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Abstract

In the following we are going to look at a generalised notion of groups. These structures, called *polystrucures* seems to fall a bit short when it comes to fit properly into the classical theme of mathematics. The cause why groups became so deeply researched was that mathematicians found close resemblance between proofs say in matrix theory and permutation theory so it was helpful to find an abstract way to do these proofs. And group theory showed to be the link. The problem with polystructures is that they lack this "linking" property.

But with the right mindset one doesn't have to discard them straight away. We are going to see that most of the important properties of groups can be exhibited in a way or another. Furthermore we will also see that the modern view of mathematics have a place booked for polystructures from the beginning, namely they form a category.

Contents

1 A way to generalize groups

Generalising concepts and trying to observe objects on a greater scale has always been one of the leading ideas of sciences. What is meant by this is we are going to put less claims on the structure and by doing so create something that is more general so to say. Some information and detail may be lost in the process but usually new aspects and viewpoints rise. An easy way to approach this is to look at the assumptions on the questioned object and *loosen up* one or some of them. Now this can lead to results not fitting properly in the original theory but their investigation is still worthwhile.

In this paper we are going to look at the concept of groups where we change the group operation to be *multivalued* or *set-valued*. So let us start with a brief discussion on multivalued mappings.
In this section we mainly follow Viros and Davvaz paper.

1.1 Set-valued mappings

A univalued map

$$f : X \to Y$$

(X and Y being arbitrary sets) can always be considered as an ordered pair that is an element of the cartesian product of the two sets

$$(x, y), \quad x \in X, \ y \in Y.$$

This can not be done with a *set-valued* mapping (though in a non-trivial sense, every mapping can be embedded in a correspodning multivalued mapping). We will write

$$f : X \to 2^Y$$

or sometimes

$$f : X \multimap Y.$$

It is thus a map where to each $x \in X$ is associated a subset $f(x) \subseteq Y$.
For the construction of polystructures we need a *multivalued binary operation* which is a multivalued map

$$f : X \times X \multimap X$$

with non-empty values, where X being an arbitrary set. Commutativity of such an operation is easy to define, it means that

$$f(x, y) = f(y, x), \quad \forall x, y \in X$$

but the need of associativity forces us the extend f to products of arbitrary collections of elements (a multivalued binary operation does not need to be associative, but we want it to be associative for some of our constructions), the extension comes almost naturally

$$2^X \times 2^X \to 2^X : (A, B) \mapsto \bigcup_{a \in A, b \in B} f(a, b).$$

This means that we may talk about associativity of f:

$$f(a, f(b, c)) = f(f(a, b), c), \qquad \forall a, b, c \in X.$$

1.2 Polystructures

1.2.1 Preliminary definitions

In this section we follow Davvaz and Comer.
From now on we have a number of ways to make our generalisation and we are going to look at them very briefly. For easing on the notation we are going to consider elements as singleton sets. What we are going to call a *hypergroupoid* is the broadest generalisation and it is defined as so

Definition 1. *Let X be a set and let $P^*(X)$ denote the set of all nonempty subsets of X. Let $\curlyvee$ be a multivalued partial function $\curlyvee : X \times X \to P^*(X)$. Then we call the system $< X, \curlyvee >$ a hypergroupoid.*

Here $\curlyvee$ is not a binary operation since it is not nececerily defined on all pairs of elements of X. As you can also see we do not postulate associativity of the operation. This is actually a straightforward generalization of a *groupoid*.

Definition 2. *A grupoid is a set X together with a unary operation $()^{-1} : X \to X$ and a partial function $* : X \times X \to X$. Here again $*$ is not a binary operation by the same reasons as above.*

If we further tighten the rules the next thing we could do is to make $\curlyvee$ associative and then we arrive at something we will call a *semihypergroup* and defines as

Definition 3. *Let X be a set and let $P^*(X)$ denote the set of all nonempty subsets of X and let $\curlyvee : P^*(X) \times P^*(X) \to P^*(X)$, satisfying*

$$a \curlyvee (b \curlyvee c) = (a \curlyvee b) \curlyvee c$$

for all $a, b, c \in X$. Then the system $< X, \curlyvee >$ is called a semihypergroup.

This is a straightforward generalization of a semigroup.

Definition 4. *A semigroup is a set X together with a associative binary operation.*

This is looking good so far, but one still feel the need something grouplike that is where we can get all the different elements by combining others.

First axiom of multigroups: Let X be a set and $\curlyvee$ multivalued binary operation which is associative and that for all $a, b \in X$ we can find x, y satisfying

$$a \curlyvee x \supseteq b, \qquad y \curlyvee a \supseteq b.$$

Later on we are going to adapt some theorems from group theory into this new system. The problem is that what we have now is way too loose and without a unit we cannot possibly hope to accomplish this adaptation. So the next step would be to add the notion of a unit element. We are mostly going to look at commutative structures but for the sake of completeness we say

Definition 5. *u is a* left(right) unit *if it fullfills*

$$u \curlyvee a \supseteq a, \quad (a \curlyvee u \supseteq a) \quad \forall a \in X$$

respectively. We call u a unit if both the above relations are true.

Mostly we are going to consider multigroups where this unit behaves even nicer namely when

Definition 6. *Whenever*

$$u \curlyvee a = a, \quad (a \curlyvee u = a), \quad \forall a \in X$$

holds, we call u left (right) scalar unit respectively. If both holds then u is called an absolute unit.

Second axiom of multigroups: There exists a unit element $u \subseteq \mathcal{P}$.
Whenever a system $\mathfrak{M} =< X, \curlyvee >$ satisfies the two axioms we call it a *multigroup*. We can define inverses of an element a in a multigroup X by the followoing

Definition 7. *Let $\mathfrak{M}$ be a multigroup with unit element u. We call $b \subseteq \mathfrak{M}$ a two sided-inverse of $a \subseteq \mathfrak{M}$ if*

(i) $b \curlyvee a \supseteq u$

(ii) $a \curlyvee b \supseteq u$

It is clear by the first axiom that inverses exist. There is a nice theorem about absolute units:

Theorem 1. *Let $< \mathfrak{M}, \curlyvee >$ be a multigroup and let it contain a left scalar unit u. If there exists a right unit in $\mathfrak{M}$ then it is unique and equal to u and u is the only left scalar unit of $\mathfrak{M}$. If $\mathfrak{M}$ contains an absolute unit then there are no other units.*

Proof. Let us call the right unit u_r. By assumption u is a left scalar unit so we have that

$$u \curlyvee u_r = u_r$$

but also u_r is a right unit that is

$$u \curlyvee u_r \supseteq u$$

so we have that $u \subseteq u_r$ and since both are singleton sets we have that $u = u_r$. Now assume that u is an absolute unit of $\mathcal{P}$, and u_l, u_r are left and right units respectively. By the above argument

$$u_l = u = u_r$$

□

1.2.2 Polygroups

If we postulate that our structure has an absolute unit then we arrive at something we are going to call a *polygroup*. This is a nice enough object, we will see that it has rich inner structure and that there are several interesting examples. We give the definition

Definition 8. *A polygroup is a system $\mathcal{P} =< X, \curlyvee, u, ()^{-1} >$ where X is a set, $u \subseteq X$ is an absolute unit, $()^{-1}$ is a unary operation, $\curlyvee$ maps $P^*(X) \times P^*(X)$ into $P^*(X)$ and the following holds for all $x, y, z \subseteq \mathcal{P}$*

1. $(x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z)$
2. $u \curlyvee x = x \curlyvee u = x$
3. $x \subseteq y \curlyvee z \Rightarrow y \subseteq x \curlyvee z^{-1}$ *and* $z \subseteq y^{-1} \curlyvee x$

This last axiom will be referred to as the reversibility property

We may characterize $()^{-1}$ in a similar way as for multigroups.

Lemma 1. *Let $< \mathcal{P}, u, *, ()^{-1} >$ be a polygroup then*

$$u \subseteq x \curlyvee y \Rightarrow x = y^{-1} \text{ or } y = x^{-1}, \quad \forall x, y \subseteq \mathcal{P}.$$

Proof. Let $\mathcal{P}$ be a polygroup with u as an absolute unit and $x, y \subseteq \mathcal{P}$. Assume

$$u \subseteq x \curlyvee y$$

by the reversibility property this gives

$$x \subseteq u \curlyvee y^{-1} = y^{-1}$$

since u is an absolute unit. Now both x och y^{-1} are singletons we get equality and similarly we get that $x^{-1} = y$. □

It is naturally true that a polygoup is a multigroup. This follows from the fact that the reversibility property implies the first axiom of multogroups. Assume that $\mathcal{P}$ is a polygroup. Given $a, b \subseteq \mathcal{P}$ we want to see that there are $x, y \subseteq \mathcal{P}$ such that

$$a \curlyvee x \supseteq b \qquad y \curlyvee a \supseteq b.$$

We have

$$b = b \curlyvee u \subseteq b \curlyvee (a^{-1} \curlyvee a) = (b \curlyvee a^{-1}) \curlyvee a$$

therefore we have an $x \subseteq b \curlyvee a^{-1}$ such that

$$b \subseteq x \curlyvee a.$$

And similarly for y.

Let us make note of some elementary facts about polygroups. For all $x, y \subseteq \mathcal{P}$ and u being an absolute unit

1. $u \subseteq x \curlyvee x^{-1} \cap x^{-1} \curlyvee x$
2. $u^{-1} = u$
3. $(x^{-1})^{-1} = x$
4. $(x \curlyvee y)^{-1} = y^{-1} \curlyvee x^{-1}$
5. $x \subseteq y \curlyvee z \Rightarrow x^{-1} \subseteq z^{-1} \curlyvee y^{-1}$

The proof of the above claims are easy and we show the proof of only one of these, say 3. We have

$$(x \curlyvee y) \curlyvee y^{-1} \curlyvee x^{-1} =$$

since $\curlyvee$ is associative

$$= x \curlyvee (y \curlyvee y^{-1}) \curlyvee x^{-1},$$

now $u \subseteq y \curlyvee y^{-1}$ so

$$x \curlyvee (y \curlyvee y^{-1}) \curlyvee x^{-1} \supseteq x \curlyvee x^{-1} \supseteq u.$$

Now we know the abstract definition of polygroup, but a concrete example of them is due. Similarly to ususal group theory we will try to capture the properties of $\curlyvee$ in a polygroup table. All the following examples will be commutative.

To begin with let us look at the smallest example we can find, a one element polygroup is just the trivial group but with two elements we can construct

Example 1. *Let* $X = \{u, a\}$ *and let* $\circledast : 2^X \times 2^X \to 2^X$ *be defined by the following rules* $x \circledast u = u \circledast x = \{x\}$ *for all* $x \subseteq X$ *and* $a \circledast a = \{a, u\}$.
This is indeed the smallest polygroup we can create and we will denote it by $\mathfrak{P}_1$.

$\circledast$	u	a
u	u	a
a	a	$\{u, a\}$

Table 1: Polygroup table of $\mathfrak{P}_1$

We can see that this structure fullfills the requirements of Defniniton 9*. The unit element behaves as postulated. The reversibility property is also fullfilled but it seems trivial due to the lack of different elements. From the table we can see that*

$$u^{-1} = u \quad \textit{and} \quad a^{-1} = a$$

Now we give some examples of polygroups of varying cardinality.

Example 2. *Let* $X = \{u, a, b\}$ *and let* $\boxdot : 2^X \times 2^X \to 2^X$ *satisfy* $u \boxdot x = x \boxdot u = x$ *for all* $x \in X$ *and* $a \boxdot b = b \boxdot a = \{u, a, b\}$*. Let us call this polygroup* $\mathfrak{P}_2$

$\boxdot$	u	a	b
u	u	a	b
a	a	a	$\{u, a, b\}$
b	b	$\{u, a, b\}$	b

Table 2: Polygroup table of $\mathfrak{P}_2$

Furthermore

$()^{-1}$	u	a	b
	u	b	a

Table 3: Inverse table of $\mathfrak{P}_2$

We can see that u behaves as needed, namely as an absolute unint. Furthermore the reversibility property is also satisfied. We give an example

$$a \in a \boxdot b \Rightarrow a \in a \boxdot b^{-1}$$

from the table we see that $b^{-1} = a$ *so*

$$a \in a \boxdot a$$

and

$$a \in a \boxdot b \Rightarrow b \in a^{-1} \boxdot a$$

and again from the table we see that $a^{-1} = b$ *so*

$$b \in b \boxdot a$$

and this is also true. It is also easy to see that associativvity is true. Let us make a calculation in this polygroup. We calculate $(a \boxdot b) \boxdot b$ *as follows*

$$(a \boxdot b) \boxdot b = (\{u, a, b\}) \boxdot b = \bigcup(u \boxdot b, a \boxdot b, b \boxdot b) =$$

$$= \bigcup(b, \{u, a, b\}, b) = \{u, a, b\} = a \boxdot (b \boxdot b)$$

Example 3. *Another way we can create examples is from linearly ordered sets. Let* X *be a linearly ordered set with order* $\prec$ *and an element* 0 *such that* $0 \prec x$ *for all* $x \subseteq X$ *differernt from* 0*. Now define a binary multivalued operation (we may refer to these as* hyperoperation *in the future) on* X *by*

$$(a, b) \mapsto a \boxplus b = \begin{cases} \max(a, b), & \text{if } a \neq b \\ \{x \in X \mid x \preceq a\} & \text{if } a = b \end{cases}$$

If we take $X = \{0, 1\}$ *and* $0 \prec 1$ *then this construction gives* $\mathfrak{P}_1$*. Let us prove the claim that constructions of these type create polygroups. So we have to show that* $\boxplus$ *is associative, we have a unit element and that the reversibility property is true as well. We have four different cases to look at*

$$a = b = c, \quad a = b \preceq c, \quad a \preceq b = c, \quad a \preceq b \preceq c$$

and since we need the same cases both for associativity and reversibility property *so we do these together so we need to show*

1. $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c$
2. $a \subseteq b \boxplus c \Rightarrow b \subseteq a \boxplus c^{-1} \quad c \subseteq b^{-1} \boxplus a$

   (a) *Let* $a = b = c$*, then*

   $$a \boxplus (b \boxplus c) = \{x \in X \mid x \preceq c\}$$

   and by similar reasoning we get that

   $$(a \boxplus b) \boxplus c = \{x \in X \mid x \preceq c\}$$

   to show that 2 also holds first we need to check if $a \subset b \boxplus c$ *is possible.*

   $$a \subseteq \underbrace{b \boxplus c}_{\{x \in X \mid x \preceq c\}}$$

   and since all a, b, c *are equal in this case we see that this assumption is true. One thing we haven't mentioned until now, namely that in this polygroup all elements are their own inverses. This means that the implication*

   $$a \subseteq b \boxplus c \Rightarrow b \subseteq a \boxplus c^{-1} \quad c \subset b^{-1} \boxplus a$$

   turns into

   $$a \subset b \boxplus c \Rightarrow b \subseteq a \boxplus c \quad c \subseteq b \boxplus a$$

   which is clearly true since $a = b = c$*.*

(b) *Let* $a = b \preceq c$*, and check first if*

$$a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c.$$

According to the definition

$$\underbrace{a \boxplus \underbrace{(b \boxplus c)}_{c}}_{c} = \underbrace{\underbrace{(a \boxplus b)}_{\{x \in X | x \preceq b\}} \boxplus c}_{c}.$$

Secondly we want to check the reversibility property, but since $a = b \preceq c$ *by assumption we can not have* $a \subseteq b \boxplus c$*, hence 2. is vacuously true.*
The reversibility property is not defined in this case since

$$a = b \preceq c$$

is not true.

(c) *Let now* $a \preceq b = c$*. For associativity we need*

$$a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c.$$

By definition

$$\underbrace{a \boxplus \underbrace{(b \boxplus c)}_{\{x \in X | x \preceq b\}}}_{b} = \underbrace{\underbrace{(a \boxplus b)}_{b} \boxplus c}_{b}.$$

For the reversibility property we have

$$a \subseteq b \boxplus c \quad \Rightarrow \quad a \subseteq \{x \in X \mid x \preceq b\}$$

and this should imply

$$b \subseteq a \boxplus c \quad \text{and} \quad c \subseteq b \boxplus a$$

both of which boils down to

$$b = c.$$

(d) *Finally assume* $a \preceq b \preceq c$*. In this case* $\boxplus$ *is clearly associative, it gives back* c *on both sides of*

$$a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c.$$

As in case (b) the reversibility property is again trivially true so we are done.

3. *We need to have unit element as well, and the "smallest" element will work perfectly.*

Example 4. *A linearly ordered set X can be turned into a different polygroup if we define the operation to be*

$$(a, b) \mapsto a \boxdot b = \begin{cases} \max(a, b), & \text{if } a \neq b \\ \{x \in X \mid x \prec a\}, & \text{if } a = b \neq 0 \\ 0, & \text{if } a = b = 0. \end{cases}$$

If the set X consists of more than 2 elements then the operation $\boxdot$ is truly multivalued.

The construction in Example 3 is called *linear order polygroup* and the one in Example 4 is called *strict linear order polygroup*.

1.3 Embedding

In this section we follow the idea of embedding by Melvin and Ore.

Our progress so far seems viable even though this generalisation may seem a bit odd, but we can still find a correlation with rings mirroring the methods we use to relate groups to group rings. By doing so we may think about polygroups as a special rings. We felt that showing this correlation will make polygroups slightly more concrete and therefore we will look into this idea now. The fact is that a polygroup can always be embedded in an algebra. We will now exhibit this embedding.

First of all we start with a field, say $\mathbb{C}$ and and a polygroup $\mathcal{P}$. Then we consider the free vector space generated by the elements of $\mathcal{P}$ and we denote this object by $\mathbb{C}^{\mathcal{P}}$. It has elements of the form

$$\sum_{i=1}^{p} \lambda_i p_i, \qquad |\mathcal{P}| = p, \quad \lambda_i \in \mathbb{C}, \quad p_i \in \mathcal{P}$$

Now $\mathbb{C}^{\mathcal{P}}$ has a vector space structure by construction so what is left to give $\mathbb{C}^{\mathcal{P}}$ a ring structure also. We omit the $\cdot$ in the above expression since in the following $\cdot$ is reserved to denote the multiplication in $\mathbb{C}$. Since $\mathcal{P}$ is a polygroup the group operation returns subsets that is

$$p_1 \curlyvee p_2 = \{q_1, q_2, \ldots\} \subseteq \mathcal{P}.$$

Definition 9.

$$\Big(\sum_{i=1}^{p} \lambda_i p_i\Big) \curlyvee \Big(\sum_{j=1}^{p} \mu_j p_j\Big) := \sum_{i,j} \sum_{q_k \in p_i \curlyvee p_j} (\lambda_i \mu_j) q_k \in \mathbb{C}^{\mathcal{P}}$$

where $\lambda_i, \mu_j \in \mathbb{C}$ *and* $p_l, q_l \in \mathcal{P}$.

Note in particular that we continue using the same symbol $\curlyvee$ for multiplication in the algebra.
So we have to show that the group operation $\curlyvee$ is associative, distributive and possesses a unit element. We start with the unit since this is easiest, we just take u, the unit of $\mathcal{P}$ to be the unit element in the ring. For associativity we have to show that

$$a \curlyvee (b \curlyvee c) = (a \curlyvee b) \curlyvee c, \quad \forall a, b, c \in \mathbb{C}^{\mathcal{P}}.$$

To see that this is true let

$$a = \sum_{i=1}^{p} \lambda_i p_i, \quad b = \sum_{j=1}^{p} \mu_j p_j, \quad c = \sum_{k=1}^{p} \nu_k p_k$$

and let us start the grunt work. Now

$$b \curlyvee c = \sum_{j=1}^{p} \mu_j p_j \curlyvee \sum_{k=1}^{p} \nu_k p_k = \sum_{j,k=1}^{p} \mu_j \cdot \nu_k p_j \curlyvee p_k$$

and this times a gives

$$\sum_{i=1}^{p} \lambda_i p_i \curlyvee \sum_{j,k=1}^{p} \mu_j \cdot \nu_k p_j \curlyvee p_k = \sum_{i,j,k=1}^{p} \lambda_i \cdot \mu_j \cdot \nu_k p_i \curlyvee p_j \curlyvee p_k.$$

So this is where the left hand side leads us to. We can clearly see that the right hand side gives the same result (the details are left to the reader).
Now for distributivity we show that

$$a \curlyvee (b + c) = a \curlyvee b + a \curlyvee c$$

we start with the left hand side (using the same notation as above)

$$b + c = \sum_{j=1}^{p} \mu_j p_j + \sum_{k=1}^{p} \nu_k p_k = \sum_{j=1}^{p} (\mu_j + \nu_j) p_j$$

and

$$a \curlyvee \sum_{j=1}^{p} (\mu_j + \nu_j) p_j = \sum_{i=1}^{p} \lambda_i p_i \curlyvee \sum_{j=1}^{p} (\mu_j + \nu_j) p_j =$$

$$= \sum_{i,j=1}^{p} \lambda_i (\mu_j + \nu_j) p_i \curlyvee p_j$$

and for the right hand side

$$a \curlyvee b + a \curlyvee c$$

becomes

$$\sum_{i=1}^{p} \lambda_i p_i \curlyvee \sum_{j=1}^{p} \mu_j p_j = \sum_{i,j=1}^{p} \lambda_i \cdot \mu_j p_i \curlyvee p_j$$

for the other term, for notational ease we change the summation index in c to be j which gives

$$\sum_{i=1}^{p} \lambda_i p_i \curlyvee \sum_{j=1}^{p} \nu_j p_j = \sum_{i,j=1}^{p} \lambda_i \cdot \nu_j p_i \curlyvee p_j.$$

Finally the sum of these two group elements is

$$\sum_{i,j=1}^{p} \lambda_i \cdot \mu_j p_i \curlyvee p_j + \sum_{i,j=1}^{p} \lambda_i \cdot \nu_j p_i \curlyvee p_j$$

which gives

$$\sum_{i,j=1}^{p} \lambda_i (\mu_j + \nu_j) p_i \curlyvee p_j$$

and this agrees with the left hand side as we wanted to. And now our embedding is ready, not forgetting that whenever $p_i \curlyvee p_j$ returns a subset we consider the sum of the elements of this using $+$ from the $\mathbb{C}$.
We can summarize the above embedding in

Proposition 1. *Given a polygroup $\mathcal{P}$ and a field k, we can define ρ which injects $\mathcal{P}$ into the free vector space over k generated by the elements of $\mathcal{P}$, in symbols*

$$\rho : \mathcal{P} \hookrightarrow k^{\mathcal{P}},$$

such that

$$\rho(a \curlyvee b) = \rho(a) \cdot \rho(b).$$

1.4 Subpolygroups

This and the following section is based on Davvaz paper.

Definition 10. *A non-empty subset $\mathcal{K}$ of a polygroup $\mathcal{P}$ is called a* subpolygroup *of $\mathcal{P}$ if $\mathcal{K}$ is also a polygroup with the same operations and unit element.*

This definition is also a straightforward generalization and as in group theory we would like to have some criterion to decide whether a set is a subpolygroup. This comes in the form of

Theorem 2. *A non-empty subset $\mathcal{K}$ of a polygroup $\mathcal{P}$ is a subpolygroup of $\mathcal{P}$ if and only if*

$$a, b \subseteq \mathcal{K} \Rightarrow a \curlyvee b \subseteq \mathcal{K}$$

and

$$a \subseteq \mathcal{K} \Rightarrow a^{-1} \subseteq \mathcal{K}$$

Another important notion is normality which can also be adapted by

Definition 11. *Let $\mathcal{P}$ be a polygroup and let $\mathcal{N}$ be a subpolygroup of $\mathcal{P}$. $\mathcal{N}$ is called normal if and only if*

$$a^{-1} \curlyvee \mathcal{N} \curlyvee a \subseteq \mathcal{N}, \quad \forall a \subseteq \mathcal{P},$$

where $a \curlyvee \mathcal{N}$ means $\bigcup a \curlyvee \{n \mid n \subseteq \mathcal{N}\}$.

A couple of nice properties are collected in the following lemma

Lemma 2. *Let $\mathcal{K}$ and $\mathcal{N}$ subpolygroups of $\mathcal{P}$ with $\mathcal{N}$ normal in $\mathcal{P}$. Then*

1. $\mathcal{N} \curlyvee a = a \curlyvee \mathcal{N}, \quad \forall a \subseteq \mathcal{P}$
2. $(\mathcal{N} \curlyvee a)(\mathcal{N} \curlyvee b) = \mathcal{N} \curlyvee a \curlyvee b, \quad \forall a, b \subseteq \mathcal{P}$
3. $\mathcal{N} \curlyvee a = \mathcal{N} \curlyvee b, \quad \forall b \subseteq \mathcal{N} \curlyvee a$
4. $\mathcal{N} \cap \mathcal{K}$ *is normal in* $\mathcal{K}$
5. $\mathcal{N} \curlyvee \mathcal{K} = \mathcal{K} \curlyvee \mathcal{N}$*is a subpolygroup of* $\mathcal{P}$
6. $\mathcal{N}$ *is normal in* $\mathcal{N} \curlyvee \mathcal{K}$

Proof. 1. Since $\mathcal{N}$ is normal we have

$$a^{-1} \curlyvee \mathcal{N} \curlyvee a \subseteq \mathcal{N} \Rightarrow \mathcal{N} \curlyvee a \subseteq a \curlyvee \mathcal{N}$$

on the other hand

$$\mathcal{N} \curlyvee a \subseteq a \curlyvee \mathcal{N} \Rightarrow \mathcal{N} \subseteq a \curlyvee \mathcal{N} \curlyvee a^{-1}$$

by the reversibility property this means

$$a \curlyvee \mathcal{N} \subseteq \mathcal{N} \curlyvee a$$

and we are done.

2. This follows from (1)

3. Assume $b \subseteq \mathcal{N} \curlyvee a$, then

$$b \subseteq \mathcal{N} \curlyvee a \Rightarrow \mathcal{N} \curlyvee b \subseteq \mathcal{N} \curlyvee a$$

and by definition

$$b \subseteq \mathcal{N} \curlyvee a \Rightarrow a \subseteq \mathcal{N} \curlyvee b \Rightarrow \mathcal{N} \curlyvee a \subseteq \mathcal{N} \curlyvee b$$

and we are done.

4. Let $a, b \subseteq \mathcal{N} \cap \mathcal{K}$ then $a, b \subseteq \mathcal{N}$ and $a, b \subseteq \mathcal{K}$ and so $a \curlyvee b \subseteq \mathcal{N}$ and $a \curlyvee b \subseteq \mathcal{K}$ therefore $a \curlyvee b \subseteq \mathcal{N} \cap \mathcal{K}$. We have the inverses since $\mathcal{N}$ and $\mathcal{K}$ are themselves groups. Now $\mathcal{N} \cap \mathcal{K}$ is normal in $\mathcal{K}$ if

$$a^{-1} \curlyvee (\mathcal{N} \cap \mathcal{K}) \curlyvee a \subseteq \mathcal{N} \cap \mathcal{K} \qquad \forall a \subseteq \mathcal{K}.$$

Take $l \subseteq \mathcal{N} \cap \mathcal{K}$ and $a \subseteq \mathcal{K}$ then

$$a^{-1} \curlyvee l \curlyvee a \subseteq \mathcal{K} \qquad \forall a \subseteq \mathcal{K}$$

furthermore $\mathcal{N}$ is normal in $\mathcal{P}$

$$a^{-1} \curlyvee l \curlyvee a \subseteq \mathcal{N} \qquad \forall a \subseteq \mathcal{K} \subseteq \mathcal{P}.$$

5. Let us show first the equality of these two. Take $n \curlyvee k \subseteq \mathcal{N} \curlyvee \mathcal{K}$ then we have

$$n \curlyvee k \subseteq \mathcal{N} \curlyvee \mathcal{K} \Rightarrow$$
$$n \curlyvee k \subseteq \mathcal{N} \curlyvee \tilde{k}, \qquad \tilde{k} \subseteq \mathcal{K}$$

By (1) this gives

$$n \curlyvee k \subseteq \tilde{k} \curlyvee \mathcal{N} \subseteq \mathcal{K} \curlyvee \mathcal{N}$$

so

$$\mathcal{N} \curlyvee \mathcal{K} \subseteq \mathcal{K} \curlyvee \mathcal{N}.$$

The other inclusion is similarly easy to see.

Now $\mathcal{N} \curlyvee \mathcal{K}$ is a subgroup if its closed under the operation and includes inverses. A product in $\mathcal{N} \curlyvee \mathcal{K}$ looks like

$$n_1 \curlyvee k_1 \curlyvee n_2 \curlyvee k_2 \qquad n_1, n_2 \subseteq \mathcal{N} \quad k_1, k_2 \subseteq \mathcal{K}.$$

Let now $x \subseteq \mathcal{P}$ be a singleton set such that $x \subseteq k_1 \curlyvee n_2$. Since $\mathcal{N}$ is normal we have $k_1 \curlyvee \mathcal{N} = \mathcal{N} \curlyvee k_1$ and so there is an n_3 such that $x \subseteq n_3 \curlyvee k_1$. That is, we have

$$n_1 \curlyvee x \curlyvee k_2 \subseteq n_1 \curlyvee n_3 \curlyvee k_1 \curlyvee k_2 \subseteq \mathcal{N} \curlyvee \mathcal{K}$$

and since x was chosen arbitrary it is clear that

$$n_1 \curlyvee k_1 \curlyvee n_2 \curlyvee k_2 \subseteq \mathcal{N} \curlyvee \mathcal{K}$$

To see that inverses are also included in $\mathcal{N} \curlyvee \mathcal{K}$ take

$$k \subseteq \mathcal{K} \qquad n \subseteq \mathcal{N}$$

so

$$k^{-1} \subseteq \mathcal{K} \qquad n^{-1} \subseteq \mathcal{N}$$

since they are polygroups. Now the inverse of an element $n \curlyvee k \subseteq \mathcal{N} \curlyvee \mathcal{K}$ is $k^{-1} \curlyvee n^{-1}$ but since $\mathcal{N} \curlyvee \mathcal{K} = \mathcal{K} \curlyvee \mathcal{N}$ we have

$$k^{-1} \curlyvee n^{-1} \subseteq \mathcal{K} \curlyvee \mathcal{N} = \mathcal{N} \curlyvee \mathcal{K}.$$

6. Let $n \subseteq \mathcal{N}$ and $a \subseteq \mathcal{N} \curlyvee \mathcal{K}$ then

$$a^{-1} \curlyvee n \curlyvee a \subseteq \mathcal{N} \subseteq \mathcal{N} \curlyvee \mathcal{K}$$

□

There is one more property of ordinary groups which one may feel the need of here, namely an equivalence relation after which we can partition $\mathcal{P}$ into cosets.

Definition 12. *If $\mathcal{N}$ is normal in $\mathcal{P}$, then we define the relation $x \equiv y \mod \mathcal{N}$ if and only if $x \curlyvee y^{-1} \cap \mathcal{N} \neq \emptyset$. This relation is denoted $x\mathcal{N}_{\mathcal{P}}y$.*

This looks promising we just have to show that $\mathcal{N}_{\mathcal{P}}$ is an equivalence relation

Theorem 3. *The relation $\mathcal{N}_{\mathcal{P}}$ is an equivalence relation*

Proof. Let $x, y, z \subseteq \mathcal{P}$. So we have to show that $\mathcal{N}_{\mathcal{P}}$ is reflexive, symmetric and transitive.

We start with with *reflexivity.* Since $u \subseteq x \curlyvee x^{-1}$ we have that for all $x \subseteq \mathcal{P}$ that $x \ \mathcal{N}_{\mathcal{P}} \ x$.

Secondly for *symmetry* assume that $x \ \mathcal{N}_{\mathcal{P}} \ y$, this means that $x \curlyvee y^{-1} \cap \mathcal{N} = \emptyset$ so we have an element $z \subseteq x \curlyvee y^{-1}$ which in turn means (since $a \subseteq b \curlyvee c \Rightarrow a^{-1} \subseteq c^{-1} \curlyvee b^{-1}$ in a polygroup) that $z^{-1} \subseteq y \curlyvee x^{-1}$ so $y \ \mathcal{N}_{\mathcal{P}} \ x$. So $\mathcal{N}_{\mathcal{P}}$ is symmetric.

To see that *transitivity* of $\mathcal{N}_{\mathcal{P}}$ assume that $x \; \mathcal{N}_{\mathcal{P}} \; y$ and $y \; \mathcal{N}_{\mathcal{P}} \; z$ which gives that we have a and b such that $a \subseteq x \curlyvee y^{-1} \cap \mathcal{N}$ and $b \subseteq y \curlyvee z^{-1} \cap \mathcal{N}$. Applying the reversibility property gives $x \subseteq a \curlyvee y$ and $z^{-1} \subseteq y^{-1} \curlyvee b$. Combining the last two yields $x \curlyvee z^{-1} \subseteq a \curlyvee y \curlyvee y^{-1} \curlyvee b \subseteq a \curlyvee b \subseteq \mathcal{N}$. Hence $x \curlyvee z^{-1} \cap \mathcal{N} \neq \emptyset$ which is the desired result. □

Now we have our equivalence relation so we have a chance to create a coset decomposition of $\mathcal{P}$. Let us denote the equivalence class of x by $\mathcal{N}_{\mathcal{P}}(x)$. Let $[P : N] = \{\mathcal{N}_{\mathcal{P}}(x) \mid x \subseteq \mathcal{P}\}$, in usual group theory this detones the relative size of $\mathcal{N}$ in $\mathcal{P}$, the number of copies of $\mathcal{N}$ that fits into $\mathcal{P}$. Let us define a hyperoperation $\odot$ on $[P : N]$ by

$$\mathcal{N}_{\mathcal{P}}(x) \odot \mathcal{N}_{\mathcal{P}}(y) = \{\mathcal{N}_{\mathcal{P}}(z) \mid z \subseteq \mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(y)\}.$$

Usually we denote the coset of a subgroup $\mathcal{N}$ by $\mathcal{N} \curlyvee x$ and let the set of cosets be denoted by $\mathcal{P}/\mathcal{N}$. What we want show now is the fact that whenever $\mathcal{N}$ is normal we have

$$\mathcal{N} \curlyvee x = \mathcal{N}_{\mathcal{P}}(x).$$

Theorem 4. *Let $\mathcal{P}$ be a polygroup and let $\mathcal{N}$ be a normal subpolygroup of $\mathcal{P}$. Then $\mathcal{N} \curlyvee x = \mathcal{N}_{\mathcal{P}}(x)$.*

Proof. Let $y \subseteq \mathcal{N} \curlyvee x$, then by definition

$$\exists n \subseteq \mathcal{N} \; s.t. \; y \subseteq n \curlyvee x \Rightarrow n \in y \curlyvee x^{-1} \Rightarrow y \curlyvee x^{-1} \cap \mathcal{N} \neq \emptyset \Rightarrow \mathcal{N} \curlyvee x \subseteq \mathcal{N}_{\mathcal{P}}(x).$$

On the other hand, let $y \in \mathcal{N}_{\mathcal{P}}(x)$ this means by definition that

$$y \curlyvee x^{-1} \cap \mathcal{N} \neq \emptyset \Rightarrow \exists n \subseteq \mathcal{N} \; s.t. \; n \subseteq y \curlyvee x^{-1} \Rightarrow y \subseteq n \curlyvee x \Rightarrow \mathcal{N}_{\mathcal{P}}(x) \subseteq \mathcal{N} \curlyvee x$$

□

So we can conclude that $[\mathcal{P} : \mathcal{N}] = \mathcal{P}/\mathcal{N}$, which is nice and *grouplike*.

Lemma 3. *Let $\mathcal{N}$ be a normal subpolygroup of $\mathcal{P}$ then for all $x, y \subseteq \mathcal{P}$, we have*

$$\mathcal{N}_{\mathcal{P}}(\mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(y)) = \mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(y)$$

Proof. By *Theorem 4* $\mathcal{N}_{\mathcal{P}}(x) = \mathcal{N} \curlyvee x$ so

$$\mathcal{N}_{\mathcal{P}}(\mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(y)) = \mathcal{N}_{\mathcal{P}}(\mathcal{N} \curlyvee x \curlyvee \mathcal{N} \curlyvee y) = \mathcal{N} \curlyvee \mathcal{N} \curlyvee x \curlyvee \mathcal{N} \curlyvee y =$$

and since $\mathcal{N}$ is normal we get

$$\mathcal{N} \curlyvee x \curlyvee \mathcal{N} \curlyvee y = \mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(y)$$

□

Another important property is captured in

Theorem 5. *Let $\mathcal{N}$ be a normal subpolygroup of $\mathcal{P}$. Then for all $x, y \subseteq \mathcal{P}$ we have*

$$\mathcal{N} \curlyvee (x \curlyvee y) = \mathcal{N} \curlyvee z, \quad \forall z \subseteq x \curlyvee y.$$

Proof. Suppose $z \subseteq x \curlyvee y$, this means $\mathcal{N} \curlyvee z \subseteq \mathcal{N} \curlyvee (x \curlyvee y)$. Let now $a \subseteq \mathcal{N} \curlyvee (x \curlyvee y)$ then we have $y \subseteq (\mathcal{N} \curlyvee x)^{-1} \curlyvee a = x^{-1} \curlyvee \mathcal{N} \curlyvee a$ multiplying on the left by x gives

$$x \curlyvee y \subseteq x \curlyvee x^{-1} \curlyvee \mathcal{N} \curlyvee a$$

Since $\mathcal{N}$ is normal we can use *Lemma 2/1* to obtain

$$x \curlyvee y \subseteq x \curlyvee \mathcal{N} \curlyvee x^{-1} \curlyvee a \subseteq \mathcal{N} \curlyvee a$$

therefore for every $z \subseteq x \curlyvee y$ we have that $z \subseteq \mathcal{N} \curlyvee a$ which means $a \subseteq \mathcal{N} \curlyvee z$ as required. □

To make the isomorphism theorems work we need to give a polygroup structure to $[\mathcal{P} : \mathcal{N}]$. This is achieved by

Proposition 2. *The system $< [\mathcal{P} : \mathcal{N}], \odot, \mathcal{N}_{\mathcal{P}}(u), ()^{-1} >$ is a polygroup, where $\mathcal{N}_{\mathcal{P}}(a)^{-1} = \mathcal{N}_{\mathcal{P}}(a^{-1})$.*

Proof. We need to show associativity of $\odot$, that we have an identity and the reversibility property. For all $a, b, c \subseteq \mathcal{P}$, we have

$$(\mathcal{N}_{\mathcal{P}}(a) \odot \mathcal{N}_{\mathcal{P}}(b)) \odot \mathcal{N}_{\mathcal{P}}(c) =$$

by definition of $\odot$

$$= \{\mathcal{N}_{\mathcal{P}}(x) \mid x \subseteq \mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(b)\} \odot \mathcal{N}_{\mathcal{P}}(c) =$$

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid y \subseteq \mathcal{N}_{\mathcal{P}}(x) \curlyvee \mathcal{N}_{\mathcal{P}}(c), x \subseteq \mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(b)\} =$$

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid y \subseteq \mathcal{N}_{\mathcal{P}}(\mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(b)) \curlyvee \mathcal{N}_{\mathcal{P}}(c)\} =$$

by *Lemma 3*

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid y \subseteq (\mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(b)) \curlyvee \mathcal{N}_{\mathcal{P}}(c)\}.$$

And

$$\mathcal{N}_{\mathcal{P}}(a) \odot (\mathcal{N}_{\mathcal{P}}(b) \odot \mathcal{N}_{\mathcal{P}}(c)) =$$

by the exact same reasoning as above

$$= \mathcal{N}_{\mathcal{P}}(a) \odot \{\mathcal{N}_{\mathcal{P}}(x) \mid x \subseteq \mathcal{N}_{\mathcal{P}}(b) \curlyvee \mathcal{N}_{\mathcal{P}}(c)\} =$$

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid y \subseteq \mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(x),\ x \subseteq \mathcal{N}_{\mathcal{P}}(b) \curlyvee \mathcal{N}_{\mathcal{P}}(c)\} =$$

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid \mathcal{N}_{\mathcal{P}}(a) \curlyvee \mathcal{N}_{\mathcal{P}}(\mathcal{N}_{\mathcal{P}}(b) \curlyvee \mathcal{N}_{\mathcal{P}}(c))\} =$$

$$= \{\mathcal{N}_{\mathcal{P}}(y) \mid y \subseteq \mathcal{N}_{\mathcal{P}}(a) \curlyvee (\mathcal{N}_{\mathcal{P}}(b) \curlyvee \mathcal{N}_{\mathcal{P}}(c))\}.$$

We see that these result in the same equivalence class. $\mathcal{N}_{\mathcal{P}}(u)$ is the unit element by *Theorem 5*, furthermore $\mathcal{N}_{\mathcal{P}}(x^{-1})$ is the inverse of $\mathcal{N}_{\mathcal{P}}(x)$. For the reversibility propery take $\mathcal{N}_\mathcal{P}(c) \subseteq \mathcal{N}_\mathcal{P}(a) \curlyvee \mathcal{N}_\mathcal{P}(b)$, then we have an $x \subseteq \mathcal{N}_\mathcal{P}(c)$ such that $\mathcal{N}_\mathcal{P}(x) = \mathcal{N}_\mathcal{P}(c)$. Therefore we have a $y \subseteq \mathcal{N}_\mathcal{P}(a)$ and $z \subseteq \mathcal{N}_\mathcal{P}(b)$ such that $x \subseteq y \curlyvee z$ and since $x, y, z \subseteq \mathcal{P}$ we have $y \subseteq x \curlyvee z^{-1}$. This means that $\mathcal{N}_\mathcal{P}(y) \subseteq \mathcal{N}_\mathcal{P}(x) \curlyvee \mathcal{N}_\mathcal{P}(z^{-1})$ which in turn means $\mathcal{N}_\mathcal{P}(a) \subseteq \mathcal{N}_\mathcal{P}(c) \odot \mathcal{N}_\mathcal{P}(b^{-1})$. And we can get the other part of the reversibility property by similar reasoning and we are done. □

Corollary 1. *If $\mathcal{N}$ is a normal subpolygroup of $\mathcal{P}$, then the system $< \mathcal{P}/\mathcal{N}, \odot, \mathcal{N}, ()^{-1} >$ is a polygroup with*

$$\mathcal{N}x \odot \mathcal{N}y = \{\mathcal{N} \curlyvee z \mid z \subseteq x \curlyvee y\}$$

and

$$(\mathcal{N} \curlyvee x)^{-1} = \mathcal{N} \curlyvee x^{-1}.$$

Now we see that the notion of a *quotient polygroup* makes sense.

1.5 Polygroup homomorphisms

In abstract algebra an important point besides defining these abstract structures is to find connections between them. This often comes in the form of homomorphisms. And we will generalise this notion so it will fit into *polystructures*. The definition is again a straightforward generalisation of the non-multivalued case.

Definition 13. *Let $\mathcal{P}$ and $\mathcal{Q}$ be polygroups. A map $f : \mathcal{P} \to \mathcal{Q}$ is called a* weak polygroup homomorphism *if*

$$f(u_\mathcal{P}) = u_\mathcal{Q}$$

and

$$f(a \curlyvee_\mathcal{P} b) \subseteq f(a) \curlyvee_\mathcal{Q} f(b), \quad \forall a, b \subseteq \mathcal{P}$$

Here we use that f – as well as $\curlyvee$ – is extended to a map $f : 2^\mathcal{P} \to 2^\mathcal{Q}$ as in the previous sections.

As with many of the concepts in this area we have a stronger notion of homomorphism as well and we give the definition now

Definition 14. *Let $\mathcal{P}$ and $\mathcal{Q}$ be polygroups. A map $f : \mathcal{P} \to \mathcal{Q}$ is called a* strong polygroup homomorphism *if*

$$f(u_\mathcal{P}) = u_\mathcal{Q}$$

and

$$f(a \curlyvee_\mathcal{P} b) = f(a) \curlyvee_\mathcal{Q} f(b), \quad \forall a, b \subseteq \mathcal{P}$$

A strong homomorphism which is injective and surjective is called an *isomorphism.*
Let $f : \mathcal{P} \to \mathcal{Q}$ be a strong homomorphism, since $\mathcal{P}$ is a polygroup

$$u_\mathcal{P} \subseteq a \curlyvee a^{-1} \qquad \forall a \subseteq \mathcal{P}$$

and since f is a strong homomorphism

$$u_{\mathcal{Q}} \subseteq f(a) \curlyvee_{\mathcal{Q}} f(a^{-1})$$

which by the reversibility property gives

$$f(a^{-1}) \subseteq (f(a))^{-1} \curlyvee u_{\mathcal{Q}} = (f(a))^{-1}$$

After taking inverses on both sides in

$$u_{\mathcal{Q}} \subseteq f(a) \curlyvee_{\mathcal{Q}} f(a^{-1})$$

we get

$$u_{\mathcal{Q}} \subseteq (f(a))^{-1} \curlyvee f(a^{-1})^{-1}$$

and again by the reversibility property we get

$$(f(a))^{-1} \subseteq u_{\mathcal{Q}} \curlyvee f(a^{-1}) = f(a^{-1})$$

and these two inclusions mean

$$f(a^{-1}) = (f(a))^{-1}$$

which is another nice property of ordinary group theory we could transfer into this new structure. We summarize it in a lemma

Lemma 4. *Let $\mathcal{P}$ be a polygroup, $a \in \mathcal{P}$, then*

$$f(a^{-1}) = (f(a))^{-1}.$$

Definition 15. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polygroups and f be a polygroup homomorphism. Then the* kernel *of f is the set*

$$\ker f = \{x \subseteq \mathcal{P}_1 \mid f(x) = u_{\mathcal{Q}}\}$$

The kernel of a homomorphism is always a subpolygroup, but usually not normal. Even with strong homomorphisms we cannot assure that the kernel be normal.

Proposition 3. *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be polygroups and let ϕ be a weak polygroup homomorphism. Let $K = ker\phi$, then K is a subpolygroup of $\mathcal{P}_1$.*

Proof. K is a subpolygroup if it is closed under the operation and has inverses. Let $k_1, k_2 \subseteq K$ then

$$\phi(k_1 \curlyvee_{\mathcal{P}_1} k_2) \subseteq \phi(k_1) \curlyvee_{\mathcal{P}_2} \phi(k_2) = u_2 \curlyvee_{\mathcal{P}_2} u_2 = u_2$$

and since u_2 is a singleton we have equality here so indeed $k_1 \curlyvee_{\mathcal{P}_1} k_2 \subseteq K$ To see that K has the needed inverses let $k \subseteq K$ then by *Lemma 4*

$$\phi(k^{-1}) = \phi(k)^{-1} = u_2^{-1} = u_2$$

which shows that $k^{-1} \subseteq K$ and we are done. ∎

Another fact we are going to need for the isomorphism theorems is

Proposition 4. *Let $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ be a strong polygroup homomorphism. Then $Im\phi$ is a subpolygroup of P_2.*

Proof. Let $a, b \subseteq \mathcal{P}_1$ and $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ be a strong polygroup homomorphism. Then

$$\phi(a) \curlyvee_{\mathcal{P}_2} \phi(b) = \phi(a \curlyvee_{\mathcal{P}_1} b) \subseteq \mathrm{Im}\phi.$$

Furthermore

$$\phi(a)^{-1} = \phi(a^{-1}) \subseteq \mathrm{Im}\phi.$$

□

Here we can mention another property of polygroups which mirrors groups namely

Theorem 6. *Let $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ be a strong polygroup homomorphism. Then ϕ is injective as a mapping of sets if and only if $\ker\phi = \{u\}$.*

Proof. First we assume that ϕ is injective. We take an $x \in \ker\phi$ then by definition

$$\phi(x) = u_2 = \phi(u_1) \Rightarrow \phi(x) = \phi(u_1) \Rightarrow x = u_1$$

where we used injectivity in the last implication. For the contrary assume that $\ker\phi = \{u_1\}$ and take y, z such that $\phi(y) = \phi(z)$. Then

$$u_2 = \phi(u_1) \subseteq \phi(y) \curlyvee_{\mathcal{P}_2} \phi(y^{-1}) = \phi(z) \curlyvee_{\mathcal{P}_2} \phi(y^{-1}) = \phi(z \curlyvee_{\mathcal{P}_1} y^{-1})$$

this means that we have an $x \in z \curlyvee_{\mathcal{P}_1} y^{-1}$ such that

$$u_2 = \phi(u_1) = \phi(x)$$

but $\ker\phi = \{u_1\}$ by assumption so $x = u_1$. But this means we have $u_1 \in z \curlyvee_{\mathcal{P}_1} y^{-1}$ which in turn implies that $y = z$ and this is the desired result. □

Let us look at some concrete examples of polygroup homomorphisms.

Example 5. *Let*

$$\mathfrak{f} : \mathfrak{P}_2 \to \mathfrak{P}_1$$

since we used the symbol a in both polygroups we will use indexes to avoid confusion

$$\mathfrak{f}(u_{\mathfrak{P}_2}) = u_{\mathfrak{P}_1}$$

$$\mathfrak{f}(a_{\mathfrak{P}_2}) = \mathfrak{f}(b) = a_{\mathfrak{P}_1}.$$

Then $\mathfrak{f}$ is a strong polygroup homomorphism.

Another, bit more abstract example of a polygroup homomorphism arises from the linear order polygroups we were talking about earlier.

Example 6. *Let X and Y be linearly ordered sets with some additional structure given by the operation $\boxplus$ as in* Example 3 *and with smallest element 0_X and 0_Y respectively. Then any monotone map $\mathfrak{f} : X \to Y$ mapping 0_X to 0_Y is a polygroup homomorphism.*
Let us verify this claim. Let X, Y be sets as above and let f be a monotone increasing mapping, that is

$$a \preceq b \Rightarrow f(a) \preceq f(b) \qquad \forall a, b \subseteq X.$$

For f to be a polygroup homomorphism we need that

$$f(0_X) = 0_Y$$

and that

$$f(a \boxplus b) = f(a) \boxplus f(b).$$

The first equality holds by definition. For the second assume first that $a \preceq b \subseteq X$ then

$$f(a \boxplus b) = f(b).$$

On the other hand since f is monotone we have $f(a) \preceq f(b)$ which means

$$f(a) \boxplus f(b) = f(b).$$

Let now $a = b \subseteq X$, then

$$f(a \boxplus b) = f(\{x \subseteq X \mid x \preceq a\}) = \{x \in X \mid f(x) \preceq f(a)\}.$$

Finally

$$f(a) \boxplus f(b) =$$

since f is monotone we get

$$= \{x \subseteq X \mid f(x) \preceq f(a)\}.$$

Now we have our morphisms between polygroups so we are in the position to make another important observation which will have its own brief section.

1.6 Category theoretical view

This short section is based on Melvin and Ore.

The absence of previous research could be caused by the fact that these polystructures lack the many intresting examples of say groups to urge a development of an abstract theory. In the mindset of trying to relate polygroups to other "more usual" algebraic objects we may find it an interesting fact that polgroups with their strong polygroup homomorphisms constitute a category. So our next goal is to show that we have a category **Pol** of polygroups.
First we need objects to be in **Pol**, let us take the class of all polygroups. We need arrows in between these and the role of the arrows can be played by strong polygroup homomorphisms. The identity arrow will be the identity morphism denote it **1**. Lastly we need to be able to compose these arrows, that is we want every triangle

$$\begin{tikzcd} A \arrow[r, "\mu"] \arrow[dr, "\phi"] & B \arrow[d, "\psi"] \\ & C \end{tikzcd}$$

to commute, where $A, B, C \in \mathrm{obj}(\mathbf{Pol})$ and $\mu, \psi, \phi \in \mathrm{Hom}_{\mathbf{Pol}}$. We can take

$$\phi = \psi \circ \mu$$

where $\circ$ means usual function composition, that is do μ first then ψ whenever this is defined we can take ϕ to be this composition. And we are actually done, which should not come as a surprise. Category theory doesn't really look at the individual elements of the objects rather how the objects interract.
The embedding into an algebra in a previous section is actually a functor

$$\mathfrak{F} : \mathbf{Pol} \Rrightarrow \mathbf{Alg}_{\mathbb{C}}$$

After this short detour into category theory we can continue with the scrutiny of polygrups.

1.7 Examples

Before we go further and show the main results, that is isomorphism theorems let us break up the monotonity of these definitions and theorems by some more complicated examples of polygroups. These examples are rooted in ordinary abstract algebra.

Example 7. *Conjgacy class polygroups. Let G be a group and let $x, y \in G$. The element y is said to be the conjugate of the element x if there exists an element $a \in G$ suc that $y = axa^{-1}$. If H and K are subgroups of G, then K is said to be a conjugate subgroup of H if there exists $a \in G$ so that $K = aHa^{-1}$. Conjugacy of elements defines an equivalence relation on G.*
So let us denote the collection of conjugacy classes of the group G by $\bar{G}$. Let us also define $\star$, a binary operation on these classes to be the set of all conjugacy classes contained in the elementwise product of the sets. Then the system $< \bar{G}, \star, \{u\}, ^{-1} >$ is a polygroup. Let us demonstrate this construction using the dihedral group D_3, that is the group of symmetries of an equilateral triangle. If we denote the counterclockwise rotation by $120°$ by a and flipping along the vertical axis by b, we find that D_3 is decomposed into the following conjugacy classes

$$C_1 := \{u\}, \quad C_2 := \{a, a^2\}, \quad C_3 := \{b, ab, a^2b\}$$

and let us exhibit the structure of $\bar{D}_3$ in a polygroup table

$\star$	C_1	C_2	C_3
C_1	C_1	C_2	C_3
C_2	C_2	C_1, C_2	C_3
C_3	C_3	C_3	C_1, C_2, C_3

To take a bit more complicated example we can do the same thing with D_4. This consists of 8 symmetries. If we denote the counterclockwise rotation with $90°$ by a and the flip along the vertical axis by b we get

$$D_4 = \{u = a^0, a, a^2, a^3, b, ba, ba^2, ba^3\}$$

If we calculate the conjugacy classes we find the following

$$C_1 = \{u\}, \quad C_2 = \{a^2\}, \quad C_3 = \{a, a^3\}, \quad C_4 = \{ba, ba^3\}, \quad C_5 = \{b, ba^2\}$$

and we can capture the structure of $\bar{D}_4$

$\star$	C_1	C_2	C_3	C_4	C_5
C_1	C_1	C_2	C_3	C_4	C_5
C_2	C_2	C_1	C_3	C_4	C_5
C_3	C_3	C_3	C_1, C_2	C_5	C_4
C_4	C_4	C_4	C_5	C_1, C_2	C_3
C_5	C_5	C_5	C_4	C_3	C_1, C_2

So these polystructures were here all the time, we just failed to recognize them due to their lack of internal beauty.

Example 8. *Double coset algebras. Let G be a group and suppose it has a subgroup H. We define the system $G//H =< \{HgH \mid g \in G\}, \star, H, ^{-1} >$ where $(HgH)^{-1} = Hg^{-1}H$ and*

$$(Hg_1H) \star (Hg_2H) = \{Hg_1hg_2H \mid h \in H\}.$$

This system is called the algebra of double cosets $G//H$ and is a polygroup.
Let us now consider a concrete example. We again will look at D_3 and use the same notation as above. Let $H = \{u, b\}$ and let us look at the double cosets it generates

$$HuH = \{u, b\}, \quad HaH = \{a, a^2, ab, a^2b\} =: K$$

furthermore

$$HuH = HbH = H, \qquad HaH = Ha^2H = HabH = Ha^2bH = K.$$

We can summarize this in a table

$\star$	H	K
H	H	K
K	K	H,K

and we readily see that this is isomorphic to $\mathfrak{P}_1$.
To take a bit more complicated example we can consider D_4 also. Again we pick a subgroup say $H = \{u, a^2, ab, a^3b\}$ and calculate the respective cosets. So

$$HuH = \{u, a^2, b, a^2b\}$$

as expected, then

$$HaH = \{a, a^3, ab, a^3b\} =: K$$

since

$$uau = a^2aa^2 = a^2bab = baa^2b = a$$
$$uaa^2 = a^2au = bab = a^2baa^2b = a^3$$
$$uab = a^2aa^2b = baa^2 = a^2bau = ab$$

and finally

$$uaa^2b = a^2ab = bau = a^2baa^2 = a^3b$$

So here again we get two cosets and after a tedious amount of work we find that

$$H = Ha^2H = HbH = HabH = Ha^3b, \quad K = Ha^3H = HaH = Ha^3bH.$$

So, to summarize we have that

$$D_3//H =< \{H, K\}, \star, H, ^{-1} >$$

and the same for D_4.

2 Isomorphism theorems

For this section we mainly used Davvaz paper. Now after this brief pause we can return to our main result namely the fundamental theorems for polygroups. We change to multiplicative notation, that is we drop $\curlyvee$ in the following.

2.1 The first isomorphism theorem

Theorem 7. *Let $\phi : \mathcal{P}_1 \to \mathcal{P}_2$ be a strong polygroup homomorphism with kernel K such that K is a normal subpolygroup of $\mathcal{P}_1$. Let u_1, u_2 be unit elements of $\mathcal{P}_1, \mathcal{P}_2$ respectively. Then*

$$\mathcal{P}_1/K \simeq Im\phi$$

Proof. Define

$$\psi : \mathcal{P}_1/K \to \mathrm{Im}\phi$$

by

$$\psi(Kp) = \phi(p), \qquad \forall p \subseteq \mathcal{P}_1.$$

First of all ψ is well defined since if we take $p, q \subseteq \mathcal{P}_1$ such that $Kp = Kq$ then

$$kp \subseteq Kp \qquad k \subseteq K, p \subseteq \mathcal{P}$$

$$\phi(kp) = \phi(k)\phi(p) = \phi(p)$$

and

$$qp \subseteq Kq \qquad q \subseteq K, p \subseteq \mathcal{P}$$

$$\phi(kq) = \phi(k)\phi(q) = \phi(q)$$

thus

$$Kp = Kq \Rightarrow \phi(p) = \phi(q)$$

To see that ψ is injective we take $p, q \subseteq \mathcal{P}_1$ with $\phi(p) = \phi(q)$. We have then

$$\phi(p)\phi(q)^{-1} = \phi(pq^{-1}) \supseteq u_2$$

that is

$$\exists k \subseteq pq^{-1} \quad \text{such that} \quad \phi(k) = u_2$$

and

$$k \subseteq pq^{-1} \Rightarrow p \subseteq kq$$

also implies

$$p \in Kq$$

and by *Lemma 2/3* this gives

$$Kp = Kq$$

as required. Finally ψ is surjective by construction. $\square$

2.2 Second isomorphism theorem

Theorem 8. *Let $\mathcal{K}$ and $\mathcal{N}$ be subpolygroups of $\mathcal{P}$, with $\mathcal{N}$ normal, then*

$$\mathcal{K}/\mathcal{N}\cap\mathcal{K}\cong(\mathcal{N}\mathcal{K})/\mathcal{N}$$

Proof. By *Lemma 2/(5)* $\mathcal{N}\mathcal{K}=\mathcal{K}\mathcal{N}$, furthermore $\mathcal{N}\mathcal{K}$ is a subpolygroup of $\mathcal{P}$ and $\mathcal{N}$ is normal in $\mathcal{N}\mathcal{K}$ so $(\mathcal{N}\mathcal{K})/\mathcal{N}$ is defined. Let

$$\phi:\mathcal{K}\to(\mathcal{N}\mathcal{K})/\mathcal{N},\quad \phi(k)=\mathcal{N}k.$$

First ϕ is strong since

$$\phi(u)=\mathcal{N}u=\mathcal{N}$$

and

$$\mathcal{N}(k_1k_2)=\mathcal{N}k_1\mathcal{N}k_2.$$

by *Lemma 2/(2)*. For surjectivity of ϕ, take an element from the image $\mathcal{N}a\subseteq\mathcal{N}\mathcal{K}/\mathcal{N}$, with $a\subseteq\mathcal{N}\mathcal{K}$, that is $a\subseteq nk$. By *Theorem 5* we have

$$\mathcal{N}a=\mathcal{N}nk=\mathcal{N}k=\phi(k)$$

as required. This means that

$$Im\phi=\mathcal{N}\mathcal{K}/\mathcal{N}$$

Now if we can show that $\ker\phi=\mathcal{N}\cap\mathcal{K}$ then by the first isomorphism theorem we are done. Let $k\subseteq\mathcal{K}$ then we have the following chain

$$k\subseteq\ker\phi \iff$$

$$\phi(k)=\mathcal{N} \iff$$
$$\mathcal{N}k=\mathcal{N} \iff$$
$$k\subseteq\mathcal{N} \iff$$
$$k\subseteq\mathcal{N}\cap\mathcal{K}$$

where the first and second implication is by the definition of ϕ, the third is by *Lemma 2/(3)*, the fourth is by assumption. And we are done. □

And last but not least

2.3 Third isomorphism theorem

Theorem 9. *If $\mathcal{K}$ and $\mathcal{N}$ are normal subpolygroups of a polygroup $\mathcal{P}$ such that $\mathcal{N}\subseteq\mathcal{K}$, then $\mathcal{K}/\mathcal{N}$ is a normal subpolygroup of $\mathcal{P}/\mathcal{N}$ and*

$$(\mathcal{P}/\mathcal{N})/(\mathcal{K}/\mathcal{N})\simeq\mathcal{P}/\mathcal{K}.$$

Proof. Obviously $\mathcal{N}$ is normal in $\mathcal{K}$ so $\mathcal{K}/\mathcal{N}$ and $\mathcal{P}/\mathcal{N}$ are defined. Since both are subpolygroups and one is contained in the other so $\mathcal{K}/\mathcal{N}$ is a subpolygroup of $\mathcal{P}/\mathcal{N}$. It is normal since

$$\mathcal{N}p^{-1}k\mathcal{N}p\mathcal{N} \subseteq \mathcal{N}p^{-1}kp\mathcal{N} = \mathcal{N}p^{-1}kp \subseteq \mathcal{K}/\mathcal{N}, \quad k \in \mathcal{K}, p \in \mathcal{P}.$$

Now define

$$\phi : \mathcal{P}/\mathcal{N} \to \mathcal{P}/\mathcal{K}$$

by

$$\phi(\mathcal{N}x) = \mathcal{K}x.$$

We have

$$\mathcal{N} \subseteq \mathcal{K}$$

by assumption, so

$$\mathcal{K}\mathcal{N} \subseteq \mathcal{K}$$

furthermore

$$\mathcal{K} \subseteq \mathcal{K}\mathcal{N} \subseteq \mathcal{K}$$

so

$$\mathcal{K}\mathcal{N} = \mathcal{K}$$

ϕ is well defined since if we take $p, q \in \mathcal{P}$ with $\mathcal{N}p = \mathcal{N}q$, then

$$\mathcal{N}p = \mathcal{N}q \Rightarrow$$

$$\mathcal{K}\mathcal{N}p = \mathcal{K}\mathcal{N}q \Rightarrow$$

$$\mathcal{K}p = \mathcal{K}q.$$

The above defined ϕ is a strong polygroup homomorphism since

$$\phi(\mathcal{N}) = \mathcal{K}$$

and for $p, q \subseteq \mathcal{P}$

$$\phi(\mathcal{N}p\mathcal{N}q) = \phi(\mathcal{N}pq)$$

since $\mathcal{N}$ is normal, and by the definition of ϕ we get

$$\phi(\mathcal{N}pq) = \mathcal{K}pq$$

on the other hand

$$\phi(\mathcal{N}p)\phi(\mathcal{N}q) = \mathcal{K}p\mathcal{K}q$$

since $\mathcal{K}$ is normal we have

$$\mathcal{K}p\mathcal{K}q = \mathcal{K}\mathcal{K}pq$$

and

$$\mathcal{K}\mathcal{K}pq = \mathcal{K}pq$$

since $\mathcal{K}$ is a polygroup.

Now ϕ is surjective by construction and $ker\phi = \mathcal{K}/\mathcal{N}$ since if we take

$$a \subseteq ker\phi \quad a = \mathcal{N}p, p \subseteq \mathcal{P}$$

then

$$\phi(a) = \phi(\mathcal{N}p) = \mathcal{K}p = \mathcal{K}$$

therefore $p \subseteq \mathcal{K}$ and then $\mathcal{N}p \subseteq \mathcal{K}/\mathcal{N}$ so we have

$$ker\phi \subseteq \mathcal{K}/\mathcal{N}.$$

To see that $\mathcal{K}/\mathcal{N} \subseteq ker\phi$ take $k \subseteq \mathcal{K}$ so $\mathcal{N}k \subseteq \mathcal{K}/\mathcal{N}$. Then

$$\phi(\mathcal{N}k) = \mathcal{K}k = \mathcal{K}$$

therefore

$$\mathcal{K}/\mathcal{N} = ker\phi$$

as required. □

3 Chromatic polygroups

This nice application comes from Comer and Davvaz.

Chromatic polygroups are obtained from a special edge coloured complete graph by making a multivalued algebra out of the set of coulours, something which is called *colour scheme*.

3.1 Colour schemes

We start by defining what a color scheme is.

Definition 16. *Suppose that we have a set of colours $\mathfrak{C}$ and that ϵ is an involution of $\mathfrak{C}$.*
A color scheme is a system $\mathcal{V} =< V, C_x >_{x\in\mathfrak{C}}$. We will think about V as a set of vertices and C_x is a set of edges (coloured the same) in a directed full graph. $\mathcal{V}$ admits the following:

1. *$\{C_x \mid x \in \mathfrak{C}\}$ partitions $V^2 = \{(a, b) \mid a, b \in V, a \neq b\}$*
2. *$C_x^{\prec} = C_{\epsilon(x)}$, for all $x \in \mathfrak{C}$, where $\prec$ denotes the inverse of a relation*
3. *for all $x \in \mathfrak{C}, a \in V$ there exists $b \in V$ such that $(a, b) \in C_x$*
4. *$C_x \cap (C_y \mathbin{/\!\!\!\backslash} C_z) \neq \emptyset$ implies $C_x \subseteq C_y \mathbin{/\!\!\!\backslash} C_z$, that is the existence of a path coloured (y, z) between two vertices joined by an edge coloured x is independent of the two verices, here $\mathbin{/\!\!\!\backslash}$ denotes the composition of two relations*

Let us look a bit closer at these axioms and translate them into terms of directed graphs. The first one means that we have a full directed graph. The second means that edges are "paired" with their inverses, that is the colour of an edge in one direction depends on the colour of the edge in the other direction. The third translates to the fact that at each node we have at least one outgoing edge of each colour. The last one means that whenever we have a path colored x between two vertices and further there is a path coloured (y, z) between the same two vertices, then whenever we find another edge coloured x between two other vertices we also have a (y, z) colored path as well between them.
Given a colour scheme $\mathcal{V}$ choose a new symbol $I \notin \mathfrak{C}$, then the colour algebra of $\mathcal{V}$ is the system $\mathcal{M}_{\mathcal{V}} =< \mathfrak{C} \cup \{I\}, *, I, ^{-1} >$, where $()^{-1} = \epsilon(x)$ for $x \in \mathfrak{C}$. Furthermore $I^{-1} = I$,

$$x * I = I * x = x, \quad \forall x \in \mathfrak{C} \cup \{I\}$$

and for $x, y, z \in \mathcal{C}$

$$x * y = \{z \in \mathfrak{C} \mid C_z \subseteq C_x \bowtie C_y\} \cup \{I \mid y = x^{-1}.\}$$

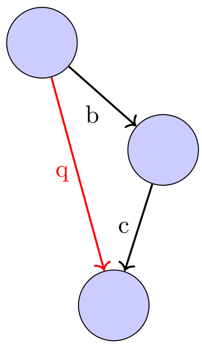
Our next goal is to show that $\mathcal{M}_{\mathcal{V}}$ is a polygroup. We have to verify the conditions of *Definition 9*. We can readily identify I as the unit element. For associativity we have to show that

$$a * (b * c) = (a * b) * c \qquad \forall a, b, c \in \mathfrak{C}.$$

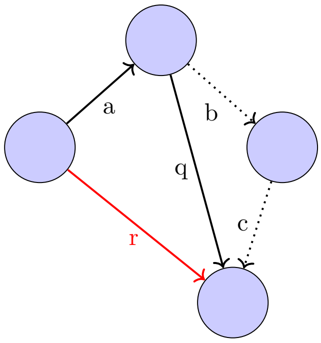
On the left hand side we have

$$b * c = \{q \in \mathfrak{C} \mid C_p \subseteq C_b \bowtie C_c\} \cup \{I \mid c = b^{-1}\}$$

that is all two vertices having a path coloured (b, c) between them also have a path colored q.

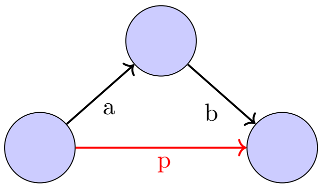


We evaluate $a * q$ which will give an edge coloured r between all nodes already connected by a (a, q) path

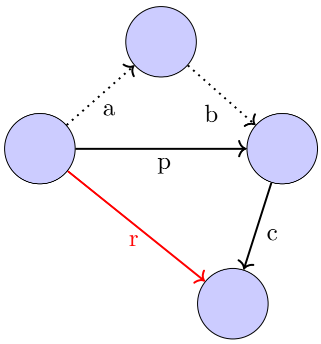


Looking at the right hand side we get

$$a * b = \{p \in \mathfrak{C} \mid Cp \subseteq C_a \Wedge C_b\} \cap \{I \mid a = b^{-1}\}$$



multiplying these p coloured edges with c gives rise to r



from the graphs it is obvious that the operation is associative. In terms or relations we have that a path coloured a, say between nodes A and B means that A is "related" to B, that is ARB. Further let B be related to C (by an egde coloured b), so BRC. Then

$$ARB * BRC = ARC.$$

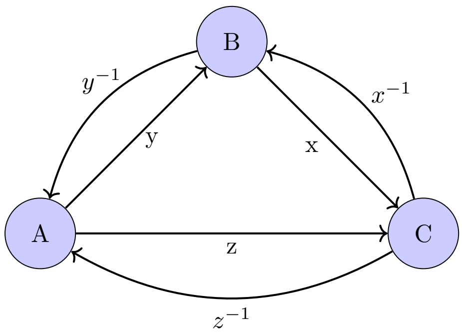
Then associativity translates to (let A, B, C, D be vertices of a directed full graph)

$$(ARB * BRC) * CRD = ARB * (BRC * CRD)$$

which obviously holds. Now the last axiom we have to check is the reversibility property. We have

$$x \subseteq y * z \Rightarrow \qquad y \subseteq x * z^{-1} \quad \text{and} \quad z \subseteq x^{-1} * y$$

Let us draw some graphs. Say, y goes between A and B, x is the colour of the edge beteween B and C and finally z between A and C. We also have edges x^{-1}, y^{-1}, z^{-1} by definition. If we look at the triangle these vertices make up we get



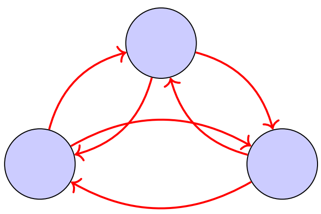
from which the above assertion follows.
In terms of relations this translates to

$$ARC \subseteq ARB * BRC \Rightarrow$$

$$ARB \subseteq ARC * CRB \quad \text{and} \quad BRC \subseteq BRA * ARC.$$

Which again holds clearly. And we are done. A polygroup is chromatic if it is isomorphic to some $\mathcal{M}_{\mathcal{V}}$. As concrete examples let us extract a polygroup from some colour schemes.

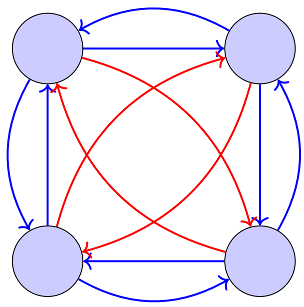
Example 9. *The simplest we can draw is*



with red arrows pointing in all possible directions. This actually gives rise to $\mathfrak{P}_1$ *from our very first example of polygroups.*

To give bit more complicated example we can try with four vertices and two colours and get

Example 10.



reading off from the graph we have (we will use b *for a blue edge and* r *for a red one)*

$$b * b = \{I, r\}$$
$$r * r = I$$
$$b * r = r * b = b$$

$*$	I	r	b
I	I	r	b
r	r	I	b
b	b	b	$\{I, r\}$

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