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Grothendieck's homotopy hypothesis and the homotopy theory of homotopy theories

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GROTHENDIECK'S HOMOTOPY HYPOTHESIS AND THE HOMOTOPY THEORY OF HOMOTOPY THEORIES

Abstract. We will present two possible models for "∞-categories": simplicial set with a horn-filling condition and Kan-complex enriched categories. We present Grothendieck's homotopy hypothesis as a "litmus test" for ∞-categories, and then develop the necessary machinery for explaining the phrase "homotopy theory of homotopy theories". We also define the maximal Kan-complex contained in a quasi-category, a generalization of the maximal groupoid contained in a category, and prove that it is an adjoint — we believe that this has not been done explicitly (in print) before.

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1. Introduction

In this text we will present *Grothendieck's homotopy hypothesis* and the slogan "homotopy theory of homotopy theories" in a self-contained way suitable as a first introduction to these two notions in particular and higher category theory in general. The prerequisites are few — some ordinary category theory (e.g. limits, adjunctions, groupoids) and the meaning of homotopy. The topics have been selected to present these ideas in a brief and understandable way — we believe that these "slogans" are often used but seldom explained in detail and so believe that a short and basic but rather complete introduction like this will fill a gap in the existing literature. The proofs in Section 3 along with Definition 21 are our own work and we hope that these will help the reader understand how work might be carried out in these settings (in the same way that they did just that for the author).

In Sections 2 and 3 we explore *simplicial sets*, which form a category **sSet**, and define a *quasi-category* as a simplicial set with a certain horn-filling property. The full subcategory of **sSet** that these quasi-categories form are denoted **QCat**. We present an important functor $N : \mathbf{Cat} \to \mathbf{sSet}$, the *nerve* functor, and show that it in fact has domain **QCat**. We then proceed to define the *fundamental* *category* functor $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ and show that this is a left adjoint to N. We finish Section 3 with a discussion of *Kan complexes*, which are simplicial sets with a stronger horn-filling property than that of quasi-categories. We show that the inclusion $\mathbf{Kan} \hookrightarrow \mathbf{QCat}$ is an analogue to the inclusion $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$. In particular we define the *maximal Kan-complex* contained in a quasi-category as a functor $\mathrm{mK} : \mathbf{QCat} \to \mathbf{Kan}$ and show that this is a right adjoint to the inclusion $\mathbf{Kan} \hookrightarrow \mathbf{QCat}$, just as $\mathrm{core} : \mathbf{Cat} \to \mathbf{Gpd}$ is a right adjoint to the inclusion $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$. This hints that Kan-complexes might be considered a ∞-dimensional analogue to groupoids.

Next is Section 4 where we discuss informally what a ∞-category might be and introduce the (n, k)-categories, which have morphisms up to dimension n but where every arrow of dimension greater than k is invertible. In this way ordinary categories are $(1, 1)$-categories, while groupoids are $(1, 0)$-categories. We explain that quasi-categories are $(\infty, 1)$-categories while Kan-complexes are $(\infty, 0)$-categories, and that in this view we really might consider Kan-complexes to be ∞-groupoids. We also discuss some difficulties in defining ∞-categories and how what we explore in this text is just one possible approach. Finally we discuss Grothendieck's hypothesis, which essentially states that ∞-groupoids "should" be equivalent to topological spaces. This is thus a "litmus test" for ∞-categories — ideally anything that claims to be a ∞-category should fulfill this hypothesis. In fact **Kan** does, but we do not show this. This is followed by a description of the *singular set* functor $\mathrm{Sing} : \mathbf{Top} \to \mathbf{sSet}$ and *geometric realization* $|-| : \mathbf{sSet} \to \mathbf{Top}$ which are adjoint $\mathrm{Sing} \vdash |-|$ that gives the equivalence that the homotopy hypothesis suggests should be there.

We proceed to Section 6 where we present the axioms of model categories. A model category can be though of as a place to "do homotopy theory", in the way that we have some class of arrows that we would like to formally invert as to make them into isomorphisms. The prime example is working with topological spaces up to homotopy — this is in essence working in **Top** with the homotopy equivalences inverted. We can do this formally with any class of arrows in any category, but this might give us set-theoretic problems. Having a model structure assures us that this does not happen. With this we understand that a "homotopy theory" is in fact a model structure on a category, and so might begin to explain "the homotopy theory of homotopy theories".

The following sections present simplicial categories, **sCat**, as an alternative way to define ∞-categories. We also present the general machinery behind "nerves and realizations" that generalize both the $N \vdash \tau_1$ and $\mathrm{Sing} \vdash |-|$ adjunctions. This gives us proofs of these adjunctions and also a *homotopy coherent nerve* $N : \mathbf{sCat} \to \mathbf{sSet}$ and it's realization. Finally we discuss the *hammock localization* that from a model category M gives us a simplicial category LM, with the property that we might recover the homotopy theory of M from LM. Together with a model structure on **sCat** itself this gives us a "homotopy theory of homotopy theories".

The history of higher category theory is quite rich. It of course came after the theory of ordinary categories, but the need for something like it can be found much earlier. Consider the (first) homotopy group of a topological space T at some basepoint t, $\pi(T, t)$. We would have preferred to construct it by taking the elements to be paths in T, and the group operation being concatenation of paths. However, the usual way to define concatenation of paths is not associative, so we instead define

$\pi(T,t)$ as *homotopy classes* of paths. Higher category theory provides an alternative work-around: we remember the "higher dimensional" information contained in the homotopies between paths, and construct a "fundamental ∞-groupoid" instead. This fundamental ∞-groupoid "should" contain all the information of a topological space — this assumption is Grothendieck's homotopy hypothesis.

The start of actual higher category theory is in the 1980s. In 1983 Grothendieck writes a letter (known by the title *Pursuing Stacks*, [Groth]) containing some of his ideas on how the higher dimensional structure of a "∞-category" should be defined (see Section 4 for some discussion on why there is no one answer to that question). The initial letter is about a dozen pages long, but Grothendieck keeps adding to it. Eventually the finial version is a few hundred pages long. The text is written in an unusual style — the reader gets to follow Grothendieck's thought, including his errors and mistakes; take a glance at it, if you haven't seen it before. The *stacks* that Grothendieck was pursuing are a notion used in algebraic geometry, but for this text the most relevant parts of *Pursuing Stacks* is the discussions on ∞-groupoids.

Notable names are Daniel Kan (1927-2013) (e.g. [DK]) and Daniel Quillen (1940-2011) (e.g. [Qui] where he introduced model categories) that were crucial to the initial developments. In more recent years higher category theory has become a rather broad area, in part since there are so many conceptually similar but formally different notions of ∞-categories (and associated objects). For the views presented in this text Jacob Lurie (see [Lurie] for a rather full discussion on quasi-categories) and André Joyal (who introduced the name *quasi-category*, see e.g. [Joyal]) have had large impacts. Of course many others have been quite important too, and we recommend that the reader take a look at the bibliography.

2. Simplicial Sets

Definition 1. A *simplicial set* is a functor $F : \Delta^{\mathrm{op}} \to \mathbf{Set}$ (where Δ is the category of finite ordinals with order-preserving maps). The category of simplicial sets is denoted $\mathbf{sSet}$.

Remark 2. The category of finite ordinals Δ has as objects the posets $[n] = \{0, 1, \cdots, n\}$, with the usual order, for $n \in \mathbb{N}$ and the morphisms are order preserving maps.

Example 3. The *standard n-simplex* $\Delta[n]$ is the simplicial set given by $\Delta[n](x) = \mathrm{Hom}_{\Delta}(x, [n])$ for objects $x \in \Delta$ and for maps $f : x \to y \in \Delta$ we let $\Delta[n](f)$ be the map from $\mathrm{Hom}_{\Delta}(y, [n])$ to $\mathrm{Hom}_{\Delta}(x, [n])$ that takes ϕ to $\phi \circ f$. These standard n-simplices will be rather useful later.

Thus a simplicial set X is a collection of sets $X_0, X_1, X_2, \ldots$ indexed by $\mathbb{N}$ (here X_n is shorthand for $X([n])$) and maps between them: there are $n+2$ maps $X_n \to X_{n+1}$ and n maps $X_n \to X_{n-1}$, so it looks something like

$$X_0 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} X_1 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} X_2 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} X_3 \begin{smallmatrix}\longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\end{smallmatrix} \cdots$$

The maps $X_n \to X_{n+1}$ are known as the *degeneracy maps* and denoted s_k^n for $0 \leq k \leq n+1$, while the maps $X_n \to X_{n-1}$ are the *face maps* and are denoted d_k^n for $0 \leq k \leq n-1$. As it is usually clear what X_n the maps operate on we often suppress the superscript, and simply write s_k or d_k.

The properties of the face and degeneracy maps come from the properties of maps in Δ^{op} and explicitly these are

$$\begin{aligned} d_i d_j &= d_{j-1} d_i && \text{if } i < j \\ d_i s_j &= s_{j-1} d_i && \text{if } i < j \\ d_i s_j &= \mathrm{id} && \text{if } i = j \text{ or } i = j+1 \\ d_i s_j &= s_j d_{i-1} && \text{if } i > j+1 \\ s_i s_j &= s_{j+1} s_i && \text{if } i < j+1 \end{aligned} \tag{2.1}$$

We think of the objects of X_n as being "n-dimensional" (and will say that an element $x \in X_n$ is a n-simplex) and the face maps as picking out a $n-1$ dimensional piece of the boundary of the object. We write, for an object $x \in X_n$, $\partial x = \left(d_0^n x, d_1^n x, \cdots, d_{n-1}^n x\right)$ and call this the *boundary* of x. In this view the degeneracy maps are seen as taking an object to a higher dimensional one where one side is "constant", e.g. taking a "point" $x \in X_0$ to a "line" $l \in X_1$ which has x as both of its endpoints.

We think of a 1-simplex x as having a direction, in the way that x goes from $d_1 x$ to $d_0 x$. We might draw this as

$$d_1 x \xrightarrow{x} d_0 x$$

If $x \in X_2$, then ∂x consists of three 1-simplices, and we might draw them all as

$$\begin{tikzcd} & d_0 d_0 x \arrow[dr, "d_0 x"] & \\ d_1 d_2 x \arrow[ur, "d_2 x"] \arrow[rr, "d_1 x"'] & & d_0 d_1 x \end{tikzcd}$$

Note that the way we write the corners in this triangle is not unique; for example, $d_1 d_2 x = d_1 d_1 x$. To simplify discussions, we will call the 0-simplex $d_1 d_2 x$ the 0-vertex of x, and in the same way $d_0 d_0 x$ will be the 1-vertex and $d_0 d_1 x$ will be the 2-vertex. In general, if $x \in X_n$, then the k-vertex of x is $\underbrace{d_0 d_0 \cdots d_0}_{k \text{ times}} d_n d_{n-1} \cdots d_{k+1}(x)$.

Example 4. Take S to be any non-empty set. We construct the *discrete simplicial set* X on S by letting $X_n = S$ for all $n \in \mathbb{N}$, and defining

$$\begin{aligned} d_i^n(s) &= s \\ s_i^n(s) &= s \end{aligned}$$

The conditions in 2.1 hold trivially, so X is a simplicial set.

Example 5. Take S to be any non-empty set. We construct the *codiscrete simplicial set* X on S. Let $X_n = \underbrace{S \times S \times \cdots \times S}_{n+1 \text{ times}}$ and let $d_i^n : X_n \to X_{n-1}$ take $(a_0, a_1, \cdots, a_n)$ to $(a_0, a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_{n-1})$, i.e. d_i just drops the i:th coordinate. In a similar manner, let $s_i^n : X_n \to X_{n+1}$ take $(a_0, a_1, \cdots, a_{n-1})$ to $(a_0, a_1, \cdots, a_{i-1}, a_i, a_i, a_{i+1}, \cdots, a_{n-1})$, i.e. s_i inserts an extra copy of the i:th coordinate. The properties in 2.1 are somewhat tedious to check, but they do hold, so X is a simplicial set.

Example 6. An *abstract simplicial complex* is a subset S of $\mathcal{P}(V)$, the power set of some set V with the property that if $x \in S$ and $y \subseteq x$, then $y \in S$. We think of V as a set of vertices and S as a set of simplices.

If we impose a total order on V we can turn S into a simplicial set S' by taking S'_n to be the set of order-preserving functions $f : [n] \to V$ such that $\mathrm{im} f \in S$. Let D^i be the map $[n-1] \to [n]$ in Δ that corresponds to the map $d_i : [n] \to [n-1]$ in Δ^{op}, and similarly for S^i. The the face map d_i in S' takes f to $f \circ D^i$ and the degeneracy map s_i takes f to $f \circ S^i$.

Thus f is a n-simplex of S', then the i-vertex of f is $f(i)$ and the boundary ∂f is $\left(f \circ D^0, f \circ D^1, \cdots, f \circ D^n\right)$.

2.1. **The nerve of a category.** There is a functor $N : \mathbf{Cat} \to \mathbf{sSet}$, which takes a small category C to the simplicial set NC with $NC_0 = \mathrm{ob}C$, $NC_1 = \mathrm{arr}C$. The map $d_1 : NC_1 \to NC_0$ is the source map, and d_0 is the target map; s_0 gives the identity arrows. NC_2 consists of triangles

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

and d_0 on the above triangle picks out g, d_1 gives $g \circ f$ and d_2 gives f. The elements of NC_2 can be though of as the composites $g \circ f$, but retaining the information on how it was constructed. Elements of NC_3 then consists of triple composites $h \circ g \circ f$ and so on.

Another way to think of NC is to view the elements of NC_k as $(k+1)$-length chains of objects and arrows in C, i.e. $A_0 \overset{f_1}{\to} A_1 \overset{f_2}{\to} A_2 \overset{f_3}{\to} \cdots \overset{f_{k-1}}{\to} A_{k-1} \overset{f_k}{\to} A_k$ where the arrows have appropriate source and targets so that the whole composition exists. Here the action of d and s are simple — d_i removes the i:th step and replaces it with the arrow $f_{i+1} \circ f_i$, while s_i inserts another copy of A_i with $f_i = \mathrm{id}_{A_i}$ — compare this with Example 5.

N is a right adjoint — the left adjoint is the *fundamental category* functor, which we will discuss in section 3.1. Also, N is full and faithful, but we will not show that.

3. Quasi-categories

In an ordinary category we may take two arrows (with suitable source and target) and from these get a (unique) third arrow, using composition. Clearly we'd like something similar for higher dimensional categories (although we won't require uniqueness), but it's not too easy to figure out what this should mean for higher dimensional arrows. With our geometric view of a simplicial set we get the following idea: if we have n-simplices $x_0, x_1, x_2, \cdots, x_n$ that could be the boundary of a $(n+1)$-simplex (if we add some n-simplex x_{n+1}), then there should indeed be at least one such possible x_{n+1}, i.e. it should be possible to fill the *horn* given by $x_1, x_2, \cdots, x_n$ into an actual $n+1$-simplex. The various possible x_{n+1}:s are though of as the compositions of $x_0, x_1, x_2, \cdots, x_n$. We will additionally restrict this property to so called *inner* horns — in an ordinary category we cannot expect to find a filling (dashed arrow) for the horn

$$\begin{tikzcd} & b \arrow[dr, "g"] & \\ a \arrow[ur, dashed] \arrow[rr, "f"'] & & c \end{tikzcd}$$

Consider for example the category with objects a, b and c, the two arrows f and g, and the identity arrows. In this category there is no arrow from a to b that could be the dotted arrow. Note that if the category is a groupoid, then such an arrow must exist — there must be an arrow $g^{-1} \circ f$, which would make the above diagram commute. This all leads to the following definitions:

Definition 7. The *n-dimensional k-horn* Λ^n_k, for $0 \leq k \leq n$ is the subset of the standard n-simplex Δ^n generated by all the $(n-1)$-dimensional faces of Δ^n containing the k-th vertex. Λ^n_k is said to be an *inner* horn if $0 < k < n$.

Remark 8. Note that there is a canonical inclusion $i^n_k : \Lambda^n_k \to \Delta^n$.

Definition 9. A simplicial set X is a *quasi-category* if every inner horn $\Lambda^n_k \to X$ can be extended along i^n_k to a map $\Delta^n \to X$.

The full subcategory of **sSet** given by the quasi-categories is denoted **QCat**.

Definition 10. A *Kan complex* is a simplicial set X where every horn $\Lambda^n_k \to X$ can be extended along i^n_k. (We simply drop the condition that the horn be an inner horn.)

Thus the "horn-filling property" of a simplicial set X is the property that given any arrow $h : \Lambda^n_k \to X$ in **sSet** there is a dashed arrow making the following diagram commute:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "h"] \arrow[d, hook, "i^n_k"'] & X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Some examples of quasi-categories are the simplicial set constructed in Example 5 and the nerve of a category:

Example 11. For any category C the nerve NC is a quasi-category. A n-simplex of NC is a list $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$ together with the information on what the various composites are. An inner k-horn is then such a list but lacking some information on the composites — however, since C is a category composition is uniquely determined by the arrows we are composing, so that information is easily regained, allowing us to fill the horn.

Note that not all horn have fillers — consider for example

$$\begin{tikzcd} & A_1 \arrow[dr] & \\ A_0 \arrow[ur, dashed] \arrow[rr] & & A_2 \end{tikzcd}$$

We cannot expect to find a dotted arrow in a general category (take again the category with objects A_1, A_2 and A_3, identities and the two solid arrows).

Another example is topological spaces - we will see in Section 5 how we can construct a quasi-category (in fact a Kan-complex) from a space.

3.1. **The fundamental category for quasi-categories.** In this section we will define the *fundamental category* of a quasi-category X, denoted $\tau_1 X$. To define this we will use the relation $\sim$ on X_1.

Definition 12. Given a simplicial set X the relation $\sim$ on X_1 is defined as $f \sim g$ if $b = d_0 f = d_0 g$, $a = d_1 f = d_1 g$ (i.e. f and g are parallel) and there is some $\sigma \in X_2$ such that $\partial\sigma = (s_0 b, f, g)$ (where $\partial\sigma$ is $(d_0\sigma, d_1\sigma, d_2\sigma)$).

Theorem 13. *$\sim$ is an equivalence relation.*

Proof. Reflexivity is immediate — take $\sigma = s_1 f$. Then using properties of arrows in Δ^{op} (equations 2.1) we see that $d_0 s_1 f = s_0 d_0 f = s_0 b$, $d_1 s_1 f = \mathrm{id} f = f$ and $d_2 s_1 f = \mathrm{id} f = f$, so indeed $\partial\sigma = (s_0 b, f, f)$.

Symmetry: in the diagram

$$\begin{tikzcd}
 & b \arrow[ddr, "s_0b"] & \\
 & b \arrow[u, "s_0b"'] \arrow[dr, "s_0b"'] & \\
a \arrow[uur, "g"] \arrow[ur, "f"'] \arrow[rr, "f"'] & & b
\end{tikzcd}$$

all three small triangles are the boundaries of something in X_2 — the upper left is $\partial\sigma$, the upper right is $\partial s_0 s_0 b$ and the lower is $\partial s_1 f$. So by horn-filling the large outer triangle is also the boundary of some simplex in X_2 and provides proof that $g \sim f$.

Transitivity: if $f \sim g$ and $g \sim h$ we get a diagram similar to the one above

$$\begin{tikzcd}
 & b \arrow[d, "s_0b"'] \arrow[ddr, "s_0b"] & \\
 & b \arrow[dr, "s_0b"'] & \\
a \arrow[uur, "f"] \arrow[ur, "g"'] \arrow[rr, "h"'] & & b
\end{tikzcd}$$

where the upper right triangle is the boundary of $s_0 s_0 b \in X_2$, and again by horn filling the outer triangle must be the boundary of something in X_2, so that $f \sim h$. $\square$

One might as well "switch the order" in the definition of $\sim$; one gets three new relations: $f \sim_1 g$ if there is a $\sigma \in X_2$ with $\partial\sigma = (s_0 b, f, g)$, $f \sim_2 g$ if $\partial\sigma = (g, f, s_0 a)$ and $f \sim_3 g$ if $\partial\sigma = (f, g, s_0 a)$. The proof above needs only small changes to show that these three are also equivalence relations.

Theorem 14. *$\sim$, $\sim_1$, $\sim_2$ and $\sim_3$ are in fact the same relation.*

Proof. That $\sim = \sim_1$ and $\sim_2 = \sim_3$ follows from symmetry. We show that $\sim = \sim_3$. The necessary diagram is

$$\begin{tikzcd}
& & b \arrow[ddrr, "s_0 b"] & & \\
& & a \arrow[u, "f"'] \arrow[drr, "f"] & & \\
a \arrow[uurr, "f"] \arrow[urr, "s_0 a"'] \arrow[rrrr, "g"'] & & & & b
\end{tikzcd}$$

Note that $s_0 f$ has the small upper left triangle as boundary, and that $s_1 f$ has the small upper right triangle as boundary, so we may use horn filling if either the small bottom triangle or the large outer triangle is the boundary of some 2-simplex and then conclude that the other one is also the boundary of some 2-simplex. But this is precisely the statement that $f \sim g \leftrightarrow f \sim_3 g$, so indeed $\sim = \sim_3$. □

Let $[x]$ and $[y]$ be two composable equivalence classes of $\sim$, i.e. $d_0 x = d_1 y$, and let σ be some filler of the horn

$$\begin{tikzcd}
& b \arrow[dr, "y"] & \\
a \arrow[ur, "x"] \arrow[rr, dashed, "z"'] & & c
\end{tikzcd}$$

We will need all z that arises from such fillers to be in the same equivalence class, and also show that the choice of representatives of $[x]$ and $[y]$ doesn't matter.

Theorem 15. *If $x \sim x'$ and $y \sim y'$, then $z \sim z'$, where z and z' come from fillers as in the diagrams:*

$$\begin{tikzcd}
& d_0 x \arrow[dr, "y"] & \\
d_1 x \arrow[ur, "x"] \arrow[rr, dashed, "z"'] & & d_0 y
\end{tikzcd}
\qquad
\begin{tikzcd}
& d_0 x' \arrow[dr, "y'"] & \\
d_1 x' \arrow[ur, "x'"] \arrow[rr, dashed, "z'"'] & & d_0 y'
\end{tikzcd}$$

Proof. The first step is to show that there is some $\rho \in X_2$ with $\partial\rho = (y, z', x)$.

$$\begin{tikzcd}
& & c & & \\
& & b \arrow[u, "y'"'] \arrow[drr, "s_0 b"] & & \\
a \arrow[uurr, "z'"] \arrow[urr, "x'"'] \arrow[rrrr, "x"'] & & & & b \arrow[uull, "y"']
\end{tikzcd}$$

In this diagram, all three small internal triangles are the boundaries of something in X_2 — the upper right and bottom one because $y \sim y'$ and $x \sim x'$ respectively, and the top left one because z' came from a horn-filling simplex. Thus we may use horn filling to conclude that the large outer triangle also is the boundary of something in X_2, which gives our ρ.

Using this we show that $z \sim z'$:

$$
\begin{tikzcd}
& c & \\
& b \arrow[u, "y'"] \arrow[dr, "y"] & \\
a \arrow[uur, "z'"] \arrow[ur, "x"] \arrow[rr, "z"'] & & c \arrow[uul, "s_0 c"']
\end{tikzcd}
$$

Here the upper left triangle is the boundary of ρ, the upper right is the boundary of something as $y \sim y'$ and the bottom one is a boundary as z came from such a filler. Thus the large outer triangle is also the boundary of some 2-simplex, and so $z \sim z'$. $\square$

Definition 16. The *fundamental category* $\tau_1 X$ of a quasi-category X is the category with X_0 as objects, and $X(x, y) / \sim$ as arrows from x to y. The above discussion shows that composition exists and is well-defined. The identity arrow id_x of an element $x \in X_0$ is the class $[s_0(x)]$.

We can make τ_1 into a functor $\tau_1 : \mathbf{QCat} \to \mathbf{Cat}$ by specifying $\tau_1(f)$ for $S \xrightarrow{f} T$ a morphism in $\mathbf{QCat}$. On objects x in $\tau_1 S$ let $\tau_1 f(x) = f(x)$, and on arrows $x \xrightarrow{[u]} y$ in $\tau_1 S$ let $\tau_1 f(u) = [f(u)]$ — this is well defined, as if $u \sim v$ in S, then there is σ in S with $\partial\sigma = (\mathrm{id}_{d_0 u}, v, u)$, and as f is a map of quasi-categories, $\partial(f\sigma) = (f\mathrm{id}_{d_0 u}, fv, fu)$ so that $fu \sim fv$.

The domain of τ_1 can be extended to the whole of $\mathbf{sSet}$. We note that τ_1 is a left adjoint, with its right adjoint being N, the nerve functor - we show this in Theorem 18.

Remark 17. For a quasi-category Q, the fundamental category $\tau_1 Q$ is the same as the so-called *homotopy category*, ho$Q$. The homotopy category arises from the general notion of *model categories*, which we will discuss in section 6.

3.2. **The fundamental category functor in full.** A category looks a lot like a simplicial set with only 0 and 1 dimensional objects — the largest difference is the lack of composition in a simplicial set. If we have a quasi-category we may use the 2-simplices to define composition (as we did above). In a general simplicial set we might not have any reasonable choice for composition — there might not be any 2-simplex that fills a given horn. So in order to make a simplicial set into a category we will need to add some new morphisms.

Given a simplicial set X we construct a directed graph X^* (which essentially will be a category without composition, which we will define later). Let the vertices of X^* be the 0-simplices of X, and let the edges be lists of composable 1-simplices in X. If for example $f, g, h \in X_1$ with $d_0 f = d_1 g$ and $d_0 g = d_1 h$ we would have edges (f), (g) and (h), and also (f, g), (g, h) and (f, g, h). In this way we guarantee that will have composites later on — we might take $g \circ f$ to be (f, g). The source of a list $(f_1, f_2, \cdots, f_n)$ is taken to be $d_1 f_1$ and the target to be $d_0 f_n$. We define composition of two composable edges $f = (f_1, f_2, \cdots, f_n)$ and $g = (g_1, g_2, \cdots, g_m)$ to be $g \circ f = (f_1, \cdots, f_n, g_1, \cdots, g_m)$; this is associative. We now create the identity morphisms, by identifying $f \circ s_0 x$, $s_0 y \circ f$ and f (where $f : x \to y$). Finally, we identify $f \circ g$ with h if there is some $\sigma \in X_2$ with $d_0\sigma = f$, $d_1\sigma = h$ and $d_2\sigma = g$.

3.3. **About τ_1 and N.**

Theorem 18. *The functor $\tau_1 : \mathbf{QCat} \to \mathbf{Cat}$ is left adjoint to $N : \mathbf{Cat} \to \mathbf{QCat}$.*

Proof. We will define two natural transformations $\varepsilon : \tau_1 N \to \mathrm{id}_{\mathbf{Cat}}$ and $\eta : \mathrm{id}_{\mathbf{sSet}} \to N\tau_1$ and show that these are the counit and unit of the adjunction, respectively. To do this we first discuss the functors $\tau_1 N$ and $N\tau_1$.

Firstly, $\tau_1 N$ is a functor from **Cat** to **Cat**. For any small category C, the objects of $\tau_1 N(C)$ are the same as the objects of C. The arrows of $\tau_1 N(C)$ are equivalence classes of arrows in $N(C)$ — in what cases are two arrows $f, g : x \to y$ of $N(C)$ equivalent? This happens only when there is some 2-simplex σ of $N(C)$, with $d_0(\sigma) = \mathrm{id}_y$, $d_1(\sigma) = g$ and $d_2(\sigma) = f$. But recall the definition of the nerve — a 2-simplex ρ of $N(C)$ is a composition $a \xrightarrow{r} b \xrightarrow{s} c$, with $d_0(\rho) = b \xrightarrow{s} c$, $d_1(\rho) = a \xrightarrow{s \circ r} c$ and $d_2(\rho) = a \xrightarrow{r} b$. We draw these two diagrams side-by-side:

$$\begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] \arrow[d, phantom, "\sigma"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & {} & y \end{tikzcd} \qquad \begin{tikzcd} & b \arrow[dr, "s"] \arrow[d, phantom, "\rho"] & \\ a \arrow[ur, "r"] \arrow[rr, "s \circ r"'] & {} & c \end{tikzcd}$$

Here we see that if $f \sim g$, then we must have $f = \mathrm{id} \circ g = g$, so not only are the objects of $\tau_1 N(C)$ the same as the objects of C, but additionally the arrows of $\tau_1 N(C)$ look the same as those of C, and from this we also see that composition is the same. Thus C and $\tau_1 N(C)$ are isomorphic, and we may take ε to be this isomorphism. Explicitly, arrows in $\tau_1 N(C)$ are equivalence classes of length one sequences, e.g. $A_0 \xrightarrow{[f]} A_1$, and ε takes this to f, which is an arrow in C.

Now we turn to the functor $N\tau_1 : \mathbf{QCat} \to \mathbf{QCat}$. If X is some quasi-category, then the objects of $\tau_1(X)$ are X_0, and so $(N\tau_1(X))_0 = \mathrm{ob}(\tau_1(X)) = X_0$. Higher dimensional simplices are sequences of elements of X_0 chained together by the maps of $\tau_1(X)$, i.e. equivalence classes of 1-simplices in X, e.g.

$$A_0 \xrightarrow{[f_1]} A_1 \xrightarrow{[f_2]} A_2 \xrightarrow{[f_3]} A_3$$

As for maps, let $s : X \to Y$. Then $N\tau_1(s)$ takes

$$A_0 \xrightarrow{[f_1]} A_1 \xrightarrow{[f_2]} A_2 \xrightarrow{[f_3]} A_3$$

to

$$sA_0 \xrightarrow{[sf_1]} sA_1 \xrightarrow{[sf_2]} sA_2 \xrightarrow{[sf_3]} sA_3$$

The naturality square for η, for any simplicial sets X and Y and any map $s : X \to Y$ is

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "\eta_X"'] & Y \arrow[d, "\eta_Y"] \\ N\tau_1(X) \arrow[r, "N\tau_1(s)"'] & N\tau_1(Y) \end{tikzcd}$$

It's reasonable to guess that η_S might take a n-simplex σ of S to the chain $\sigma(0) \xrightarrow{[f_1]} \sigma(1) \xrightarrow{[f_2]} \cdots \xrightarrow{[f_n]} \sigma(n)$, where $\sigma(k)$ is the k:th corner of σ and f_k is the 1-simplex from $\sigma(k-1)$ to $\sigma(k)$ in σ. This clearly makes the naturality square commute, so η is a natural transformation.

Now we need only show that for any small category C and simplicial set S we have $\mathrm{id}_{\tau_1 S} = \varepsilon_{\tau_1 S} \circ \tau_1(\eta_S)$ and $\mathrm{id}_{NC} = N(\varepsilon_C) \circ \eta_{NC}$. For the first of these, note that $\tau_1(\eta_S)$ goes from $\tau_1 S$ to $\tau_1 N \tau_1 S$, and takes objects to themselves and arrows $x \xrightarrow{[f]} y$ to themselves, too. Then ε also takes both objects and arrows to themselves, and so the composition is indeed the identity morphism.

For the second equation, η_{NC} takes an n-simplex $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} \cdots \xrightarrow{f_n} A_n$ to $A_0 \xrightarrow{[f_1]} A_1 \xrightarrow{[f_2]} A_2 \xrightarrow{[f_3]} \cdots \xrightarrow{[f_n]} A_n$, and as NC is a nerve, the equivalence classes are all singletons, which are taken to their only element by ε, so that the composition is again the identity. This shows the theorem. □

From this discussion it's not hard to draw the conclusion that a simplicial set X is the nerve of some category if and only if it has *unique* horn-fillers. We also note that as τ_1 is a left adjoint it preserves colimits. Additionally, it can be shown that τ_1 preserves finite products, although we wont do that.

Theorem 19. *The functor* $\tau_1 : \mathbf{sSet} \to \mathbf{Cat}$ *is left adjoint to* $N : \mathbf{Cat} \to \mathbf{sSet}$*.*

Proof. We again use the unit-counit notion of adjunction.

Note that if $\tau_1 N(C)$ has the same objects as $C \in \mathbf{Cat}$. The arrows of $\tau_1 N(C)$ are lists of 1-simplces of $N(C)$, which are in turn arrows of C. We thus let ε take objects to themselves and take an arrow $(f_0, f_1, f_2, \cdots, f_n)$ to the arrow $f_n \circ \cdots \circ f_2 \circ f_1 \circ f_0$ (note that this is well-defined since it respects the composition in $\tau_1 N(C)$). This makes ε into a natural transformation.

For $N\tau_1(S)$ the 0-simplices are the same as in S. The 1-simplices are list of length one of arrows in $\tau_1 S$, which in turn are lists of compatible 1-simplices of S. So we let η take 0-simplices to themselves, a 1-simplex f to (f), a 2-simplex σ to $(d_2\sigma, d_0\sigma)$ etc. This is well-defined since composition in $\tau_1 S$ was defined using the 2-simplices of S. Then η is also a natural transformation.

We need to show that for any $S \in \mathbf{sSet}$ we have $\mathrm{id}_{\tau_1 S} = \varepsilon_{\tau_1 S} \circ \tau_1(\eta_S)$. But η_S embeds S into $N\tau_1(S)$, and so $\tau_1(\eta_S)$ embeds $\tau_1 S$ into $\tau_1 N \tau_1(S)$, and then $\varepsilon_{\tau_1 S}$ composes the lists of $\tau_1 N(\tau_1 S)$ so that they become actual arrows in $\tau_1 S$ — all this preserves the actual arrows, so that the equation holds.

The second equation we need to confirm is $\mathrm{id}_{NC} = N(\varepsilon_C) \circ \eta_{NC}$. Here η_{NC} embeds NC into $N\tau_1(NC)$. As ε_C composes lists into single arrows, $N(\varepsilon_C)$ composes together the list corresponding to a side in a simplex. E.g. $N(\varepsilon_C)$takes $(f_0, f_1, \cdots, f_n) : A_0 \to A_1$ in $\tau_1 N(C)$ to $f_n \circ \cdots \circ f_1 \circ f_0 : A_0 \to A_1$ in NC. So in all the right hand side of the equation changes nothing. Thus ε and η shows that τ_1 and N are adjoint. □

3.4. **Other functors.** Every Kan complex is of course a quasi-category, and so there is a natural inclusion $\mathbf{Kan} \hookrightarrow \mathbf{QCat}$, and as it turns out this corresponds to the inclusion of groupoids into categories, $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$. There is thus good reason to consider Kan complexes to be "quasi-groupoids".

Theorem 20. *For any category* C*,* NC *is a Kan-complex if and only if* C *is a groupoid.*

Proof. Recall that a groupoid is a category where every morphism is invertible.

Let NC be a Kan-complex, and let f be any morphism $x \xrightarrow{f} y$ in C . Consider the following two (outer) 2-horns

$$\begin{tikzcd} & y \arrow[dr, dashed, "f_2"] & \\ y \arrow[ur, "\mathrm{id}_y"] \arrow[r, dashed, "f_1"'] & x \arrow[u, "f"] \arrow[r, "\mathrm{id}_x"'] & x \end{tikzcd}$$

Here we know that there are fillers f_1 and f_2 that makes the diagram commute — we can also see that $f_1 = \mathrm{id}_x \circ f_1 = f_2 \circ \mathrm{id}_y = f_2$, and that $f_2 \circ f = \mathrm{id}_x$ and $f \circ f_1 = \mathrm{id}_y$, so that $f_1 = f_2$ is an inverse to f.

Let C be a groupoid. NC is a quasi-category (see Example 11). Following the process in that example we see that any outer n-horn for $n \geq 3$ has a filler. It only remains to show for outer 2-horns.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, dashed] \arrow[rr, "f"'] & & z \end{tikzcd} \qquad \begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & z \end{tikzcd}$$

In the case of Λ_0^2 (the right diagram) take $f \circ g^{-1}$ to be the dotted arrow. For Λ_2^2 (the left diagram) take instead $g^{-1} \circ f$ as the dotted arrow. $\square$

The inclusion $i : \mathbf{Kan} \hookrightarrow \mathbf{QCat}$ is a left adjoint, with the right adjoint being $\mathrm{mK} : \mathbf{QCat} \to \mathbf{Kan}$ which takes a quasi-category to the largest Kan complex contained in it.

Definition 21. For a quasi-category S, let $\mathrm{mK}(S)$ be the simplicial set with S_0 as objects, and where 1-simplices are those $f \in S_1$ such that there are two 2-simplices σ and π in S_2 with boundaries as below:

$$\begin{tikzcd} & d_0 f \arrow[dr, "f^l"] & \\ d_1 f \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_{d_1 f}"'] & & d_1 f \end{tikzcd} \qquad \begin{tikzcd} & d_0 f \arrow[dr, "f"] & \\ d_1 f \arrow[ur, "f^r"] \arrow[rr, "\mathrm{id}_{d_1 f}"'] & & d_1 f \end{tikzcd}$$

(We think of f^l and f^r as left and right inverses to f, respectively. We also note that these need not be unique.)

For $n > 1$, construct inductively $\mathrm{mk}(S)_n$ as the elements σ of S_n where every entry in $\partial\sigma$ is in $\mathrm{mK}(S)_{n-1}$.

If $f : X \to Y$ in $\mathbf{QCat}$, we let $\mathrm{mK}(f) = f|_{\mathrm{mK}(X)}$ — this turns mK into a functor $\mathbf{QCat} \to \mathbf{Kan}$.

Theorem 22. *For any quasi-category S, the simplicial set $\mathrm{mK}(S)$ is a Kan-complex.*

Proof. Step one: show that $\mathrm{mK}(S)$ is a quasi-category. Assume that f and g are 1-simplices in $\mathrm{mK}(S)$, and that there is a 2-simplex σ in S with $\partial\sigma = (g, h, f)$, for some $h \in S_1$. We need to show that h has inverses h^l and h^r — we do this for h^l only, the proof for h^r is similar.

Take h' to be d_1 of some 2-simplex that is a filler of the horn

$$\begin{tikzcd} & y \arrow[dr, "f^l"] & \\ z \arrow[ur, "g^l"] \arrow[rr, dashed, "h'"'] & & x \end{tikzcd}$$

(Fix some choice of g^l and f^l.) Now we show that there is a 2-simplex ρ in S with $\partial\rho = \left(h', f^l, g\right)$. We draw the following diagram:

$$
\begin{tikzcd}
& y \arrow[ddr, "f^l"] & \\
& z \arrow[u, "g^l"'] \arrow[dr, "h'"'] & \\
y \arrow[uur, "\mathrm{id}_y"] \arrow[ur, "g"'] \arrow[rr, "f^l"'] & & x
\end{tikzcd}
$$

The upper triangles and the large outer one are all the borders of things in S_2. The upper left triangle comes from the choice of g^l, the right upper triangle from the choice of h', and the large outer as $f^l \sim f^l$. Thus by horn-filling in S there is such a ρ.

Now we show that h' is in fact a left inverse of h. The diagram is

$$
\begin{tikzcd}
& z \arrow[ddr, "h'"] & \\
& y \arrow[u, "g"'] \arrow[dr, "f^l"'] & \\
x \arrow[uur, "h"] \arrow[ur, "f"'] \arrow[rr, "\mathrm{id}_x"'] & & x
\end{tikzcd}
$$

Here every small triangle is the border of some 2-simplex of S — the bottom one from our choice of f^l, the top left one from the definition of h and the top right one is the border of ρ.

Horn-filling for higher dimensions follows from the construction of the higher dimensions of $\mathrm{mK}(S)$. In particular, σ is in $\mathrm{mK}(S)_2$, so that the horn given by f and g does have a filler.

Step two: show that in this quasi-category every outer horn has a filler. Say we have an outer horn (the proof for the other outer horn is similar)

$$
\begin{tikzcd}
& y \arrow[dr, dashed] & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & & z
\end{tikzcd}
$$

Take some filler $\sigma \in \mathrm{mK}(S)_2$ of

$$
\begin{tikzcd}
& x \arrow[dr, "g"] & \\
y \arrow[ur, "f^l"] \arrow[rr, dashed, "h"'] & & z
\end{tikzcd}
$$

We claim that there is some $\rho \in S_2$ with $\partial\rho = (h, g, f)$ so that ρ is the required filler (it will be in $\mathrm{mK}(S)_2$ as its boundary is in $\mathrm{mK}(S)_1$). The diagram to consider is

$$\begin{tikzcd}
 & x \arrow[ddr, "g"] & \\
 & y \arrow[u, "f^l"] \arrow[dr, "h"'] & \\
x \arrow[uur, "\mathrm{id}_x"] \arrow[ur, "f"'] \arrow[rr, "g"'] & & z
\end{tikzcd}$$

Here the two upper small triangles and the large outer triangle are borders of 2-simplices of $\mathrm{mK}\,(S)$. The upper left from the choice of f^l, the upper right from choice of h, and the outer one as $g \sim g$. Thus by horn-filling there is something that fills the lower triangle — take this as our ρ. Again, in higher dimensions the results follow from the construction from S. $\square$

Theorem 23. *The functor* $\mathrm{mK} : \mathbf{QCat} \to \mathbf{Kan}$ *is right adjoint to the inclusion functor* $i : \mathbf{Kan} \to \mathbf{QCat}$.

Proof. We need natural transformations $\varepsilon : i \circ \mathrm{mK} \to \mathrm{id}_{\mathbf{QCat}}$ (the counit) and $\eta : \mathrm{id}_{\mathbf{Kan}} \to \mathrm{mK} \circ i$ (the unit). Note that, since every morphism in a Kan complex has both left and right inverses (as in Definition 21), $\mathrm{mK}\,(K) = K$ if K is a Kan complex. Thus $\mathrm{mK} \circ i = \mathrm{id}_{\mathbf{Kan}}$, and we may take η_K to be the identity on K. We need ε to, given $f : X \to Y$ in $\mathbf{QCat}$, satisfy the naturality square

$$\begin{tikzcd}
i \circ \mathrm{mK}\,(X) \arrow[r, "i \circ \mathrm{mK}(f)"] \arrow[d, "\varepsilon_X"'] & i \circ \mathrm{mK}\,(Y) \arrow[d, "\varepsilon_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Note that every simplex of $i \circ \mathrm{mK}\,(X)$ is a simplex of X, so we may take ε_X to be the inclusion $\mathrm{mK}\,(X) \hookrightarrow X$.

We now need to verify the unit-counit equations $\mathrm{id}_{i(Y)} = \varepsilon_{i(Y)} \circ i\,(\eta_Y)$ and $\mathrm{id}_{\mathrm{mK}(X)} = \mathrm{mK}\,(\varepsilon_X) \circ \eta_{\mathrm{mK}(X)}$ for $X \in \mathbf{QCat}$ and $Y \in \mathbf{Kan}$. For the first, note that as Y is a Kan complex, both $\varepsilon_{i(Y)}$ and $i\,(\eta_Y)$ are the identity on Y, and so the composition is the identity on Y and thus on $i\,(Y)$, and for the second equation the situation is similar — both arrows on the right hand side are the identity on $\mathrm{mK}\,(X)$, and so the equation holds. This shows that $\mathrm{mK} \vdash i$. $\square$

The adjunction $\mathrm{mK} \vdash i$ corresponds to the adjunction between the inclusion $\mathbf{Gpd} \hookrightarrow \mathbf{Cat}$ and mG, where mG takes a category to the maximal groupoid contained in it (i.e. mG throws away all morphisms that are not isomorphisms). These functors are summarized in the following picture:

$$\begin{tikzcd}
\mathbf{QCat} \arrow[r, shift left, "\tau_1"] \arrow[r, phantom, "\perp"] \arrow[d, shift left, "\mathrm{mK}"] \arrow[d, phantom, "\dashv"] & \mathbf{Cat} \arrow[l, shift left, "N"] \arrow[d, shift right, "\mathrm{mG}"'] \arrow[d, phantom, "\vdash"] \\
\mathbf{Kan} \arrow[u, shift left, "i"] \arrow[r, shift right, "\tau_1"'] \arrow[r, phantom, "\top"] & \mathbf{Gpd} \arrow[l, shift right, "N"'] \arrow[u, shift right, "i"']
\end{tikzcd}$$

The inclusions **Gpd** $\hookrightarrow$ **Cat** and **Kan** $\hookrightarrow$ **QCat** also have left adjoints, where we instead of removing the offending morphisms/simplices we add new ones. There is a functor fG : **Cat** $\to$ **Gpd** that adds an inverse for every morphism that lacks one, and thus turns a category into a groupoid, and a similar functor fK : **QCat** $\to$ **Kan** that adds fillers to all outer horns that lack them.

4. Grothendieck's Homotopy Hypothesis

Note: This section contains a high-level, informal discussion about what higher category theory should, or could, be.

The fundamental group of a topological space X at a point $x \in X$, $\pi_1(X, x)$, should be familiar. By considering not only loops, but any path between any two points of X, we may similarly construct the fundamental groupoid of X.

Definition 24. Let S be a topological space. The *fundamental groupoid* $\Pi_1(S)$ of S has as objects the points of S, and as arrows homotopy classes of paths in S, i.e. a path p gives an arrow $[p]$ from $p(0)$ to $p(1)$.

This is clearly a groupoid, as every path can be reversed.

However, with our newfound knowledge of higher dimensional categories there is no need to consider *classes* of paths — we could take points of S as objects, paths as arrows, homotopies as arrows between arrows, etc.

4.1. Different kinds of homotopies.

What precisely is an "arrow between arrows"? The perhaps most intuitive way to draw them is as

$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$$

This is how we think about homotopies — maps between two parallel maps. What we've been working with in quasi-categories are 2-simplices, which we might draw as

$$\begin{tikzcd} & y \arrow[dr] \arrow[d, phantom, "\Downarrow"] & \\ x \arrow[ur] \arrow[rr] & {} & z \end{tikzcd}$$

This kind of 2-arrow is suitable when we'd like to talk about e.g. composition, as we have done. Taking $y = z$ and the arrow between them to be id_y we recover the kind of 2-arrow we had above — this is what we did when we constructed the τ_1 functor using the relation $\sim$.

We could of course go further, to get e.g.

$$\begin{tikzcd} & b \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & \\ a \arrow[ur] \arrow[dr] & & c \\ & d \arrow[ur] & \end{tikzcd}$$

Note that homotopies are symmetric — say that we have a quasi-category X and a 2-simplex σ in X; then σ provides a homotopy from $d_0\sigma \circ d_2\sigma$ to $d_1\sigma$, but also in the other direction, from $d_1\sigma$ to $d_0\sigma \circ d_2\sigma$

4.2. **Invertible homotopies.** Coming from the direction of topology it seems reasonable that homotopies should go both ways — if a path p can be transformed into q, we expect the reverse to hold as well. But recall that in our topological space even the paths can be reversed, and that is clearly not a reasonable assumption for categories in general — perhaps we ought not take every property suggested by topological spaces.

Quasi-categories gives a structure where 1-simplices are not necessarily invertible, but simplices of higher order are. If we instead consider how we might add a second level of morphisms to an ordinary category by enriching it over **Cat**, this object would not necessarily have invertible 2-arrows.

There is a common terminology used to discuss these various ways of modeling higher dimensional categories. We speak of (n,k)-categories, which are categories with morphisms up to dimension n, where every morphism of dimension greater than k is invertible. Thus ordinary categories are $(1,1)$-categories, quasi-categories are $(\infty,1)$-categories, groupoids are $(1,0)$-categories and quasi-groupoids are $(\infty,0)$-categories.

We've seen one example each of $(\infty,1)$- and $(\infty,0)$-categories — both as simplicial sets with certain conditions, but there are other objects that might fill these roles. These different models of high level categories of course have different properties.

4.3. **The homotopy hypothesis.** Thus there are many different alternative choices for what a ∞-category should be, but from the first discussion we'd expect the ∞-groupoids to correspond to topological spaces. With this background we state *Grothendieck's homotopy hypothesis* as

There should be an equivalence between **Top** *and* ∞**Gpd**.

On one hand the equivalence should be given by the fundamental groupoid,

$$\Pi_\infty : \mathbf{Top} \to \infty\mathbf{Gpd}$$

and on the other by so called geometric realization,

$$|-| : \infty\mathbf{Gpd} \to \mathbf{Top}$$

As it turns out, in the case we have studied where ∞**Gpd** are Kan complexes, this is a provable theorem.

4.4. **Topological $(\infty,1)$-categories.** If one starts out with the aim of having the homotopy hypothesis as a theorem, one might be inclined to define $(\infty,1)$-categories using some topological notion. This is fully possible, and gives *topological categories.* Stated briefly, a topological category is a category enriched over the category **CGHaus** of compactly generated Hausdorff spaces. (We need to use **CGHaus** instead of **Top** as the later isn't Cartesian closed.) This in essence means that instead of Hom-sets we have Hom-spaces, where points correspond to arrows, and paths to homotopies etc. In a specific sense these topological categories and quasi-categories are equivalent models of $(\infty,1)$-categories.

5. Geometric Realization and the Singular Set

In Section 4 we discussed informally how we might expect a relation between topological spaces and ∞-categories. In this chapter we explore this connection further and more formally (for the case where we take quasi-categories to be our $(\infty, 1)$-categories).

Well need the following notation:

Definition 25. Given a map $\phi : [n] \to [m]$ in Δ, define the map $\phi^* : |\Delta^n| \to |\Delta^m|$ by letting $\phi(t_0, t_1, \cdots, t_n)$ have $\sum_{j \in \phi^{-1}(i)} t_j$ as it's i:th coordinate.

Definition 26. Given a map $\phi : [n] \to [m]$ in Δ and a simplicial set X, let ϕ_* denote the map $X(\phi)$ in X.

5.1. **The singular set.** We will use $|\Delta^n|$ to denote the standard n-simplex, a topological space given by $|\Delta^n| = \{(t_0, t_1, \cdots, t_n) : \sum_{i=0}^n t_i = 1, t_i \geq 0\} \subseteq \mathbb{R}^{n+1}$ with the subspace topology.

Definition 27. Let T be a topological space. Then the *singular set* $S(T)$ of T (sometimes denoted Sing$T$) is a simplicial set defined by

$$S(T)_n = \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, T)$$

For a map $\phi : [n] \to [m]$ in Δ, let $S(T)(\phi)$ be the map from $\mathrm{Hom}_{\mathbf{Top}}(|\Delta^m|, T)$ to $\mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, T)$ that takes f to $f \circ \phi^*$.

Remark 28. S is in fact a functor — if $f : X \to Y$ is an arrow in **Top**, then Sf is a map from SX to SY by composition, taking maps $|\Delta^n| \to X$ to $|\Delta^n| \to X \xrightarrow{f} Y$.

5.2. **Geometric realization.** Geometric realization of a simplicial set S assigns a n-simplex to each element of S_n, and then glues these together in a nice way, somewhat similarly to how a CW-complex is constructed.

We use ϕ^* and ϕ_* to associate maps between the standard topological simplices and the face and degeneracy maps in a simplicial set. If for example $\phi : [1] \to [3]$ is the map $\phi(i) = i + 1$, then ϕ^* picks out the edge between the 1 and 2 vertices in $|\Delta^3|$, while $\phi_* = d_0 \circ d_3$ (recall that simplicial sets are contravariant functors).

Definition 29. Given a simplicial set X, the *geometric realization* of X, $|X|$, is defined as

$$\left(\coprod_{n=0}^{\infty} X_n \times |\Delta^n| \right) / \sim$$

with the quotient topology. Here $\sim$ is the relation defined by $(x, \phi^*(t)) \sim (\phi_*(x), t)$ for every map $\phi \in \mathrm{arr}\Delta$.

(The product $X_n \times |\Delta^n|$ is given in **Top**, by giving X_n the discrete topology.)

We note that geometric realization is left adjoint to the singular set functor. This will follow from Theorem 60.

6. Model Categories

Definition 30. A *model category* is a category C together with three classes of morphisms called *fibrations*, *cofibrations* and *weak equivalences* that satisfies the following axioms:

(0) Any composition of two fibrations, cofibrations or weak equivalences are again respectively a fibration, cofibration or weak equivalence. Equivalently, the three classes are closed under composition.
(1) The category C is complete and cocomplete.

(2) If f and g are composable morphisms, and any two of f, g and $f \circ g$ are weak equivalences, then so is the third. (This is sometimes called the "two out of three-rule".)

(3) If we have a diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"] & Y \arrow[r, "j"] \arrow[d, "g"] & X \arrow[d, "f"] \\ A \arrow[r, "m"'] & B \arrow[r, "n"'] & A \end{tikzcd}$$

where $j \circ i = \mathrm{id}_X$ and $n \circ m = \mathrm{id}_A$, then if g is a fibration, cofibration or weak equivalence, then f is too.

(4) If, in the diagram below, f is a cofibration and g is a fibration and at least one of them is a weak equivalence, then there is a dotted arrow making the diagram commute.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

(5) Any map $x : X \to Y$ in C can be factored as $x = f \circ c$, where f is a fibration and c is a cofibration, in two ways — one where f is a weak equivalence and one where c is.

$$\begin{tikzcd} X \arrow[rr, "x"] \arrow[dr, "c"'] & & Y \\ & A \arrow[ur, "f"'] & \end{tikzcd}$$

A model category is though of as a place to "do homotopy theory". By formally inverting all weak equivalences in C, so that they become isomorphisms, we get a homotopy category, ho$C$ — for example, the category of homotopy classes of topological spaces arises in this way. This can be seen as a way of "reducing" a category, by loosening the conditions necessary for two objects to be isomorphic — in **Top** we might be satisfied by homotopy equivalences rather than actual homeomorphisms.

(Given any category and some arrows we could of course formally invert them, but this might force us to create very many new arrows — so many that the collection of arrows between two objects might not be a set anymore. Model categories does not have this size problem if we invert the weak equivalences.)

There are trivial model structures on any co- & complete category — take the weak equivalences to be the isomorphisms, and let every morphism be both a fibration and a cofibration. **Top** admits a non-trivial model structure that allows the ordinary homotopy theory of CW-complexes (where the fibrant arrows are fibrant maps, hence the name), but this is not the only choice. Additionally, **sSet** admits a non-trivial model structure, which turns out to be closely related to the one on **Top**.

Remark 31. Model categories and model structure were introduced by Daniel Quillen (1940-2011), and much of what is discussed here is sometimes referred to as "Quillen –", e.g. "Quillen equivalence of model categories", "Quillen model structure" etc.

There is much more to say about model categories than what we mention here — see for example [Hov], where another approach to the homotopy category is used.

6.1. Basic facts about model categories.

Lemma 32. *If C is a model category and id_x is the identity arrow for some $x \in \mathrm{ob}C$, then id_x is a weak equivalence.*

Proof. By the fifth axiom of model categories, we can factor id_x as $\mathrm{id}_x = f \circ c$, where $f : y \to x$ is a fibration and weak equivalence, and $c : x \to y$ is a fibration. The the diagram

$$\begin{tikzcd} x \arrow[r, "c"] \arrow[d, "\mathrm{id}_x"] & y \arrow[r, "f"] \arrow[d, "f"] & x \arrow[d, "\mathrm{id}_x"] \\ x \arrow[r, "\mathrm{id}_X"'] & x \arrow[r, "\mathrm{id}_x"'] & x \end{tikzcd}$$

commutes, and also fulfills the requirements of the third axiom, so that id_x is a weak equivalence. □

Definition 33. Some common terminology for a model category C:

- An object $x \in \mathrm{ob}C$ is *fibrant* if the unique arrow from x to the terminal object is a fibration. (A terminal object exists as C is complete.)
- Similarly, an object $x \in \mathrm{ob}C$ is *cofibrant* if the unique arrow from the initial object to x is a cofibration. (A initial object exists as C is cocomplete.)
- An arrow that is both a weak equivalence and a (co)fibration is known as a *trivial (co)fibration.* (These are occasionally called *acyclic* rather than *trivial* (co)fibrations.)

A common way to classify the (trivial) (co)fibrations of a model category is by *lifting properties* relative to the other classes.

Definition 34. If for the two maps $f : x \to y$ and $p : a \to b$, any commutative diagram of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & a \arrow[d, "p"] \\ y \arrow[r] \arrow[ur, dashed] & b \end{tikzcd}$$

has a dotted arrow making it commute, then f has the *left lifting property (LLP)* with respect to p, and p has the *right lifting property (RLP)* with respect to f.

Theorem 35. *In a model category C,*

(1) *The fibrations are the maps of C that have the RLP with respect to the trivial cofibrations*
(2) *The trivial fibrations are the maps of C that have the RLP with respect to the cofibrations*
(3) *The cofibrations are the maps of C that have the LLP with respect to the trivial fibrations*
(4) *The trivial cofibrations are the maps of C that have the LLP with respect to the fibrations*

Proof. Note that 3 and 4 follows from 1 and 2 by duality. Note that the fourth axiom of model categories tells us that every fibration has the RLP with respect to trivial cofibrations, and that trivial fibrations has the RLP with respect to cofibrations.

We need to show that if $f : x \to y$ has the RLP with respect to every trivial cofibration, then f is a fibration. Use the fifth axiom to factor f as $f = p \circ q$, where $p : z \to y$ is a fibration and $q : x \to z$ is a trivial cofibration. Then there is a f' making the following diagram commute:

$$\begin{tikzcd} x \arrow[r, "\mathrm{id}_x"] \arrow[d, "q"'] & x \arrow[d, "f"] \\ z \arrow[r, "p"'] \arrow[ur, "f'"] & y \end{tikzcd}$$

and so the diagram

$$\begin{tikzcd} x \arrow[r, "q"] \arrow[d, "f"'] & z \arrow[r, "f'"] \arrow[d, "p"'] & x \arrow[d, "f"'] \\ y \arrow[r, "\mathrm{id}_y"'] & y \arrow[r, "\mathrm{id}_y"'] & y \end{tikzcd}$$

commutes, and so by the third model category axiom f must be a fibration. The proof of 2 is the same, but take p to be trivial instead of q. □

We'll need this lemma later, and it provides another good example of working in model categories:

Lemma 36. *If C is a model category and $f : x \to y$ is a trivial fibration in C, then any pullback along f is also a trivial fibration.*

Proof. Take s to be any map from x' to y in C. Construct the pullback

$$\begin{tikzcd} p \arrow[r, "s'"] \arrow[d, "f'"'] & x \arrow[d, "f"] \\ x' \arrow[r, "s"'] & y \end{tikzcd}$$

We need to show that f' is a trivial fibration. By Theorem 35, it's enough to show that f' has the RLP with respect to any cofibration $c : a \to b$ in C. Since f is a trivial fibration it has the RLP with respect to c, i.e. there is a map α making

$$\begin{tikzcd} a \arrow[r] \arrow[d, "c"'] & x \arrow[d, "f"] \\ b \arrow[r] \arrow[ur, "\alpha"'] & y \end{tikzcd}$$

commute, whenever the square commutes. Thus for any commutative rectangle

$$\begin{tikzcd} a \arrow[r] \arrow[d, "c"'] & p \arrow[r] \arrow[d, "f'"] & x \arrow[d, "f"] \\ b \arrow[r] & x' \arrow[r] & y \end{tikzcd}$$

we can use the UMP of the pullback with the maps $b \to x'$ and $\alpha : b \to x$ to conclude that there is a unique map $b \to p$ making the diagram commute — this is a right lift from c to f', and as c was a arbitrary cofibration, f' is a trivial fibration. $\square$

Remark 37. Just as in Theorem 35 there are really four versions of this lemma — in addition to the above any pullback along a fibration is a fibration, and any pushout along a (trivial) cofibration is a (trivial) cofibration. The proof for the first statement is very similar, and the two other statements follow by duality.

6.2. **Homotopy on model categories.** We will define homotopy between arrows, and then use this to define the homotopy equivalence classes. In the next subsection this will be used to define the homotopy category of a model category.

Definition 38. If C is a model category and $x \in \mathrm{ob}C$, then

(1) A *path object* p is a factorization
$$\Delta_x : x \xrightarrow{f} p \xrightarrow{(p_1,p_2)} x \times x$$
of the diagonal $\Delta_x : x \xrightarrow{(\mathrm{id},\mathrm{id})} x \times x$, where f is a weak equivalence.
(2) A *cylinder object* c is a factorization
$$\nabla_x : x \sqcup x \xrightarrow{c_1+c_2} c \xrightarrow{f} x$$
of the codiagonal $\nabla_x : x \sqcup x \xrightarrow{\mathrm{id}+\mathrm{id}} x$, where f is a weak equivalence.

Note that by the last axiom of model categories every object has at least one path object and one cylinder object.

Definition 39. If C is a model category, $x, y \in \mathrm{ob}C$ and $f, g : x \to y$, then

(1) A *right homotopy,* $\eta : f \Rightarrow_{\mathrm{R}} g$, is an arrow $\eta : x \to p$ to some path object p of y, such that the following diagram commutes:
$$\begin{tikzcd} & x \arrow[dl, "f"'] \arrow[d, "\eta"] \arrow[dr, "g"] & \\ y & p \arrow[l, "p_1"] \arrow[r, "p_2"'] & y \end{tikzcd}$$
(2) A *left homotopy,* $\eta : f \Rightarrow_{\mathrm{L}} g$, is an arrow $\eta : c \to y$ from some cylinder object c of x, such that the following diagram commutes:
$$\begin{tikzcd} x \arrow[r, "c_1"] \arrow[dr, "f"'] & c \arrow[d, "\eta"] & x \arrow[l, "c_2"'] \arrow[dl, "g"] \\ & y & \end{tikzcd}$$

We will from now on only discuss the right versions of the statements — the left sided versions follow by duality.

Lemma 40. *If y is fibrant and $y \xrightarrow{\alpha} p \xrightarrow{(p_1,p_2)} y \times y$ is a path object for y, where the map (p_1, p_2) is a fibration, then the maps p_1 and p_2 are trivial fibrations.*

Proof. Note that $\mathrm{id}_y = p_1 \circ \alpha$, and α is a weak equivalence by definition of path object, and by Lemma 32 so is id_y, and so by the second axiom p_1 is also a weak equivalence, and similarly p_2 is too. Recall that the product $y \times y$ is the pullback of $y \to * \leftarrow y$ (which are fibrations, as y is fibrant), and so by Remark 37 the projections $\pi_1, \pi_2 : y \times y \to y$ are fibrations. But $p_1 = \pi_1 \circ (p_1, p_2)$ and so p_1 is a fibration, and similarly for p_2. □

The following is a very useful lemma that tells us that if there is a homotopy then there is a "nice" homotopy too.

Lemma 41. *Let $f, g : x \to y$. If there is a right homotopy $\eta : f \Rightarrow_{\mathrm{R}} g$, then there is a right homotopy $\theta : f \Rightarrow_{\mathrm{R}} g$ with path object $y \to q \xrightarrow{(q_1,q_2)} y \times y$ such that the map (q_1, q_2) is a fibration.*

Proof. Let the path object associated to η be $y \xrightarrow{\pi} p \xrightarrow{(p_1,p_2)} y \times y$. Use the fifth axiom to factor (p_1, p_2) as $p \xrightarrow{c} q' \xrightarrow{d} y \times y$, where c is a weak equivalence and cofibration, and d is a fibration. As c and π both are weak equivalences, so is $c \circ \pi$, thus $y \xrightarrow{c\circ\pi} q' \xrightarrow{d} y \times y$ is a path object with the required properties. □

Corollary 42. *If f and g are parallel arrows in a model category and there is a right homotopy $\eta : f \Rightarrow_{\mathrm{R}} g$, then there is a right homotopy $\theta : f \Rightarrow_{\mathrm{R}} g$ with path object $y \to q \xrightarrow{(q_1,q_2)} y \times y$ where the maps q_1 and q_2 are trivial fibrations.*

If there is *any* right homotopy from f to g, then we say that f and g are *homotopic*. Being homotopic defines a relation R on $\mathrm{Hom}_C(x, y)$ for any objects x and y in C, by fRg if there exists a right homotopy $f \Rightarrow_{\mathrm{R}} g$.

Theorem 43. *If y is fibrant, then R is an equivalence relation on* $\mathrm{Hom}_C(x, y)$.

Proof. Let $f, g, h : x \to y$ in some model category C.

For symmetry, if η is a right homotopy from f to g, where the associated path object is $y \to p \xrightarrow{(p_1,p_2)} y \times y$, then taking instead the path object to be $y \to p \xrightarrow{(p_2,p_1)} y \times y$ makes η into a right homotopy from g to f.

For reflexivity, we need to show that there is $\eta : f \Rightarrow_{\mathrm{R}} f$. Take the path $y \xrightarrow{\mathrm{id}_y} y \xrightarrow{(\mathrm{id}_y,\mathrm{id}_y)} y \times y$, and take $\eta = f$.

For transitivity, let $\eta : f \Rightarrow_{\mathrm{R}} g$ and $\theta : g \Rightarrow_{\mathrm{R}} h$ be two homotopies and let the associated path objects be $y \xrightarrow{\alpha} p \xrightarrow{(p_1,p_2)} y \times y$ respectively $y \xrightarrow{\beta} q \xrightarrow{(q_1,q_2)} y \times y$. Using Corollary 42 we can assume that p_1, p_2, q_1 and q_2 are trivial fibrations. Let

$$\begin{tikzcd} r \arrow[r, "p_2'"] \arrow[d, "q_1'"'] & q \arrow[d, "q_1"] \\ p \arrow[r, "p_2"'] & y \end{tikzcd}$$

be a pullback square. Then as q_1 and p_2 are trivial fibrations Lemma 36 tells us that both q_1' and p_2' are trivial fibrations. Since r is a pullback, and $p_2 \circ \alpha = \mathrm{id}_y = q_1 \circ \beta$ (as α and β come from the path objects) there is a unique map $\gamma : y \to r$ such that the diagram

$$\begin{tikzcd}
y \arrow[dr, "\gamma"] \arrow[drr, bend left, "\beta"] \arrow[ddr, bend right, "\alpha"'] & & \\
& r \arrow[r, "p_2'"] \arrow[d, "q_1'"'] & q \arrow[d, "q_1"] \\
& p \arrow[r, "p_2"'] & y
\end{tikzcd}$$

commutes. Now, α is a weak equivalence as p is a path object and q_1' is also a weak equivalence, so by the second axiom γ is also a weak equivalence (as one part of the diagram is $q_1' \circ \gamma = \alpha$). Now, $q_1' \circ \gamma = \alpha$ and $p_2' \circ \gamma = \beta$, so $p_1 \circ q_1' \circ \gamma = \mathrm{id}_y = q_2 \circ p_2' \circ \gamma$, and so

$$y \xrightarrow{\gamma} r \xrightarrow{(p_1 \circ q_1', q_2 \circ p_2')} y \times y$$

is a path object for y. Now, the homotopy diagrams gives that $q_1 \circ \theta = g = p_2 \circ \eta$, so again use the UMP of pullbacks to conclude that there is a unique map $\zeta : x \to r$ such that the diagram

$$\begin{tikzcd}
x \arrow[dr, "\zeta"] \arrow[drr, bend left, "\theta"] \arrow[ddr, bend right, "\eta"'] & & \\
& r \arrow[r, "p_2'"] \arrow[d, "q_1'"'] & q \arrow[d, "q_1"] \\
& p \arrow[r, "p_2"'] & y
\end{tikzcd}$$

commutes. Note that $q_1' \circ \zeta = \eta$, so $p_1 \circ q_1' \circ \zeta = p_1 \circ \eta = f$, and similarly $q_2 \circ p_2' \circ \zeta = h$, so ζ is a homotopy from f to h, with the above path object. Thus R is transitive. $\square$

Lemma 44. *If x is cofibrant and y is fibrant, then the relations R and L on* $\mathrm{Hom}\,(x, y)$ *coincide.*

Proof. We show that if y is fibrant and $f \Rightarrow_{\mathrm{R}} g$ (for $f, g : x \to y$) then $f \Rightarrow_{\mathrm{L}} g$. Then duality gives us the full lemma.

By Corollary 42 there is a homotopy $\eta : f \Rightarrow_{\mathrm{R}} g$ with path object $y \xrightarrow{\alpha} p \xrightarrow{(p_1,p_2)} y \times y$ where p_1 and p_2 are trivial fibrations. Use the fifth axiom to factor the codiagonal $\nabla : x \sqcup x \to x$ as $x \sqcup x \xrightarrow{c_1+c_2} c \xrightarrow{\beta} x$ with $c_1 + c_2$ a cofibration and β a trivial fibration — this makes c into a cylinder object for x. The diagram

$$\begin{tikzcd}
x \sqcup x \arrow[r, "\alpha \circ f + \eta"] \arrow[d, "c_1 + c_2"'] & p \arrow[d, "p_1"] \\
c \arrow[r, "f \circ \beta"'] & y
\end{tikzcd}$$

commutes as $f \circ \beta \circ (c_1 + c_2) = f \circ (\mathrm{id} + \mathrm{id}) = f + f$ and $p_1 \circ (\alpha \circ f + \eta) = (p_1 \circ \alpha \circ f + p_1 \circ \eta) = f + f$. The fourth axiom applies and gives us a map $l : c \to p$ making the diagram commute. Let $\theta = p_2 \circ l$. Then $\theta \circ c_1 = p_2 \circ l \circ c_1 = p_2 \circ \alpha \circ f = \mathrm{id} \circ f = f$ while $\theta \circ c_2 = p_2 \circ l \circ c_2 = p_2 \circ \eta = g$, so θ is a homotopy $f \Rightarrow_{\mathrm{L}} g$. $\square$

When Lemma 44 holds we call the relation R simply "homotopy", and we write $f \simeq g$ for fRg.

6.3. **The homotopy category of a model category.** Given a object x of a model category C, we can use the fifth axiom to factor the map $\emptyset \to x$ from the initial object to x as $\emptyset \to x^* \xrightarrow{p_x} x$ where p_x is a fibration and weak equivalence. The map $x \to 1$ can be factored as $x \xrightarrow{q_x} x_* \to 1$ where q_x is a cofibration and weak equivalence.

Note that x^* is cofibrant and x_* is fibrant. If x is itself cofibrant we agree to pick $x^* = x$ and if x is fibrant we pick $x_* = x$. We think of x^* and x_* as respectively cofibrant and fibrant "replacements" for x.

Lemma 45. *If $f : x \to y$ in a model category C, then there is a map $f^* : x^* \to y^*$ such that the following diagram commutes:*

$$\begin{tikzcd} x^* \arrow[r, "f^*"] \arrow[d, "p_x"'] & y^* \arrow[d, "p_y"] \\ x \arrow[r, "f"'] & y \end{tikzcd}$$

Further, f^* is a weak equivalence if and only if f is.

Proof. The diagram

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & y^* \arrow[d, "p_y"] \\ x^* \arrow[r, "f \circ p_x"'] \arrow[ur, dashed, "f^*"] & y \end{tikzcd}$$

commutes since $\emptyset$ is the initial object of C. Since p_y was a weak equivalence and fibration and x^* is cofibrant we can use the fourth axiom to get a map $f^* : x^* \to y^*$ making the diagram commute. Clearly $p_y \circ f^* = f \circ p_x$.

From the second axiom, the fact that p_x and p_y are weak equivalences and $f \circ p_x = p_y \circ f^*$ we see that the weak equivalence of either of f and f^* implies that the other one also is a weak equivalence. □

There is of course a dual to the above lemma, which gives us a map f_*.

Definition 46. If C is a model category, let C_c/R denote the category which has as object the cofibrant objects of C and where the arrows are the right homotopy classes of arrows in C. Dually, let C_f/L consist of the fibrant objects of C and left homotopy classes of arrows of C.

Definition 47. There is a functor $P : C \to C_c/R$ that takes an object x to x^* and a map $f : x \to y$ to $[f^*]$, a right homotopy class of arrows from x^* to y^*. Dually, there is a functor $Q : C \to C_f/L$.

For this to be a functor we need $[f \circ g] = [f' \circ g']$ for any representatives f' and g' of $[f]$ and $[g]$, respectively. Also note that if $f = \mathrm{id}_x$ in Lemma 45 we might take $f^* = \mathrm{id}_{x^*}$, so we need every choice of f^* to be homotopic to id_{x^*} (so that the functor preserves identities). Both of these statements follow from

Lemma 48. *If x is fibrant and $c : p \to q$ is a trivial cofibration then precomposition with c gives a bijection from $c' : \mathrm{Hom}\,(q, x)\,/R \to \mathrm{Hom}\,(p, x)\,/R$.*

Proof. First note that if $f, g : q \to x$ and $\eta : f \Rightarrow_{\mathrm{R}} g$ then $\eta \circ c : f \circ c \Rightarrow_{\mathrm{R}} g \circ c$ so that precomposition with c actually preserves homotopy classes and thus c' is a function.

Take any class $[f]$ in $\mathrm{Hom}(p, x)/R$. The diagram

$$\begin{tikzcd} p \arrow[r, "f"] \arrow[d, "c"'] & x \arrow[d] \\ q \arrow[r] \arrow[ur, dashed, "l"'] & 1 \end{tikzcd}$$

commutes since 1 is the terminal object. By assumption the fourth axiom applies and gives a map $l : q \to x$ with the property that $l \circ c = f$, so c' is surjective.

Now take $f, g : q \to x$ such that $c'([f]) = c'([g])$. This means that there is a right homotopy $\eta : f \circ c \Rightarrow_{\mathrm{R}} g \circ c$, and by Lemma 41 there is a right homotopy θ from $f \circ c$ to $g \circ c$ with path object $x \to t \xrightarrow{(t_1, t_2)} x \times x$ where the map (t_1, t_2) is a fibration. The diagram

$$\begin{tikzcd} p \arrow[r, "\theta"] \arrow[d, "c"'] & t \arrow[d, "{(t_1,t_2)}"] \\ q \arrow[r, "{(f,g)}"'] \arrow[ur, dashed, "l"'] & x \times x \end{tikzcd}$$

commutes, and the fourth axiom applies to give us a map $l : q \to t$. Note that l is a right homotopy from f to g, so $[f] = [g]$, and thus c' is injective. $\square$

With this lemma we can conclude that $-^*$ preserves left homotopy (by the dual of Lemma 48, $-\circ p_x$ and $-\circ p_y$ in the proof of Lemma 45 preserves homotopy). The proof of the dual of Lemma 44 then shows that, since x^* is cofibrant, left homotopy implies right homotopy. Thus P is a well-defined functor.

Definition 49. Let $C_{cf}/\simeq$ be the category which has as objects the objects of C that are both fibrant and cofibrant and where the arrows are homotopy classes of arrows in C.

Lemma 50. *The restriction of the functor P to C_f gives a functor $P' : C_f/L \to C_{cf}/\simeq$. The restriction of the functor Q to C_c gives a functor $Q' : C_c/R \to C_{cf}/\simeq$.*

Proof. We only need to show that if x and y are fibrant objects of C and if $f, g : x \to y$ such that $[f]_L = [g]_L$ (note that since L is not necessarily a equivalence relation, since y might not be cofibrant, these are the classes *generated* by L) then $P(f) = P(g)$, then the rest follows from duality. It is enough to show that this holds if there actually is a left homotopy from f to g (since the rest will follow), but this follows from our discussion following Lemma 48. $\square$

Definition 51. Given a model category C, the *homotopy category* $\mathrm{ho}C$ has the same objects as C and where $\mathrm{Hom}_{\mathrm{ho}C}(x, y) = \mathrm{Hom}_{C_{cf}/\simeq}(Q'(x^*), Q'(y^*))$.

There is a functor $\pi : C \to \mathrm{ho}C$ that takes objects to themselves and weak equivalences to isomorphisms.

We could imagine $\mathrm{ho}C$ to be $C_{cf}/\simeq$ — this is certainly a category; we have successfully avoided the size-problem of simply inverting the weak equivalences. This is not equal to $\mathrm{ho}C$, but $C_{cf}/\simeq$ is equivalent to $\mathrm{ho}C$.

The following theorem is another characterization of $\mathrm{ho}C$:

Theorem 52. *Let B be a category, C a model category and $F : C \to B$ be a functor that takes weak equivalences to isomorphisms. Then there is a* unique *functor* $\mathrm{ho}F : \mathrm{ho}C \to B$ *such that* $(\mathrm{ho}F) \circ \pi = F$.

For the proof, see [Hov, Lemma 1.2.2].

7. Simplicial Categories

A *simplicial category* is, to put it briefly, a category enriched over **sSet**. This essentially means that we replace the *sets* $\mathrm{Hom}(x, y)$ with *simplicial sets*, i.e. we allow for them to have more structure than just sets do. We provide an unrelated example to demonstrate enrichment:

Example 53. Let C be the category of vector spaces over some field k, with k-linear transformations as arrows. Given two vector spaces U and V, take two transformations $u, v \in \mathrm{Hom}_C(U, V)$ — these provide a new arrow $u + v \in \mathrm{Hom}_C(U, V)$, by defining $(u + v)(x) = u(x) +_k v(x)$. Also, there is the constant map $w(x) = 0$, which is in every homset. This is one example where the category has homsets with a natural extra structure — in this case the structure of an abelian group. Thus our category C could be considered a category enriched over **Ab**.

For the formal definition, we need to do some work first.

7.1. Monoidal structure of sSet.

We will in this section verify that **sSet** has a so-called *monoidal structure*, although for brevity we won't specify precisely what that means (think of it as a category with products). First we need a product of simplicial sets — the product $S \times T$ is simply taken pointwise, so that $(S \times T)_n = S_n \times T_n$.

Remark 54. Simplicial sets are so called *presheaves*, and so **sSet** is a category of presheaves — by a general result such categories are both complete and cocomplete (by taking the limits pointwise), and in particular have products.

7.1.1. *Properties of the product.* Let $*$ denote the simplicial set with $*_n = \{0\}$ for each $n \in \mathbb{N}$. The face and degeneracy maps are then determined. Note that for any simplicial set S we trivially have isomorphisms $S \times * \cong S \cong * \times S$, so that $*$ acts as a kind of unit for the product in **sSet** (another way to see this is to note that $*$ is a terminal object in **sSet** — note that any terminal object would do). Denote the first isomorphism idR_S and the second idL_S (they are simply the projection on the first and second element, respectively).

Also, $\times$ is associative (in a non-strict sense) as there is an isomorphism $(A \times B) \times C \cong A \times (B \times C)$ for simplicial sets A, B and C. Denote this isomorphism by $\mathrm{as}_{A,B,C}$.

Finally, we have for any simplicial sets A, B, C and D two commutative diagrams

$$\begin{tikzcd}
& (A \times (B \times C)) \times D \arrow[dr, "\mathrm{as}_{A,B\times C,D}"] & \\
((A \times B) \times C) \times D \arrow[ur, "\mathrm{as}_{A,B,C}\times \mathrm{id}_D"] \arrow[d, "\mathrm{as}_{A\times B,C,D}"'] & & A \times ((B \times C) \times D) \arrow[d, "\mathrm{id}_A \times \mathrm{as}_{B,C,D}"] \\
(A \times B) \times (C \times D) \arrow[rr, "\mathrm{as}_{A,B,C\times D}"'] & & A \times (B \times (C \times D))
\end{tikzcd}$$

$$\begin{tikzcd}
(A \times I) \times B \arrow[rr, "\mathrm{as}_{A,I,B}"] \arrow[dr, "\mathrm{idR_A} \times \mathrm{id_B}"'] & & A \times (I \times B) \arrow[dl, "\mathrm{id}_A \times \mathrm{idL}_B"] \\
& A \times B &
\end{tikzcd}$$

These two diagrams essentially tells us that $\times$ really behaves as an associative operation with a unit as we'd expect.

All this shows that $(\mathbf{sSet}, \times, *)$ really is a monoidal category, so that we may speak of "categories enriched over **sSet**".

7.2. **Simplicial enrichment.** For clarity, the identity morphisms in **sSet** will be denoted 1 in this section, rather than id.

Definition 55. A *simplicial category* C (or *category enriched over* **sSet**) consists of

- A class ob$C$, called the *objects* of C
- For every pair of objects x and y in C as simplicial set $C(x,y)$ (also denoted $\mathrm{Hom}_C(x,y)$)
- For each object x of C an arrow $\mathrm{id}_x : * \to C(x,x)$ in **sSet**, called the *identity of* x.
- For any three objects x, y and z of C, an arrow $\circ_{x,y,z} : \mathrm{Hom}_C(y,z) \times \mathrm{Hom}_C(x,y) \to \mathrm{Hom}_C(x,z)$ in **sSet**, called *composition,* such that for any objects x, y, z and w the following three diagrams commute:
  (1) Left unity

$$\begin{tikzcd}
* \times C(x,y) \arrow[rr, "\mathrm{id}_y \times 1_{C(x,y)}"] \arrow[dr, "\mathrm{idL}_{C(x,y)}"'] & & C(y,y) \times C(x,y) \arrow[dl, "\circ_{x,y,y}"] \\
& C(x,y) &
\end{tikzcd}$$

  (2) Right unity

$$\begin{tikzcd}
C(x,y) \times * \arrow[rr, "1_{C(x,y)} \times \mathrm{id}_x"] \arrow[dr, "\mathrm{idR}_{C(x,y)}"'] & & C(x,y) \times C(x,x) \arrow[dl, "\circ_{x,x,y}"] \\
& C(x,y) &
\end{tikzcd}$$

  (3) Associativity

$$\begin{tikzcd}
(C(z,w) \times C(y,z)) \times C(x,y) \arrow[d, "\mathrm{as}_{C(z,w),C(y,z),C(x,y)}"'] \arrow[dr, "\circ_{y,z,w} \times 1_{C(x,y)}"] & \\
C(z,w) \times (C(y,z) \times C(x,y)) \arrow[d, "1_{C(z,w)} \times \circ_{x,y,z}"'] & C(y,w) \times C(x,y) \arrow[d, "\circ_{x,y,w}"] \\
C(z,w) \times C(x,z) \arrow[r, "\circ_{x,z,w}"'] & C(x,w)
\end{tikzcd}$$

7.3. **About simplicial categories.** Simplicial categories have a higher-dimensional structure — we can think of the 0-simplices of $C(x,y)$ as morphisms, the 1-simplices as morphisms between morphisms etc. Recall from Section 4 that $(\infty,1)$-categories should have all morphisms of dimension 2 and higher invertible, which means that the simplicial sets $C(x,y)$ should have invertible morphisms from dimension 1 and up. We know from Section 3 that Kan complexes are just those simplicial set. Thus we'd expect simplicial categories where the hom-simplical sets are Kan complexes to model $(\infty,1)$-categories. However, it turns out that that restriction is not necessary, and so simplicial categories (with no conditions on $\mathrm{Hom}(x,y)$) are model of $(\infty,1)$-categories. (See [Lurie] for details.)

The category of simplicial categories is denoted **sCat**, and can be given a nontrivial model structure, which we will need for Section 9, but this structure is to complex too describe here.

7.4. **More structure on sSet.** Here we are going to use simplicial categories and model categories to show some extra structure on **sSet**, to illustrate how "nice" that category really is. This is a side-track an can be skipped, and to keep it brief we will omit proofs.

The category **sSet** can be seen as a simplicial category.

Above we showed that **sSet** has a monoidal structure — this is enough to guarantee that there is an *internal* hom. In **Set**, given two objects U and V, the set of functions from U to V, i.e. $\mathrm{Hom}_{\mathbf{Set}}(U,V)$, is itself a set, and thus an object of **Set**; thus in **Set** the *internal* hom is the same as the *external* one — in general this might not be true: there might not even be any sensible object to take as the "internal hom". For **sSet** there is a way to do this however.

Definition 56. Given two objects x and y in **sSet**, the *internal hom* $\mathrm{hom}_{\mathbf{sSet}}(x,y)$ (denoted with a lower case h) is the simplicial set given by $\mathrm{hom}_{\mathbf{sSet}}(x,y)_n = \mathrm{Hom}_{\mathbf{sSet}}(x\times\Delta[n],y)$ where $\Delta[n]$ is the standard n-simplex defined in Example 3.

By replacing the ordinary hom-sets of **sSet** with the internal hom-set (which are in fact simplicial sets) we get a simplicial category of simplicial sets. For the remainder of this section we will use **sSet** to mean this simplicial category rather than the ordinary one.

In any simplicial category, and thus in **sSet**, it is easy to define homotopy. Homotopy is "supposed" to mean that two maps can (in a sufficiently nice way) be deformed to each other. The higher dimensional structure of a simplicial category gives this without any trouble. For simplicity we restrict ourselves to simplicial categories where the hom-sets are Kan complexes (since we expect homotopies to be invertible).

Definition 57. Given two objects x and y of a simplicial category C, two maps $f,g\in\mathrm{Hom}_C(x,y)_0$ are said to be *homotopic*, $f\simeq g$, if there is some $\sigma\in\mathrm{Hom}_C(x,y)_1$ with $d_1\sigma=f$ and $d_0\sigma=g$.

Now note that we this way have *two* different notions of homotopy in **sSet** — the one we just described, and the that arises from some model structure on **sSet**. The nice thing here is that with the Quillen model structure on **sSet** these two homotopies coincide.

8. Nerves and Realizations

We have encountered two pairs of adjoints that both have, as it turns out, a lot in common. These are the nerve and fundamental category adjoints, $\tau_1 \dashv N$ and the singular set and geometric realization pair, $|-| \dashv \mathrm{Sing}$. Both of these fit into a general pattern of "tensors and hom" or "nerve and realization". To explain this we first need a "structure theorem" for simplicial sets.

8.1. Simplicial sets as colimits.

Definition 58. Given a simplicial set S, there is a *simplex category* $\Delta \downarrow S$ which has as elements maps $\sigma : \Delta[n] \to S$, $n \in \mathbb{N}$, (where $\Delta[n]$ are the standard n-simplices from Example 3) in **sSet**. An arrow $f : \sigma \to \tau$ in $\Delta \downarrow S$ is a map making the following diagram commute

$$\begin{tikzcd} \Delta[n] \arrow[rr, "f"] \arrow[dr, "\sigma"'] & & \Delta[m] \arrow[dl, "\tau"] \\ & S & \end{tikzcd}$$

The identity arrows are the identities on the $\Delta[n]$:s and composition in $\Delta \downarrow S$ is the same as in **sSet**.

There is a functor $F_S : \Delta \downarrow S \to \mathbf{sSet}$ that takes an object $\sigma : \Delta[n] \to S$ to $\Delta[n]$ and maps $f : \sigma \to \tau$ to themselves.

Theorem 59. *For a simplicial set S, we have* $\mathrm{colim} F_S = S$.

Proof. There is a trivial cocone (S, λ_σ) where the maps $\lambda_\sigma = \sigma$. Let (T, η_σ) be another cocone of F_S.

Note that $\Delta[n]$ has precisely one non-degenerate simplex, which we will denote $[n]$. (Non-degenerate means that it is not the image of any degeneracy map.)

There is at most one map $(S, \lambda_\sigma) \to (T, \eta_\sigma)$. For the sake of contradiction, assume that there are two different such maps, say f and g. Then there is some simplex x, say of dimension n, in S for which $f(x) \neq g(x)$. There is precisely one map $\phi : \Delta[n] \to S$ such that ϕ takes $[n]$ to x. But then $f \circ \phi \neq g \circ \phi$, contradicting the assumption that f and g were maps of cocones.

There is at least one map $f : (S, \lambda_\sigma) \to (T, \eta_\sigma)$. For a simplex $x \in S_n$, let $\sigma_x : \Delta[n] \to S$ be the map that takes $[n]$ to x and let $f(x) = \eta_{\sigma_x}([n])$. It's quick to check that this is indeed a map of simplicial sets, and the construction makes it trivial to realize that this makes it into a map of cocones

In conclusion, (S, λ_σ) is an initial cocone, and thus the colimit of F_S, as claimed. □

8.2. Creating nerves. Given a category C, a functor $\Delta^{\mathrm{op}} \to C$ is known as a *simplicial object*, and we often substitute "object" for the name of the category, e.g. a functor $\Delta^{\mathrm{op}} \to \mathbf{Grp}$ would be a *simplicial group*, while for **Set** we get simplicial sets. (There is a naming clash here, as a functor $\Delta^{\mathrm{op}} \to \mathbf{Cat}$ and a **sSet**-enriched category could both be called *simplicial category*, but we will only use the to mean a **sSet**-enriched category.) The "standard simplices" we have encountered in various categories form such simplicial objects; in for example **Top** there is a simplicial object $|\Delta_\bullet|$ given by $|\Delta_\bullet|(n) = |\Delta^n|$. A *cosimplicial object* is then similarly a functor $\Delta \to C$.

Given a cocomplete category C and a cosimplicial object $s^{\bullet}$ of C there is a functor $N : C \to \mathbf{sSet}$, given by $N(x)_n = \mathrm{Hom}_C(s^{\bullet}(n), x)$. For a map $f : x \to y$ in C, take $N(f)$ to be the map that takes a simplex $x' \in N(x)_n$ to $f \circ x'$.

If we have $C = \mathbf{Top}$ and with the correct choice of $s^{\bullet}$ (i.e. $s^{\bullet}(n) = |\Delta^n|$), the functor N is the singular set functor from Section 5, and that we might similarly retrieve the nerve functor from Section 2. Thus this construction can be seen as a generalized nerve functor. This is a right adjoint.

8.3. **Realizing simplicial sets.** If C is a cocomplete category and $s_{\bullet}$ a simplicial object of C we may use the $s_{\bullet}(n)$:s as "standard simplices", and use these to glue together a an object of C following the specification of a simplicial set. Let $\mathbf{sSet}\Delta$ be the full subcategory of $\mathbf{sSet}$ with the $\Delta[n]$ as objects. Then there is a functor $\phi : \mathbf{sSet}\Delta \to C$ given by $\Delta[n] \mapsto s_{\bullet}(n)$, and using Theorem 59 we can extend this to a functor from the entire category $\mathbf{sSet}$ to C, by taking

$$|S| = \mathrm{colim}\phi \circ F_S$$

since the image of F_S is in $\mathbf{sSet}\Delta$. This functor is a generalization of the geometric realization from Section 5. It is also a left adjoint, and with the right adjoin being N from above. (If the simplicial and cosimplicial objects $s_{\bullet}$ and $s^{\bullet}$ agree on objects.)

Theorem 60. *Given a category C and the functors N and $|-|$ as above, then N is right adjoint to $|-|$.*

Proof. We show a natural bijection between $\mathrm{Hom}_C(|X|, Y)$ and $\mathrm{Hom}_{\mathbf{sSet}}(X, NY)$ for $X \in \mathbf{sSet}$ and $Y \in C$.

We have $\mathrm{Hom}_C(|X|, Y) = \mathrm{Hom}(\mathrm{colim}\phi \circ F_X, Y)$ by definition of $|-|$, and from the definition of colimit we see that this is naturally isomorphic to

$$\operatorname*{colim}_{x \in \Delta \downarrow X} \mathrm{Hom}_C(\phi \circ F_X(x), Y)$$

Note that for $x \in \Delta \downarrow X$ we have $\phi \circ F_X(x) = s_{\bullet}(n)$ for some n, so we have $\mathrm{Hom}_C(\phi \circ F_X(x), Y) = N(Y)_n$, and as the limit is taken over maps $\Delta[n] \to X$ we get a natural isomorphism with $\mathrm{Hom}_{\mathbf{sSet}}(X, NY)$, which is what we wanted. □

8.4. **The homotopy coherent nerve.** We're going to need more standard n-simplices, this time in $\mathbf{sCat}$. To define the simplicial sets that will be our Hom it will be helpful to have to following notation:

Given two natural numbers i and j, with $i \leq j$, let a (i,j)-set be a subset S of $\mathbb{N}$ with the property that i and j both are in S, and if $s \in S$ then $i \leq s \leq j$. Let $P_{i,j}$ denote the set of (i,j)-sets. Then $P_{i,j}$ is partially ordered by inclusion. Let $CP_{i,j}$ denote the ordinary categorification of a poset, i.e. the objects of $CP_{i,j}$ are the elements of $P_{i,j}$ and there is an arrow $p \to p'$ if and only if $p \subseteq p'$.

Definition 61. The *standard n-simplex* $[\![\Delta^n]\!]$ in $\mathbf{sCat}$ is the simplicial category with $\mathrm{ob}\,[\![\Delta^n]\!] = \{0, 1, 2, \cdots, n\}$ and where $\mathrm{Hom}_{[\![\Delta^n]\!]}(i,j) = \emptyset$ if $i > j$, and otherwise $\mathrm{Hom}_{[\![\Delta^n]\!]}(i,j) = N(CP_{i,j})$, the nerve of $CP_{i,j}$.

This definition is a bit difficult to digest. But note that, as we'd expect, there's no arrows from larger objects to smaller, and $P_{i,i}$ has precisely one object, so $\mathrm{Hom}(i,i)$ have only identity arrows in every dimension. If we instead consider $\mathrm{Hom}(i, i+2)$, then there are two objects of $CP_{i,i+2}$, corresponding to $\{i, i+2\}$ and $\{i, i+1, i+2\}$, and in the category $[3]$ we have the diagram

$$\begin{tikzcd} \bullet \arrow[r] \arrow[rr, bend right] & \bullet \arrow[r] & \bullet \end{tikzcd}$$

we might think of $\{i, i+2\}$as the lower arrow, while $\{i, i+1, i+2\}$ would be the composition of the two upper arrows. Then the 1-dimensional simplex generated by $\{i, i+2\} \subseteq \{i, i+1, i+2\}$ in $\mathrm{Hom}\,(i, i+2)$ gives a homotopy showing that the two different maps $i \to i+2$ are indeed "the same".

These standard n-simplices gives us both a simplicial object and cosimplicial object in **sCat**, by taking $\Delta^{\bullet} = [\![\Delta^n]\!]$ (and the correct maps), and so by the machinery developed above we get a nerve-functor and a realization functor. Concretely, we have

Definition 62. Given a simplicial category C, the *homotopy coherent nerve*, NC (intentionally the same notation as for the ordinary nerve), is a simplicial set given by

$$NC_n = \mathrm{Hom}_{\mathbf{sSet}}\,(\Delta^n, NC) = \mathrm{Hom}_{\mathbf{sCat}}\,([\![\Delta^n]\!], C)$$

where Δ^n denotes the standard n-dimension simplex in the corresponding category.

Note: it is safe to confuse the ordinary nerve $N : \mathbf{Cat} \to \mathbf{sSet}$ and the homotopy coherent nerve $N : \mathbf{sCat} \to \mathbf{sSet}.$ If we have a category C we might turn it into a simplicial category C' by taking $\mathrm{Hom}_{C'}\,(x, y)$ to be the discrete simplicial set (from Example 4) on $\mathrm{Hom}_C\,(x, y)$. If we do this then $NC \cong NC'$.

9. Homotopies of homotopies

9.1. **Localization of model categories.** We will here present the so called "hammock localization" of a model category, which assigns to every model category a simplicial category.

In what follows, C is a model category and W is the class of weak equivalences in C.

Definition 63. Given C and W as above, a *hammock* of length n and height k between two objects x and y of C is a commutative diagram with the shape

$$\begin{tikzcd}
x \arrow[r, dash] & c_{1,1} \arrow[r, dash] \arrow[d] & c_{1,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & c_{1,n} \arrow[r, dash] \arrow[d] & y \\
x \arrow[r, dash] & c_{2,1} \arrow[r, dash] \arrow[d] & c_{2,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & c_{2,n} \arrow[r, dash] \arrow[d] & y \\
x \arrow[r, dash] & c_{3,1} \arrow[r, dash] \arrow[d] & c_{3,2} \arrow[r, dash] \arrow[d] & \cdots \arrow[r, dash] & c_{3,n} \arrow[r, dash] \arrow[d] & y \\
 & \vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \\
x \arrow[r, dash] & c_{k,1} \arrow[r, dash] & c_{k,2} \arrow[r, dash] & \cdots \arrow[r, dash] & c_{k,n} \arrow[r, dash] & y
\end{tikzcd}$$

where the horizontal dashes represents arrows that may point in either direction, that fulfills the following properties:

(1) All vertical maps are in W
(2) All arrows in a column go in the same direction
(3) All arrows pointing to the left are in W

There are some simplifications possible — for example a column with only identity arrows contains no information, so we may omit them. Additionally, if the arrows in two columns next to each other point in the same direction we may replace them with their composition. Applying these two methods we get *reduced hammocks*, which then have the additional rules

(4) No column contains only identity morphisms
(5) The arrows in two adjacent columns point in different directions

Let $H_C^k(x,y)$ denote the set of all reduced hammocks of any (finite) length and height k between x and y. These have a simplicial structure, in the sense that there is a simplicial set $H(x,y)$ with $H(x,y)_k = H_{C,W}^k(x,y)$. The face maps d_i remove the i:th row (replacing the vertical arrows with the composition) and the degeneracies s_i repeat it (with the new vertical arrows being the identity arrows).

Definition 64. For a model category C, the *hammock localization* of C is a simplicial category $\mathrm{HL}(C)$ with the same objects as C and where $\mathrm{Hom}_{\mathrm{HL}(C)}(x,y) = H(x,y)$.

Theorem 65. $\mathrm{HL}(C)$ *is a simplicial category.*

Proof. It remains to specify id_x (for $x \in \mathrm{obHL}(C)$) and composition $\circ_{a,b,c} : \mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b) \to \mathrm{Hom}(a,c)$ and check that conditions of Definition 55 holds. Take $\mathrm{id}_x(*_n)$ to be the hammock of length 0, i.e. where the only maps are id and height n. For composition, given $(H,G) \in \mathrm{Hom}(b,c) \times \mathrm{Hom}(a,b)$, i.e. a hammock H in $\mathrm{Hom}(b,c)$ and G in $\mathrm{Hom}(a,b)$ of the same height, we can construct a hammock from a to c by placing G to the left of H (and reducing, if necessary).

Some rather tedious but not complicated computations show that the conditions do indeed hold. We only write down the first one, which is to check that $(\circ_{a,b,b}) \circ \left(\mathrm{id}_b \times 1_{\mathrm{Hom}(a,b)}\right) = \mathrm{idL}_{\mathrm{Hom}(a,b)}$ as maps from $* \times \mathrm{Hom}(a,b)$ to $\mathrm{Hom}(a,b)$. Here $\mathrm{idL}_{\mathrm{Hom}(a,b)}$ is simply the projection on the second element. But id_b picks out a hammock of $\mathrm{Hom}(b,b)$ that has only identities from b to b, so when we then apply the composition we change nothing, and so the left hand side is also the projection on the second element. □

9.2. **Localization preserves homotopy.** First we need to be able to speak of the components of a simplicial set.

Definition 66. Given a simplicial set S, we can consider its *connected components*, $\pi_0 S$. Define an equivalence relation $\sim$ on S_0 generated by $x \sim y$ if there is some $f \in S_1$ with $d_0 f = y$ and $d_1 f = x$. Then we define $\pi_0 S = S_0/\sim$.

The statement that "localization preserves homotopy" is then

Theorem 67. *Given a model category C and objects x and y, there is a one-to-one correspondence between maps from x to y in* $\mathrm{ho}C$ *and the components of the simplicial set* $\mathrm{Hom}_{\mathrm{HL}(C)}(x,y)$*, i.e.* $\mathrm{Hom}_{\mathrm{ho}C}(x,y) \cong \pi_0 \mathrm{Hom}_{\mathrm{HL}(C)}(x,y)$.

The original proof by Dwyer and Kan uses some more advanced machinery than what we have developed (in particular *homotopy (co)limits*), and so we omit it. It can be found in [DK].

9.3. **Homotopy theory of homotopy theories.** Recall from Section 6 that we think of model categories as "homotopy theories". But by the above, the functor HL "preserves homotopy", in the sense that for a model category C, ho$C$ might be retrieved from HLC as π_0HLC. Thus we might consider **sCat** as a category of homotopy theories. But **sCat** itself admits a model structure, and thus presents a homotopy theory — in this case we might in this sense view it as a "homotopy theory of homotopy theories".

10. Conclusion

We have presented the notion of $(\infty, 1)$-categories and presented two structures that might fit the bill, namely

- Simplicial sets with a horn-filling condition
- Simplicial categories where $\mathrm{Hom}\,(x, y)$ is a Kan-complex for any objects x and

Both of these are "nice" in the sense that they satisfy Grothendieck's homotopy hypothesis. The hypothesis really is a statement about $(\infty, 0)$-categories, and says that we might expect a $(\infty, 0)$-category to be "the same" as a topological space. For quasi-categories this equivalence is concertized by the two adjoint functors geometric realization $|-| : \mathbf{sSet} \to \mathbf{Top}$ and singular set $\mathrm{Sing} : \mathbf{Top} \to \mathbf{Kan}$.

We have see (amongst others) the following functors:

- $|-|$ and S, between **Top** and **sSet**.
- $|-|$ and N between **sCat** and **sSet**
- τ_1 and N between **Cat** and **sSet**.

This diagram shows some functors between the carious categories we have seen. Note that it is *not* commutative!

$$\begin{tikzcd}
 & \mathbf{sSet} \arrow[ddl, "|-|"'] \arrow[dr, "\tau_1"] & \\
 & \mathbf{QCat} \arrow[u, hook] & \mathbf{Cat} \arrow[l, "N"] \\
\mathbf{Top} \arrow[r, leftrightarrow, "\cong"'] & \mathbf{Kan} \arrow[u, hook] &
\end{tikzcd}$$

Model categories have been defined and we have shown how they are carriers of homotopy theories. Using the hammock localization we can turn a model category M into a simplicial category S in such a way that the maps in the homotopy category ho$M$ correspond one-to-one with the connected components of the simplicial hom-sets in S. In this way the category **sCat** "contains" all other homotopy theories, and since **sCat** itself is a model category we can speak of "homotopy theory of homotopy theories".

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