left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \mathrm{d}\lambda = O(\epsilon), \tag{105}$$

(that is, an ϵ can be chosen so that I_1 is arbitrarily close to 0) where λ is Lebesgue measure on $\mathbb{C}$. It is appropriate to split the integral I_1 into two integrals and deal with each one separately:

$$I_1 = \int_U \left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \mathrm{d}\lambda + \int_{\mathcal{D}_\rho\setminus U}\left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \mathrm{d}\lambda =: I_2 + I_3. \tag{106}$$

Since U is the union of $\deg Q$ compact subsets of $\mathcal{D}_\rho \setminus \mathrm{Vor}^{\mathrm{B}}_{\mathrm{S}}$, it follows from the uniform convergence in Proposition 4.1 that there exists a number N such that $n \geq N$ implies that $\left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \leq \epsilon$ if $z \in U$. Hence

$$I_2 := \int_U \left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \,\mathrm{d}\lambda \leq \pi\rho^2\epsilon = O(\epsilon). \tag{107}$$

The integral I_3 is appropriately bounded by the triangle inequality:

$$\int_{\mathcal{D}_\rho\setminus U} \left|\widetilde{\mathcal{L}}_{\mu_n}(z) - \Psi(z)\right| \,\mathrm{d}\lambda \leq \int_{\mathcal{D}_\rho\setminus U} \left|\widetilde{\mathcal{L}}_{\mu_n}(z)\right| \,\mathrm{d}\lambda + \int_{\mathcal{D}_\rho\setminus U} |\Psi(z)| \,\mathrm{d}\lambda =: I_5 + I_4. \tag{108}$$

If $M_1 := \max\{\Psi(z),\ z \in \mathcal{D}_\rho\}$, the last integral satisfies

$$I_4 \leq M_1 \lambda(\mathcal{D}_\rho \setminus U) \leq 2\ell\epsilon M_1, \tag{109}$$

where ℓ denotes the length of $\mathrm{Vor}^{\mathrm{B}}_{\mathrm{S}} \cap \mathcal{D}_\rho$. Thus, $I_4 = O(\epsilon)$.

To deal with the last integral I_5, we write

$$\widetilde{\mathcal{L}}_{\mu_n}(z) = \frac{1}{m_n}\left(\sum_{k=1}^{m_n} \log|z-\alpha_k| - \log n!\right) = \widetilde{\mathcal{L}}^o_{\mu_n}(z) + \widetilde{\mathcal{L}}^i_{\mu_n}(z), \tag{110}$$

where $\widetilde{\mathcal{L}}^o_{\mu_n}(z) := (1/m_n)\left(\sum_{|\alpha_k|\geq\rho+1} \log|z-\alpha_k| - \log n!\right)$ and $\widetilde{\mathcal{L}}^i_{\mu_n}(z) := (1/m_n)\left(\sum_{|\alpha_k|<\rho+1} \log|z-\alpha_k|\right)$. Thus, by using the triangle inequality again,

$$I_5 \leq \int_{\mathcal{D}_\rho\setminus U} \left|\widetilde{\mathcal{L}}^o_{\mu_n}(z)\right| \,\mathrm{d}\lambda + \int_{\mathcal{D}_\rho\setminus U} \left|\widetilde{\mathcal{L}}^i_{\mu_n}(z)\right| \,\mathrm{d}\lambda =: I_6 + I_7. \tag{111}$$

Consequently, for such ρ,

$$0 \leq \log|z-\alpha_k| \leq \log(\rho + |\alpha_k|) \leq \log(\rho+1) + \log|\alpha_k|, \quad \text{if } |z| < \rho,\ |\alpha_k| \geq \rho+1,$$

so it follows that

$$\begin{aligned} I_6 &= \int_{\mathcal{D}_\rho\setminus U} \left|\widetilde{\mathcal{L}}^o_{\mu_n}(z)\right| \,\mathrm{d}\lambda \\ &\leq \frac{1}{m_n}\int_{\mathcal{D}_\rho\setminus U} \left|\sum_{|\alpha_k|\geq\rho+1} (\log(\rho+1) + \log|\alpha_k|) - \log n!\right| \,\mathrm{d}\lambda \\ &\leq \int_{\mathcal{D}_\rho\setminus U} \left|\log(\rho+1) + \frac{1}{m_n}\log\left(\frac{|z_{\mathcal{D}^c_{\rho+1},n}|}{n!}\right)\right| \,\mathrm{d}\lambda \\ &\leq (\log(\rho+1) + \max\{|C_2|, |C_3|\})\,\lambda(\mathcal{D}_\rho \setminus U) = O(\epsilon), \end{aligned} \tag{112}$$

where the last inequality holds for all sufficiently large n due to Lemma 4.4. (Also note that the inequality $\log(\rho + |\alpha_k|) \leq \log(\rho+1) + \log|\alpha_k|$ corrects a minor mistake in [6], where the corresponding, incorrect inequality was $\log(\rho + |\alpha_k|) \leq \log\rho + \log|\alpha_k|$.)

Finally, if in addition to $|z| < \rho$ and $|\alpha_k| < \rho + 1$ we also have $|z - \alpha_k| > \epsilon$, then $|\log|z - \alpha_k|| < \max\{-\log\epsilon, \log(2\rho+1)\}$. This leads to the inequalities

$$\begin{aligned}
&\int_{\mathcal{D}_\rho \setminus U} |\log|z-\alpha_k||\,\mathrm{d}\lambda \\
&< \int_{|z-\alpha_k|\leq\epsilon} |\log|z-\alpha_k||\,\mathrm{d}\lambda + \max\{-\log\epsilon, \log(2\rho+1)\}\lambda(\mathcal{D}_\rho \setminus U) \\
&\leq 2\pi(1/2 - \log\epsilon)(\epsilon^2/2) + \max\{-\log\epsilon, \log(2\rho+1)\}\epsilon = o(1).
\end{aligned} \tag{113}$$

Consequently, from (113),

$$\begin{aligned}
I_7 &= \int_{\mathcal{D}_\rho \setminus U} \left|\widetilde{\mathcal{L}}^i_{\mu_n}(z)\right| \mathrm{d}\lambda \\
&= \frac{1}{m_n} \int_{\mathcal{D}_\rho \setminus U} \left| \sum_{|\alpha_k|<\rho+1} \log|z-\alpha_k| \right| \mathrm{d}\lambda \\
&\leq \frac{1}{m_n} \sum_{|\alpha_k|<\rho+1} \left(\int_{\mathcal{D}_\rho \setminus U} |\log|z-\alpha_k||\,\mathrm{d}\lambda \right) = o(1),
\end{aligned} \tag{114}$$

where the last inequality in (114) follows because the sum has at most m_n terms.

As a result, part (iii) of Theorem 4.1 (except for the statement of Corollary 4.2 below, which needs to be dealt with separately) follows from the fact that the upper bounds in (107), (109), (112), and (114) go to 0 when ϵ goes to 0.

Corollary 4.2. *$\Psi(z) = L(z) - D$, where $L(z) := \int_{\mathbb{C}} \log|z-\zeta|\,\mathrm{d}\mu_S(\zeta)$ is the logarithmic potential of μ_S and $D := (\log(|d_t|t))/(q+t-1)$.*

Proof. We will first prove that $L(z) := \int_{\mathbb{C}} \log|z-\zeta|\,\mathrm{d}\mu_S(\zeta)$ is well-defined as a L^1_{loc}–function. Let $l_{ij} = \{z: \ |z-z_i| = |z-z_j|\}$, and use the notation of Proposition 4.2. Then, for a compact set $K \subset \mathbb{C}$,

$$\int_K |L(z)| d\lambda(z) \leq \sum_{i,j} \int_{l_{ij}} \left(\int_K |\log|z-\zeta||\,\mathrm{d}\lambda(z) \right) \mathrm{d}\delta_{ij}(\zeta).$$

Now fix a line l_{ij}. An affine change of coordinates transforms l_{ij} into the real axis, and then δ_{ij} is given by $\frac{1}{\pi}\frac{1}{1+t^2}\,\mathrm{d}t$. Hence it suffices to prove that

$$\int_{\mathbb{R}} \left(\int_K \frac{|\log|z-t||}{1+t^2}\,\mathrm{d}\lambda(z) \right) \mathrm{d}t$$

is finite. This is clear, since for large $|t|$, the integrand is approximately $\lambda(K)\log|t|/t^2$.

Secondly, we will prove that $L(z)$ has the property that

$$\lim_{|z|\to\infty} \left(L(z) - \frac{q-1}{q+t-1}\log|z| \right) = 0. \tag{115}$$

Since $\Psi(z)$, by inspection from (77), has the property that

$$\lim_{|z|\to\infty} \left(\Psi(z) - \frac{q-1}{q+t-1}\log|z| \right) = \frac{\log\left(|d_t|t\right)}{q+t-1} = D,$$

it will follow that $\Psi(z) - L(z)$ is bounded. However, $\Psi(z)$ and $L(z)$ have by definition the same Laplacian, and hence $\Psi(z) - L(z)$ is harmonic. By a standard theorem due to Harnack, this implies that $\Psi(z) - L(z)$ is constant, and hence by taking the limit as $|z| \to \infty$, this difference is equal to $-D$.

Now to prove (115) as above, using that the total mass of μ_s is $\frac{q-1}{q+t-1}$,

$$\left| L(z) - \frac{q-1}{q+t-1}\log|z| \right| \le \sum_{i,j} \int_{l_{ij}} \left| \log\left|1 - \frac{\zeta}{z}\right| \right| \mathrm{d}\delta_{ij}(\zeta),$$

Again, after an affine transformation, it is enough to consider

$$\int_{\mathbb{R}} \frac{\left|\log\left|1-\frac{t}{z}\right|\right|}{1+t^2}\,\mathrm{d}t,$$

which is easily seen to have the limit 0 as $|z| \to \infty$. □

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Appendix A

Definition A.1.

1. *A collection τ of subsets of a set X is said to be a topology in X if τ has the following three properties:*
   (a) $\emptyset \in \tau$ *and* $X \in \tau$.
   (b) *If* $V_k \in \tau$ *for* $k = 1, \ldots, n$, *then* $\bigcap_{k=1}^{n} V_k \in \tau$.
   (c) *If* $\{V_\alpha\}$ *is an arbitrary collection of members of* τ *(finite, countable, or uncountable), then* $\bigcup_\alpha V_\alpha \in \tau$.
2. *If* τ *is a topology in* X, *then* X *is called a topological space, and the members of* τ *are called the open sets in* X.
3. *If* X *and* Y *are topological spaces and if* f *is a mapping of* X *into* Y, *then* f *is said to be continuous provided that* $f^{-1}(V)$ *is an open set in* X *for every open set* V *in* Y.

Definition A.2.

1. *A collection* Σ *of subsets of a set* X *is said to be a* σ*-algebra in* X *if* Σ *has the following properties:*
   (a) $X \in \Sigma$.
   (b) *If* $A \in \Sigma$, *then* $A^c \in \Sigma$, *where* A^c *is the complement of* A *relative to* X.
   (c) *If* $A = \bigcup_{n=1}^{\infty} A_n$ *and if* $A_n \in \Sigma$ *for* $n = 1, 2, 3, \ldots$, *then* $A \in \Sigma$.
2. *If* Σ *is a* σ*-algebra in* X, *then* X *is called a measurable space, and the members of* Σ *are called the measurable sets in* X.
3. *If* X *is a measurable space,* Y *is a topological space, and* f *is a mapping of* X *into* Y, *then* f *is said to be measurable provided that* $f^{-1}(V)$ *is a measurable set in* X *for every open set* V *in* Y.

Definition A.3.

1. *A positive measure is a function* μ, *defined on a* σ*-algebra* Σ, *whose range is in* $[0, \infty]$ *and which is countably additive. This means that if* $\{A_k\}$ *is a disjoint countable collection of members of* Σ, *then*

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k). \tag{116}$$

   To avoid trivialities, we shall also assume that $\mu(A) < \infty$ *for at least one* $A \in \Sigma$.

2. *A measure space is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.*

3. *A complex measure is a complex-valued countably additive function defined on a σ-algebra.*

Remark A.1. *A positive measure is frequently referred to as a measure. Furthermore, it follows from Definition A.3 that if μ is a positive measure on a σ-algebra Σ, then $\mu(\emptyset) = 0$.*

Definition A.4. *The support of a complex function f on a topological space X is the closure of the set $\{x : f(x) \neq 0\}$.*

Theorem A.1 (Gauss-Lucas)**.** *Let P be a non-constant polynomial in one variable. Then all the zeros of its derivative lie in the convex hull H of the zeros of P.*

Proof. Let $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ denote the polynomial, and assume that ζ is a zero of P' such that $P(\zeta) \neq 0$. Furthermore, let $z_1, z_2, \ldots, z_k$ denote all distinct zeros of P with multiplicities $m_1, m_2, \ldots, m_k$ respectively. By using the fundamental theorem of algebra, we see that

$$P(z) = a_n \prod_{j=1}^{k} (z - z_j)^{m_j}. \tag{117}$$

By taking the derivative of (117) using the Leibniz product rule, followed by division by P, yields that

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{k} \frac{m_j}{z - z_j}. \tag{118}$$

Substituting ζ in (118) yields, since $\overline{0} = 0$,

$$\begin{aligned} 0 = \frac{P'(\zeta)}{P(\zeta)} &= \sum_{j=1}^{k} \frac{m_j}{\zeta - z_j} = \overline{\sum_{j=1}^{k} \frac{m_j}{\zeta - z_j}} = \sum_{j=1}^{k} \frac{m_j}{\overline{\zeta} - \overline{z_j}} \\ &= \sum_{j=1}^{k} \frac{m_j |\zeta - z_j|^2}{(\overline{\zeta} - \overline{z_j})|\zeta - z_j|^2} = \sum_{j=1}^{k} \frac{m_j}{|\zeta - z_j|^2}(\zeta - z_j). \end{aligned} \tag{119}$$

Consequently

$$0 = \sum_{j=1}^{k} \frac{m_j}{|\zeta - z_j|^2}(\zeta - z_j) = \sum_{j=1}^{k} c_j(\zeta - z_j), \tag{120}$$

where

$$c_j := \frac{m_j}{|\zeta - z_j|^2} > 0, \quad j = 1, 2, \ldots, k. \tag{121}$$

Solving (120) for ζ, we get

$$\zeta = \frac{1}{\sum_{j=1}^{k} c_j} \left(\sum_{j=1}^{k} c_j z_j \right) = \sum_{j=1}^{k} d_j z_j, \tag{122}$$

where

$$d_j := \frac{c_j}{\sum_{r=1}^{k} c_r}, \quad j = 1, 2, \ldots, k. \tag{123}$$

We note that $0 < d_j \leq 1, j = 1, 2, \ldots, k$, and $\sum_{j=1}^{k} d_j = 1$. Consequently, the weighted sum in (122) shows that ζ is in the convex hull H. □

Finally, we provide a few definitions from graph theory.

Definition A.5. *Let V be a finite nonempty set, and let $E \subseteq V \times V$. The pair (V, E) is then called a directed graph (on V), or digraph (on V), where V is the set of vertices, or nodes, and E is its set of (directed) edges or arcs. We write $G = (V, E)$ to denote such a graph.*

When there is no concern about the direction of any edge, we still write $G = (V, E)$. But now E is a set of unordered pairs of elements taken from V, and G is called an undirected graph.

Remark A.2. *If we say that G is a graph, it is implied that G is an undirected graph.*

Definition A.6. *Let $G = (V, E)$ be a graph with $a \in V$. An edge $(a, a) \in E$ is called a loop. If G has no loops, it is said to be loop-free.*

Definition A.7. *Let x, y be (not necessarily distinct) vertices in a graph $G = (V, E)$. An $x - y$ walk in G is a (loop-free) finite alternating sequence*

$$x = x_0,\, e_1,\, x_1\, e_2,\, x_2,\, e_3, \ldots, e_{n-1},\, x_{n-1},\, e_n,\, x_n = y$$

of vertices and edges from G, starting at vertex x and ending at vertex y and involving the n edges $e_i = (x_{i-1}, x_i)$, where $1 \leq i \leq n$.

The length of this walk is n, the number of edges in the walk. Any $x - y$ walk where $x = y$ (and $n > 1$) is called a closed walk. Otherwise the walk is called open.

Definition A.8. *Consider any $x - y$ walk in a graph $G = (V, E)$.*

1. *If no edge in the $x - y$ walk is repeated, then the walk is called an $x - y$ trail. A closed $x - x$ trail is called a circuit.*

2. *If no vertex of the $x - y$ walk occurs more than once, then the walk is called an $x - y$ path. When $x = y$, the term cycle is used to describe such a closed path.*

Definition A.9. *Let $G = (V, E)$ be an undirected graph. We call G connected if there is a path between any two distinct vertices of G.*

Let $G = (V, E)$ be a directed graph. Its associated undirected graph is the graph obtained from G by ignoring the directions on the edges. If more than one undirected edge results for a pair of distinct vertices in G, then only one of these edges is drawn in the associated undirected graph. When this associated graph is connected, we consider G connected.

A graph that is not connected is called disconnected.

Definition A.10. *A graph G is called planar if G can be drawn in the plane with its edges intersecting only at vertices of G. Such a drawing of G is called an embedding of G in the plane.*

Definition A.11. *Let $G = (V, E)$ be an graph. For each vertex $v \in V$, the degree of v, written $\deg(v)$, is the number of edges in G that are incident with v. Here a loop at a vertex v is considered as two incident edges for v.*

Definition A.12. *Let $G = (V, E)$ be a loop-free, undirected graph. The graph G is called a forest if G contains no cycles. A forest that is connected is called a tree.*

Definition A.13. *Let $T = (V, E)$ be a tree. A vertex $v \in V$ with $\deg(v) = 1$ is called a leaf vertex.*