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Koszul Algebras

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Abstract

This thesis is an introduction to the study of Koszul algebras. We give enough theory to state the definition of Koszul algebras. After giving the definition we discuss Koszul duality, we compare two ways of defining the Koszul dual algebra of a quadratic algebra and show that they are equivalent. One important result for Koszul algebras is that their Hilbert series and the Hilbert series of their Koszul dual algebras satisfy a certain relation which we prove in this thesis.

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1 Introduction

In this thesis we give a first introduction to Koszul algebras which are special quadratic algebras first introduced by Stewart Priddy. In Section 2 we start by going over some definitions and properties from linear algebra but to get the most out of this thesis one should have seen most of this before. There are many ways to define Koszul algebras but we will follow [3] and say that a quadratic algebra is Koszul if a certain chain complex associated to the algebra is exact. We also give an introduction to Koszul duality and here we present two ways of defining the Koszul dual of a quadratic algebra and show that they are in fact equivalent. The main theorem which we will prove in this thesis relates the dimensions of the graded parts of locally finite dimensional Koszul algebras to those of their dual algebras. We also prove that a quadratic algebra is Koszul if and only if its Koszul dual algebra is. Koszul algebras have been studied since the 70's and they occur in different fields of mathematics such as, algebraic geometry, topology, noncommutative algebra and more.

2 Basics

In this section we define the space of linear transformations from a vector space to another and the tensor product of two vector spaces. We define associative algebras and the free associative algebra over a finite dimensional vector space. We discuss chain complexes and exact sequences. Some claims in this section are left unproved and the discussions are short, there is much more to say about the topics of this section. The operations and structures defined below are important when discussing Koszulness and they will be frequently used throughout this thesis.
Let us denote the base field by $\mathbb{K}$.

Definition 2.1. *Let U end V be vector spaces over $\mathbb{K}$. We denote by $Hom(U,V)$ the vector space of linear maps from U to V. Addition is defined by $(f+g)(u) = f(u) + g(u)$ and scalar product by $(k \cdot g)(u) = k \cdot g(u)$ for all $u \in U$ and $k \in \mathbb{K}$. A special case is when $V = \mathbb{K}$. We then write*

$$U^* := Hom(U, \mathbb{K})$$

and we call U^* the linear dual space of U.

When V is a finite dimensional vector space then we have $dim(V) = dim(V^*)$ and starting with a basis $\{e_1, ...e_n\}$ for V we get what is called the dual basis $\{e_1^*, ..., e_n^*\}$ where e_i^* is the functional defined by

$$e_i^*(e_j) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

[1, p 96].
We can also take the linear dual of a linear map in the following way. Let $f : U \to V$ be a linear map between vector spaces U and V. Then the dual of f, f^* is a linear map from V^* to U^* by $f^*(v^*)(u) = v^*(f(u))$ for any $v^* \in V^*$ and $u \in U$

Starting with two vector spaces, there are many ways to obtain a new one. We have seen that $Hom(U,V)$ is a vector space and below we will define the tensor product of two vector spaces which is also a vector space.

Definition 2.2. *Let U and V be vector spaces over a field $\mathbb{K}$. The tensor product $U \otimes V$ is a vector space together with a bilinear map $t : U \times V \to U \otimes V$ satisfying the following universal property: For any bilinear map $h : U \times V \to W$ from the cartesian product of U and V to some vector space W there is a unique linear map $\tilde{h} : U \otimes V \to W$ such that $h = \tilde{h} \circ t$*

$$\begin{tikzcd} U \times V \arrow[r, "t"] \arrow[dr, "h"'] & U \otimes V \arrow[d, dashed, "\exists!\tilde{h}"] \\ & W \end{tikzcd}$$

One can show that the tensor product always exists and is unique up to isomorphism [1, p 362].
Now I present one way to construct the tensor product of two finite dimensional vector spaces U and V over a field $\mathbb{K}$.

Proposition 2.3. *Let U and V be finite dimensional vector spaces with bases $\{u_1, ..., u_n\}$ and $\{v_1, ..., v_m\}$ respectively. Let T be the vector space spanned by the formal basis vectors $\{u_i \otimes v_j\}_{1\leq i\leq n, 1\leq j\leq m}$ and consider the bilinear map*

$$t : U \times V \to T$$
$$(\sum_i \alpha_i u_i, \sum_j \beta_j v_j) \mapsto \sum_{i,j} \alpha_i \beta_j u_i \otimes v_j.$$

Then T together with t satisfies the universal property in Definition 2.2. and thus it is the tensor product of V and U.

Proof. Let h be any bilinear map from $U \times V$ to some vector space W. If the map $\tilde{h}$ from Definition 2.2. exists, then it should satisfy

$$\tilde{h}(u_i \otimes v_j) = h(u_i, v_j).$$

But by extending linearly this condition uniquely defines a linear map from T to W. We check that this makes the diagram in Definition 2.2. commute:

$$\tilde{h} \circ t(\sum_i \alpha_i u_i, \sum_j \beta_j v_j) = \tilde{h}(\sum_{i,j} \alpha_i \beta_j u_i \otimes v_j) =$$

$$= \sum_{i,j} \alpha_i \beta_j \tilde{h}(u_i \otimes v_j) = \sum_{i,j} \alpha_i \beta_j h(u_i, v_j) = h(\sum_i \alpha_i u_i, \sum_j \beta_j v_j).$$

This shows that T is indeed the tensor product of U and V. □

One can also take tensor product of linear maps. Consider the finite dimensional vector spaces U, V, W and M, then for f in $Hom(U, V)$ and g in $Hom(W, M)$ the tensor product $f \otimes g$ is defined as the linear map in $Hom(U \otimes W, V \otimes M)$ that maps $u \otimes w$ to $f(u) \otimes g(w)$.
By considering multilinear maps rather than bilinear maps, one can define tensor products of more than two vector spaces similarly to how we did for two [1, p 382] and it is common to omit the tensor symbol in this case. That is, we will often write $\sum_i v_{i_1} ... v_{i_n}$ instead of $\sum_i v_{i_1} \otimes ... \otimes v_{i_n}$ for elements of $V^{\otimes n}$. The tensor product can be used to define bilinear operations which we will do below.

Definition 2.4. *An algebra A over a field $\mathbb{K}$ is a vector space equipped with a linear map $\mu : A \otimes A \to A$. We say that A is an associative algebra if the the following diagram commutes*

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{1\otimes\mu} & A \otimes A \\ \downarrow{\scriptstyle \mu\otimes 1} & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow{\mu} & A \end{array}.$$

Note that this is the same as saying that A has a binary operation, called the product, which is bilinear and satisfies $(v_1v_2)v_3 = v_1(v_2v_3)$.

Definition 2.5. *We call an associative algebra unital if it is equipped with a linear map $u : \mathbb{K} \to A$ such that the following diagram commutes.*

$$\begin{tikzcd}
\mathbb{K} \otimes A \arrow[r, "u\otimes 1"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K} \arrow[l, "1\otimes u"'] \arrow[dl, "\cong"] \\
 & A &
\end{tikzcd}$$

where μ is the multiplication of A and the isomorphisms are the natural ones mapping $k \otimes a \in \mathbb{K} \otimes A$ to ka and $a \otimes k \in A \otimes \mathbb{K}$ to ka respectively.

All the algebras in this thesis will be assumed to be unital so we will only define algebra morphisms for such algebras.

Definition 2.6. *An algebra morphism is a linear map $f : A \to A'$ between unital algebras which respects the multiplication and the unit. That is $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$*

Because algebras are so central in this thesis we will give a few more definitions of special algebra structures which we are interested in and define ideals of algebras.

Definition 2.7. *We say that a unital algebra is augmented if it is equipped with an algebra homomorphism $\epsilon : A \to \mathbb{K}$. In this case A can be written as a direct sum $\mathbb{K} \oplus ker(\epsilon)$. We call $ker(\epsilon)$ the augmentation ideal.*

Definition 2.8. *An algebra A is said to be graded if it can, as a vector space, be written as a direct sum*

$$A = \bigoplus_{i \geq 0} A_0$$

and if the multiplication then satisfies $A_iA_j \subset A_{i+j}$.

Definition 2.9. *A two sided ideal of an algebra is a subspace of the underlying vector space which is closed under multiplication by any element of the algebra from the right or from the left.*

With the help of tensor product and direct sum of vector spaces we can construct what is called the *free associative algebra* over a vector space V. It is the graded, unital, augmented algebra

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus ...$$

with the associative product $\mu(v_1v_2...v_i \otimes v_1'v_2'...v_j') := v_1v_2...v_iv_1'v_2'...v_j'$ for $v_1v_2...v_i \in V^{\otimes i}$ and $v_1'v_2'...v_j' \in V^{\otimes j}$ and extending linearly. The free associative algebra over a vector space V satisfies the following universal property [3]:

Let $i : V \to T(V)$ be the inclusion. Then for any linear map $f : V \to A$ from V to a unital associative algebra there is a unique algebra morphism $\tilde{f}$ making the following diagram commute

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, dashed, "\exists! \tilde{f}"] \\ & A \end{tikzcd}$$

We are also interested in the structures that are dual to algebras in the sense that they have a reverse multiplication.

Definition 2.10. *A coalgebra is a vector space C equipped with a linear map $\Delta : C \to C \otimes C$, called the coproduct, we call C a coassociative coalgebra if the following diagram commutes*

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "1\otimes\Delta"] \\ C \otimes C \arrow[r, "\Delta\otimes 1"] & C \otimes C \otimes C \end{tikzcd}$$

and we say it is counital if it is equipped with a linear map $\epsilon : C \to \mathbb{K}$ making the following diagram commute.

$$\begin{tikzcd} & C \arrow[dl, "\cong"] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ C \otimes \mathbb{K} & C \otimes C \arrow[l, "1\otimes\epsilon"] \arrow[r, "\epsilon\otimes 1"] & \mathbb{K} \otimes C \end{tikzcd}$$

where the isomorphisms are the natural ones mapping $c \in C$ to $c \otimes 1$ and $1 \otimes c$ respectively.

Just like for algebras we are interested in coalgebras with a graded structure. Let us give a precise definition.

Definition 2.11. *We say that a coalgebra is graded if it can be written as a direct sum of vector spaces*

$$\bigoplus_{i \geq 0} C_i$$

and if the coproduct then satisfies $\Delta(C_k) \subset \bigoplus_{s+t=k} C_s \otimes C_t$

Algebras and coalgebras are closely related in the following way. If we start with a finite dimensional algebra A with multiplication $\mu : A \otimes A \to A$. Then A^* becomes a coalgebra with the following composition of maps as comultiplication

$$A^* \xrightarrow{\mu^*} (A \otimes A)^* \xrightarrow{\eta^{-1}} A^* \otimes A^* \ .$$

Here η is the natural embedding $A^* \otimes A^* \hookrightarrow (A \otimes A^*)$ defined by $\eta(f \otimes g)(a \otimes b) = f(a)g(b)$ for any $f \otimes g \in A^* \otimes A^*$ and $a \otimes b \in A \otimes A$ and extended linearly. It is an isomorphism in the finite dimensional case[1, p 374]. Similarly if C is a coalgebra with comultiplication Δ, then C^* becomes an algebra with the following composition of maps as multiplication

$$C^* \otimes C^* \xrightarrow{\eta} (C \otimes C)^* \xrightarrow{\Delta^*} C^* \ .$$

Because η is an embedding there is no need to assume finite dimension in this case.

Finally we will define special types of sequences of vector spaces and linear maps called chain complexes. They are central in this thesis since we use them in defining Koszul algebras.

Definition 2.12. *A chain complex is a family of vector spaces $\{V_i\}_{i \in \mathbb{Z}}$ together with linear maps $\{d_i : V_i \to V_{i-1}\}$ that satisfy $Im(d_i) \subset ker(d_{i-1})$.*

$$\ldots \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \xrightarrow{d_{i-1}} V_{i-2} \xrightarrow{d_{i-2}} \ldots$$

We can also consider a chain complex as a graded vector space $\bigoplus_{i \in \mathbb{Z}} V_i$ with a linear map d called the differential defined by $d(v) = d_i(v)$ for $v \in V_i$ which then satisfies $d^2 = 0$.
If we have $Im(d_{i+1}) = ker(d_i)$ then we say that the chain complex is exact at V_i. If this holds for all i then the chain complex is an exact sequence.

One way of proving that a chain complex is exact is by finding what is called a homotopy. That is, a family of linear maps $h = \{h_j : V_{j-1} \to V_j\}_j$ such that $(h_j \circ d_j)(v) + (d_{j+1} \circ h_{j+1})(v) = v$ for any $v \in V_j$. Indeed, if we have such a homotopy and v is in the kernel of d_j then we get

$$v = (h_j \circ d_j)(v) + (d_{j+1} \circ h_{j+1})(v) = (d_{j+1} \circ h_{j+1})(v)$$

which shows that $ker(d_j) \subset Im(d_{j+1})$ which then proves that the chain complex is exact.

3 Koszul Algebras

We begin by defining quadratic algebras. Then we define the Koszul dual coalgebra of a quadratic algebra and show how a certain linear map from the Koszul dual coalgebra to the original quadratic algebra gives rise to a chain complex. We then use this chain complex to give a definition of Koszul algebras. There are many equivalent definitions of Koszul algebras but we will only state one.

3.1 Quadratic algebras and their Koszul dual coalgebras

In the introduction we mentioned that Koszul algebras are special quadratic algebras. So we begin by defining quadratic algebras.

Definition 3.1. *For any graded augmented algebra $A = \mathbb{K} \oplus (\bigoplus_{i=1}^{\infty} A_i)$, consider the inclusion $i : A_1 \to A$. By the universal property of $T(A_1)$, this induces a unique algebra morphism $p : T(A_1) \to A$. We say that A is quadratic if p is surjective and $ker(p)$ is generated, as a two sided ideal, by elements in $A_1 \otimes A_1$.*

This means that for a quadratic algebra A we have

$$A = \mathbb{K} \bigoplus V \bigoplus (V \otimes V)/R \bigoplus (V \otimes V \otimes V)/(V \otimes R + R \otimes V) \bigoplus ...$$

for the vector space $V = A_1$ and subspace $R = ker(p|_{A_1 \otimes A_1}) \subset V \otimes V$. The pair (V, R) is called the quadratic data of A and it is enough to uniquely determine the quadratic algebra A up to isomorphism. We call any basis $\{r_1, ..., r_n\}$ for R a set of relations.

Let $T(V)$ be the free associative algebra over some vector space V. We can construct what is called the *tensor coalgebra*, denoted $T^C(V)$. As a vector space it is the same as $T(V)$, but instead of the usual product μ we equip it with a coproduct $\Delta : T^C(V) \to T^C(V) \otimes T^C(V)$ which maps the element

$$v_1 v_2 ... v_n \in T^C(V)$$

to

$$\sum_{i=0}^{n} v_1 ... v_i \otimes v_{i+1} ... v_n \in T^C(V) \otimes T^C(V)$$

and extends linearly. Note that in the sum above the first term is $1 \otimes v_1 ... v_n$ and the last is $v_1 ... v_n \otimes 1$.

Proposition 3.2. *The tensor coalgebra $T^C(V)$ is a coassociative counital graded coalgebra with counit ϵ which acts as the identity on $\mathbb{K} \subset T^C(V)$ and is zero on everything else.*

Proof. First we check that the coproduct Δ is coassociative.

$$(1\otimes\Delta)\circ\Delta(a_1a_2...a_n) = (1\otimes\Delta)\big(\sum_{i=0}^{n} a_1...a_i\otimes a_{i+1}...a_n\big) =$$

$$\sum_{i=0}^{n} a_1...a_i\otimes\Delta(a_{i+1}...a_n) =$$

$$\sum_{i=0}^{n} a_1...a_i\otimes\big(1\otimes a_{i+1}...a_n + \sum_{j=i+1}^{n} a_{i+1}...a_j\otimes a_{j+1}...a_n\big) =$$

$$\sum_{i=0}^{n} a_1...a_i\otimes 1\otimes a_{i+1}...a_n + \sum_{0\le i<j\le n} a_1...a_i\otimes a_{i+1}...a_j\otimes a_{j+1}...a_n. \quad (1)$$

$$(\Delta\otimes 1)\circ\Delta(a_1a_2...a_n) = (\Delta\otimes 1)\big(\sum_{i=0}^{n} a_1...a_i\otimes a_{i+1}...a_n\big) =$$

$$\sum_{i=0}^{n}\Delta(a_1...a_i)\otimes a_{i+1}...a_n = \sum_{i=0}^{n}\big((\sum_{j=0}^{i-1} a_1...a_j\otimes a_{j+1}...a_i) + a_1...a_i\otimes 1\big)\otimes a_{i+1}...a_n\big) =$$

$$\sum_{i=0}^{n} a_1...a_i\otimes 1\otimes a_{i+1}...a_n + \sum_{1\le j<i\le n} a_1...a_j\otimes a_{j+1}...a_i\otimes a_{i+1}...a_n. \quad (2)$$

By comparing (1) and (2) we see that Δ is indeed coassociative. Next, we check that ϵ is a counit for $T^C(V)$.

$$1\otimes\epsilon\circ\Delta(a_1...a_n) = 1\otimes\epsilon\big(\sum_{i=0}^{n} a_1...a_i\otimes a_{i+1}...a_n\big) = a_1...a_n\otimes 1$$

$$\epsilon\otimes 1\circ\Delta(a_1...a_n) = \epsilon\otimes 1\big(\sum_{i=0}^{n} a_1...a_i\otimes a_{i+1}...a_n\big) = 1\otimes a_1...a_n$$

This shows that the following diagram commutes

$$\begin{tikzcd} & T^C(V) \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\ T^C(V)\otimes\mathbb{K} & T^C(V)\otimes T^C(V) \arrow[l, "1\otimes\epsilon"] \arrow[r, "\epsilon\otimes 1"'] & \mathbb{K}\otimes T^C(V) \end{tikzcd}$$

and then ϵ is a counit. The grading of $T^C(V)$ is the same (as vector spaces) as the grading for $T(V)$ and from how we defined the coproduct Δ it is clear that this grading is compatible with the coproduct.

□

We have seen that a quadratic data uniquely defines a quotient algebra of the tensor algebra. In a similar fashion we will soon be able to construct a subcoalgebra of the tensor coalgebra, starting with some quadratic data.

Definition 3.3. *Let C be a coalgebra with comultiplication Δ and let $Q \subset C$ be a subspace of the underlying vector space structure of C. We call Q a subcoalgebra if Q is a coalgebra with the coproduct Δ inherited from C. That is, if*

$$\Delta(Q) \subset Q \otimes Q$$

Proposition 3.4. *Let A be a quadratic algebra with quadratic data (V, R). The subspace*

$$\mathbb{K} \bigoplus V \bigoplus R \bigoplus R \otimes V \cap V \otimes R \bigoplus R \otimes V \otimes V \cap V \otimes R \otimes V \cap V \otimes V \otimes R \bigoplus ... \subset T^C(V)$$

is a subcoalgebra of $T^C(V)$ which we call the Koszul dual coalgebra of A and denote by $A^{¡}$.

Proof. Because Δ is linear it is enough to see what it does to elements that consist of only one term. So let $v_1...v_n$ be such an element of $A^{¡}$. We apply the coproduct Δ to it

$$\Delta(v_1...v_n) = \sum_{i=0}^{n} v_1...v_i \otimes v_{i+1}...v_n.$$

We want to show that the sum above is in $A^{¡} \otimes A^{¡}$. But we know that $1 \in \mathbb{K} \subset A^{¡}$, $v_1, v_n \in V \subset A^{¡}$ and from the way $A^{¡}$ was constructed, it follows that for all $1 \leq j \leq n-1$ we have $v_j v_{j+1} \in R$. Then it follows that any term of the sum is in a subspace of $A^{¡} \otimes A^{¡}$ of one of the following forms

$$\mathbb{K} \otimes \Big(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)$$

$$V \otimes \Big(\bigcap_{i+2+j=n-1} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)$$

$$\Big(\bigcap_{i+2+j=k} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big) \otimes \Big(\bigcap_{i+2+j=n-k} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)$$

$$\Big(\bigcap_{i+2+j=n-1} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big) \otimes V$$

$$\Big(\bigcap_{i+2+j=n} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big) \otimes \mathbb{K}$$

□

Note that as a vector space we have $A^{¡} = \bigoplus_{i \geq 0} (T^C(V)_i \cap A^{¡})$. This makes $A^{¡}$ into a graded coalgebra. The fact that Δ is compatible with this grading is clear from how the coproduct was defined.

3.2 Convolution product

Let C be a coassociative coalgebra and A an associative algebra with coproduct Δ and product μ respectively. If f and g are linear maps from C to A (considered as vector spaces) we define the convolution product of f and g as follows

$$f \star g := \mu \circ (f \otimes g) \circ \Delta$$

which again is a linear map from C to A.
Something else we can do starting with a linear map $f : C \to A$ is to extend it to a linear map from $C \otimes A$ to itself, we call this extended map d_f and define it as

$$d_f := (1 \otimes \mu) \circ (1 \otimes f \otimes 1) \circ (\Delta \otimes 1)$$

Proposition 3.5. *For f and g in $Hom(C, A)$ where C is a coassociative coalgebra and A is an associative algebra, we have the following relation*

$$d_f \circ d_g = d_{f \star g}$$

Proof.

$$\begin{aligned} d_f \circ d_g = (1 \otimes \mu) \circ (1 \otimes f \otimes 1) \circ (\Delta \otimes 1) \circ (1 \otimes \mu) \circ (1 \otimes g \otimes 1) \circ (\Delta \otimes 1) = \\ (1 \otimes \mu) \circ (1 \otimes f \otimes 1) \circ (\Delta \otimes \mu) \circ (1 \otimes g \otimes 1) \circ (\Delta \otimes 1) = \\ (1 \otimes \mu) \circ (1 \otimes f \otimes 1) \circ (\Delta \otimes \mu \circ (g \otimes 1)) \circ (\Delta \otimes 1) = \\ (1 \otimes \mu) \circ ((1 \otimes f) \circ \Delta \otimes \mu \circ (g \otimes 1)) \circ (\Delta \otimes 1) = \\ (1 \otimes \mu) \circ (1 \otimes 1 \otimes \mu) \circ (1 \otimes f \otimes g \otimes 1) \circ (\Delta \otimes 1 \otimes 1) \circ (\Delta \otimes 1) = \\ (1 \otimes \mu) \circ (1 \otimes \mu \otimes 1) \circ (1 \otimes f \otimes g \otimes 1) \circ (1 \otimes \Delta \otimes 1) \circ (\Delta \otimes 1) = \end{aligned} \tag{3}$$

$$(1 \otimes \mu) \circ (1 \otimes (\mu \circ (f \otimes g) \circ \Delta) \otimes 1) \circ (\Delta \otimes 1) = d_{f \star g}.$$

We used the associativity of μ and the coassociativity of Δ to get (3). □

As a consequence $f \star f = 0$ implies $d_f \circ d_f = 0$ which we will use in the next section when defining the Koszul complex associated to a quadratic algebra A.

3.3 Definition of Koszul algebras

We have seen that for any quadratic algebra A with quadratic data (V, R) we can construct the Koszul dual coalgebra $A^{¡}$. Now we define a linear map $\alpha : A^{¡} \to A$ as the following composition of maps

$$A^{¡} \xrightarrow{p} (A^{¡})_1 \xrightarrow{i} A \ .$$

Since $(A^{¡})_1 = A_1 = V$ the definition makes sense. As we saw in the previous section this induces a linear map $d_\alpha : A^{¡} \otimes A \to A^{¡} \otimes A$. We will soon see that this map makes the following sequence into a chain complex

$$... \xrightarrow{d_\alpha} (A^{¡})_{n-i} \otimes A_i \xrightarrow{d_\alpha} (A^{¡})_{n-i-1} \otimes A_{i+1} \xrightarrow{d_\alpha} ... \ . \tag{4}$$

The fact that $d_\alpha((A^{¡})_{n-i} \otimes A_i) \subset (A^{¡})_{n-i-1} \otimes A_{i+1}$ can be realised by direct checking. To see that $d_\alpha^2 = 0$ it is, according to the result from the previous section, enough to show that $\alpha \star \alpha = 0$:

$$\alpha \star \alpha(r_1 r_2 ... r_k) = \mu \circ \alpha \otimes \alpha \circ \Delta(r_1 r_2 ... r_k) =$$
$$= \mu \circ \alpha \otimes \alpha(\sum_{i=0}^{k} r_1 ... r_i \otimes r_{i+1} ... r_k) = \begin{cases} 0, & k \neq 2 \\ \mu(r_1 \otimes r_2), & k = 2 \end{cases} .$$

But $\mu(r_1 \otimes r_2) = 0$ because $r_1 r_2 \in R$.
So (4) is really a chain complex and we are ready to give a definition of Koszul algebras.

Definition 3.6. *We say that the quadratic algebra A is Koszul if the chain complex* (4) *is an exact sequence for all positive integers n.*

We have noticed that we get one chain complex for every positive integer n. We can align these complexes on top of each other and take the direct sum over the columns to get a big chain complex:

$$\begin{array}{lcccccccccccc}
 & ... & & ... & & ... & & ... & & ... & & ... \\
 & & & \bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \\
n=3 & ... & \longrightarrow & A^{¡}_3 \otimes \mathbb{K} & \longrightarrow & A^{¡}_2 \otimes A_1 & \longrightarrow & A^{¡}_1 \otimes A_2 & \longrightarrow & \mathbb{K} \otimes A_3 & \longrightarrow & 0 \\
 & & & \bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \\
n=2 & ... & \longrightarrow & 0 & \longrightarrow & A^{¡}_2 \otimes \mathbb{K} & \longrightarrow & A^{¡}_1 \otimes A_1 & \longrightarrow & \mathbb{K} \otimes A_2 & \longrightarrow & 0 \\
 & & & \bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & & \\
n=1 & ... & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A^{¡}_1 \otimes \mathbb{K} & \longrightarrow & \mathbb{K} \otimes A_1 & \longrightarrow & 0
\end{array} .$$

We call the big chain complex described above the Koszul complex associated to the quadratic algebra A and we denote it $(A^{¡} \otimes A, d_\alpha)$. We call the n'th row of the Koszul complex the weight n component of the Koszul complex. Often, one adds a bottom row

$$n = 0 \quad ... \longrightarrow 0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathbb{K} \xrightarrow{1} \mathbb{K}$$

to the Koszul complex because then, whenever A is Koszul this becomes a free resolution of $\mathbb{K}$ over A. We will however not go further in to this subject in this thesis.

4 Koszul duality and Hilbert series

In this section we define the Koszul dual algebra of a quadratic algebra A and show how to construct it from the quadratic data for A. We also define the Hilbert series of a graded algebra and show how the Hilbert series of a Koszul algebra and its Koszul dual algebra are related to each other.

4.1 Koszul dual algebra

We begin by defining the Koszul dual algebra of a quadratic algebra by its relation to the Koszul dual coalgebra. After that we will show that there is another, in some ways nicer way, to interpret the Koszul dual algebra.

Definition 4.1. *The Koszul dual algebra of a quadratic algebra A is denoted $A^!$ and defined by $A^! = \bigoplus_{i\geq 0}(A_i^{¡})^*$.*

The multiplication of $A^!$ is very natural. For $f \in (A^{¡}_n)^*$ and $g \in (A^{¡}_m)^*$ we get $fg \in (A^{¡}_{n+m})^*$ by setting

$$fg(v_1...v_{n+m}) := f(v_1...v_n)g(v_{n+1}...v_{n+m}).$$

Before we move on to the main proposition of this section where we present the alternative way of interpreting $A^!$ we will define what is called the annihilator of a subspace and present some of its properties.

Definition 4.2. *Let V be a finite dimensional vector space and let N be a subspace of V. Then we define the annihilator of N as follows*

$$N^\circ := \{f \in V^* | f(N) = 0\}$$

Proposition 4.3. *If V is a finite dimensional vector space and M and N are subspaces of V then the annihilator satisfy the following properties.*
i) $(M + N)^\circ = M^\circ \cap N^\circ$
ii) $(M \cap N)^\circ = M^\circ + N^\circ$
iii) $dim(M^\circ) = dim(V) - dim(M)$

For the proof of these properties we refer to [1, p 102].

We know that for finite dimensional V there is a natural isomorphism

$$V^* \otimes V^* \cong (V \otimes V)^*$$

under which $f \otimes g \in V^* \otimes V^*$ is identified with the functional mapping $u \otimes v \in V \otimes V$ to $f(u)g(v)$[1, p 374]. With this isomorphism in mind we define, given a subspace $R \subset V \otimes V$, the subspace of orthogonal relations $R^\perp$ as follows

$$R^\perp := \{f \in V^* \otimes V^* | f(R) = 0\}$$

In other words, $R^\perp$ is the image of R° under the isomorphism $(V \otimes V)^* \cong V^* \otimes V^*$.

Proposition 4.4. *The Koszul dual algebra is a quadratic algebra with quadratic data* $(V^*, R^\perp)$ *where* $R^\perp$ *is the subspace of orthogonal relations.*

Proof. To see this, we look at $(A^{\text{i}}_k)^*$. It is the space of functionals on A^{i}_k. However, $(A^{\text{i}})_k \subset V^{\otimes k}$ so let us for a moment consider all the functionals $f : V^{\otimes k} \to \mathbb{K}$. By restricting these to A^{i}_k we get a surjective map

$$\phi_k : (V^{\otimes k})^* \to (A^{\text{i}}_k)^*.$$

The kernel of this map is $\ker(\phi_k) = \{f \in (V^{\otimes k})^* | f(A^{\text{i}}_k) = 0\} = (A^{\text{i}}_k)^\circ$. But since

$$A^{\text{i}}_k = \bigcap_{i+2+j=k} V^{\otimes i} \otimes R \otimes V^{\otimes j}$$

is an intersection of subspaces of $V^{\otimes k}$, repeated use of Proposition 4.3. (i) gives us

$$ker(\phi_k) = \Big(\bigcap_{i+2+j=k} V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)^\circ = \sum_{i+2+j=k} \Big(V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)^\circ.$$

Now I claim that

$$(V^{\otimes i} \otimes R \otimes V^{\otimes j})^\circ$$

is precisely the image of

$$(V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j}$$

under the isomorphism $\eta : (V^*)^{\otimes k} \cong (V^{\otimes k})^*$. To see that

$$\eta \Big((V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j} \Big) \subset (V^{\otimes i} \otimes R \otimes V^{\otimes j})^\circ$$

let

$$f_1 ... f_i h_1 h_2 g_1 ... g_j \in (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j}$$

and

$$v_1 ... v_i r_1 r_2 w_1 ... w_j \in V^{\otimes i} \otimes R \otimes V^{\otimes j}.$$

Then we see that

$$\begin{aligned} \eta(f_1 ... f_i h_1 h_2 g_1 ... g_j)(v_1 ... v_i r_1 r_2 w_1 ... w_j) = \\ f_1(v_1) ... f_i(v_i) h_1(r_1) h_2(r_2) g_1(w_1) ... g_j(w_j) = \\ f_1(v_1) ... f_i(v_i) \cdot 0 \cdot g_1(w_1) ... g_j(w_j) = 0 \end{aligned}$$

which is enough to show the inclusion. To show that the inclusion is in fact an equality, we will show that the subspaces in question have the same dimension. That is

$$dim \Big(\eta \Big((V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j} \Big) \Big) = dim \Big((V^{\otimes i} \otimes R \otimes V^{\otimes j} \Big)^\circ \Big).$$

Put $d = dim(V)$ and $r = dim(R)$. Since V is finite dimensional we then have $dim(V^*) = d$ and from how we defined $R^\perp$ it is clear that we have $dim(R^\perp) = dim(R^\circ)$ and Proposition 4.3 (iii) then tells us that

$$dim(R^\perp) = dim(V \otimes V) - dim(R) = d^2 - r.$$

Now we use the fact that isomorphisms preserve dimensions, to calculate

$$\begin{aligned} dim\Big(\eta\Big((V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j}\Big)\Big) &= \\ = dim\Big((V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j}\Big) &= \\ d^{i+j}(d^2 - r). \end{aligned}$$

Now we use Proposition 4.3 (iii) to calculate

$$\begin{aligned} dim\Big((V^{\otimes i} \otimes R \otimes V^{\otimes j})^\circ\Big) &= \\ = dim(V^{\otimes i+2+j}) - dim(V^{\otimes i} \otimes R \otimes V^{\otimes j}) &= \\ = d^{i+2+j} - d^{i+j}r = d^{i+j}(d^2 - r). \end{aligned}$$

We have shown that the surjective linear map

$$\phi_k \circ \eta : (V^*)^{\otimes k} \to (A^!_k)^*$$

has kernel

$$\sum_{i+2+j=k} (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j}$$

which means that $A^!$ has the vector space structure corresponding to the quadratic data $(V^*, R^\perp)$.
Let us see that the multiplications are the same. Let

$$[f_1...f_n] \in (V^*)^{\otimes n} / \Big(\sum_{i+2+j=n} (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j} \Big)$$

and

$$[g_1...g_m] \in (V^*)^{\otimes m} / \Big(\sum_{i+2+j=m} (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j} \Big).$$

Then the following is the usual multiplication in a quadratic algebra

$$[f_1...f_n]\cdot[g_1...g_m] = [f_1...f_n g_1...g_m] \in (V^*)^{\otimes n+m} / \Big(\sum_{i+2+j=n+m} (V^*)^{\otimes i} \otimes R^\perp \otimes (V^*)^{\otimes j} \Big).$$

Now let $f = \phi_n \circ \eta(f_1...f_n)$ and $g = \phi_m \circ (g_1...g_m)$ and $v_1...v_n v_{n+1}...v_{n+m} \in A^!_{n+m}$. We calculate

$$(f \cdot g)(v_1...v_n v_{n+1}...v_{n+m}) = f(v_1...v_n) g(v_{n+1}...v_{n+m}) =$$

$$f_1(v_1)...f_n(v_n)g_1(v_{n+1})...g_m(v_{n+m}) = \phi_{n+m}\circ\eta\Big([f_1...f_ng_1...g_m]\Big)(v_1...v_nv_{n+1}...v_{n+m}).$$

This shows that $A^!$ has the algebra structure corresponding to the quadratic data $(V^*, R^\perp)$.

$\square$

So if A is a quadratic algebra, then so is $A^!$. Then we want to know of course what is $(A^!)^!$. This we answer in the following proposition.

Proposition 4.5. *Let A be a quadratic algebra. Then $(A^!)^!$ is isomorphic to A.*

Proof. If (V, R) is the quadratic data corresponding to A, then by Proposition 4.4., $(A^!)^!$ has quadratic data $(V^{**}, (R^\perp)^\perp)$. Let us denote by I and J the ideals generated by R and $(R^\perp)^\perp$ in $T(V)$ and $T(V^{**})$ respectively. Recall that for a finite dimensional vector space V there is an isomorphism $V \cong V^{**}$ under which an element $v \in V$ is mapped to $\hat{v} \in V^{**}$ defined by $\hat{v}(f) = f(v)$ for any $f \in V^*$. This gives an algebra isomorphism $\eta : T(V^{**}) \cong T(V)$ which maps an element $\hat{v}_1...\hat{v}_n \in (V^{**})^{\otimes n}$ to $v_1...v_n$ and extends by linearity. Now let us consider the composition of canonical maps

$$T(V^{**}) \xrightarrow{\eta} T(V) \xrightarrow{q} T(V)/I$$

where q is the quotient map. If we call this composition ϕ then the lemma will follow if we can show that $ker(\phi) = (J)$ and because R generates I and $(R^\perp)^\perp$ generates J it is enough to show that $\eta((R^\perp)^\perp) = R$.

($\supset$) Let $\sum_i v_iw_i$ be any element in R. We need to show that $\sum_i \hat{v}_i\hat{w}_i \in (R^\perp)^\perp$. So let $f = \sum_j f_jg_j \in R^\perp$, then

$$(\sum_i \hat{v}_i\hat{w}_i)(f) = (\sum_i \hat{v}_i\hat{w}_i)(\sum_j f_jg_j) =$$
$$\sum_{i,j} f_j(v_i)g_j(w_i) = f(\sum_i v_iw_i) = 0$$

which means that $\sum_i \hat{v}_i\hat{w}_i \in (R^\perp)^\perp$.

($\subset$) Now let $\sum_i \hat{r}_i\hat{q}_i$ be any element in $(R^\perp)^\perp$. Then for any $f = \sum_j f_jg_j \in R^\perp$ we have that

$$f(\eta(\sum_i \hat{r}_i\hat{q}_i)) = f(\sum_i r_iq_i) =$$
$$\sum_{i,j} f_j(r_i)g_j(q_i) = (\sum_i \hat{r}_i\hat{q}_i)(f) = 0.$$

So $f(\eta(\sum_i \hat{r}_i\hat{q}_i)) = 0$ for all $f \in R^\perp$. To see that this implies that $\eta(\sum_i \hat{r}_i\hat{q}_i) \in R$ let $\mathscr{R}$ be a basis for R and extend this to a basis $\mathscr{B}$ for $V \otimes V$. Furthermore let $\bar{\mathscr{B}}$ be the basis for $V^* \otimes V^*$ corresponding to the dual basis $\mathscr{B}^*$ of $(V \otimes V)^*$. Now if $\sum_i r_iq_i \notin R$ then it can be written as a unique linear combination of the elements of $\mathscr{B}$ where not all terms are in $\mathscr{R}$. But then, from how the dual bases are defined, it is clear that there is at least one element in $\bar{\mathscr{B}}$ which is zero on R, hence it is in $R^{\perp}$, but it is not zero on $\sum_i r_i q_i$ which is a contradiction. So $\eta((R^{\perp})^{\perp}) = R$ and then A and $(A^!)^!$ are isomorphic.

□

This is nice because then Koszul duality is really a *duality* in the sense that taking the dual of the dual is the same as doing nothing. Soon we will use this to prove something even stronger, namely that a quadratic algebra A is Koszul if and only if its Koszul dual algebra is Koszul. For this however we will need the following two lemmas.

Lemma 4.6. *Consider the following chain complex*

$$... \longrightarrow V_{i-1} \xrightarrow{d_{i-1}} V_i \xrightarrow{d_i} V_{i+1} \longrightarrow ... \ .$$

If it is exact at V_i then the following sequence is exact at V_i^*

$$... \longleftarrow V_{i-1}^* \xleftarrow[d_{i-1}^*]{} V_i^* \xleftarrow[d_i^*]{} V_{i+1}^* \longleftarrow ... \ .$$

Proof. We have to show that $ker(d_{i-1}^*) = im(d_i^*)$.
($\supset$) We pick a $f \in im(d_i^*)$. Then $f = d_i^*(h)$ for some $h \in V_{i+1}^*$. Then for any $x \in V_{i-1}$ we have

$$d_{i-1}^*(f)(x) = d_{i-1}^*(d_i^*(h))(x) = d_i^*(h)(d_{i-1}(x)) = h(d_i(d_{i-1}(x))) = h(0) = 0$$

which means $f \in ker(d_{i-1}^*)$.
($\subset$) For the other inclusion we pick $g \in ker(d_{i-1}^*)$ and we want to show that $g = q \circ d_i$ for some $q \in V_{i+1}^*$. To do this, let $\mathscr{U} = \{u_1, ..., u_t\}$ be a basis for $im(d_i) \subset V_{i+1}$ and extend this to a basis $\mathscr{W} = \{u_1, ..., u_t, w_1, ..., w_r\}$ for V_{i+1}. I claim that we can define a linear map $q : V_{i+1} \to \mathbb{K}$ on the basis elements as follows, $q(u_j) = g(z)$ for $u_j \in \mathscr{U}$ where z is any element in $d_i^{-1}(u_j)$ and $q(w_s) = 0$ for $w \in \mathscr{W} \setminus \mathscr{U}$. First we check that q is well defined. If $d_i(z_1) = d_i(z_2) = u_j$ then $z_1 - z_2 \in ker(d_i)$. Then, since $ker(d_i) = im(d_{i-1})$ there is an element $x \in V_{i-1}$ such that $z_1 - z_2 = d_{i-1}(x)$. But then because $g \in ker(d_{i-1}^*)$ we get

$$g(z_1 - z_2) = g(d_{i-1}(x)) = d_{i-1}^*(g)(x) = 0.$$

This means that if $d_i(z_1) = d_i(z_2)$ then $g(z_1) = g(z_2)$ which means that q is well defined. That q is linear follows from the linearity of d_i and g. From how we defined q it is clear that $g = q \circ d_i$ which means that $g \in im(d_i^*)$ and then the lemma follows.

□

Lemma 4.7. *For finite dimensional vector spaces and linear maps*

$$N \otimes L \xrightarrow{h \otimes l} S \otimes T$$

we have $(h \otimes l)^* = h^* \otimes l^*$ if we make the usual identification between $N^* \otimes L^*$ and $(N \otimes L)^*$ and between $S^* \otimes T^*$ and $(S \otimes T)^*$.

Proof. It is enough to show that the following diagram commutes

$$\begin{tikzcd} N^* \otimes L^* \arrow[d, "\eta_{N,L}"] & S^* \otimes T^* \arrow[l, "h^*\otimes l^*"'] \arrow[d, "\eta_{S,T}"] \\ (N \otimes L)^* & (S \otimes T)^* \arrow[l, "(h\otimes l)^*"'] \end{tikzcd} .$$

So let $n \otimes i$ be a term in an element of $N \otimes L$ and let $\phi \otimes \psi \in S^* \otimes T^*$. Then we check

$$(\eta_{N,L} \circ (h^* \otimes l^*(\phi \otimes \psi)))(n \otimes i) =$$
$$(\eta_{N,L}((\phi \circ h) \otimes (\psi \circ l)))(n \otimes i) = \phi(h(n))\psi(l(i))$$

and

$$((h \otimes l)^* \circ \eta_{S,T}(\phi \otimes \psi))(n \otimes i) = (\eta(\phi \otimes \psi))(h(n) \otimes l(i)) =$$
$$\phi(h(n))\psi(l(i))$$

and we see that the diagram does indeed commute.

□

Theorem 4.8. *A quadratic algebra A is Koszul if and only if its Koszul dual algebra $A^!$ is Koszul.*

Proof. Let us denote the multiplication of A by μ and the comultiplication of $A^{¡}$ by Δ. Then, if we use the isomorphism discussed earlier in this section to identify $(V^{\otimes n})^*$ with $(V^*)^{\otimes n}$, we see that the multiplication of $A^! = (A^{¡})^*$ is Δ^*. Then, using Lemma 1, we see that $(A^!)^{¡} = ((A^!)^!)^* = A^*$ so $(A^!)^{¡}$ has comultiplication μ^*. Now if A is Koszul, then the following sequence is exact

$$\dots \xrightarrow{d_\alpha} (A^{¡})_{n-i} \otimes A_i \xrightarrow{d_\alpha} (A^{¡})_{n-i-1} \otimes A_{i+1} \xrightarrow{d_\alpha} \dots .$$

But then it follows from Lemma 4.6. that the following sequence is exact as well

$$\dots \xleftarrow[d_\alpha^*]{} ((A^{¡})_{n-i} \otimes A_i)^* \xleftarrow[d_\alpha^*]{} ((A^{¡})_{n-i-1} \otimes A_{i+1})^* \xleftarrow[d_\alpha^*]{} \dots .$$

Now we recall from section 3.2 that $d_\alpha = (1 \otimes \mu) \circ (1 \otimes \alpha \otimes 1) \circ (\Delta \otimes 1)$. Then, from how the linear dual works with composition of linear maps, and by Lemma 4.7 we get

$$d_\alpha^* = ((1 \otimes \mu) \circ (1 \otimes \alpha \otimes 1) \circ (\Delta \otimes 1))^* = (\Delta \otimes 1)^* \circ (1 \otimes \alpha \otimes 1)^* \circ (1 \otimes \mu)^* =$$
$$(\Delta^* \otimes 1) \circ (1 \otimes \alpha^* \otimes 1) \circ (1 \otimes \mu^*).$$

Now let us make explicit what α^* is. Recall that we defined α as the composition of a projection and an inclusion

$$A \xrightarrow{p} A_1 \xrightarrow{i} A^{¡} .$$

Then α^* is the composition

$$(A^{¡})^* \xrightarrow{i^*} A_1^* \xrightarrow{p^*} A^* \ .$$

Since $i^*(f)(A_1) = f(i(A_1)) = f(A_1^{¡})$ for any $f \in (A^{¡})^*$ we see that i^* is in fact the projection of $(A^{¡})^*$ onto $(A_1^{¡})^*$ or $A^!$ onto $A_1^!$. Similarly since $p^*(g)(A) = g(p(A)) = g(A_1)$ for any $g \in A_1^*$ we see that p^* is actually just the inclusion of A_1^* into A^* which we know to be $(A^!)^{¡}$. In conclusion α^* is the composition of a projection and an inclusion

$$(A^!) \xrightarrow{\bar{p}} A_1^! \xrightarrow{\bar{i}} (A^!)^{¡} \ .$$

But now we recognise d_α^* to be the differential of the Koszul complex associated to the quadratic algebra $A^!$ and then this is the weight n component of the Koszul complex

$$\dots \xleftarrow[d_\alpha^*]{} ((A^{¡})_{n-i} \otimes A_i)^* \xleftarrow[d_\alpha^*]{} ((A^{¡})_{n-i-1} \otimes A_{i+1})^* \xleftarrow[d_\alpha^*]{} \dots \ .$$

Because we already said it was exact and because n was an arbitrary positive number we draw the conclusion that $A^!$ must be Koszul.
For the converse, suppose $(A^!)$ is Koszul, then the same argument tells us that $(A^!)^!$ is Koszul and then A is Koszul by Proposition 4.5. $\square$

4.2 Hilbert series

A graded algebra A can be infinite dimensional as a vector space but locally finite dimensional in the sense that each graded part of A has finite dimension. We are interested in the dimensions of the graded parts of such algebras.

Definition 4.9. *The Hilbert series of a locally finite dimensional quadratic algebra A is the formal power series*

$$h^A(x) = \sum_{i=0}^{\infty} dim(A_i)x^i.$$

Lemma 4.10. *Given an exact sequence of the following form*

$$0 \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \xrightarrow{d_{n-1}} \dots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \xrightarrow{d_0} 0 \ .$$

the alternating sum of the dimensions of these vector spaces is zero. That is

$$\sum_{i=0}^{n} (-1)^i dim(V_i) = 0.$$

Proof. We use the rank plus nullity Theorem from linear algebra and write $dim(V_i) = dim(ker(d_i)) + dim(im(d_i))$ which gives

$$\sum_{i=0}^{n}(-1)^i dim(V_i) = \sum_{i=0}^{n}(-1)^i \Big(dim\big(ker(d_i)\big) + dim\big(im(d_i)\big)\Big)$$

and by exactness we get

$$\sum_{i=0}^{n}(-1)^i \Big(dim\big(ker(d_i)\big)+dim\big(im(d_i)\big)\Big) = \sum_{i=0}^{n}(-1)^i \Big(dim\big(im(d_{i+1})\big)+dim\big(im(d_i)\big)\Big).$$

This last sum is a telescoping sum and it reduces to

$$dim\big(im(d_0)\big) + (-1)^n dim\big(im(d_{n+1})\big)$$

but both of these terms are zero and then the proof is complete. □

Note that having zeros in the beginning and end of the chain complex in the previous lemma is the same as saying that d_n is injective and d_1 is surjective.

Theorem 4.11. *If the quadratic algebra A is Koszul, then the Hilbert series of A and its Koszul dual algebra $A^!$ satisfy the following relation*

$$h^A(x)h^{A^!}(-x) = 1.$$

Proof. First note that since $(A^!)_i = (A^{¡})_i^*$ and because the dimension does not change when we take the linear dual of something finite dimensional it follows that $h^{A^!}(x) = h^{A^{¡}}(x)$. Now we want to know what the coefficient a_n of x^n in $h^A(x)h^{A^{¡}}(-x)$ is. We get

$$a_n = \sum_{i=0}^{n}(-1)^{n-i} dim(A^{¡}_{n-i})dim(A_i).$$

But since the dimension of the tensor product of finite dimensional vector spaces is the product of the dimensions we get that the last expression is equal to

$$\sum_{i=0}^{n}(-1)^i dim(A^{¡}_{n-i} \otimes A_i).$$

When n is greater than zero this is however the alternating sum of the dimensions of the vector spaces in the part of the generalised Koszul complex associated to A and the positive integer n:

$$0 \xrightarrow{d_{n+1}} (A^{¡})_n \xrightarrow{d_n} (A^{¡})_{n-1} \otimes A_1 \xrightarrow{d_{n-1}} ... \xrightarrow{d_2} (A^{¡})_1 \otimes A_{n-1} \xrightarrow{d_1} A_n \xrightarrow{d_0} 0 \ .$$

But since A is Koszul this complex is an exact sequence and it follows from the previous lemma that $a_n = 0$ for $n > 0$. For a_0 we get

$$a_0 = dim(A_0)dim(A_0^{¡}) = dim(\mathbb{K})^2 = 1$$

and this finishes the proof. □

We should mention that the other implication is not true. That is, there are quadratic algebras whose Hilbert series satisfy the relation in the previous theorem but which are not Koszul. An example of such an algebra is constructed in [4].

5 Examples

We give examples of some quadratic algebras that are Koszul and some that are not. We want to illustrate the result from the previous section so we are particularly interested in the degrees of the graded parts of the algebras and their Koszul dual algebras.

Example 1: The tensor algebra

Our first example is the tensor algebra itself. Let V be a finite dimensional vector space with basis $\{v_1, ..., v_m\}$ and let $R = \langle 0 \rangle$. Since $0 \in V \otimes V$ we see that the tensor algebra $T(V)$ is indeed a quadratic algebra. Its Koszul dual coalgebra is

$$T(V)^{¡} = \mathbb{K} \oplus V.$$

The weight n component of the Koszul complex will look as follows

$$0 \longrightarrow V \otimes V^{\otimes n-1} \xrightarrow{d_\alpha} V^{\otimes n} \longrightarrow 0 \ . \tag{7}$$

Let us see where d_α takes an element $w \otimes v_1 \cdots v_{n-1} \in V \otimes V^{\otimes n-1}$

$$\begin{aligned} d_\alpha(w \otimes v_1 \cdots v_{n-1}) &= (1 \otimes \mu \circ 1 \otimes \alpha \otimes 1 \circ \Delta \otimes 1)(w \otimes v_1 \cdots v_{n-1}) = \\ &= (1 \otimes \mu \circ 1 \otimes \alpha \otimes 1)(1 \otimes w \otimes v_1 \cdots v_{n-1} + w \otimes 1 \otimes v_1 \cdots v_{n-1}) = \\ &= (1 \otimes \mu \otimes 1)(1 \otimes w \otimes v_1 \cdots v_{n-1}) = 1 \otimes wv_1 \cdots v_{n-1}. \end{aligned}$$

Because $\mathbb{K} \otimes V^{\otimes n}$ is canonically isomorphic to $V^{\otimes n}$ we can omit the 1 and we see that d_α is just the multiplication of $T(V)$. Now to see that (7) is exact it is enough to show that d_α is injective and surjective and since $dim(V \otimes V^{\otimes n-1}) = dim(V^{\otimes n})$ it suffices to show that d_α is surjective [1, p 63]. However, any element of $V^{\otimes n}$ is a sum of elements of the form $v_1 \cdots v_n$ which is the image of $v_1 \otimes v_2 \cdots v_n \in V \otimes V^{\otimes n-1}$ so d_α is surjective and injective which shows that (7) is exact.
Now we know that the tensor algebra $T(V)$ is Koszul, so Theorem 4.11. tells us that

$$h^{T(V)}(x) h^{T(V)^!}(-x) = 1$$

but let us still check explicitly that this relation holds. Recall that the dimension of the tensor product of finite dimensional vector spaces is the product of their dimensions. Then if we put $m = dim(V)$ we get

$$dim(T(V)_n) = dim(V^{\otimes n}) = m^n$$

and

$$h^{T(V)}(x) = 1 + mx + m^2x^2 + m^3x^3 + ... = \frac{1}{1 - mx}.$$

Since $T(V)^! = (T(V)^¡)^* = \mathbb{K} \oplus V$ we get

$$h^{T(V)^!}(x) = 1 + mx$$

and we see that

$$h^{T(V)}(x)h^{T(V)^!}(-x) = \frac{1-mx}{1-mx} = 1$$

as expected.

Example 2: The polynomial ring in two variables

Our second example of a Koszul algebra is one of the most natural quadratic algebras, the polynomial ring in two variables. First we show that it is a quadratic algebra.

Proposition 5.1. *Let $V = \langle x, y \rangle$ be the vector space spanned by the formal basis vectors x and y, then*

$$\mathbb{K}[X, Y] \cong T(V)/I$$

where I is the two sided ideal generated by $R = \langle xy - yx \rangle \subset V^{\otimes 2}$.

Proof. To see this, consider the algebra morphism

$$\phi : T(V) \to \mathbb{K}[X, Y]$$

which maps the generators x and y in $T(V)$ to X and Y respectively. We want to show that the kernel of this morphism is I. Because $\phi(xy - yx) = XY - YX = XY - XY = 0$ it follows that $I \subset ker(\phi)$. For the other inclusion suppose $\phi(f) = 0$ for some

$$f = f_0 + f_1 + f_2 + ...f_m \in T(V)$$

where $f_i \in V^{\otimes i}$ for $i = 0, 1, 2..., m$. Some $f_i's$ might be zero. However, because terms of different degree in $T(V)$ are mapped to different monomials it follows that $\phi(f_i) = 0$ for $i = 1, 2, ..., m$ so we can assume f to be homogenous of degree n. So f is a sum of terms looking like

$$a \underbrace{xxxyyxxyy...yyx}_{\text{n factors}}.$$

We can write

$$f = h_0 + h_1 + ... + h_n$$

where h_i consists of all the terms with precisely i x-factors. However, since terms with different numbers of x-factors are mapped to different monomials it follows that $\phi(h_i) = 0$ for $i = 1, 2, ..., n$ and then we can assume that f is homogenous and consists only of terms with the same number of x-factors. That is

$$f = a_1g_1 + ...a_kg_k$$

where the g_i's are monic of the same degree and have the same number of x-factors (and y-factors). But then the g_i's are all mapped to the same monic monomial in $\mathbb{K}[X,Y]$ and we see that

$$\begin{aligned}\phi(f) = \phi(a_1g_1 + ... + a_kg_k) = a_1\phi(g_1) + ... + a_k\phi(g_k) = \\ = (a_1 + ... + a_k)\phi(g_1)\end{aligned}$$

which is zero only if $a_1 + ... + a_k = 0$. So we can write $a_1 = -a_2 - ... - a_k$ and we get

$$\begin{aligned}f = (-a_2 - ... - a_k)g_1 + a_2g_2 + ... + a_kg_k = \\ = a_2(g_2 - g_1) + ... + a_k(g_k - g_1). \qquad (5)\end{aligned}$$

But g_i and g_1 have the same number of x- and y-factors ($i = 2, 3, 4, ..., k$) and we can, in a finite number of steps, get from g_i to g_1 by successively switching places between an x and a y that were next to each other. In other words, if we put $g_i^{(0)} = g_i$ and $g_i^{(r+1)} = g_1$, then there exists $g_i^{(1)}, ..., g_i^{(r)}$ such that $g_i^{(s)}$ and $g_i^{(s+1)}$ are the same except that some x- and y-factors that were next to each other have switched places. We can then write any term in the sum (5) as follows

$$a_i(g_i - g_1) = a_i((g_i - g_i^{(1)}) + (g_i^{(1)} - g_i^{(2)}) + ... + (g_i^{(r_i-1)} - g_i^{(r_i)}) + (g_i^{(r_i)} - g_1)) \in I.$$

□

Now we know that $\mathbb{K}[X,Y]$ is a quadratic algebra, so let us show that it is also Koszul. To do this we study the Koszul complex that we used to define Koszulness in the previous section.

Proposition 5.2. *Let $\mathbb{K}$ be a characteristic 0 field. The polynomial ring in two variables $\mathbb{K}[X,Y]$ is Koszul.*

Proof. First note that $A_i^{¡} = 0$ for $i \geq 3$. To see this, suppose we have

$$f \in A_3^{¡} = V \otimes R \cap R \otimes V.$$

Since $\{x, y\}$ is a basis for V and $\{xy - yx\}$ is a basis for R we can write f in the following way where a, b and c are scalars

$$f = v \otimes r = (ax + by) \otimes c(xy - yx) = ac \cdot xxy - ac \cdot xyx + bc \cdot yxy - bc \cdot yyx.$$

But f can also be expressed in the following way, this time α, β and ω scalars

$$f = r' \otimes v' = \alpha(xy - yx) \otimes (\beta x + \omega y) = \alpha\beta \cdot xyx + \alpha\omega \cdot xyy - \alpha\beta \cdot yxx - \alpha\omega \cdot yxy.$$

Then, keeping in mind that $A_3^{¡} \subset V^{\otimes 3}$ and that $\{xxy, yyx, xyy, yxx, xyx, yxy, xxx, yyy\}$ is a basis for $V^{\otimes 3}$, we deduce by comparing the two expressions for f that f must be zero. Now we consider the chain complex that we introduced in the previous chapter. For $n = 1$ we get

$$0 \longrightarrow V \xrightarrow{d_\alpha} V \longrightarrow 0\ .$$

However, $d_\alpha(v) = \mu \circ (1 \otimes \alpha) \circ \Delta(v) = \mu \circ (1 \otimes \alpha)(1 \otimes v + v \otimes 1) = \mu(1 \otimes v) = v$ is the identity on V and then the chain complex is exact.
Because $A^{¡}_i = 0$ for $i \geq 3$ we get, for n greater than 1, a chain complex of the form

$$0 \xrightarrow{d_\alpha} R \otimes (\mathbb{K}[X,Y])_{n-2} \xrightarrow{d_\alpha} V \otimes (\mathbb{K}[X,Y])_{n-1} \xrightarrow{d_\alpha} (\mathbb{K}[X,Y])_n \longrightarrow 0 \ . \tag{6}$$

To show that this is exact we will construct a homotopy between the zero map and the identity $h = \{h_i\}$

$$0 \xleftarrow{h_3} R \otimes (\mathbb{K}[X,Y])_{n-2} \xleftarrow{h_2} V \otimes (\mathbb{K}[X,Y])_{n-1} \xleftarrow{h_1} (\mathbb{K}[X,Y])_n \xleftarrow{h_0} 0 \ .$$

We see that h_3 and h_0 must be the zero maps. Furthermore, we define

$$h_1(X^iY^{n-i}) = n^{-1}i(x \otimes X^{i-1}Y^{n-i}) + (1 - n^{-1}i)(y \otimes X^iY^{n-i-1}).$$

Here n^{-1} makes sense because we are working over a characteristic zero field. Since $\{X^iY^{n-i}\}_{0 \leq i \leq n}$ is a basis for $(\mathbb{K}[X,Y])_n$ this is enough to define the linear map h_1. Next we define

$$h_2(x \otimes X^iY^{n-i-1}) = (1 - n^{-1}i - n^{-1})(xy - yx \otimes X^iY^{n-i-2})$$

and

$$h_2(y \otimes X^iY^{n-i-1}) = -in^{-1}(xy - yx \otimes X^{i-1}Y^{n-i-1}).$$

Since $\{x \otimes X^iY^{n-i-1}\}_{0 \leq i \leq n-1} \cup \{y \otimes X^iY^{n-i-1}\}_{0 \leq i \leq n-1}$ is a basis for $V \otimes (\mathbb{K}[X,Y])_{n-1}$ this is enough to define the linear map h_2. One can check that this will give $dh + hd = 1$ in the n'th complex and then the complexes (6) are exact.

□

Now Theorem 4.11. tells us that $h^A(x)hA^!(-x) = 1$ where $A = \mathbb{K}[X,Y]$ but again, let us check explicitly that this holds. The n'th graded part of $\mathbb{K}[X,Y]$ is the subspace of all homogenous polynomials of degree n. It has a basis $\{x^{n-i}y^i\}_{0 \leq i \leq n}$ so it has dimension $n+1$. This gives

$$h^A(x) = 1 + 2x + 3x^2 + 4x^3 + \dots \ .$$

Using formal power series we get

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots = h^A(x).$$

Now we want to find $h^{A^!}(x)$. Since the graded parts of $A^!$ and $A^{¡}$ have the same dimension we consider $h^{A^{¡}}(x)$.

$$dim(A^{¡}_0) = dim(\mathbb{K}) = 1$$

$$dim(A^{¡}_1) = dim(V) = 2$$

$$dim(A^{¡}_2) = dim(R) = 1$$

This gives

$$h^{A^!}(x) = h^{A^{¡}}(x) = 1 + 2x + x^2 = (1+x)^2$$

and then we see that

$$h^A(x)h^{A^!}(-x) = 1$$

as expected.

It turns out that the Koszul dual algebra of the polynomial ring is an other interesting graded algebra, namely the exterior algebra. We look for the subspace of orthogonal relations $R^\perp \subset V^* \otimes V^*$. Let $x^*, y^* \in V^*$ be the dual basis. We check that

$$(x^*)^2(xy - yx) = x^*(x)x^*(y) - x^*(y)x^*(x) = 1 \cdot 0 - 0 \cdot 1 = 0$$

$$(y^*)^2(xy - yx) = y^*(x)y^*(y) - y^*(y)y^*(x) = 0 \cdot 1 - 1 \cdot 0 = 0$$

$$(x^*y^* + y^*x^*)(xy - yx) =$$

$$x^*(x)y^*(y) + y^*(x)x^*(y) - x^*(y)y^*(x) - y^*(y)x^*(x) = 1 + 0 - 0 - 1 = 0$$

$$x^*y^*(xy - yx) = x^*(x)y^*(y) - x^*(y)y^*(x) = 1 - 0 = 1 - 0 = 1$$

and since $\{(x^*)^2, (y^*)^2, x^*y^* + y^*x^*, x^*y^*\}$ is a basis for $V^* \otimes V^*$ we conclude that $R^\perp = \langle (x^*)^2, (y^*)^2, x^*y^* + y^*x^* \rangle$. Then we know from Proposition 6 in the previous chapter that

$$A^! = T(V^*)/J$$

where J is the two sided ideal generated by $R^\perp$. Let us see that this is indeed the exterior algebra $\wedge V^*$, the proof of this is similar to the proof we gave showing that the polynomial ring is a quadratic algebra. We consider the algebra homomorphism ψ that maps the generators x^* and y^* of $T(V^*)$ to the generators X^* and Y^* of $\wedge V^*$. I claim that the kernel of this map is precisely J. It is clear from the definition of the exterior algebra [1, p 400] that $\psi(J) = 0$. By considering the Hilbert series for $A^{¡}$ and keeping in mind that $dim(A^{¡}_i) = dim(A^!_i)$ for all i we see that $T(V^*)_i \subset J$ for $i \geq 3$. Now suppose $p \in ker(\psi)$. Because terms of different degree are mapped to different degree elements in $\wedge V^*$ we can suppose that p is homogenous. Because any element of degree 3 or larger is in J let us assume $deg(p) \leq 2$. Also it is clear that any nonzero element of degree 0 or 1 can not be in the kernel so we assume $deg(p) = 2$. Then because $\{(x^*)^2, (y^*)^2, x^*y^* + y^*x^*, x^*y^*\}$ is a basis for $V^* \otimes V^*$ p can be written in a unique way as a linear combination

$$a(x^*)^2 + b(y^*)^2 + c(x^*y^* + y^*x^*) + dx^*y^*$$

and since $p \in ker(\psi)$ and $J \subset ker(\psi)$ we get

$$\psi(a(x^*)^2 + b(y^*)^2 + c(x^*y^* + y^*x^*) + dx^*y^*) = \psi(dx^*y^*) = 0$$

which implies that $d = 0$ and then $p \in J$.

Example 3: A quadratic algebra that is not Koszul

Our next example will be an algebra which is not Koszul. We will use that Theorem 4.11. is a necessary condition for Koszulness in order to show that the following algebra is not Koszul. Let

$$A = \mathbb{K}[x_1, x_2, x_3, x_4]/I$$

where

$$I = (x_1^2 - x_2^2 - x_1x_3, x_1x_2 + x_3x_4 + x_4^2, x_1x_4 - x_3^2, x_2x_3 + x_2x_4 + x_3^2, x_2x_3 - x_4^2).$$

In Example 2 we showed that the polynomial ring in 2 variables is quadratic and the same is actually true for any number of variables[2]. Hence, A is also a quadratic algebra.
Now I claim that $A_3 = 0$. To see this we look at the five relations that generate I. If we multiply these by x_1, x_2, x_3, x_4 respectively we get 20 elements in $(\mathbb{K}[x_1, x_2, x_3, x_4])_3$. By considering the following basis $\{x_ix_jx_k\}_{1\leq i\leq j\leq k\leq 4}$, for $(\mathbb{K}[x_1, x_2, x_3, x_4])_3$, and expressing the twenty elements with coordinates in this basis one can check that they are actually linearly independent. Then, since $(\mathbb{K}[x_1, x_2, x_3, x_4])_3$ has dimension $\binom{4}{3} = 20$, it follows that they span all of $(\mathbb{K}[x_1, x_2, x_3, x_4])_3$. That is $(\mathbb{K}[x_1, x_2, x_3, x_4])_3 \subset I$ and then it follows that $(\mathbb{K}[x_1, x_2, x_3, x_4])_n \subset I$ for $n \geq 3$ which means that $A_n = 0$ for $n \geq 3$. It is now easy to calculate the Hilbert series for A since it only has three terms. To get the dimension of A_2 we apply the rank plus nullity theorem to the natural projection

$$(\mathbb{K}[x_1, x_2, x_3, x_4])_2 \to (\mathbb{K}[x_1, x_2, x_3, x_4])_2/R$$

where R is the subspace spanned by the 5 generators of I. The dimensions of A_0 and A_1 is 1 and 4 which gives

$$h^A(x) = 1 + 4x + 5x^2.$$

Now if A is Koszul, then we could use Theorem 4.11. to recursively calculate the coefficients of $h^{A^!}(-x)$ which would give

$$h^{A^!}(-x) = 1 - 4x + 11x^2 - 24x^3 + 41x^4 - 44x^5 - 29x^6 + ... \qquad .$$

This would however mean that $Dim(A_6^!) = -29$ which is a contradiction. Hence A is not Koszul.

6 Summary and Discussion

We have discussed what Koszul duality means for quadratic algebras and we have seen that there are special quadratic algebras called Koszul algebras. If a finite dimensional quadratic algebra A has the Koszul property we showed that the Hilbert series of A and its dual algebra $A^!$ satisfy the following relation

$$h^A(x)h^{A^!}(-x) = 1.$$

This formula can be used to calculate the Hilbert series for a Koszul algebra given that you know the Hilbert series of its dual algebra. It can also be used to prove that a quadratic algebra is not Koszul which we did in Example 3. We also showed that the two important algebras $T(V)$ and $\mathbb{K}[X, Y]$ are Koszul algebras. The homological properties of Koszul algebras have not been discussed here. There is much to say on this subject and many results can be found in [2].

References

[1] Steven Roman, *Advanced Linear Algebra*, Springer 2008

[2] Ralf Fröberg, *Koszul Algebras*, Advances in commutative ring theory (1997)

[3] Jean-Louis Loday and Bruno Vallette, *Algebraic Operads*, Springer 2012

[4] Positselskii, *Relation Between the Hilbert Series of Quadratic Dual Algebras does not Imply Koszulity*, Funct Anal Its Appl (1995)