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Chladni figures in circular membranes

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Abstract

If a drum membrane is sprinkled with fine sand and a speaker emits sound at certain frequencies sufficiently close, the sand will gather into special and interesting patterns. This thesis explores the mathematics to understand this phenomena.

We solve the *wave equation*

$$\frac{\partial^2}{\partial x^2}u(x,y,t) + \frac{\partial^2}{\partial y^2}u(x,y,t) = c^2\frac{\partial^2}{\partial t^2}u(x,y,t),$$

over the unit disc, which describes the motion of the membrane.

The solution we find is a series involving trigonometric functions and *Bessel functions*. In particular we see how the motion of the membrane can be described in terms of *eigenfunctions*, or fundamental shapes, vibrating at their own *eigenfrequency*. They, as a superposition, completely describe the motion of a drum.

Acknowledgements

I wish to extend my gratiude to my supervisor Rikard Bögvad for his thorough readings and incredibly helpful comments.

Contents

1 Introduction

In 1787 Ernst Chladni published his book *Entdeckungen über die Theorie des Klanges* (Discoveries in the Theory of Sound), in which the reader finds a rather interesting construction and observation. If one mounts a steel plate on a rod, covered lightly with fine sand, and then draws a bow over this, interesting and complex patterns will emerge. As the plate resonates it will divide into regions separated by nodal lines. The sand will gather in these nodal lines, hence forming the patterns seen.

The construction is by no means restricted to steel plates and bows, and the phenomenon can be observed in drum heads and other rigid surfaces. In Figure 1 we see the phenomenon occuring on a resonating circular steel plate. It is interesting to compare the picture in Figure 1 to the mathematically derived patterns shown in Figure 4 on page 12. The resemblance is remarkable.

Figure 1: Chladni figures occuring on a drum head. Image courtesy of [Kui11].

The topic of this work is Chladni figures in drum heads so we shall explore this from a mathematical point of view, both experimentially and analytically, in circular membranes. In particular, one can go online to look at videos of Chladni figures in drums. The construction is in general some kind of external force acting on the drum, like a speaker producing a sine wave, which makes it resonate. We shall try to give an explanation of the phenomenon seen.

Already in 1766 Euler published his investigations on vibrations in circular membranes [Eul66][1]. He was led to study the equation

$$\frac{1}{e^2}\frac{d^2z}{dt^2} = \frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} + \frac{1}{r^2}\frac{d^2z}{d\phi^2}$$

in which e is a constant depending on the characteristics of the membrane and z is the vertical displacement at time t at the point (r, ϕ) in polar coordinates. When solving this Euler performed a subsitution of z involving the function $u(r)$, and the constants α and β, to arrive at the differential equation

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\frac{\alpha^2}{e^2} - \frac{\beta^2}{r^2}\right)u = 0$$

which we will come to know as *Bessel's equation of order* β. He also gave a solution which is finite at the origin and we will get to know these solutions as *Bessel functions*, $J_\beta(x)$.

[1]For those who are interested, this can be read at `https://books.google.se/books/about/Novi_commentarii_Academiae_scientiarum_i.html?id=UVY-AAAAcAAJ&redir_esc=y`

In 1822 Fourier published his researches on heat conduction, *Théorie Analytique de la Chaleur* (The Analytical Theory of Head) in which he considered expansions of arbitrary functions $f(x)$ in terms of *Bessel functions of the first kind of order 0*,

$$f(x) = \sum_{m=1}^{\infty} a_m J_0(j_m x).$$

Lommel demonstrated a more general result of this kind of expansion involving Bessel functions of the first kind [Wat66]. We shall consider expansions of this kind together with trigonometric Fourier series.

When reading this paper, a rather comprehensive understanding of the phenomenon with resonance and Chladni figures can be obtained by reading chapters 3, 4, and 7. Chapters 5 and 6 are mainly included to provide some mathematical rigor to the solution obtained. Finally, the final chapter 8 is merely included for some amusement. It does not provide much additional insight, but introduce some more functions occuring in mathematical physics.

2 Short preliminaries: Fourier series and Fourier method

2.1 Fourier series

This section is taken from [PZ97], and slightly modified. The Fourier series of a function f on $[-L, L]$ is the series

$$\mathcal{F}[f](\omega) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \Big[a_m \cos(\frac{m\pi}{L}\omega) + b_m \sin(\frac{m\pi}{L}\omega) \Big]$$

with the coefficients a and b defined by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx,$$
$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{m\pi}{L}x)dx,$$
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{m\pi}{L}x)dx.$$

For Fourier series we have the following theorem

Theorem 1. *If f is continuous on $[-L, L]$, $f(-L) = f(L)$ and f' is piecewise continuous, then the Fourier series of f converges uniformly to f on $[-L, L]$.*

Proof. The theorem is stated and proved for the particular case $L = \pi$ as Theorem 2.15 in [PZ97]. In Chapter 2 section 10 of [PZ97] they show how this result is extended to the more general interval $L > 0$. □

In this case we write $f = \mathcal{F}[f]$ and call $\mathcal{F}[f]$ the Fourier expansion of f (on $[-L, L]$).

When given a continuous function on an interval $[a, a + 2L]$ for $a, L \in \mathbb{R}$ with $L > 0$, this method can be applied aswell. By considering the function $g(x) = f(x+a+L)$ which is a continuous function on $[-L, L]$, the given Fourier expansion of g can be translated in a similar way to give the Fourier expansion of f.

2.2 Fourier method (Separation of variables)

A technique used when solving partial differential equations is *separation of variables*, also called the *Fourier method*. Here it will be demonstrated with an example.

We shall look at the vibrating string problem. Picture a string in the 2D-plane fixed at the origin and $x = 1$, see Figure 2. We assume that it has uniform mass and density, equal to 1. We can model the free vibrating motion by a partial differential

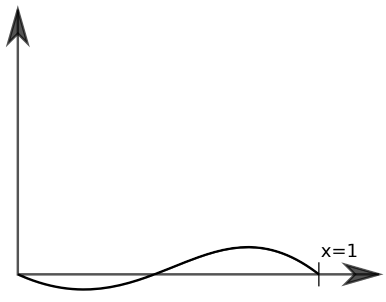


Figure 2: A string in the 2D-plane fixed at the origin and $x = 1$.

equation, derived in [CH53]. The differential equation is the wave equation given by

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

where $y(t,x)$ is the *height* of the string at time t and position x. We will also have an initial condition at time $t = 0$, which is a function f of x such that $f(0) = f(1) = 0$. Imagine that the initial position is a plucked guitar string as in Figure 3. The figure also shows how y is to be interpreted. To know how y evolves in time we must solve the differential equation above. To do so, we assume that y can be written as the product $y(t,x) = T(t)X(x)$. Then the differential equation becomes

$$T''(t)X(x) = T(t)X''(x)$$

which we can rewrite it as $\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$ Since the variables of each side are independent, the sides must both be equal to some constant $-\lambda = \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}$. We have obtained the two differential equations

$$T''(t) + \lambda T(t) = 0,$$
$$X''(x) + \lambda X(x) = 0.$$

Since we assumed that $y(t,x) = T(t)X(x)$ we can derive conditions for the two differential equations given. We have that $y(0,x) = T(0)X(x)$ is non-zero, hence $T(0)$ is a non-zero constant. Assume that $y_t(0,x) = 0$, which implies $T'(0) = 0$. These initial data can be interpreted as the string held still in a position and then released at time $t = 0$. Further $y(t,0) = y(t,1) = 0$ implies that $X(0) = X(1) = 0$.

By the theory of ordinary differential equations the solutions to these equations are

$$X(x) = c_1 e^{\mu_1 x} + c_2 e^{\mu_2 x}$$

and

$$T(t) = k_1 e^{\mu_1 t} + k_2 e^{\mu_2 t}$$

where μ_1 and μ_2 are the roots of $\xi^2 + \lambda = 0$. Using that $X(0) = X(1) = 0$ and this, it is easy to verify that there are no non-trivial solutions for $\lambda \leq 0$, hence $\lambda > 0$. In particular, it follows from $X(0) = k_1 + k_2 = 0$ and

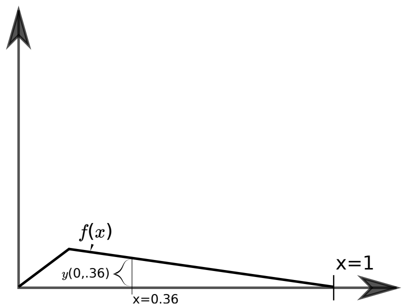


Figure 3: A plucked guitar string in the 2D-plane at time $t = 0$.

$X(1) = k_1(e^{i\sqrt{\lambda}} - e^{-i\sqrt{\lambda}}) = 2ik_1 \sin(\sqrt{\lambda}) = 0$ that $\lambda = n^2\pi^2$ for $n = 1, 2, \ldots.$ So it makes sense to denote the solution of $X(x)$ as $X_n(x)$ for $\lambda = n^2\pi^2$, and after some simplification the solution $X_n(x)$ is given by

$$X_n(x) = k_n \sin(n\pi x)$$

and in a similar way we find that the solution $T_n(t)$ is given by

$$T_n(t) = c_n \cos(n\pi t)$$

for complex constants c_n and k_n. Hence $y(t,x) = T_n(t)X_n(x)$ satisfies all the conditions except for $y(0,x) = f(x)$. To find the general solution use infinite sums

$$y(t,x) = \sum_{n=1}^{\infty} A_n \cos(n\pi t)\sin(n\pi x).$$

At this point, the theory of Fourier seris can be applied to find the constant A_n for each n via the condition $y(0,x) = \sum_n A_n \sin(n\pi x) = f(x)$. Extend $f(x)$ to an odd function on the interval $[-1,1]$. Assuming that f' is piecewise continuous and since $f(-1) = f(1) = 0$ this have a Fourier expansions in sine function only

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x).$$

This determines A_n:

$$\begin{aligned} A_n &= \int_{-1}^{1} f(x)\sin(n\pi x)dx \\ &= 2\int_{0}^{1} f(x)\sin(n\pi x)dx. \end{aligned}$$

3 The two dimensional wave equation

By [CH53] we know that an ideal two-dimensional membrane, i.e. one with uniform mass and tension, with no external forces acting on it changes in time according to the differential equation

$$\xi_{tt}(x,y,t) = c^2(\xi_{xx}(x,y,t) + \xi_{yy}(x,y,t)), \tag{1}$$

where $\xi(x,y,t)$ is the horizontal displacement at the point (x,y) at time t, and the constant c in the above equation depends on the characteristics of the membrane, such as tension and mass.

3.1 The two dimensional wave equation in polar coordinates

When considering a circular membrane it is natural to transform the wave equation to polar coordinates with the substitutions

$$x = r\cos(\theta),$$
$$y = r\sin(\theta).$$

If we set $u(r,\theta,t) = \xi(r\cos(\theta), r\sin(\theta), t) = \xi(x,y,t)$ we get

$$u_r(r,\theta,t) = \xi_x(x,y,t)x_r + \xi_y(x,y,t)y_r,$$
$$u_\theta(r,\theta,t) = \xi_x(x,y,t)x_\theta + \xi_y(x,y,t)y_\theta.$$

Differentiating once more gives

$$\begin{aligned} u_{rr}(r,\theta,t) =& \xi_{xx}(x,y,t)x_r^2 + \xi_{yy}(x,y,t)y_r^2 + 2\xi_{xy}(x,y,t)x_r y_r + \\ & \xi_x(x,y,t)x_{rr} + \xi_y(x,y,t)y_{rr}, \\ u_{\theta\theta}(r,\theta,t) =& \xi_{xx}(x,y,t)x_\theta^2 + \xi_{yy}(x,y,t)y_\theta^2 + 2\xi_{xy}(x,y,t)x_\theta y_\theta + \\ & \xi_x(x,y,t)x_{\theta\theta} + \xi_y(x,y,t)y_{\theta\theta}, \end{aligned}$$

which we simplify to

$$\begin{aligned} u_{rr} =& \xi_{xx}\cos^2(\theta) + \xi_{yy}\sin^2(\theta) + 2\xi_{xy}\cos(\theta)\sin(\theta), \\ u_{\theta\theta} =& \xi_{xx}r^2\sin^2(\theta) + \xi_{yy}r^2\cos^2(\theta) - 2\xi_{xy}r^2\sin(\theta)\cos(\theta) - r^2(\xi_x\cos(\theta) + \xi_y\sin(\theta). \end{aligned}$$

From this it is not hard to verify

$$\xi_{xx} + \xi_{yy} = u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2}.$$

Since $u_{tt} = \xi_{tt}$ we can now conclude that the differential equation 1 is transformed to

$$u_{tt}(r,\theta,t) = c^2\left(u_{rr}(r,\theta,t) + \frac{u_r(r,\theta,t)}{r} + \frac{u_{\theta\theta}(r,\theta,t}{r^2}\right) \tag{2}$$

in polar coordinates, i.e. for circular membranes.

4 Solving the two dimensional wave equation for a circular membrane

To solve the differential equation (2) we will use the method of separation of variables, like we did in Section 2.2. To read more on this, the reader is advised to check [BC11]. But before we begin the process of solving the differential equation we must decide upon initial and boundary value conditions.

4.1 Reasonable initial and boundary value conditions

We shall examine a circular drum head which we assume is evenly tense and of uniform mass. Further we may assume that the radius is 1, and since we can imagine it as clamped to a rim, the boundary is stationary. Thus we require that $u(1, \theta, t) \equiv 0$.

The membrane is obviously circular, and as the angle is measured in radians we require that $u(r, 0, t) = u(r, 2\pi, t)$ and $u_\theta(r, 0, t) = u_\theta(r, 2\pi, t)$.

Finally, we need some initial conditions regarding the shape, or position, and velocity of the membrane. Let $f(r, \theta)$ be a continuous function on $[0, 1] \times [0, 2\pi]$ such that $f(1, \theta) \equiv 0$ and $f(r, 0) = f(r, 2\pi)$, so that it represents a reasonable initial shape of the membrane. Further, let $g(r, \theta)$ be a similar function describing the initial velocity of the membrane. We shall also pose the condition that not both of f and g be identical to 0.

We conclude this subsection by collecting the conditions:

$$u(1, \theta, t) \equiv 0 \qquad \theta \in [0, 2\pi] t \in [0, \infty), \tag{3}$$

$$u(r, 0, t) = u(r, 2\pi, t),\ u_\theta(r, 0, t) = u_\theta(r, 2\pi, t) \qquad r \in [0, 1] t \in [0, \infty), \tag{4}$$

$$u(r, \theta, 0) = f(r, \theta),\ u_t(r, \theta, 0) = g(r, \theta) \qquad r \in [0, 1] \theta \in [0, 2\pi]. \tag{5}$$

Now we are in a position to start solving the wave equation in polar coordinates by separation of variables.

4.2 Separating the first variable

We are searching for a non-trivial solution u on the form

$$u = \sum_m \sum_n R_{m,n}(r) A_{m,n}(\theta) T_{m,n}(t) \tag{6}$$

for some unknown functions R, A and T. However, for now we will let u be written on the form $u(r, \theta, t) = v(r, \theta)T(t)$ and come back to the form in (6) later.

With u in the form $u(r, \theta, t) = v(r, \theta)T(t)$ the differential equation (2) becomes

$$T''(t)v = c^2 \left(v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} \right) T(t),$$

which we rewrite as

$$\frac{T''(t)}{c^2 T(t)} = \frac{v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2}}{v}.$$

Since the left hand side depend only on t and the right hand side only on r and θ, both sides must be equal to some common constant, $-\lambda^2$. Later it will become clear why this number shall be negative. Therefore we get the equations

$$\frac{T''(t)}{c^2T(t)} = -\lambda^2$$

and

$$\frac{v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2}}{v} = -\lambda^2$$

which we rewrite as

$$T''(t) + c^2\lambda^2 T(t) = 0, \text{ and} \tag{7}$$

$$v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} + \lambda^2 v = 0 \tag{8}$$

Considering the boundary condition (3), we have $u(1,\theta,t) = T(t)v(1,\theta) \equiv 0$ which implies that $v(1,\theta) \equiv 0$, and from considering (4) we must have $v(r,0) = v(r,2\pi)$ and $v_\theta(r,0) = v_\theta(r,2\pi)$. From the initial condition (5) we get that $T(0)v = f$ and $T'(0)v = g$, which implies that either $f \equiv 0$ of $g \equiv 0$ or that there is some constant c such that $f = c \cdot g$. We shall go with $g \equiv 0$, so that $T'(0) = 0$. This assumption can be interpreted as holding the membrane in a static position and releasing it at time $t = 0$. Now, we may take $T(0) = 1$. In conclusion the boundary and initial value conditions are now

$$T(0) = 1,$$
$$T'(0) = 0,$$

and

$$v(1,\theta) \equiv 0, \tag{9}$$

$$v(r,0) = v(r,2\pi), v_\theta(r,0) = v_\theta(r,2\pi). \tag{10}$$

Now we will solve for the function T, and we have the system

$$\begin{cases} T''(t) + c^2\lambda^2 T(t) = 0, & \\ T(0) = 1, & \\ T'(0) = 0, & t \in [0,\infty). \end{cases}$$

A short motivational discussion of the values of λ is perhaps needed. Since $u = T(t)v$, where v will be decided by the function f in the initial condition (5), we may expect some property of T, namely that it oscillates as the drum head moves up and down as time progresses. Therefore the only reasonable values for the constant $-\lambda^2$ above is that it is negative, as then the general solution is given by $T(t) = c_1\cos(c\lambda t) + c_2\sin(c\lambda t)$. Later on, we will come back to the exact values of λ.

By plugging in the initial conditions in $T(t) = c_1\cos(c\lambda t) + c_2\sin(c\lambda t)$ we get $c_1 = 1$ and $c_2 = 0$. So $T(t)$ is given by

$$T(t) = \cos(c\lambda t),$$

with λ undetermined.

4.3 Separating the final variables

Now we will consider the system formed by the equation (8) with the boundary and initial conditions in (9) and (10), i.e.

$$\begin{cases} v_{rr} + \frac{v_r}{r} + \frac{v_{\theta\theta}}{r^2} + \lambda^2 v = 0, \\ v(1,\theta) = 0, \\ v(r,0) = v(r,2\pi), \\ v_\theta(r,0) = v_\theta(r,2\pi), \qquad r \in [0,1], \quad \theta \in [0,2\pi]. \end{cases}$$

Let $v(r,\theta) = R(r)A(\theta)$, then the equation (8) is written as

$$R''(r)A(\theta) + \frac{R'(r)}{r}A(\theta) + \frac{R(r)}{r^2}A''(\theta) + \lambda^2 R(r)A(\theta) = 0,$$

which, after we multiply through by $\frac{r^2}{R(r)A(\theta)}$, separates as

$$\frac{r^2R''(r) + rR'(r)}{R(r)} + r^2\lambda^2 = -\frac{A''(\theta)}{A(\theta)} = \omega^2.$$

The constant on the right hand side of this equation must be positive as is seen by the solutions below. From this we get the two equations

$$R'' + \frac{R'}{r} + (\lambda^2 - \frac{\omega^2}{r^2})R = 0, \tag{11}$$
$$A''(\theta) + \omega^2 A(\theta) = 0. \tag{12}$$

The boundary condition (9) implies that

$$R(1) = 0 \tag{13}$$

and the conditions (10) imply

$$\begin{cases} A(0) = A(2\pi), \\ A'(0) = A'(2\pi). \end{cases} \tag{14}$$

Combining the equation (11) with the boundary value (13) gives the system

$$\begin{cases} R'' + \frac{R'}{r} + (\lambda^2 - \frac{\omega^2}{r^2})R = 0, \\ R(1) = 0, \qquad r \in [0,1] \end{cases} \tag{15}$$

and likewise, the equation (12) with the conditions in (14) gives the system

$$\begin{cases} A'' + \omega^2 A = 0, \\ A(0) = A(2\pi), \\ A'(0) = A'(2\pi). \end{cases} \tag{16}$$

4.4 Solving the system related to θ

We will not go into much detail about solving the system (16), but verify that it has the solution

$$\begin{gathered} A(\theta) = c_1\cos(\omega\theta) + c_2\sin(\omega\theta), \\ \text{for the values } \omega = 0, 1, 2, 3... \end{gathered} \tag{17}$$

where c_1 and c_2 are constants to be decided upon later.

Clearly $A''(\theta)i = -\omega^2(c_1\cos(\omega\theta) + c_2\sin(\omega\theta)) = -\omega^2 A(\theta)$, hence $A''(\theta) + \omega^2 A(\theta) = 0$. The conditions $A(0) = A(2\pi)$ and $A'(0) = A'(2\pi)$ means

$$\begin{aligned} c_1 &= c_1\cos(2\pi\omega) + c_2\sin(2\pi\omega),\\ c_2 &= c_2\cos(2\pi\omega) - c_1\sin(2\pi\omega), \end{aligned}$$

from which we derive

$$\begin{aligned} (c_1^2 + c_2^2) &= (c_1^2 + c_2^2)\cos(2\pi\omega),\\ 0 &= (c_1^2 + c_2^2)\sin(2\pi\omega). \end{aligned}$$

Since we require that not both c_1 and c_2 be equal to zero, these conditions imply that $1 = \cos(2\pi\omega)$ and $0 = \sin(2\pi\omega)$. It follows that $\omega = 0, 1, 2, ...$

4.5 Solving the final system

Now we turn to something a little more interesting and less obvious - solving the final system in (15):

$$\begin{aligned} R'' + \frac{R'}{r} + (\lambda^2 - \frac{\omega^2}{r^2})R &= 0,\\ R(1) &= 0, \end{aligned}$$

for $r \in [0, 1]$ with $\omega = 0, 1, 2, ...$ and λ so far undetermined.

Perform the substitution $s = \lambda r$ so it becomes the system

$$R''(s) + \frac{R'(s)}{s} + (1 - \frac{\omega^2}{s^2})R(s) = 0, \tag{18}$$

$$R(\lambda) = 0. \tag{19}$$

The equation in (18) is called the Bessel equation of order (or index) ω. The solutions to this equation are called *Bessel functions* and more specifically the two solutions to this equation are *the Bessel function of the first kind of index* ω and *the Bessel function of the second kind of index* ω. We shall look at the former in a little more detail in the next section. For now, we will only say that they are denoted by J_ω and N_ω respectively. Hence $R(s) = c_1 J_\omega(s) + c_2 N_\omega(s)$. The function $R(s)$ can be further simplified in our case, because we shall always require R to be well defined throughout the whole interval, in particular at the origin. However, $N_\omega(s) \to -\infty$ as $s \to 0$ (cf. Section 3·5 of [Wat66][2].) so we are required to have $c_2 = 0$. Therefore

$$R(s) = c_1 J_\omega(s)$$

with $R(\lambda) = 0$. Thus, λ shall equal the zeros of R of which there are infinitely (but countably) many. In Section 5.1 this fact is stated for when the index ω is an integer, which we shall see later is all that we need. The n'th zero of J_ω is denoted by $j_{\omega n}$ for $n = 1, 2, ...$ with $j_{\omega 1} < j_{\omega 2}....$ By substituting back $s = \lambda r = j_{\omega n} r$ we get

$$R(r) = c_1 J_\omega(j_{\omega n} r)$$

for some undetermined constant c_1.

[2] Note that Watson use the notation Y_ω for N_ω

4.6 Conclusion

From the two previous sections we know that $\omega = m$ for $m = 0, 1, 2, ...$ and $\lambda = j_{mn}$ for $n = 1, 2, 3, ...$ We found in Section 4.2 that $T(t) = \cos(c\lambda t)$ so

$$T_{m,n}(t) = \cos(cj_{mn}t).$$

In Section 4.4 we verified that $A(\theta) = c_1 \cos(\omega\theta) + c_2 \sin(\omega\theta)$ so we write

$$A_m(\theta) = c_{m1} \cos(m\theta) + c_{m2} \sin(m\theta).$$

In the previous section the conclusion was that $R(r) = c_1 J_\omega(j_{\omega n} r)$ hence

$$R_{mn}(r) = k_{mn} J_m(j_{mn} r).$$

From this we know that the product $A_m(\theta)R_{mn}(r)T_{mn}(t)$ satisfies the original system in (2) and therefore every linear combination of these products also solves it. It follows that the complete expression for u is formally written as

$$u(t, r, \theta) = \sum_{m=0} \sum_{n=1} c_{mn} J_m(j_{mn} r)(a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta)) \cos(cj_{mn}t). \tag{20}$$

4.7 Modes of the circular membrane

The function $c_{mn} J_m(j_{mn} r)(a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta))$ in the formal expression (20) is called the (m, n)-mode of $u(t, r, \theta)$. Note that the term $\cos(cj_{mn}t)$ is periodic with frequency $\frac{cj_{mn}}{2\pi}$. We say that the (m, n)-mode resonates at $\frac{cj_{mn}}{2\pi}$ Hz. We shall look more at what this means in Section 7, but for now we shall just use the term very informally.

In Figure 4 the nodal lines of the first few (m, n)-modes, that is the lines where the modes are 0. Compare this to the true physical phenomenon pictured in Figure 1 on page 1. The nodal lines of the (m, n)-modes that we compute are connected to the Chladni figures observed in circular membranes in a simple way. The sand that we put on the membrane will collect were the membrane lies absolutely still. This is precisely where $u \equiv 0$ at all times t. The circles we see come from the Bessel function $J_m(j_{mn} r)$, so they are described by the set $\{(x, y) \in \mathbb{R}^2 : \sqrt{x^2 + y^2} = \frac{j_{mk}}{j_{mn}} \text{ for } \quad k = 1, 2, ..., n\}$. The rays, which we see when $m > 0$, come from the trigonometric functions $a_{mn} \cos(m\theta) + b_{mn} \sin(m\theta)$. This may be rewritten in the form $c \cos(m\theta + \varphi)$ where $c = \sqrt{a_{mn}^2 + b_{mn}^2}$ and $\tan \varphi = \frac{-b}{a}$. The solutions to $\cos(m\theta + \varphi) = 0$ are found by solving $m\theta + \varphi = \pm\pi + 2\pi k$. Hence

$$\theta = \frac{-\varphi \pm \pi}{m} + 2\pi \frac{k}{m}$$

for $k = 0, 1, ..., m - 1$.

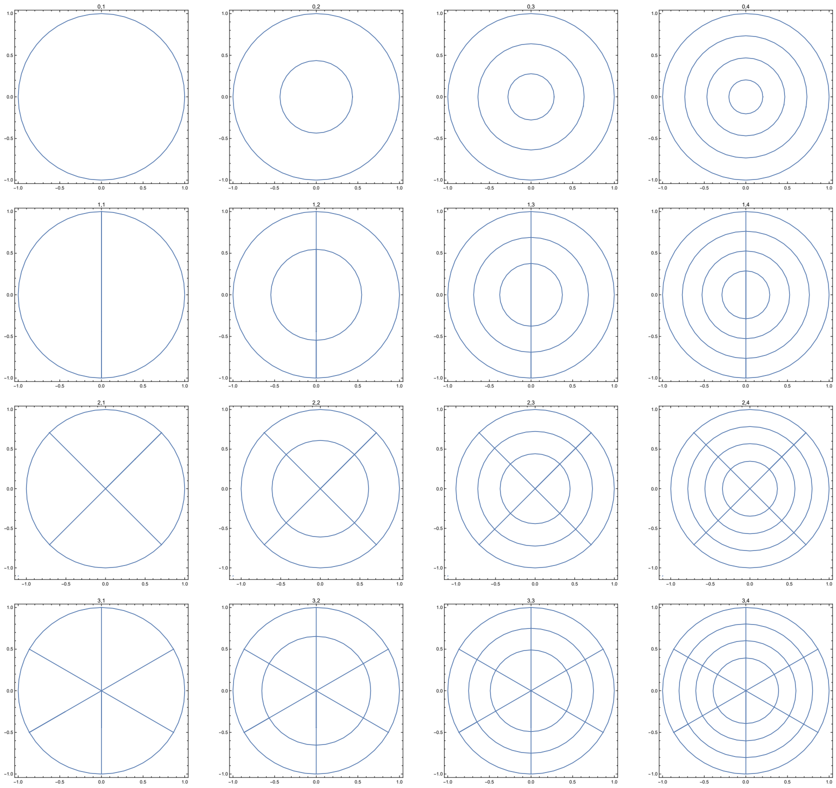


Figure 4: The first few modes of the circular membrane. On row $m+1$ and column n the (m,n)-mode is pictured for $m = 0, \cdots, 3$ and $n = 1, \cdots, 4$.

5 Bessel Functions

In this section we shall look at properties of the Bessel functions. The starting point is the Bessel equation, which is the differential equation

$$y'' + \frac{y'}{x} + \left(1 - \frac{n^2}{x^2}\right) y^2 = 0.$$

As it is a second order differential equation, it has a basis of two solutions as expected. One of which tends to $-\infty$ as $x \to 0^+$. That function is generally called the Bessel function of the second kind, Weber functions or Neumann functions. We shall not be concerned with Neumann functions as we will require that the expansion u and initial condition f are non-singular everywhere.

This section is based almost entirely on [Wat66].

5.1 Definitions

The Bessel functions are functions that solve the Bessel equation (18). The Bessel function of the first kind of complex index ω, usually denoted J_ω, have a couple of different representations as power series or integrals. We will only consider the case with integer index, $\omega = n$ for integers n.

Definition 1 (Bessel function of the first kind of index n)**.** Let n be some integer greater than or equal to zero and let x lie in the interval $[0, c]$ for some constant $c > 0$. We define the *Bessel function of the first kind of the index n of the argument x*, denoted $J_n(x)$, by

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n} m!(m+n)!}.$$

Note that this series converges for arbitrary x, by the root test (Cauchy's criterion):

$$\begin{aligned}
\lim_{m\to\infty} \sqrt[m]{\frac{|x|^{2m+n}}{2^{2m+n} m!(m+n)!}} &= \lim_{m\to\infty} \sqrt[m]{\left(\left(\frac{|x|}{2}\right)^2\right)^m \cdot \left(\frac{|x|}{2}\right)^n \cdot \frac{1}{m!(m+n)!}} \\
&= \left(\frac{|x|}{2}\right)^2 \cdot 1 \cdot 0 \\
&= 0,
\end{aligned}$$

since $\lim_{m\to\infty} \sqrt[m]{m!} = \infty$. We say that the radius of convergence is infinite.

We shall verify

Property 1. *The function $J_m(x)$ satisfies the Bessel equation in equation* (18).

Proof. For readability, denote the coefficients $\frac{(-1)^k}{2^{2k+m} m!(k+m)!}$ by c_k. The radius of convergence is infinite, so Theorem 8.1 of [Rud76] implies that the series is differentiable and the derivative is given by termwise differentiation. Termwise differentiation of $J_m(x)$ gives

$$J'_m(x) = \sum_{k=0}^{\infty} c_k (2k+m) x^{2k+m-1},$$
$$J''_m(x) = \sum_{k=0}^{\infty} c_k (2k+m)(2k+m-1) x^{2k+m-2},$$

and when we put this back into the left hand side of (18) we get

$$\sum_{k=0}^{\infty} c_k((2k+m)(2k+m-1) + (2k+m) - m^2) x^{2(k-1)+m} + \sum_{k=0}^{\infty} c_k x^{2k+m}.$$

This simplifies to

$$\sum_{k=0}^{\infty} c_k 4k(k+m) x^{2(k-1)+m} + \sum_{k=0}^{\infty} c_k x^{2k+m},$$

where the first term in the left hand sum vanishes, so we can shift that sum, so that it now reads

$$\sum_{k=0}^{\infty} c_{k+1} 4(k+1)(k+m+1) x^{2k+m} + c_k x^{2k+m}.$$

Finally, the coefficient $c_{k+1}4(k+1)(m+k+1) + c_k$ is equal to zero, since $c_{k+1}4(k+1)(m+k+1) = -c_k$, so the whole sum vanishes, as required by equation (18). □

The first few Bessel functions are plotted in Figure 5. As one may guess from the figure, we have the following property of Bessel functions

Property 2. *For non-negative integers m, there are countably infinitely many zeros of J_m. That is, there are infinitely many values x such that $J_m(x) = 0$. We shall order these in ascending order and denote them j_{mn} for the n'th zero.*

Proof. See Section 15·2 of [Wat66]. □

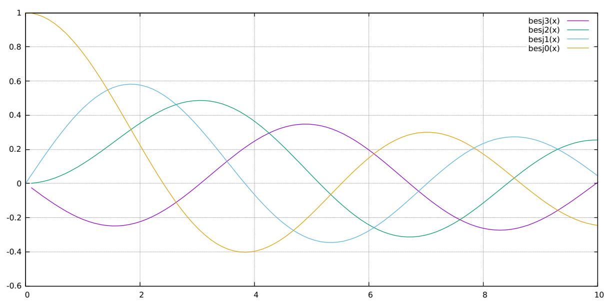


Figure 5: The first four Bessel functions of non-negative integral index.

An amusing property of the zeros is the following

Property 3. *For $m \geq 0$ we have*

$$j_{m1} < j_{(m+1)1} < j_{m2} < j_{(m+1)2} < \dots$$

That is to say that the zeros of any two Bessel functions of consecutive order are interlaced.

Proof. The proof is given in [Wat66] 15·22. $\square$

One thing to note in particular is that $J_0(0) = 1$ while $J_m(0) = 0$ for all $m > 0$. Just to avoid any confusion we shall not count 0 as a zero of J_m, so $j_{m1} > 0$.

To finish this section of basic properties we shall state and prove the recurrence relations satisfied by the Bessel equations J.

Property 4. *For $x \geq 0$ and $m \geq 0$ we have*

$$xJ'_m(x) - mJ_m(x) = -xJ_{m+1}(x).$$

Proof. Consider the series expansion of the function $J_m(x)$,

$$J_m(x) = \sum_{k=0}^{\infty} c_k x^{2k+m},$$

where $c_k = \frac{(-1)^k}{2^{2k+m}k!(k+m)!}$. From the proof of Property 1 we know that $J'_n(x)$ is given by termwise differentiation,

$$J'_m(x) = \sum_{k=0}^{\infty} c_k(2k+m)x^{2k+m-1}$$

so that the expression $xJ'_m(x) - mJ_m(x)$ is given by

$$\sum_{k=0}^{\infty} 2kc_k x^{2k+m}.$$

The first term is zero, so this is equal to

$$\sum_{k=1}^{\infty} 2kc_k x^{2k+m} = \sum_{k=0}^{\infty} 2(k+1)c_{k+1}x^{2k+m+2},$$

and with the current definition of c_k each term in this sum is given by $\frac{(-1)^{k+1}x^{2k+m+2}}{2^{2k+m+1}k!(k+m+1)!}$.

Now consider the series expansion of the function $J_{m+1}(x)$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+m+1}}{2^{2k+m+1}k!(k+m+1)!},$$

from which we see that each term for $-xJ_{m+1}(x)$ is given by $\frac{(-1)^{k+1}x^{2k+m+2}}{2^{2k+m+1}k!(k+m+1)!}$.

Since each term of $-xJ_{m+1}(x)$ and $xJ'_m(x) - mJ_m(x)$ are precisely equal, it follows that

$$xJ'_m(x) - mJ_m(x) = -xJ_{m+1}(x).$$

We did not handle the special case $m = 0$, where $J_0(x)$ is a sum on the form $1 + \sum_{k=1}^{\infty} c_k x^{2k}$, because this case is seen by following the procedure above with only slight modification. $\square$

5.2 Bessel functions in a function space

Let E be the set of all real valued functions that are continuous and of bounded variation on $[0,1]$, with $f(1)=0$. Recall that a function of bounded variation on a compact set is characterized as a function which may be written as the difference between two positive montonic, increasing functions (cf. Chapter 6, Jordan's Theorem in [RF10]). Define the inner product, $<\cdot,\cdot>$, of two functions f and g of E by

$$<f,g>=\int_0^1 tf(t)g(t)dt.$$

Then E is an inner product space, as is easy to verify and left to the reader. Let $\|\cdot\|$ denote the induced norm, that is

$$\|f\|=\sqrt{<f,f>}.$$

For the remainder of this section we assume that functions belong to E unless stated otherwise.

Property 5. *For any integer $m \geq 0$ the collection $\{\frac{1}{\|J_m(j_{mn}x)\|}J_m(j_{mn}x)\}_{n=1}^{\infty}$ is an orthonormal system of functions in E, with*

$$\|J_m(j_{mn}x)\|=\left(\int_0^1 tJ_m^2(j_{mn}t)dt\right)^{\frac{1}{2}}=\frac{|J_{m+1}(j_{mn})|}{\sqrt{2}}.$$

Proof. To show orthogonality, we need to show that $<J_m(j_{mn}x),J_{m'}(j_{m'n'}x)>=0$ when $n \neq n'$. From Property 1 we know that $J_m(x)$ satisfies the differential equation

$$J_m''(x)+\frac{J_m'(x)}{x}+\left(1-\frac{m^2}{x^2}\right)J_m(x)=0.$$

Perform the substitution $x=j_{mn}s$ and multiply by s^2 to obtain the equivalent equation

$$s^2J_m''(j_{mn}s)+sJ_m'(j_{mn}s+\left(s^2j_{mn}^2-m^2\right)J_m(j_{mn}s)=0.$$

For simplicity, we shall denote the functions of the equation by u and v for now, so we get:

$$s\frac{d}{ds}(su')+\left(s^2j_{mn}^2-m^2\right)u=0$$
$$s\frac{d}{ds}(sv')+\left(s^2j_{mn'}^2-m^2\right)v=0$$

Multiply the first equation by v and the second by u and subtract them to obtain

$$(j_{mn'}^2-j_{mn}^2)suv=\frac{d}{ds}(svu'-suv'). \tag{21}$$

Integrate from 0 to 1, we get:

$$(j_{mn'}^2-j_{mn}^2)\int_0^1 suvds=[svu'-suv']_0^1=0.$$

Hence, for $n \neq n'$ the integral $\int_0^1 sJ_m(j_{mn}s)J_m(j_{mn'}s)ds$ is equal to 0.

To evaulate the integral when $n = n'$, substitute $u(s) = J_m(j_{mn}s)$ and $v(s) = J_(j_{mn'}s)$ in (21), and differentiate both sides with respect to j_{mn}:

$$-2j_{mn}sJ_m(j_{mn}s)J_m(j_{mn'}s) + (j_{mn'}^2 - j_{mn}^2)s^2J_m'(j_{mn}s)J_m(j_{mn'}s) =$$

$$\frac{d}{ds}(sJ_m(j_{mn'}s)J_m'(j_{mn}s) + s^2J_m(j_{mn'}s)j_{mn}J_m''(j_{mn}s) - s^2j_{mn'}J_m'(j_{mn}s)J_m'(j_{mn'}s)).$$

Put $n = n'$ and integrate from 0 to 1:

$$-2j_{mn}\int_0^1 sJ_m^2(j_{mn}s)ds = -j_{mn}(J_m'(j_{mn}))^2.$$

Hence $\int_0^1 sJ_m^2(j_{mn}s) = \frac{J_m'(j_{mn})^2}{2}$. By Property 4, $xJ_m'(x)-mJ_m(x) = -xJ_{m+1}(x)$, we put $x = j_{mn}$ and see that $J_m'(j_{mn})^2 = J_{m+1}(j_{mn})^2$.

We conclude that

$$\int_0^1 sJ_m(j_{mn}s)J_m(j_{mn'}s)ds = \begin{cases} 0 & n \neq n', \\ \frac{J_{m+1}^2(j_{mn})}{2} & n = n'. \end{cases}$$

□

Similarly to Fourier series, we shall consider series on the form

$$\sum_n \frac{< f, J_m(j_{mn}x) >}{\|J_m(j_{mn}x)\|^2}J_m(j_{mn}x) = \sum_n \frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 tf(t)J_m(j_{mn}t)dtJ_m(j_{mn}x).$$

These are called *Fourier-Bessel* series. In the next section we shall see that for $f \in E$ the corresponding Fourier-Bessel series indeed converges to f, that is

$$\sum_n \frac{< f, J_m(j_{mn}x) >}{\|J_m(j_{mn}x)\|^2}J_m(j_{mn}x) = f(x).$$

5.3 Fourier-Bessel series

The main result of this section is to show that Fourier-Bessel series converges uniformly throughout the closed interval $[0, 1]$. However, the origin is not directly considered for series of index $m = 0$ because it seems to be too complicated with our approach.

Definition 2 (Fourier-Bessel series)**.** Given a function $f(x)$ the Fourier-Bessel series of f is a series of Bessel functions

$$\sum_{n=1}^{\infty} a_nJ_m(j_{mn}x),$$

with coefficients given by

$$a_n = \frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 tf(t)J_m(j_{mn}t)dt.$$

This series is called *the Fourier-Bessel series associated with* $f(x)$. We shall sometimes use the shorthand expression *series of order* m to emphasize that the series is made up of J_m. If, in addition, the series above converges to $f(x)$ for some value of x, the series is called *the Fourier-Bessel expansion of f(x).*

Before proceeding, we shall look at the convergence of the series to some functions. In Figure 6 we see how the partial sums for a series of index 0 get closer to the function $f(x) = x^2 - 1$ for increasing number of terms. In Figure 7

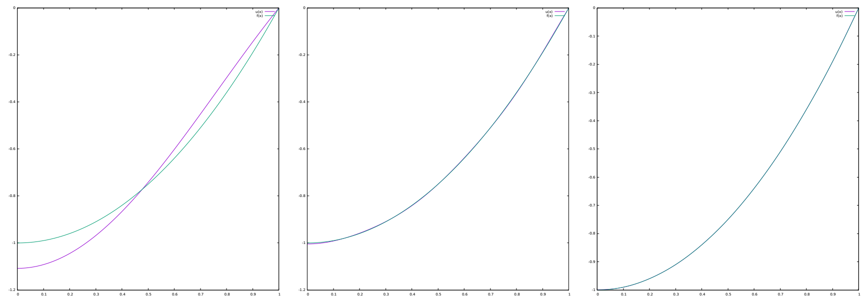


Figure 6: Shows the partial sums for a Fourier Bessel series of index 0. They have 1, 5 and 10 terms respectively. The function f is given by $f(x) = x^2 - 1$.

we see how the partial sums for a series of index 1 get closer to the function $f(x) = x^2 - 1$ for increasing number of terms aswell. However since $J_1(0) = 0$ the series don't converge to f at the origin so we see some overshoot right after the origin. This looks a lot like the well known Gibb's phenomenon for Fourier series (cf. Section 3.7 of [PZ97] and [FK03][3]). In Figure 8 we see the partial

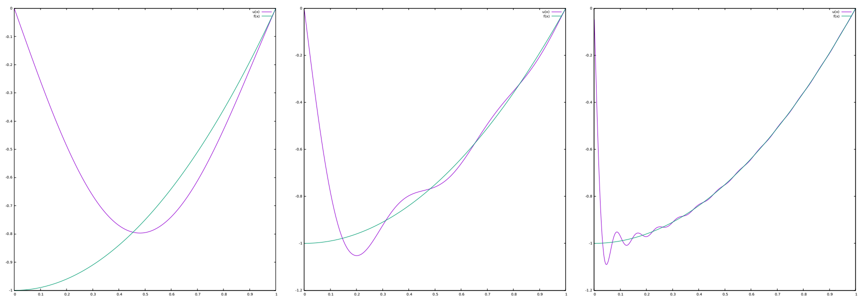


Figure 7: Shows the partial sums for a Fourier Bessel series of index 1. They have 1, 5 and 25 terms respectively. The function f is given by $f(x) = x^2 - 1$.

sums for series of index 5 with 1, 5 and 25 terms respectively. Note that the terms of this series look a bit "flatter" than the ones for series of smaller index.

In Figure 9 and Figure 10 we see how some partial sums approximate a function f which is 0 at the origin for series of index 1 respectively 5. They are quite close already for a few terms. In Figure 11 we see how some partial sums of the series of index 0 approximate f. The partial sums have 1, 5 and 25 terms respectively.

The functions we have looked at now look somewhat like Bessel functions, they have continuous derivative and not a large amplitude. We shall now look

[3]While this is not actually conclusive, it contains some reason to belive that the Gibb's phenomenon is true for Fourier-Bessel series.

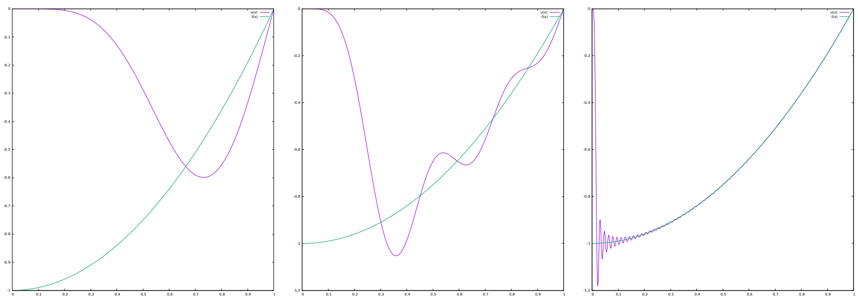


Figure 8: Shows the partial sums for a Fourier Bessel series of index 5. They have 1, 5 and 125 terms respectively. The function f is given by $f(x) = x^2 - 1$.

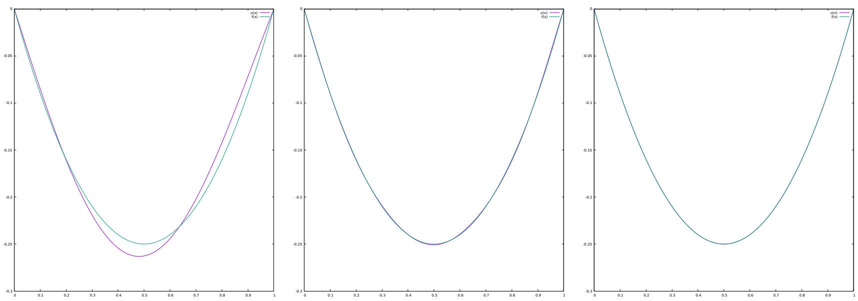


Figure 9: Shows the partial sums for a Fourier Bessel series of index 1. They have 1, 5 and 25 terms respectively. The function f is given by $f(x) = -x^2$.

at functions which are not as nice. They will only have piecewise continuous derivative and change character a lot. We begin with

$$f_1(x) = \begin{cases} 10x & x < 0.1 \\ 1 - 1000(x - 0.1)(x - 0.3) & x < 0.3 \\ 1 + 100x(x - 0.3) & x < 0.7 \\ (93 + \frac{1}{3})(1 - x) & x \leq 1 \end{cases}$$

and in Table 1 the distance from a series approximation of f of index m with n terms to the function f. The distance is given by

$$\sqrt{\int_0^1 |f(x) - g(x)| dx}.$$

In Figure 12 we see how different indexed series behave when approximating the function f_1.

It is also of interest to know how the series behave around discontinuities, so in Figure 13 we see series approximations of the piecewise continuous function $f(x) = -x$ for $x < 0.5$ and $f(x) = 1 - x$ for $x \leq 1$. We can see that there is not much difference between a series of index 0 and one of index 5 from Figure 13a and Figure 13b. From Figure 13a and Figure 13c we see how the series get closer to f and there's less overshoot at the discontinuity at $x = 0.5$. However since the series are composed of continuous functions we can not expect the

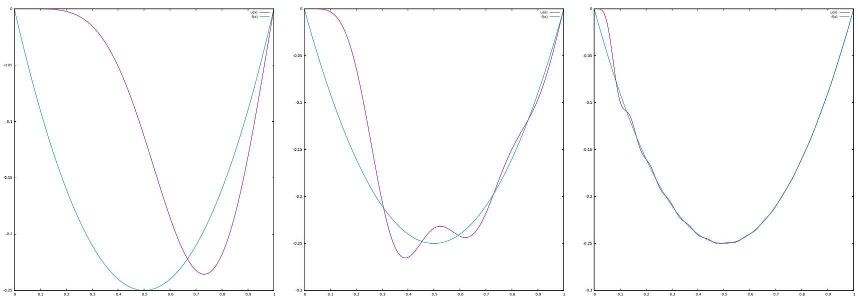


Figure 10: Shows the partial sums for a Fourier Bessel series of index 5. They have 1, 5 and 25 terms respectively. The function f is given by $f(x) = -x^2$.

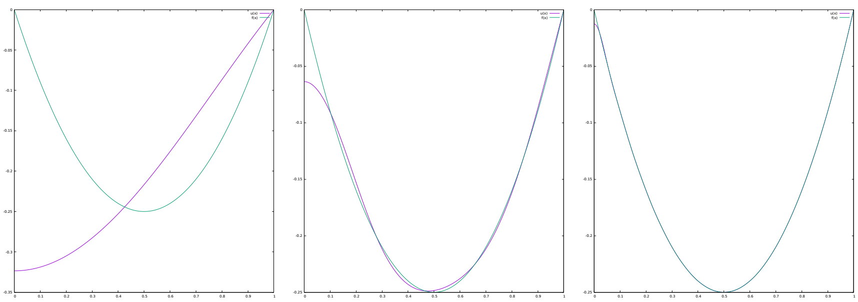


Figure 11: Shows the partial sums for a Fourier Bessel series of index 0. They have 1, 5 and 25 terms respectively. The function f is given by $f(x) = -x^2$.

series expansion be identical to f around $x = 0.5$, just as for regular Fourier series. In [Wat66] it is shown that Fourier-Bessel series obey much the same kind of properties as Fourier series and in particular it is shown that the series converge to $\frac{f(x+)+f(x-)}{2}$ pointwise. Now we shall state the relevant theorems from [Wat66].

5.4 Convergence of Fourier-Bessel series

An important result is the following, stated in section 18·24 of [Wat66]. In subsequent sections the theorems regarding uniform convergence is stated.

Theorem 2. *Let f be a function defined arbitrarily in an interval $[0, 1]$ such that $\int_0^1 \sqrt{t} f(t) dt$ exists and is absolutely convergent*[4] *. Moreover f is assumed to have* bounded variation[5] *in $(0, 1)$. Then if $x \in (0, 1)$ the Fourier-Bessel series*

[4] When we evaluate the Fourier-Bessel series we write it as

$$\sum_n a_n J_m(j_{mn}x) = \int_0^1 tf(t)\Big(\sum_n \frac{2}{J_{m+1}^2(j_{mn})} J_m(j_{mn}t) J_m(j_{mn}x)\Big)dt.$$

This condition is sufficient to show that only the immediate vicinity of the point x contributes to the integral as n grows. See section 18·23 of [Wat66] for more details.

[5] Watson actually writes limited total fluctuation, which I interpret as the same as bounded variation. The property we want from this condition is that f can be written as the difference of two positive increasing monotonic functions.

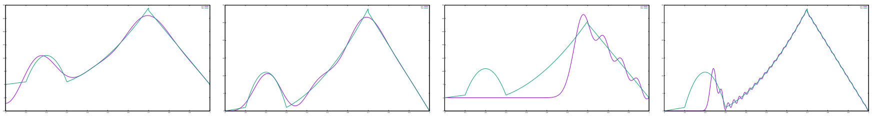


Figure 12: Shows the partial sums for a Fourier Bessel series of index 0, 4, 64 and 64 with 8 terms each except for the last one with 64 terms, for the function f_1.

m	n	distance	m	n	distance	m	n	distance	m	n	distance
1	1	2.5611	4	1	1.6987	16	1	2.8519	64	64	1.1244
1	4	1.2419	4	4	1.3808	16	4	1.6606	80	64	1.2468
1	16	0.5283	4	16	0.4936	16	16	1.2485	0	4	1.6933
1	64	0.1717	4	64	0.1757	16	64	0.2425	0	64	0.1698

Table 1: The distance between a partial sum of n terms of index m to the function f_1.

$\sum_{n=1}^{\infty} a_n J_m(j_{mn}x)$ *converges to* $\frac{f(x+)+f(x-)}{2}$, *when* $m \geq 0$.

Hence, for continuous functions f the Fourier-Bessel series associated with f converge pointwise in the interval $(0,1)$. Now we want to know if, or when, we have uniform convergence of

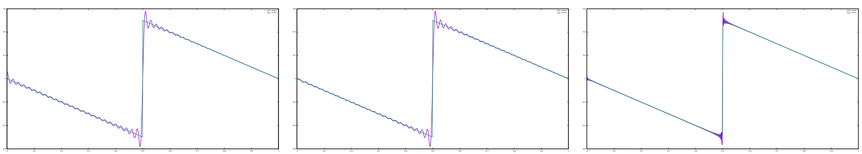
$$f(x) = \sum_{n=1} a_n J_m(j_{mn}x).$$

The following theorem from [Wat66] shows that this is indeed the case in the closed interval $[\Delta, 1]$, with $\Delta > 0$, when $f(1-) = 0$.

Theorem 3. *If f is continuous in $[0,1]$ and $f(1-) = 0$ in addition to the constraints in Theorem 2, then the Fourier Bessel expansion converge uniformly to f throughout the interval $[\Delta, 1]$ for some positive Δ.*

The point at the origin is a bit more complicated, and generally we can only say the following

Theorem 4. *Assume that $t^{1/2}f(t)$ have bounded variation in the interval* $[0,1]$,



(a) Series of index 0, the 100 first terms.

(b) Series of index 5, the 100 first terms.

(c) Series of index 0, the 500 first terms.

Figure 13: Approximations of a piecewise continuous function. This clearly demonstrates overshoot at the jump points, so that one may suspect an analogue of the Gibb's phenomenon.

and the conditions from previous theorems hold. Then the series

$$\sum_{n=1} a_n x^{1/2} J_m(j_{mn}x)$$

converges uniformly to $x^{1/2}f(x)$ *in the interval* $[0, 1-\Delta]$ *with* $0 < \Delta < 1$.

This is in line with intuition after looking at the figures of the previous section since $J_m(0) = 0$ for $m > 0$ so that this series must converge to something with value 0 at the origin.

6 Verifying the expansion u

We are now in a position to look at when our solution u is correct. By construction, u satisfies the wave equation in polar coordinates. So we must examine when

$$u(0,r,\theta)=f(r,\theta),$$

i.e. when the partial sums of u converge to f. Most certainly, we will not have this case for any arbitrary function f, since both the trigonometric functions and the Bessel functions of first the first kind are continuous functions.

6.1 Notation and properties

Let $\mathcal{A}$ denote the function space of real valued functions $f(r,\theta)$ on the rectangle $[0,1]\times[0,2\pi]$ such that

- f is continuous,
- $\frac{\partial}{\partial\theta}f$ exists and is piecewise continuous for each r,
- $f(r,\theta)$ is of bounded variation in the variable θ for each r,
- $f(1,\theta)=0$ for all θ,
- $\frac{\partial}{\partial r}f$ exists and is continuous for each θ,
- $\frac{\partial^2}{\partial r^2}f$ exists and is piecewise continuous for each θ,
- $f(r,0)=f(r,2\pi)$ for all r,
- $f(0,\theta)$ is constant.

Remark: If one considers a smooth surface $f^*(x,y)$ in the xy-plane defined on the unit disk such that f^* vanishes on the boundary, the function $f(r,\theta)=f^*(r\cos(\theta),r\sin(\theta))$ is one such function. This could provide some nice intuition, as it is easy to visualize.

Define the inner product of two functions f and g from $\mathcal{A}$ as

$$<f,g>=\frac{1}{\pi}\int_0^{2\pi}\int_0^1 rf(r,\theta)g(r,\theta)drd\theta.$$

Then $\mathcal{A}$ is an inner product space. Let E be an orthonormal system (possibly infinite) therein. If we let $\|\cdot\|$ denote the induced norm, i.e. $\|v\|=\sqrt{<v,v>}$, then Theorem 1.17 of [PZ97] asserts that for any f in $\mathcal{A}$ the vector $\sum_{e\in E}<f,e>e$ is closest to f in the span of E.

For our purposes we shall take the set E to be $\{\varphi_{mn},\psi_{mn}\}_{m=0,n=1}^{\infty}$ where

$$\varphi_{0n}(r,\theta)=\frac{1}{\sqrt{2}}\sqrt{\frac{2}{J_1^2(j_{0n})}}J_0(j_{0n}r),$$

$$\varphi_{mn}(r,\theta)=\cos(m\theta)\sqrt{\frac{2}{J_{m+1}^2(j_{mn})}}J_m(j_{mn}r),$$

$$\psi_{mn}(r,\theta)=\sin(m\theta)\sqrt{\frac{2}{J_{m+1}^2(j_{mn})}}J_m(j_{mn}r).$$

Since $\mathcal{A}$ is an inner product space, and in particular a vector space, linear transformations appear naturally. For our concerns, the linear transformation $L : \mathcal{A} \to \mathcal{B}$ defined by

$$Lf = c^2(f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2})$$

where c is some constant, is of special interest as equation (8) can be written in the form

$$Lv = -\lambda^2 v.$$

This form is familiar - it is the eigenvalue problem. The function space $\mathcal{B}$ is some suitable function space in which eigenvalue problems are meaningful and which contains $\mathcal{A}$, but we shall not specify it further. The solutions v are called *eigenfunctions* and the corresponding values λ^2 *eigenvalues.* In [CH53] these kinds of problems are studied extensively and on page 369 they state: *Every function $\omega(x, y)$ which satisfy the boundary conditions and possesses continuous derivaties up to the second order may be expanded in terms of the eigenfunctions in an absolutely and uniformly convergent series $\omega = \sum_{n=1}^{\infty} c_n v_n(x, y)$ with $c_n = \int\int_G \rho\omega v_n dxdy$. Thus the normalized eigenfunctions $\sqrt{\rho} v_n$ form a complete orthonormal system.* The approach used by Courant-Hilbert is quite complicated, I think, and their results are much more general than the ones we get. I think that anyone interested in mathematical physics would benefit a lot from reading just the first few chapters of this book.

In the following sections we shall take the coefficients a_{mn} and b_{mn} to be

$$a_{mn} = < f, \varphi_{mn} >,$$
$$b_{mn} = < f, \psi_{mn} >,$$

as motivated by [PZ97] and consider the resulting Fourier-Bessel series

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn}\varphi_{mn} + b_{mn}\psi_{mn}$$

for different classes of initial states f.

We will need to assume that the function u, satisfying $u_{t=0} = f$, has second order partial derivatives u_{tt}, u_{rr} and $u_{\theta\theta}$ that are given by termwise differentiation. We will not discuss what functions f give such an expansion.

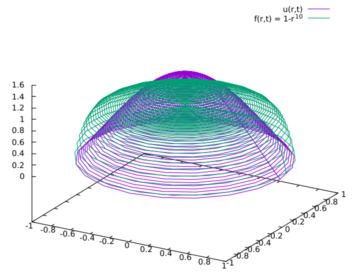
Before attacking the general case, we shall look at two examples that are special conditions of f where the Fourier coefficients become nice to evaluate.

6.2 Example 1: Radially symmetric initial condition

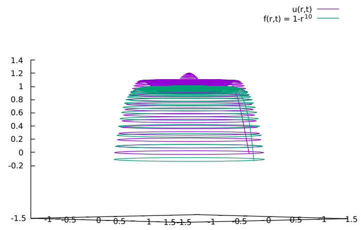
In Figure 14 we see some plots of a radially symmetric function $f \in \mathcal{A}$ and some partial sums of its expansion $u|_{t=0}$. They clearly show the convergence of u to f, and below we shall see why only Bessel functions of 0'th order is needed to describe the function u.

By equation (20) in Section 4.6 the general solution u of the wave equation is

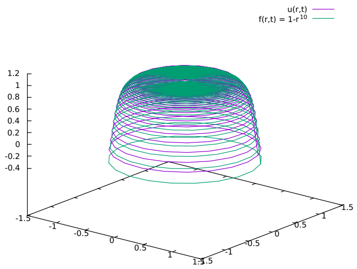
$$u(t, r, \theta) = \sum_{m=0} \sum_{n=1} c_{mn} J_m(j_{mn}r)(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta))\cos(cj_{mn}t).$$



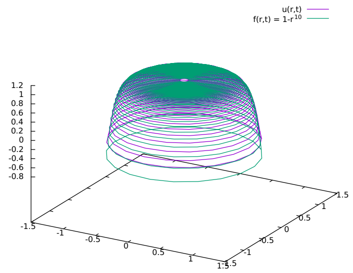
(a) With one term of the 0'th Bessel function.



(b) With three terms of the 0'th Bessel function.



(c) With ten terms of the 0'th Bessel function.



(d) With 50 terms of the 0'th Bessel function.

Figure 14: Plots of $f(r,\theta) = 1 - r^{10}$ and its expansion with different numbers of terms.

Let $A_{mn} = c_{mn}a_{mn}$ and $B_{mn} = c_{mn}b_{mn}$ so that we may write $u(0,r,\theta)$ as

$$
\begin{aligned}
u|_{t=0} &= \sum_{n=1} \sqrt{J_1^2(j_{0n})} A_{0n}\varphi_{0n} \\
&\quad + \sum_{m=1}\sum_{n=1} \sqrt{\frac{J_{m+1}^2(j_{mn})}{2}} A_{mn}\varphi_{mn} + \sqrt{\frac{J_{m+1}^2(j_{mn})}{2}} B_{mn}\psi_{mn}.
\end{aligned}
$$

Now we may decide the coefficients A_{mn} and B_{mn} by results of the previous section:

$$
\begin{aligned}
\sqrt{J_1^2(j_{0n})} A_{0n} &= < f, \varphi_{0n} >, \\
\sqrt{\frac{J_{m+1}^2(j_{mn})}{2}} A_{mn} &= < f, \varphi_{mn} >, \\
\sqrt{\frac{J_{m+1}^2(j_{mn})}{2}} B_{mn} &= < f, \psi_{mn} > .
\end{aligned}
$$

Since f is assumed to be radially symmetric it is on the form $f(r,\theta) = g(r)$ for

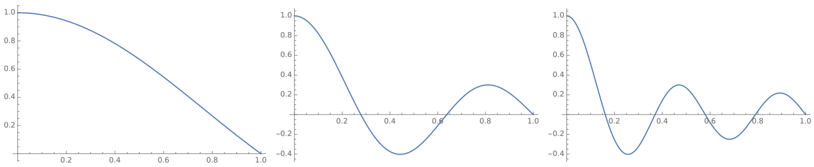


Figure 15: Plots of $J_0(j_{0n}x)$ for $n = 1, 3, 5$ over $[0, 1]$.

some function g and therefore if $m > 0$

$$< f, \varphi_{mn} >= \sqrt{\frac{2}{J_{m+1}^2(j_{mn})}} \frac{1}{\pi} \int_0^{2\pi} \cos(m\theta) d\theta \int_0^1 rg(r) J_m(j_{mn}r) dr = 0.$$

Hence $A_{mn} = 0$ and in the same way $B_{mn} = 0$ is deduced. For A_{0n} the trigonometric function is not present, and we find that the terms that remain of $u(0, r, \theta)$ are

$$u|_{t=0} = \sum_{n=1} \sqrt{J_1^2(j_{0n})} A_{0n} \varphi_{0n}.$$

Some rewriting of the terms give

$$u(0, r, \theta) = \sum_{n=1} \frac{1}{J_1^2(j_{0n})} \int_0^1 tg(t) J_0(j_{0n}t) dt J_0(j_{0n}r).$$

By Section 5 this sum converges to the function $g(r)$, which was precisely the initial condition.

This means that the general solution of the wave equation will look like

$$u(t, r, \theta) = \sum_{n=1} a_n J_0(j_{0n}r) \cos(cj_{0n}t),$$

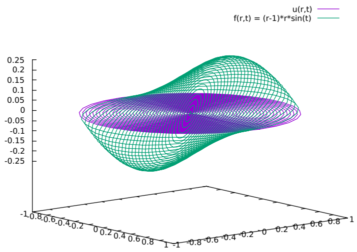
with $a_n = \frac{1}{J_1^2(j_{0n})} \int_0^1 tg(t) J_0(j_{0n}t) dt$.

Remark. This particular case shows that a radially symmetric initial condition only gives rise to circular Chladni figures. The Chladni figure connected to the term $a_n J_0(j_{0n}r) \cos(cj_{0n}t)$ is the circular membrane with $n - 1$ circles in the membrane. Looking at the plots in Figure 15, we can actually see these circles embedded. By considering these plots as drawn in the xz-plane and imagining that we rotate them 360 degrees around the z-axis, we get the initial shape for the term $a_n J_0(j_{0n}r) \cos(cj_{0n}t)$. That means that the zeros that we see in these plots are precisely the zeros for the membrane, that will lead to Chladni figures.

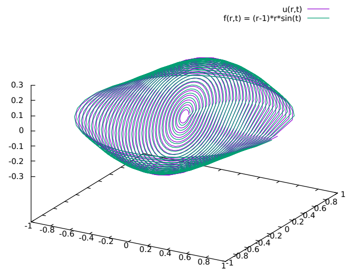
Thus, the Chladni patterns are directly given by the zeros of the Bessel function of index 0.

6.3 Example 2: The initial condition as a product

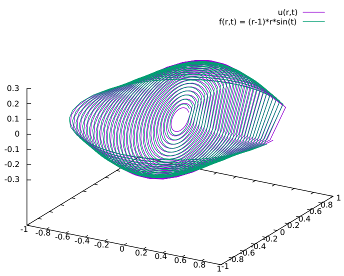
In Figure 16 we see how the expansion converges to $f(r, t) = (1 - r) r \sin(t)$.



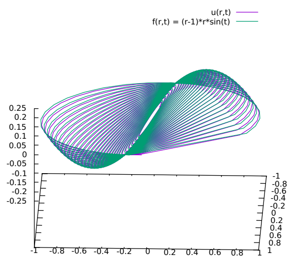
(a) With one term of the 0'th Bessel function.



(b) With one term each of the 0'th and 1'st Bessel functions.



(c) With two terms each of the 0'th and 1'st Bessel functions.



(d) With ten terms each of the 0'th and 1'st Bessel functions.

Figure 16: Plots of $f(r,t) = (1-r)r\sin(t)$ and its expansion with different number of terms.

When $f \in \mathcal{A}$ is on the form $f(r,\theta) = g(r)h(\theta)$ the integral in a_{mn} and b_{mn} is nice to evaluate. We get

$$\begin{aligned}
a_{0n} &= \langle f, \varphi_{0n} \rangle \\
&= \frac{1}{\pi}\sqrt{\frac{1}{J_1^2(j_{0n})}} \int_0^1 \int_0^{2\pi} r f(r,t) d\theta dr \\
&= \sqrt{\frac{1}{J_1^2(j_{0n})}} \int_0^1 r g(r) J_0(j_{0n} r) dr \frac{1}{\pi} \int_0^{2\pi} h(\theta) d\theta, \\
a_{mn} &= \langle f, \varphi_{mn} \rangle = \\
&= \frac{1}{\pi} \int_0^{2\pi} h(\theta)\cos(m\theta) d\theta \sqrt{\frac{2}{J_{m+1}^2(j_{mn})}} \int_0^1 r g(r) J_m(j_{mn} r) dr, \\
b_{mn} &= \langle f, \psi_{mn} \rangle, \\
&= \frac{1}{\pi} \int_0^{2\pi} h(\theta)\sin(m\theta) d\theta \sqrt{\frac{2}{J_{m+1}^2(j_{mn})}} \int_0^1 r g(r) J_m(j_{mn} r) dr.
\end{aligned}$$

Let the coefficients of sin and cos be denoted FS_m and FC_m, respectively, i.e.

$$FC_m = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos(m\theta)d\theta,$$
$$FS_m = \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin(m\theta)d\theta,$$

then we can rewrite the expansion u as

$$u(0,r,\theta) = \sum_{n=1} \frac{c_{0n}^2}{2}\int_0^1 rg(r)J_0(j_{0n}r)dr\int_0^{2\pi} h(\theta)J_0(j_{0n}r)+$$
$$\sum_{m=n=1}\int_0^1 rg(r)J_m(j_{mn}r)drJ_m(j_{mn}r)\left[FC_m\cos(m\theta)+FS_m\sin(m\theta)\right].$$

The expression within the square brackets is independent of n, so we can move it outside the innermost summation to get

$$u(0,r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta\sum_{n=1}\frac{2}{J_1^2(j_{0n})}\int_0^1 rg(r)J_0(j_{0n}rdrJ_0(j_{0n}r)$$
$$+\sum_{m=1}(FC_m\cos(m\theta)+FS_m\sin(m\theta))\sum_{n=1}\frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 rg(r)J_m(j_{mn}r)drJ_m(j_{mn}r).$$

From Section 5 we can conclude that the summation over n tends to the function g uniformly:

$$\sum_{n=1}\frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 rg(r)J_m(j_{mn}r)dr\cdot J_m(j_{mn}r) = g(r)$$

for $m \geq 0$.

Therefore we have

$$\begin{aligned}u(0,r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(\theta)d\theta\cdot g(r)\\ &+\sum_{m=1}(FC_m\cos(m\theta)+FS_m\sin(m\theta))\cdot g(r)\\ &= g(r)\mathcal{F}[h](\theta).\end{aligned}$$

By Theorem 1 in the preliminaries we know that $\mathcal{F}[h]$ converges uniformly to h when $f\in\mathcal{A}$. It follows that

$$u(0,r,\theta) = g(r)h(\theta).$$

Hence, the expansion $u|_{t=0}$ tends to f uniformly.

6.4 The general initial condition

The previous two examples gave some hints on how to examine the series u when the initial condition f is any function in $\mathcal{A}$. Define the functions F_m and

F_m^* by

$$F_0(r) = \frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)d\theta,$$
$$F_m(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(m\theta)d\theta,$$
$$F_m^*(r) = \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(m\theta)d\theta.$$

Then the expansion $u_{t=0}$ is given by

$$\begin{aligned} u(0,r,\theta) &= \sum_{n=1}^{\infty} rF_0(r)J_0(j_{0n}r)drJ_0(j_{0n}r) \\ &+ \sum_{m=1}^{\infty}\left(\cos(m\theta)\left[\sum_{n=1}^{\infty}\frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 rF_m(r)J_m(j_{mn}r)drJ_m(j_{mn}r)\right]\right. \\ &+ \left.\sin(m\theta)\left[\sum_{n=1}^{\infty}\frac{2}{J_{m+1}^2(j_{mn})}\int_0^1 rF_m^*(r)J_m(j_{mn}r)drJ_m(j_{mn}r)\right]\right). \end{aligned}$$

For $r > 0$ the summations over n can be evaluated by Theorem 3, so the above converges to

$$u(0,r,\theta) = F_0(r) + \sum_{m=1}^{\infty}\left(F_m(r)\cos(m\theta) + F_m^*\sin(m\theta)\right)$$

which is rewritten as

$$\begin{aligned} u(0,r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} h(r,\theta)d\theta \\ &+ \sum_{m=1}^{\infty}\left(\frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\cos(m\theta)d\theta\cos(m\theta) + \frac{1}{\pi}\int_0^{2\pi} f(r,\theta)\sin(m\theta)d\theta\sin(m\theta)\right). \end{aligned}$$

The expression should be recognized as the trigonometric Fourier series, which converges uniformly for each $r > 0$ to $f(r,\theta)$. For the special case $r = 0$ the computations are easy. The expansion is reduced to just

$$u(0,0,\theta) = \sum_{n=1}^{\infty}\frac{2}{J_1^2(j_{0n})}\int_0^1 rF_0(r)J_0(j_{0n}r)drJ_0(0) = F_0(0).$$

Since $f(0,\theta)$ is constant we have

$$u(0,0,\theta) = F_0(0) = \frac{1}{2\pi}\int_0^{2\pi} f(0,\theta)d\theta = f(0,\theta).$$

We conclude that

$$u(0,r,\theta) = f(r,\theta).$$

7 Resonance and Chladni figures

To connect with the initial motivation of the thesis, we shall now study what happens with the membrane when exposed to an external force. In particular, we shall look at a periodic external force. From [CH53] we get the *non-homogeneous* wave equation

$$\xi_{tt}(x, y, t) - Q(x, y, t) = c^2(\xi_{xx}(x, y, t) + \xi_{yy}(x, y, t)).$$

Transforming it to polar coordinates in the same manner as in section 3 we get

$$u_{tt}(r, \theta, t) - Q(r, \theta, t) = c^2 \left(u_{rr}(r, \theta, t) + \frac{u_r(r, \theta, t)}{r} + \frac{u_{\theta\theta}(r, \theta, t)}{r^2} \right). \tag{22}$$

With $Q \equiv 0$ for all t we get the *homogeneous* wave equation considered in the previous sections

$$u_{tt}(r, \theta, t) = c^2 \left(u_{rr}(r, \theta, t) + \frac{u_r(r, \theta, t)}{r} + \frac{u_{\theta\theta}(r, \theta, t)}{r^2} \right).$$

In order to simplify writing we will use $c = 1$ and $L[f] = f_{rr} + \frac{f_r}{r} + \frac{f_{\theta\theta}}{r^2}$.

The method we use to solve this problem is outlined in [CH53] pages 289-290, with some minor adaptions to our case. In order to solve (22) we shall assume that we can express the solution u on the form

$$u(r, \theta, t) = \sum_{m=0} \sum_{n=1} [q_{mn}(t) J_m(j_{mn} r) \cos(m\theta) + q^*_{mn}(t) J_m(j_{mn} r) \sin(m\theta)]$$

and also expressing the external force Q as

$$Q(r, \theta, t) = \sum_{m=0} \sum_{n=1} [Q_{mn}(t) J_m(j_{mn} r) \cos(m\theta) + Q^*_{mn}(t) J_m(j_{mn} r) \sin(m\theta)] .$$

The functions Q_{mn} and Q^*_{mn} are given by

$$Q_{0n}(t) = \sqrt{\frac{2}{J^2_{m+1}(j_{mn})\pi}} \int_0^{2\pi} \frac{1}{2} \int_0^1 r Q(r, \theta, t) J_0(j_{0n} r) dr d\theta,$$

$$Q_{mn}(t) = \sqrt{\frac{2}{J^2_{m+1}(j_{mn})\pi}} \int_0^{2\pi} \int_0^1 r Q(r, \theta, t) J_m(j_{mn} r) \cos(m\theta) dr d\theta,$$

$$Q^*_{mn}(t) = \sqrt{\frac{2}{J^2_{m+1}(j_{mn})\pi}} \int_0^{2\pi} \int_0^1 r Q(r, \theta, t) J_m(j_{mn} r) \sin(m\theta) dr d\theta.$$

To satisfy the differential equation (22) we shall solve the infinite number of systems

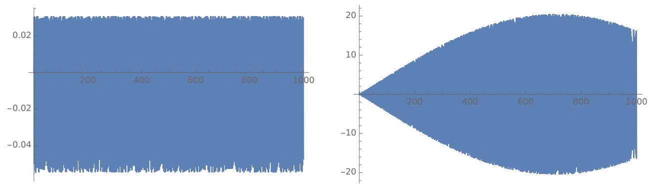
$$L[q_{mn}(t) J_m(j_{mn} r) \cos(m\theta)] = \ddot{q}_{mn}(t) J_m(j_{mn} r) \cos(m\theta) - Q_{mn}(t) J_m(j_{mn} r) \cos(m\theta),$$
$$L[q^*_{mn}(t) J_m(j_{mn} r) \sin(m\theta)] = \ddot{q}^*_{mn}(t) J_m(j_{mn} r) \sin(m\theta) - Q^*_{mn}(t) J_m(j_{mn} r) \sin(m\theta).$$

The steps taken to solve each of these are precisely the same, so we will only consider the first one. Since L does not affect the variable t we get

$$L[q_{mn}(t) J_m(j_{mn} r) \cos(m\theta)] = q_{mn}(t) L[J_m(j_{mn} r) \cos(m\theta)]$$

so
$$\frac{L[J_m(j_{mn}r)\cos(m\theta)]}{J_m(j_{mn}r)\cos(m\theta)} = \frac{\ddot{q}_{mn}(t) - Q_{mn}(t)}{q_{mn}(t)}$$
which must be equal to some constant $-\lambda^2$ as the left and right sides are dependent on different variables. We get the two equations
$$\begin{aligned} L[J_m(j_{mn}r)\cos(m\theta)] + \lambda^2 J_m(j_{mn}r)\cos(m\theta) = 0, \\ \ddot{q}_{mn}(t) + \lambda^2 q_{mn}(t) = Q_{mn}(t). \end{aligned}$$
To satisfy the first of these equations we know that the constant λ must be given by j_{mn} (cf. page 10). The solution to the second equation is given by
$$q_{mn}(t) = c_1\cos(mt) + c_2\sin(mt) + \frac{1}{\lambda}\int_0^t Q_{mn}(\tau)\sin(\lambda(t-\tau))d\tau,$$
with the coefficients c_1 and c_2 to be decided by initial conditions. If the inital conditions are $q_{mn}(0) = \dot{q}_{mn}(t) = 0$ then the solution is
$$q_{mn}(t) = \frac{1}{\lambda}\int_0^t Q_{mn}(\tau)\sin(\lambda(t-\tau))d\tau.$$
These intial conditions can be interpreted as the membrane is at rest with $u(0,x,y) = 0$ at time zero. Therefore we shall use these conditions, and thus we find the two sought solutions $q_{mn}(t)$ and $q^*_{mn}(t)$ as
$$\begin{aligned} q_{mn}(t) &= \frac{1}{j_{mn}}\int_0^t Q_{mn}(\tau)\sin(j_{mn}(t-\tau))d\tau, \\ q^*_{mn}(t) &= \frac{1}{j_{mn}}\int_0^t Q^*_{mn}(\tau)\sin(j_{mn}(t-\tau))d\tau. \end{aligned}$$
If the external force Q is periodic with frequency ω and $Q \equiv 0$ at time $t = 0$, then it can be expressed as
$$Q(x,y,t) = \varphi(x,y)\sin(\omega t).$$
Then the functions Q_{mn} and Q^*_{mn} can be written as
$$Q_{mn}(t) = \sin(\omega t)C_{mn}, \tag{23}$$
$$Q^*_{mn}(t) = \sin(\omega t)D_{mn}, \tag{24}$$
for some constants C_{mn} and D_{mn} arising from the integral over the region $[0,2\pi]\times[0,1]$ in the definitions of Q_{mn} and Q^*_{mn}.

Thus, the solution u is given by
$$u(r,\theta,t) = \sum_{m=0}\sum_{n=1} J_m(j_{mn}r)(q_{mn}(t)\cos(m\theta) + q^*_{mn}(t)\sin(m\theta)),$$
with
$$\begin{aligned} q_{mn}(t) &= \frac{1}{j_{mn}}\int_0^t Q_{mn}(\tau)\sin(j_{mn}(t-\tau))d\tau, \\ q^*_{mn}(t) &= \frac{1}{j_{mn}}\int_0^t Q^*_{mn}(\tau)\sin(j_{mn}(t-\tau))d\tau. \end{aligned}$$



(a) The development of $q(t)$ when $\omega \neq j_{mn}$.

(b) The development of $q(t)$ when $\omega = j_{mn} + \delta$.

Figure 17: Development of $q(t)$ in two cases with non-resonance: When ω is not near j_{mn} and when ω is near j_{mn} but not equal, respectively.

What is interesting to examine is how the functions q_{mn} and q^*_{mn} develop over time, as they describe the amplitude of the corresponding (m, n)-mode.

Now we shall look at how the functions $q_{mn}(t)$ and $q^*_{mn}(t)$ behave for different values of ω. When applicable, we shall assume that the relevant constant C_{mn} and/or D_{mn} in equations (23) and (24), respectively, are non-zero. Consider the expression

$$q(t) = \frac{C}{j_{mn}} \int_0^t \sin(\omega\tau)\sin(j_{mn}(t-\tau))d\tau,$$

for some given (m, n). If $\omega \neq j_{mn}$ the integral evaluates to

$$q(t) = \frac{C}{j_{mn}} \left[\frac{j_{mn}\sin(\omega t) - \omega\sin(j_{mn}t)}{j_{mn}^2 - \omega^2} \right],$$

and if $\omega = j_{mn}$

$$q(t) = \frac{C}{2j_{mn}} \left[-t\cos(j_{mn}t) + \frac{\sin(j_{mn}t)}{j_{mn}} \right].$$

The particular case when $\omega = j_{mn}$ is called *resonance* and we see that $|q(t)|$ becomes unbounded and linearly growing. This case is plotted in Figure 18. For $\omega = j_{mn} \pm \delta$ for some "small" δ we see that $q(t)$ is still periodic but the maximum of $|q(t)|$ can be arbitratily large by choosing δ sufficiently small. When the frequency ω is not near j_{mn} the maximum of $|q(t)|$ is quite small, if interpreted as the amplitude of a drum membrane. These two cases are plotted in Figure 17.

The function $q(t)$ describe exactly how the functions $q_{mn}(t)$ and $q^*_{mn}(t)$ behave, except for some constant multiple.

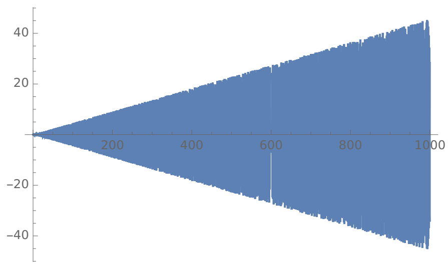


Figure 18: The development of $q(t)$ when we have resonance, i.e. $\omega = j_{mn}$.

7.1 Conclusion

My interpretation and conclusion of this train of thought is the following: If we have an external force acting on a drum membrane with a period equal to some eigenfrequency j_{mn}, then the amplituted of the correspoding eigenfunction (or *fundamental shape*) $J_m(j_{mn}r)(a\cos(m\theta) + b\sin(m\theta))$ grows linearly. All other shapes' amplitudes remain bounded. Hence, in time the solution u becomes asymptotically

$$u(r, \theta, t) \sim CtJ_m(j_{mn}r)(a\cos(m\theta) + b\sin(m\theta))\cos(j_{mn}t + \beta).$$

This is why Chladi figures appear, since this solution have nodal patterns which remain constant for all times. In these nodal patterns, sand will gather and form Chladni figures.

8 In three dimensions

We shall briefly look at a very similar problem, namely the wave equation over a sphere in three space dimensions. We will keep the boundary stationary as before, while letting the interior move along a fourth dimension. This the same idea as considering the membrane in two space dimensions, with the boundary fixed and letting the interior move along a third dimension, which is the problem considered up to now in this thesis.

When considering a membrane in two dimensions it is natural to transform the equation by introducing polar coordinates. The analogue in three dimensions is to transform the equation by introducing spherical coordinates. The wave equation in spherical coordinates is given by [CH53], Chapter IV Section §8,

$$u_{tt} = \frac{1}{r^2}(r^2 u_r)_r + \frac{1}{r^2 \sin(\theta)}(\sin(\theta) u_\theta)_\theta + \frac{1}{r^2 \sin^2(\theta)} u_{\varphi\varphi}.$$

In order to solve this, we will as before use separation of variables. So, we assume that the solution u can be written as $u(t, r, \theta, \varphi) = T(t)R(r)\Theta(\theta)\Phi(\varphi)$. The wave equation splits into the equations

$$\begin{aligned} T'' - k^2 T &= 0, \\ r^2 R'' + 2rR' + (k^2 r^2 - l(l+1))R &= 0, \\ \Phi'' + m^2 \Phi &= 0, \\ \frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Theta') + (l(l+1) - \frac{m^2}{\sin^2\theta})\Theta &= 0. \end{aligned}$$

for some constants l and m to be decided later. The time part is the same as considered previously, and is for our purposes not interesting, so we will not give it any attention. The radial part is interesting, however, so we will now focus our attention to the second equation, involving r.

If we substitute $R(r)$ for $R(r) = \frac{Z(r)}{\sqrt{kr}}$ in

$$r^2 R'' + 2rR' + (r^2 k^2 - l(l+1))R = 0$$

we get

$$r^2 Z'' + rZ' + (k^2 r^2 - (l(l+1) + \frac{1}{4}))Z = 0$$

which, if you recall, is quite similar to the Bessel equation in (18). The major difference is the term $(l(l+1) + \frac{1}{4})$, which we rewrite as a quadratic expression. We write

$$r^2 Z'' + rZ' + (k^2 r^2 - (l + \frac{1}{2})^2)Z = 0.$$

Solving this is now no different from earlier, the solution is given by

$$Z(r) = AJ_{l+\frac{1}{2}}(kr) + BN_{l+\frac{1}{2}}(kr)$$

where J and N denote the Bessel function of first and second kind, respectively (cf. Section 4.5). With k decided by

$$Z(1) = AJ_{l+\frac{1}{2}}(k) + BN_{l+\frac{1}{2}}(k) = 0,$$

the solution R is given by

$$R(r) = \frac{1}{\sqrt{kr}}(AJ_{l+\frac{1}{2}}(kr) + BN_{l+\frac{1}{2}}(kr)).$$

In this case we also have $\lim_{r\to 0^+} N_{l+\frac{1}{2}}(kr) = -\infty$, therefore we will take $B = 0$. In Definition 1 we only considered the case of integer index, but the definition and following properties is equally valid for any real index $\omega = v \in \mathbb{R}$, see [Wat66] Chapter III. Note that from the power series definition it is easy to see $\lim_{r\to 0} \frac{J_v(r)}{\sqrt{r}} = \infty$ when $|v| < \frac{1}{2}$, and bounded otherwise.

In the unit ball we have the boundary condition $R(1) = 0$, therefore $k_{l,i} = j_{l+\frac{1}{2},i}$. So in conclusion, the radial part of u, R, is given by

$$R_{l,i}(r) = \frac{1}{\sqrt{j_{l+\frac{1}{2},i}r}}AJ_{l+\frac{1}{2}}(j_{l+\frac{1}{2},i}r).$$

Now, we turn our attention to the angular parts of u. We shall consider the equations

$$\Phi'' + m^2\Phi = 0,$$
$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\Theta') + (l(l+1) - \frac{m^2}{\sin^2\theta})\Theta = 0.$$

The first equation has the general solution

$$\Phi(\varphi) = A\mathrm{e}^{im\varphi} + B\mathrm{e}^{-im\varphi},$$

for complex constants A and B, to be decided later by the inital conditions.

The second equation is quite a bit trickier, and we will not go into any depth of solving it. The solution is given by

$$\Theta(\theta) = CP_l^m(\cos(\theta)).$$

The function P_l^m is the so called *associated Legendre Polynomial of m-th order* [CH53][6], where $l = 0, 1, 2, ...$ and m runs over the integers from $-l$ to l. Details and further references of the procedure used here can be found at [Wik17].

One usually combines Θ and Φ to form the spherical harmonics, Y_l^m, so that

$$Y_l^m(\theta,\varphi) = \Theta(\theta)\Phi(\varphi) = CP_l^m(\theta)(A\mathrm{e}^{im\varphi} + B\mathrm{e}^{-im\varphi}).$$

Combining the angular parts and the radial part we have arrived at a solution for the initial system excluding the time dependent equation when the constants l, m and i are given:

$$f_{l,m,i}(r,\theta,\varphi) = DJ_{l+\frac{1}{2}}(j_{l+\frac{1}{2},i}r)Y_l^m(\theta,\varphi),$$

where $l = 0, 1, 2, \cdots$, $m = -l, -l+1, \cdots, l-1, l$ and $i = 1, 2, \cdots$. The function $f_{l,m,i}$ is called the (l, m, i)-mode following the same train of thought as for the circular membrane. Therefore the general solution u is given by summing over l, m and i

$$u(t,r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{i=1}^{\infty} f_{l,m,i}(r,\theta,\varphi)(a_{l,m,i}\cos(j_{l+\frac{1}{2},i}t) + b_{l,m,i}\sin(j_{l+\frac{1}{2},i}t))$$

[6] Chapter V, Section §10

When we examined the circular membrane we looked at the nodal lines of each (m, n)-mode. The idea now is to do the same for some (l, m, i)-modes. Just as the nodal lines were the zeros of the (m, n)-mode we shall call the zeros of the (l, m, i)-modes *nodal surfaces*. For finding the nodal surfaces, we need to solve $f_{l,m,i} = 0$. This happens when $Y_l^m(\theta, \varphi) = 0$ or $J_{l+\frac{1}{2}}(j_{l+\frac{1}{2},i}r) = 0$. We have

$$Y_l^m(\theta, \varphi) = CP_l^m(\cos\theta)(A\mathrm{e}^{im\varphi} + B\mathrm{e}^{-im\varphi}),$$

and thus for $m > 0$ the function is generally complex valued. In addition, the zeros with regards to φ are given by

$$\varphi = \frac{2\pi n - i\log(-\frac{B}{A})}{2m},$$

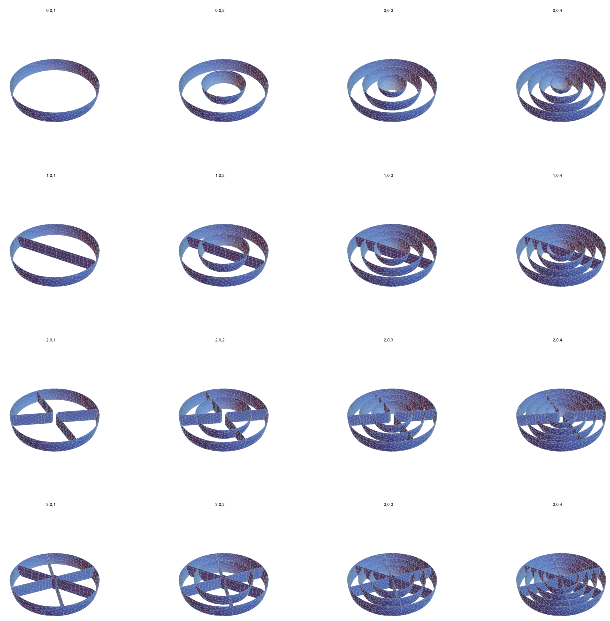
for integers n, when $|\frac{B}{A}| \neq 1$. However, we consider φ to be real as it denotes an angle and therefore we can draw the conclusion that the zeroes of $Y_l^m(\theta, \varphi)$ is independent of φ when $m > 0$. Because of this we will stick to the case when $m = 0$, which may be interpreted as the initial position and velocity of the sphere is independent of φ.

All in all, the surfaces we will be looking at are satisfying

$$f_{l,0,i} = DJ_{l+\frac{1}{2}}(j_{l+\frac{1}{2}}r)P_l^0(\cos(\theta)) = 0.$$

8.1 The plots

Below is a table of the plots. To be able to see the nodal surfaces I have cut away part of it, so that the z-values run from -0.3 to 0, but it is actually in the unit sphere. An interesting pattern also occur in the plots, I believe this is because of *Mathematica*'s inability to discretize the regions correctly, as I have entered them. It is clear in the plot of mode $2, 0, 1$ that there is a hole in the middle. This is not supposed to be there, but there should be complete planes as in the plot above $(1, 0, 1)$.


0,0,1
0,0,2
0,0,3
0,0,4
1,0,1
1,0,2
1,0,3
1,0,4
2,0,1
2,0,2
2,0,3
2,0,4
3,0,1
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3,0,3
3,0,4

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