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Differential forms and Riemannian geometry

- An application to general relativity and gravitational waves

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Abstract

The remarkable theory of general relativity is fundamental for understanding many physical properties of our Universe. The theory connects curvature of 4-dimensional spacetime to gravity. This thesis focuses on the mathematical foundation of curvature, a property of shapes and geometries. The curvature of shapes in Euclidean geometry, i.e. shapes in $\mathbb{R}^3$, are particularly easy to analyse since $\mathbb{R}^3$ has zero curvature. The generalisation of Euclidean geometry is called Riemannian geometry. Useful concepts in Riemannian geometry are defined and derived. Then, using differential forms (multilinear antisymmetric tensor fields), Cartan's structural equations and the Riemann curvature tensor, it is shown how to calculate curvature. This is applied to general relativity and used to illustrate how the existence of gravitational waves can be predicted in theory. Such a prediction was verified in 2016 with the detection of gravitational waves from two merging black holes in a galaxy far, far away.

Keywords: Cartan's structural equations, differential forms, general relativity, gravitational waves, Riemann curvature tensor, Riemann geometry.

I. Introduction

Since Albert Einstein first introduced his theory of general relativity gravity has become synonymous with *curved spacetime*. A popular way to illustrate gravity is to place a heavy ball on a stretched smooth blanket and roll a marble around the curved blanket. But the true nature of gravity is far from as intuitive as this illustration suggests. There is *curvature*, a concept in geometry and it needs a mathematical approach described by differential forms. *Spacetime* can be treated as a four dimensional manifold, three spatial dimensions and one time dimension. General relativity (GR) connects the curvature of spacetime to matter and energy, and for this one needs an approach founded in physics.

Clearly, the subject of differential forms and GR is not an easy one, so this thesis focuses on mathematics and not on physics. However, some GR is needed and these parts will be explained, but not fully derived nor proved, in order for the reader to follow. The ambition is that those with previous knowledge in elementary differential geometry (in $\mathbb{R}^n$), the basics of curvature in $\mathbb{R}^3$ and with a personal interest in physics will follow with ease and hopefully build a natural next step in the understanding of curvature and its application to GR.

The motivation behind this thesis comes from the fact that many courses in GR taught to physics students appear, at least to the experience of the author, to neglect the existence of differential forms and work exclusively with classical tensor calculus. As a consequence, the calculations can become more complex and lengthy than necessary. There are some beautiful results in differential geometry, for instance Cartan's structural equations, that are useful for reducing complexity and should be included in every physicist's toolbox.

Section II reviews curvature in $\mathbb{R}^3$, defining concepts such as ON-basis and intrinsic curvature in Euclidean geometry. Sections III and IV introduce Riemannian geometry and generalise many of the concepts of curvature in this setting. The last two sections V and VI connects curvature in Riemannian geometry to GR, which is used to show how the existence of gravitational waves can be predicted.

Notational conventions

A list of the notational conventions used throughout the thesis follows here. For a more detailed background see [Munkres, 1997], [Lee, 2013] and [Dray, 2015] where much of the inspiration is drawn from.

> **Convention.** Only C^∞ manifolds, maps, vector fields and forms are considered unless anything else is explicitly stated.

> **Convention.** Only finite vector spaces are considered. Hence, the explicit index notation $\{\cdot\}_{i=1}^n$ for sets of coordinates or bases is omitted and simply written $\{\cdot\}$.

Convention. The *Einstein summation convention*, where a sum is implied by repeated indices, is used. Lower-case index letters are used for 1-forms and upper-case letters for k-forms where $k > 1$. For example, given coordinates $\{x^i\}$ on a 3-dimensional space. If $\{dx^i\}$ is a basis for 1-forms then the 1-form α can be expressed as

$$\alpha = \alpha_i dx^i = \alpha_1 dx^1 + \alpha_2 dx^2 + \alpha_3 dx^3, \tag{1}$$

where the coefficients α_i are 0-forms.

Remark. The theory in this thesis is mostly local in the sense that many concepts are defined pointwise. This pointwise dependence is not always explicitly written out, but rather implied or clear from previous definitions.

II. Review of curvature in $\mathbb{R}^3$

A natural start is to consider flat Euclidean space, $\mathbb{R}^n$. The meaning of *flat* is of course ambiguous so far, but this will become clearer later on. Most of the content here will be defined for $\mathbb{R}^n$, but curvature will only be considered for 1- and 2-dimensional objects in $\mathbb{R}^3$. This section should be familiar to the reader, therefore proofs will be omitted but referenced instead. Nonetheless, these concepts are good for understanding curvature in a more abstract setting and worth a review.

Tangent spaces and differential forms in $\mathbb{R}^n$

A full review of tensors and differential forms are not given here, so for a complete background see either [Munkres, 1997] or [Lee, 2013].

One definition of the tangent space to $\mathbb{R}^n$ at a point $x_0 \in \mathbb{R}^n$ is given by

$$\mathbb{R}^n_{x_0} = \{x_0\} \times \mathbb{R}^n = \{(x_0\,;\,v) \,:\, v \in \mathbb{R}^n\}. \tag{2}$$

A tangent vector, denoted by either $(x_0\,;\,v)$ or v_{x_0}, is an element of this set. Adding the operations $v_{x_0} + w_{x_0} = (v+w)_{x_0}$ and $c(v_{x_0}) = (cv)_{x_0}$, where c is a real constant, for the elements of $\mathbb{R}^n_{x_0}$ makes this into a vector space with the standard basis $\{\bar{\mathbf{e}}_i\}$[1]. The vector part of v_{x_0} can then be written $v = v^i\,\bar{\mathbf{e}}_i$, where v^i are scalars. Tangent spaces to submanifolds in $\mathbb{R}^n$ can be considered as subsets of $\mathbb{R}^n_{x_0}$. An example is the 2-sphere in $\mathbb{R}^3$, where a tangent space is defined as the set of tangent vectors orthogonal to the radial unit vector through a point on the sphere. This construction is possible since submanifolds in Euclidean space inherits the dot product, from which orthogonality can be defined. However, a problem with orthogonality would arise if this definition were to be carried over to a non-Euclidean manifold where, in lack of a natural inner product, orthogonality is undefined. Therefore, the idea of a tangent space is developed further.

Definition 1. Consider a point $x_0 \in \mathbb{R}^n$. A map $w \,:\, C^\infty(\mathbb{R}^n) \to \mathbb{R}$ which is linear over $\mathbb{R}$ and satisfy the product rule

$$w(fg) = (wf)g(x_0) + f(x_0)(wg) \tag{3}$$

for $f, g \in C^\infty(\mathbb{R}^n)$ is a *derivation at* x_0. The set of all such derivations at x_0 is denoted $T_{x_0}(\mathbb{R}^n)$.

Under the operations $w_1 f + w_2 f = (w_1 + w_2)f$ and $c(w)f = (cw)f$, $T_{x_0}(\mathbb{R}^n)$ is a vector space. The following proposition binds the two definitions in $\mathbb{R}^n$ together.

[1] The standard basis is the usual basis $\bar{\mathbf{e}}_1 = (1, 0, \ldots, 0)$, $\bar{\mathbf{e}}_2 = (0, 1, \ldots, 0)$, etc.

Proposition 1. For each tangent vector $v_{x_0} \in \mathbb{R}^n_{x_0}$, the directional derivative map $D_v|_{x_0} : C^\infty(\mathbb{R}^n) \to \mathbb{R}$, given by

$$D_v|_{x_0} f = \left.\frac{d}{dt}\right|_{t=0} f(x_0 + tv) \quad \text{or} \tag{4}$$

$$D_v|_{x_0} f = v^i \frac{\partial f}{\partial x^i}(x_0) \quad \text{in the standard basis } \{\bar{\mathbf{e}}_i\}, \tag{5}$$

is a derivation at x_0. Moreover, the map $v_{x_0} \mapsto D_v|_{x_0}$ is an isomorphism from $\mathbb{R}^n_{x_0}$ onto $T_{x_0}(\mathbb{R}^n)$.

Due to the above proposition, from here on the *tangent space to* $\mathbb{R}^n$ *at* x_0 refers to $T_{x_0}(\mathbb{R}^n)$ and *tangent vectors* to elements of this set.

Example 1. The tangent space to the 2-sphere $\mathbb{S}^2$ at a point p can be calculated using the function f which maps from spherical coordinates (θ, ϕ) to the Cartesian (x, y, z) by

$$f(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi), \tag{6}$$

where $\theta \in (0, 2\pi)$ and $\phi \in (0, \pi)$. The basis vectors for $T_p(\mathbb{S}^2)$ are calculated by $D_v|_p$ where v_p is taken to be one of the basis vectors $\hat{\boldsymbol{\theta}}$ or $\hat{\boldsymbol{\phi}}$.

$$v_p = \hat{\boldsymbol{\theta}} \quad \Rightarrow \quad D_v|_p f = \frac{\partial f}{\partial \theta}(p) = (-\sin\theta \sin\phi, \cos\theta \sin\phi, 0)|_p \tag{7}$$

$$v_p = \hat{\boldsymbol{\phi}} \quad \Rightarrow \quad D_v|_p f = \frac{\partial f}{\partial \phi}(p) = (\cos\theta \cos\phi, \sin\theta \cos\phi, -\sin\phi)|_p. \tag{8}$$

The tangent space $T_p(\mathbb{S}^2)$, a plane in $\mathbb{R}^3$, is spanned by the above vectors after normalisation. //

Definition 2. The set of all k-tensors on a vector space V is denoted $\mathcal{L}^k(V)$. The subspace of all alternating k-tensors on V is denoted $\mathcal{A}^k(V)$, where $k \geq 0$

The concept of differential forms is based on tensor fields and tangent spaces.

Definition 3. Let U be an open set in $\mathbb{R}^n$. A *k-tensor field* in U is a function η that for each point $x \in U$ assigns a k-tensor defined on the vector space $T_x(\mathbb{R}^n)$. That is, $\eta(x) \in \mathcal{L}^k(T_x(\mathbb{R}^n))$ for all x.

If the assigned k-tensor is alternating for each point $x \in U$, η is called a *differential k-form*, a *k-form* or simply a *form*. So for each point x it follows that
$\eta(x) \in \mathcal{A}^k(T_x(\mathbb{R}^n)) \subset \mathcal{L}^k(T_x(\mathbb{R}^n))$ [2].

[2] In [Lee, 2013], $\mathcal{L}^k(T_x(\mathbb{R}^n))$ is referred to the *covariant k-tensor field* and has a quite different notation. The rationale behind the notational difference can be found in [Munkres, 1997] p.235.

For a general submanifold M in $\mathbb{R}^n$ where $p \in M$, the bundle of all differential k-forms on M is then

$$\mathcal{A}^k(T(M)) = \bigcup_{p \in M} \mathcal{A}^k(T_p(M)). \tag{9}$$

Definition 4. Let M be a manifold in $\mathbb{R}^n$. The sum of two k-forms of the bundle $\mathcal{A}^k(T(M))$ is also a k-form, and so is the product of a k-form and a scalar. Thus, under these two operations $\mathcal{A}^k(T(M))$ is a vector space. Since this vector space is constantly in use it will be denoted $\bigwedge^k(M)$ or $\bigwedge^k$ and referred to as the *linear space of k-forms* on M.

In order to get a sense for the structure of these spaces let $\dim(T_p(M)) = n$. The dimension of $\bigwedge^k(M)|_p$, at a given point p, is then given by the binomial coefficient, $\dim(\bigwedge^k(M)|_p) = \binom{n}{k}$.

Coordinate basis for differential forms

The simplest case is $k = 0$.

Definition 5. Let A be an open subset of $\mathbb{R}^n$. A function $f : A \to \mathbb{R}$ where $f \in C^\infty(\mathbb{R}^n)$ is a *scalar field* or a *0-form*.

This definition implies that for manifolds in $\mathbb{R}^n$ the set $\bigwedge^0(M)$ is the set of all scalar fields over M.

For $k = 1$, consider first the tangent space $T_{x_0}(\mathbb{R}^n)$. A basis for this space can be defined by calculating $D_{\bar{\mathbf{e}}_i}|_{x_0}$, $i = 1, 2, \ldots, n$. This gives the *coordinate basis* $\{\frac{\partial}{\partial x^i}|_{x_0}\}$ for $T_{x_0}(\mathbb{R}^n)$. For differential forms, the corresponding coordinate basis is the basis dual to $\{\frac{\partial}{\partial x^i}|_{x_0}\}$.

Definition 6. Let M be a manifold in $\mathbb{R}^n$ and $\{x^i\}$ coordinates on an open subset $U \subset M$. For each $p \in U$ the dual basis to $\{\frac{\partial}{\partial x^i}|_p\}$ is $\{dx^i|_p\}$, also called a *coordinate basis*. The basis elements dx^i are called *differentials*.

Considering that $T_p(U)$ is n-dimensional, the dual basis $\{dx^i|_p\}$ must also span an n-dimensional vector space which is the space of *1-forms* at p, i.e. $\bigwedge^1(U)|_p$.

A fundamental theorem in differential geometry defines this differential operator.

Theorem 1. Let M be a manifold in $\mathbb{R}^n$. There exists a unique linear transformation

$$d : \bigwedge^k(M) \to \bigwedge^{k+1}(M), \tag{10}$$

for all $k \geq 0$ called the *exterior differential*. The following properties hold for d:

(1) If f is a 0-form, then df is the 1-form

$$df = \frac{\partial f}{\partial x^i} dx^i. \tag{11}$$

(2) If $\alpha = f\, dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ is a k-form, then $d\alpha$ is the $k+1$ form

$$d\alpha = df \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k} \tag{12}$$

(3) If β and γ are k and l forms, then

$$d(\beta \wedge \gamma) = d\beta \wedge \gamma + (-1)^k \beta \wedge d\gamma. \tag{13}$$

(4) For every form α or f

$$d(d\alpha) = d(df) = 0. \tag{14}$$

The proof can be found in [Munkres, 1997] or [Lee, 2013].

For $k > 1$, take for example $k = 2$ and $n = 3$. Let M be a manifold in $\mathbb{R}^3$ and $\{x^i\} = \{x^1, x^2, x^3\}$ coordinates on an open subset $U \subset M$. The vector space $\bigwedge^2(U)|_p$ has dimension 3, so the basis must consist of three independent basis elements. It is clear that for each $p \in U$, the 1-forms $\{dx^1, dx^2, dx^3\}|_p$ span $\bigwedge^1(U)|_p$. Moreover, the wedge product of two 1-forms α and β is a 2-form $\alpha \wedge \beta$. In a coordinate basis,

$$\alpha \wedge \beta = \alpha_i\, dx^i \wedge \beta_j\, dx^j = \alpha_i \beta_j\, dx^i \wedge dx^j. \tag{15}$$

Since $dx^i \wedge dx^i = 0$ and $dx^i \wedge dx^j = -dx^j \wedge dx^i$, a natural choice of a basis for $\bigwedge^2(U)|_p$ is the coordinate basis

$$\{dx^1 \wedge dx^2, dx^1 \wedge dx^3, dx^2 \wedge dx^3\}|_p. \tag{16}$$

The more general definition is

Definition 7. Let M be a manifold in $\mathbb{R}^n$ and $\{x^i\}$ coordinates on an open subset $U \subset M$. For each $p \in U$ the coordinate basis

$$\{dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_k}|_p \,:\, 1 \leq i_1 < i_2 \cdots < i_k \leq n\} \tag{17}$$

spans $\bigwedge^k(U)|_p$. For $k > 1$ the notation dx^I is used. Capital I denotes the index set $I = (i_1, \ldots, i_k)$ of length k.

Before the next part, let's introduce an additional convention.

Convention. The coordinate basis $\{dx^i|_p\}$ or $\{dx^I|_p\}$ is always pointwise defined. From here on, this dependence on p is implicit when $\{dx^i\}$, $\{dx^I\}$ or $\bigwedge^k(U)$ are written.

ON-basis for differential forms

For tangent vectors in $T_{x_0}(\mathbb{R}^n)$, an orthonormal basis can always be ensured by the dot product and the Gram-Schmidt theorem ([Holst & Ufnarovski, 2014]), but the same tools do not work on forms. The construction of an orthonormal basis of forms requires an inner product that can act on forms, and this inner product is yet to be defined. A good starting point is the following theorem from [Munkres, 1997].

Theorem 2. Let A be an open set in $\mathbb{R}^n$, f a scalar field in A and

$$F(x) = \sum_{i=1}^{n} f_i(x)\bar{\mathbf{e}}_i \tag{18}$$

a vector field in A. There exist a vector space isomorphism, α, between vector fields in A and $\bigwedge^1(A)$, given by

$$\alpha(F) = f_i\, dx^i. \tag{19}$$

Based on this theorem one can identify the standard basis vectors $\{\bar{\mathbf{e}}_i\}$ in $\mathbb{R}^n$ with the coordinate basis $\{dx^i\}$. Since the former is an orthonormal basis it is reasonable to define an inner product on 1-forms where $\{dx^i\}$ is an orthonormal basis as well.

Definition 8. A function $g : \bigwedge^1 \times \bigwedge^1 \to \mathbb{R}$ which satisfy the inner product properties, i.e. g is linear, symmetric and non-degenerate (positive definiteness is not a necessary criteria), and for which

$$g(dx^i, dx^j) = \pm\delta^{ij} \tag{20}$$

holds is called an *inner product on* 1*-forms*.

Definition 9. An *orthonormal basis* (*ON-basis*) for 1-forms is a basis $\{\sigma^i\}$ which is orthonormal under an inner product on 1-forms.

By definition, $\{dx^i\}$ is both a coordinate basis and an ON-basis for 1-forms. However, the following example illustrates that this is not always true in other coordinate systems.

Example 2. Consider the spherical coordinates (r, θ, ϕ). They are related to the Cartesian coordinates by

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{21}$$

The set $\{dr, d\theta, d\phi\}$ is a coordinate basis for $\bigwedge^1$ and can be calculated in terms of dx^i using

Theorem 1.

$$dr = \frac{1}{r}(x\,dx + y\,dy + z\,dz), \tag{22}$$

$$d\theta = \frac{1}{r^2}\left(\frac{xz}{\sqrt{x^2+y^2}}\,dx + \frac{yz}{\sqrt{x^2+y^2}}\,dy - \sqrt{x^2+y^2}\,dz\right), \tag{23}$$

$$d\phi = \frac{1}{x^2+y^2}(-y\,dx + x\,dy). \tag{24}$$

Normality does not hold and this can be seen using the the fact that $\{dx^i\}$ is an ON-basis and with linearity of the inner product g, for example

$$g\,(d\phi, d\phi) = \frac{1}{(x^2+y^2)^2}\,g\,((-y\,dx + x\,dy),(-y\,dx + x\,dy)) = \frac{1}{x^2+y^2}, \tag{25}$$

$$g\,(d\phi, dr) = \frac{1}{r(x^2+y^2)}\,g\,((-y\,dx + x\,dy),(x\,dx + y\,dy + z\,dz)) \tag{26}$$

$$= \frac{1}{r(x^2+y^2)}\,(-xy\,g\,(dx,\,dx) + xy\,g\,(dy,\,dy)) = 0. \tag{27}$$

Instead, one can define the ON-basis $\{dr, r\,d\theta, r\sin\theta\,d\phi\}$. An easy check with g verifies the ON-property. //

An ON-basis for higher order forms can be constructed in a similar manner.

Definition 10. Let $\{\alpha_i\}_{i=1}^k$, $\{\beta_i\}_{i=1}^k$ be 1-forms and consider the k-forms

$$\alpha_1 \wedge \cdots \wedge \alpha_k \quad \text{and} \quad \beta_1 \wedge \cdots \wedge \beta_k. \tag{28}$$

These two k-forms are called *decomposable forms* and one can show that any non-decomposable form can be expressed as a linear combination of decomposable forms. Since an inner product need to be linear, it is enough to define this over decomposable k-forms. Thus, the function $g : \bigwedge^k \times \bigwedge^k \to \mathbb{R}$, $k > 1$, given by the $k \times k$ determinant

$$g(\alpha_1 \wedge \cdots \wedge \alpha_k,\, \beta_1 \wedge \cdots \wedge \beta_k) = \begin{vmatrix} g(\alpha_1,\, \beta_1) & g(\alpha_1,\, \beta_2) & \cdots & g(\alpha_1,\, \beta_k) \\ g(\alpha_2,\, \beta_1) & g(\alpha_2,\, \beta_2) & \cdots & g(\alpha_2,\, \beta_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(\alpha_k,\, \beta_1) & g(\alpha_k,\, \beta_2) & \cdots & g(\alpha_k,\, \beta_k) \end{vmatrix}, \tag{29}$$

is called an *inner product on k-forms*. Moreover, the inner product of two forms of different degrees are zero.

Definition 11. An *orthonormal basis* (*ON-basis*) for k-forms, where $k > 1$, is a basis $\{\sigma^I\}$ which is orthonormal under the inner product on k-forms.

Based on the definition of an inner product on 1-forms, one can show that

$$g(dx^I, dx^J) = \pm\delta^{IJ}, \tag{30}$$

i.e. the coordinate basis $\{dx^I\}$ is also an ON-basis.

Vector-valued differential forms in $\mathbb{R}^n$

For a manifold M in $\mathbb{R}^n$, a differential k-form at $p \in M$ is an alternating k-tensor. An alternating k-tensor is, in turn, a real-valued function. A generalisation of an alternating k-tensor is when the function assigned at p is, instead of real-valued, vector-valued.

Definition 12. Let M be a manifold in $\mathbb{R}^n$ and E a smooth (i.e. class C^∞) vector bundle[3] over M. The *linear space of E-valued k-forms* is defined as

$$\bigwedge^k (M,\, E) = E \otimes \mathcal{A}^k(T(M)). \tag{31}$$

In this notation, ordinary k-forms are elements of the linear space of $\mathbb{R}$-valued k-forms, $\bigwedge^k (M,\, \mathbb{R})$.

Vector-valued forms are also pointwise defined. Essentially, this means that for all $p \in M$ they behave like ordinary forms, but in a local basis that span $E|_p$. The full structure of these vector bundles are too complex to cover in this thesis. Therefore, assume that a local basis of $E|_p$ always exist, is finite and orthonormal.

Definition 13. Consider a vector-valued k-form $\beta \in \bigwedge^k (M,\, E)$ for a manifold M in $\mathbb{R}^n$ and a vector bundle E. For $p \in M$ let $\{\hat{\mathbf{e}}_i|_p\}$ denote the local orthonormal basis of $E|_p$ and $\{\sigma^J\}$ an ON-basis for k-forms. Locally, β can then be written

$$\beta = \alpha^i\, \hat{\mathbf{e}}_i = \alpha^i{}_J\, \sigma^J\, \hat{\mathbf{e}}_i, \tag{32}$$

where α^i are k-forms.

Again, the explicit p-dependence for E and $\{\hat{\mathbf{e}}_i\}$ is from here on made implicit.

A geometric introduction to connection forms

In order to proceed with curvature in $\mathbb{R}^3$ the concept of a connection 1-form need to be introduced. This concept relies on exterior differentiation of vector-valued k-forms and since we do not yet have access to any formal theory about this, only heuristic arguments are given here. In the two following sections these concepts will be more rigorously introduced.

[3]See [Lee, 2013] p.249 for the definition of a vector bundle.

To understand why a connection 1-form is needed, consider Euclidean geometry and more specifically $\mathbb{R}^3$. The dot product induces the standard Euclidean norm which in turn makes $\mathbb{R}^3$ a normed vector space. It is possible to define orthonormality and the standard basis $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$. The structure is then naturally passed on to any manifold M in $\mathbb{R}^3$.

Due to this structure on M, the space of k-forms, $\bigwedge^k(M)$, can be defined as in Definition 4. Moreover, since we're dealing with differential geometry, it is natural to extend this structure with a differential operator

$$d \; : \; \bigwedge^k(M) \to \bigwedge^{k+1}(M), \tag{33}$$

as in Theorem 1. Now, instead of using $\bigwedge^k(M)$, a more general vector space is the space of vector-valued k-forms, defined as in Definition 12. For example, $\bigwedge^k(M, \mathbb{R})$ is the space of $\mathbb{R}$-valued k-forms (which is equal to $\bigwedge^k(M)$) and $\bigwedge^k\left(M, \mathbb{R}^3\right)$ is the space of $\mathbb{R}^3$-valued k-forms, which is a k-form in the standard basis $\{\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2, \bar{\mathbf{e}}_3\}$. Again, it is natural to extend this structure with a differential operator on vector-valued k-forms, but without any formal theory this concept can only be investigated heuristically.

Three reasonable assumptions about this differential operator, also denoted by d, are that

$$d \; : \; \bigwedge^k\left(M, \mathbb{R}^3\right) \to \bigwedge^{k+1}\left(M, \mathbb{R}^3\right), \tag{34}$$

it is linear and obeys the product rule. Thus, from Definition 13

$$d\beta = d(\alpha^i\,\hat{\mathbf{e}}_i) = d\alpha^i\,\hat{\mathbf{e}}_i + (-1)^k\alpha^i \wedge d\hat{\mathbf{e}}_i. \tag{35}$$

Since d is linear over α^i, the first term $d\alpha^i\,\hat{\mathbf{e}}_i$ is well defined if the rules for d over ordinary k-forms coincides with those in Theorem 1, which is also reasonable to assume. Moreover, the structure of the first term is a vector of $k+1$ forms in the basis $\{\hat{\mathbf{e}}_i\}$. Consequently, the second term $\alpha^i \wedge d\hat{\mathbf{e}}_i$ must match this structure. This implies that the objects $d\hat{\mathbf{e}}_j$ are vector-valued 1-forms in the same basis $\{\hat{\mathbf{e}}_i\}$, i.e.

$$d\hat{\mathbf{e}}_j = \omega^i{}_j\,\hat{\mathbf{e}}_i. \tag{36}$$

The 1-forms $\omega^i{}_j$ are called *connection 1-forms*.

For example, the exterior differential (using the above) of an $\mathbb{R}^3$-valued 1-form in the standard basis is

$$
\begin{aligned}
d(\alpha^i\,\overline{\mathbf{e}}_i) &= d\alpha^i\,\overline{\mathbf{e}}_i + (-1)^1\alpha^i \wedge d\overline{\mathbf{e}}_i && (37)\\
&= d\alpha^1\,\overline{\mathbf{e}}_1 + d\alpha^2\,\overline{\mathbf{e}}_2 + d\alpha^3\,\overline{\mathbf{e}}_3 - \Big(\alpha^1 \wedge d\overline{\mathbf{e}}_1 + \alpha^2 \wedge d\overline{\mathbf{e}}_2 + \alpha^3 \wedge d\overline{\mathbf{e}}_3\Big) && (38)\\
&= d\alpha^1\,\overline{\mathbf{e}}_1 + d\alpha^2\,\overline{\mathbf{e}}_2 + d\alpha^3\,\overline{\mathbf{e}}_3 - \Big(\alpha^1 \wedge (\omega^1_{\ 1}\overline{\mathbf{e}}_1 + \omega^2_{\ 1}\overline{\mathbf{e}}_2 + \omega^3_{\ 1}\overline{\mathbf{e}}_3) && (39)\\
&\quad + \alpha^2 \wedge (\omega^1_{\ 2}\overline{\mathbf{e}}_1 + \omega^2_{\ 2}\overline{\mathbf{e}}_2 + \omega^3_{\ 2}\overline{\mathbf{e}}_3) + \alpha^3 \wedge (\omega^1_{\ 3}\overline{\mathbf{e}}_1 + \omega^2_{\ 3}\overline{\mathbf{e}}_2 + \omega^3_{\ 3}\overline{\mathbf{e}}_3)\Big) && (40)\\
&= \Big(d\alpha^1 - \alpha^1 \wedge \omega^1_{\ 1} - \alpha^2 \wedge \omega^1_{\ 2} - \alpha^3 \wedge \omega^1_{\ 3}\Big)\,\overline{\mathbf{e}}_1 && (41)\\
&\quad + \Big(d\alpha^2 - \alpha^1 \wedge \omega^2_{\ 1} - \alpha^2 \wedge \omega^2_{\ 2} - \alpha^3 \wedge \omega^2_{\ 3}\Big)\,\overline{\mathbf{e}}_2 && (42)\\
&\quad + \Big(d\alpha^3 - \alpha^1 \wedge \omega^3_{\ 1} - \alpha^2 \wedge \omega^3_{\ 2} - \alpha^3 \wedge \omega^3_{\ 3}\Big)\,\overline{\mathbf{e}}_3. && (43)
\end{aligned}
$$

The explicit calculations of the connection 1-forms depends on the geometry and additional work is needed before it is possible to construct these. The geometrical interpretation is easier to grasp. From (35) and (36) it is clear that the connection 1-forms describe the infinitesimal change of one basis vector in terms of the basis itself when moving from one point to another, along a manifold. It gives an explicit and well defined way to connect the local frame spanned by $\{\hat{\mathbf{e}}_i\}$ from a point p to a close point q. In $\mathbb{R}^3$ this frame is easy to visualise as a cubic "box" spanned by $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, see Figure 1 for two examples. Curvature will force this frame to twist and turn in various ways and this will manifest itself as non-zero connection 1-forms. The following lemma and examples illustrates this.

Two algorithms for calculating an ON-basis of k-forms and the connection 1-forms in $\mathbb{R}^n$ are given by the following lemma ([O'Neill, 1997]).

Lemma 1. Let $\{\hat{\mathbf{e}}_i\}$ be an orthonormal vector basis in $\mathbb{R}^n$ expressed in the standard Cartesian vector basis

$$\hat{\mathbf{e}}_i = \sum_j a^i_{\ j}\,\overline{\mathbf{e}}_j. \tag{44}$$

The corresponding ON-basis $\{\sigma^i\}$ and the connection 1-forms $\omega^i_{\ j}$ can be calculated in the following way

$$\sigma^i = a^i_{\ j}\,dx^j, \tag{45}$$

$$\omega^i_{\ j} = a^k_{\ j}\,da^i_{\ k}, \tag{46}$$

where d is the exterior derivative for ordinary k-forms. Moreover, it holds that

$$\omega^i_{\ j} = -\omega^j_{\ i}, \tag{47}$$

$$\omega^i_{\ i} = 0. \tag{48}$$

Example 3. In $\mathbb{R}^3$, the geometrical meaning of the connection 1-forms translates to measuring the rate of rotation[4] of the frame field (the cubic "box") along a path. Consider three different paths, (i) along a straight line, (ii) along a spherical surface and (iii) along a general curved line. All of these paths have natural coordinate systems for which there exist a natural choice of frame field. Take note that all paths are embedded in $\mathbb{R}^3$ with the Cartesian vector basis $\{\bar{\mathbf{e}}_i\} = \{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$.

(i) Along a straight line the usual Cartesian coordinates are the natural choice, that is $\{\hat{\mathbf{e}}_i\} = \{\bar{\mathbf{e}}_i\}$. Applying Lemma 1, the $a^i{}_j$ coefficients are given by the matrix a

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{49}$$

This implies that the exterior derivative $d\hat{\mathbf{e}}_i$ is zero along all directions (exterior derivative of a constant vector is zero), i.e. this frame field does not rotate.

(ii) Along a spherical surface, spherical coordinates

$$x = r \sin\theta \cos\phi \tag{50}$$
$$y = r \sin\theta \sin\phi \tag{51}$$
$$z = r \cos\theta, \tag{52}$$

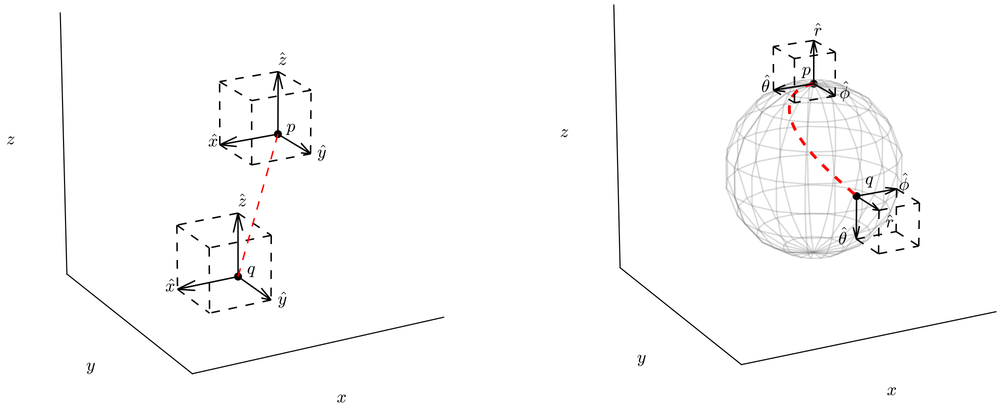
are the natural choice. Thus, in the $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$-frame the $a^i{}_j$ coefficients are given by matrix a, from which the connection 1-forms can be calculated.

$$a = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\theta & 0 \end{pmatrix}, \tag{53}$$

$$(\omega^i{}_j) = \begin{pmatrix} 0 & d\theta & \sin\theta\, d\phi \\ -d\theta & 0 & \cos\theta\, d\phi \\ -\sin\theta\, d\phi & -\cos\theta\, d\phi & 0 \end{pmatrix}. \tag{54}$$

Clearly, this frame field will rotate due to curvature of the path, manifested by non-zero connection 1-forms. Example (i) and (ii) are illustrated in Figure 1 below.

[4]This is *not* the same as the rotation, rot(F), of a vector field F in $\mathbb{R}^3$.



(a) *No rotation along a straight line.*

(b) *A frame spanned by the spherical unit vectors will rotate as it travels along the surface of a sphere.*

Figure 1: *A geometrical interpretation of connection* 1*-forms in* $\mathbb{R}^3$*. In this context, the connection* 1*-forms gives an explicit expression for the rate of rotation of a frame field along a curve.*

(iii) Along a general curve Γ the preferred coordinates are given by parametrising the curve, $\Gamma(t)$. Thereafter, a natural frame field is built up by constructing the tangent vector $T(t)$, the normal vector $N(t)$ and the binormal vector $B(t)$. From this, the connection 1-forms can be calculated. //

Curvature in $\mathbb{R}^3$

Calculating curvature in $\mathbb{R}^3$ relies on Cartan's structural equations [Dray, 2015]. No derivations of these formulas will be given here. These are in fact special cases in Euclidean space and more general derivations will be made in section IV.

Theorem 3 (*Cartan's structural equations in* $\mathbb{R}^n$**).** Let $\{\sigma^j\}$ be an ON-basis to a vector-valued 1-form and $\omega^i{}_j$ its corresponding connection 1-forms. Then, in Euclidean space $\mathbb{R}^n$

(1) the first structure equation (*torsion*) states that

$$0 = d\sigma^i + \omega^i{}_j \wedge \sigma^j, \tag{55}$$

(2) the second structure equation (*curvature*) states that

$$0 = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j. \tag{56}$$

There exist many definitions that involve curvature, e.g. principal curvature, mean curvature, Serret-Frenet formulas (for extrinsic curvature of parametrised curves), but the reason why intrinsic curvature or Gaussian curvature remains the most important follows from Gauss's Theorema Egregium. Two versions are given, a translation of the original Latin text into English and a more modern one.

Theorem 4 (*Theorema Egregium*)**.**

(I) If a curved surface is developed upon any other surface whatever, the measure of [Gaussian] curvature in each point remains unchanged. Also, any finite part whatever of the curved surface will retain the same integral [Gaussian] curvature after development upon another surface.

(II) Let $\Phi : M_1 \to M_2$ be a local isometry between regular surfaces $M_1, M_2 \subset \mathbb{R}^3$. Denote the Gaussian curvatures of M_1 and M_2 by K_1 and K_2. Then

$$K_1 = K_2 \circ \Phi. \tag{57}$$

The proof of (I) can be found in [Gauss, 1827] [5], and (II) in [Gray, 1999]. Based on these two theorems it is now possible with the help of Lemma 1 to calculate the intrinsic curvature of different geometrical objects in $\mathbb{R}^3$. Collectively, these can be investigated as general 1- and 2-dimensional submanifolds in $\mathbb{R}^3$.

Intrinsic curvature of curves in $\mathbb{R}^3$

The simplest object is a smooth parametrised curve $\Gamma(t)$. It will also turn out to be the least interesting object. The Frenet formulas ([Kühnel & Hunt, 2015]) describe the extrinsic curvature for Γ and these can be used to calculate the intrinsic curvature. The frame field is

$$\hat{\mathbf{e}}_1 = T(t) = T^x(t)\,\hat{\mathbf{x}} + T^y(t)\,\hat{\mathbf{y}} + T^z(t)\,\hat{\mathbf{z}}, \tag{58}$$
$$\hat{\mathbf{e}}_2 = N(t) = N^x(t)\,\hat{\mathbf{x}} + N^y(t)\,\hat{\mathbf{y}} + N^z(t)\,\hat{\mathbf{z}}, \tag{59}$$
$$\hat{\mathbf{e}}_3 = B(t) = B^x(t)\,\hat{\mathbf{x}} + B^y(t)\,\hat{\mathbf{y}} + B^z(t)\,\hat{\mathbf{z}}. \tag{60}$$

Omitting the t dependence and applying Lemma 1 gives the ON-basis

$$\sigma^1 = T^x\,dx + T^y\,dy + T^z\,dz, \tag{61}$$
$$\sigma^2 = N^x\,dx + N^y\,dy + N^z\,dz, \tag{62}$$
$$\sigma^3 = B^x\,dx + B^y\,dy + B^z\,dz. \tag{63}$$

[5] For the interested reader, this is a text translated into English 1902

Note that $x(t)$, $y(t)$, $z(t)$ are simply parametrised by t, so $dx = x'(t)\,dt$ and similar for dy and dz. Hence, by the Frenet formulas

$$\sigma^1 = (T^x\,x'(t) + T^y\,y'(t) + T^z\,z'(t))\,dt = \langle T\,,\,\Gamma'(t)\rangle\,dt = \langle T\,,\,T\rangle\,dt = dt, \tag{64}$$
$$\sigma^2 = \langle N\,,\,T\rangle\,dt = 0, \tag{65}$$
$$\sigma^3 = \langle B\,,\,T\rangle\,dt = 0. \tag{66}$$

As seen in Example 3 (i) the exterior derivative of the Cartesian vector basis is zero in all directions. This can be used to calculate the connection 1-forms by noting that

$$dT = d(T^i)\,\overline{\mathbf{e}}_i + T^i\,d\overline{\mathbf{e}}_i = d(T^i)\,\overline{\mathbf{e}}_i \tag{67}$$
$$= ((T^x)'\,dt\,,\,(T^y)'\,dt\,,\,(T^z)'\,dt) \tag{68}$$

Hence, the connection 1-forms are

$$\omega^1_{\ 2} = N^x\,d(T^x) + N^y\,d(T^y) + N^z\,d(T^z) \tag{69}$$
$$= N^x(T^x)'\,dt + N^y(T^y)'\,dt + N^z(T^z)'\,dt \tag{70}$$
$$= \langle N\,,\,T'\rangle\,dt \overset{\text{Frenet}}{=} \langle N\,,\,\kappa N\rangle\,dt = \kappa(t)\,dt, \tag{71}$$
$$\omega^3_{\ 2} = \langle B\,,\,N'\rangle\,dt \overset{\text{Frenet}}{=} \langle B\,,\,-\kappa T + \tau B\rangle\,dt = \tau(t)\,dt, \tag{72}$$
$$\omega^3_{\ 1} = \langle B\,,\,T'\rangle\,dt \overset{\text{Frenet}}{=} \langle B\,,\,\kappa N\rangle\,dt = 0. \tag{73}$$

where κ and τ are extrinsic measures of curvature and torsion of Γ. However, the intrinsic geometry which is analysed by applying Cartan's structural equations only produce zeros, for example $d\omega^1_{\ 2} = \kappa'(t)\,dt \wedge dt = 0$. Hence, the conclusion is that a 1-dimensional submanifold is not particularly interesting in terms of intrinsic curvature.

Gaussian curvature of surfaces in $\mathbb{R}^3$

From Theorema Egregium, the Gaussian curvature of surfaces are known to be independent of the embedding. Therefore, Cartan's second structure equation can be applied from two different point of views and the result should be the same. First, consider a surface (2-dimensional) embedded in *flat* Euclidean geometry $\mathbb{R}^3$. Still, the tools for defining flatness are not developed, but for the case of argument take flatness to be zero curvature of the geometry itself, i.e. specifically not for shapes in the geometry. For the surface, the frame field is obviously spanned in three dimension and for $d\omega^1_{\ 2}$

$$0 = d\omega^1_{\ 2} + \omega^1_{\ k} \wedge \omega^k_{\ 2} = d\omega^1_{\ 2} + \omega^1_{\ 3} \wedge \omega^3_{\ 2}. \tag{74}$$

Second, consider instead the surface by itself, i.e. not embedded in a surrounding geometry. This is perhaps where this review start to become unfamiliar, because now the surface is its own geometry. Unlike the flat Euclidean geometry the surface's own geometry need not to be flat, it can be curved in different ways. From this point of view, Cartan's structural equations, Theorem 3, now becomes a special case for flat geometries. More on this in section III and IV, but for a general surface

$$0 \neq d\omega^1{}_2 + \omega^1{}_k \wedge \omega^k{}_2 = d\omega^1{}_2, \tag{75}$$

where the intrinsic frame field is only spanned in 2 dimensions and the wedged connection 1-forms vanishes completely. Then, as a consequence of Theorema Egregium, (74) and (75) must contain the same information about curvature and, thus, be equal. From this conclusion the following result can be derived ([Dray, 2015])

$$d\omega^1{}_2 = -\omega^1{}_3 \wedge \omega^3{}_2 = -K\,\sigma^1 \wedge \sigma^2, \tag{76}$$

where K is the Gaussian curvature of the surface and an adapted frame field is used (an adapted frame field is when the third basis vector of the frame field, $\hat{\mathbf{e}}_3$, equals the normal unit vector to M).

Example 4. The curvature of the unit 2-sphere can be calculated using the result from Example 3 (ii), Lemma 1 and (76). The normal vector field is given by $\hat{\mathbf{r}}$ (this is clear from Figure 1b). So the adapted frame-field is $\{\hat{\mathbf{e}}_1 = \hat{\boldsymbol{\theta}},\, \hat{\mathbf{e}}_2 = \hat{\boldsymbol{\phi}},\, \hat{\mathbf{e}}_3 = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}}\}$.

Calculating the ON-basis and their wedge product gives the right-hand side of (76). The connection 1-form $\omega^1{}_2$ and the exterior derivative $d\omega^1{}_2$ gives the left-hand side.

$$\sigma^1 = d\theta, \quad \sigma^2 = \sin\theta\, d\phi \quad \Rightarrow \quad \sigma^1 \wedge \sigma^2 = \sin\theta\, d\theta \wedge d\phi, \tag{77}$$

$$\omega^1{}_2 = \cos\theta\, d\phi \quad \Rightarrow \quad d\omega^1{}_2 = -\sin\theta\, d\theta \wedge d\phi. \tag{78}$$

Equating these two gives

$$-\sin\theta\, d\theta \wedge d\phi = -K \sin\theta\, d\theta \wedge d\phi, \tag{79}$$

and the constant Gaussian curvature $K = 1$. //

The intrinsic curvature of volumes in $\mathbb{R}^3$ require the development of Riemann curvature, which is the corresponding measure of intrinsic curvature in higher dimensions and in other geometries. This is a natural step to take in section IV, after introducing Riemannian geometry.

III. Riemannian geometry

Manifolds in $\mathbb{R}^n$ inherit the dot product as the natural inner product, a claim to be proven later. This inner product induces a metric which allow measurements of e.g. distance, volume, angle and consequently the construction of orthonormality. For general manifolds without a well defined inner product it is pointless to try develop any structure for curvature. Hence, it is necessary to generalise the concepts, such as an inner product, from Euclidean geometry. This generalisation is called *Riemannian geometry*. Riemannian geometry is an extensive subject, so the focus will be to develop the tools for curvature. Much of the content is inspired by the books of [Lee, 2013], [Munkres, 1997], [Godinho & Natário, 2014] and [Dray, 2015].

Differentiable manifolds and their fundamental structure

By removing the dependence on Euclidean geometry some structure of the manifold, noticeably the Cartesian coordinate system and the dot product, are lost. This new, more abstract manifold is called a differentiable manifold. In order to define a differentiable manifold it is necessary to redefine the notion of a coordinate patch.

Definition 14. Let U_i be open subsets in $\mathbb{R}^k$ and V_i open subsets in a metric space M such that $\{V_i\}$ covers M. A collection of functions $\varphi_i : U_i \to V_i$, where φ_i and φ_j are

(i) bijective,

(ii) continuous,

(iii) has a continuous inverse, and

(iv) the transition function $\varphi_i^{-1} \circ \varphi_j$ is C^∞ (if the intersection is non-empty),

is called a *collection of coordinate patches*[6] and φ_i a *coordinate patch*.

Definition 15. Let M be a metric space. If there exist a collection of coordinate patches $\{\varphi_i\}$ where φ_i is defined as in Definition 14, the pair $(M, \{\varphi_i\})$ is called a *differentiable k-manifold* or simply a *differentiable manifold*.

It is possible to add additional structure in order to generalise concepts such as tangent spaces, differential forms and coordinate bases, and many of these transfers easily from $\mathbb{R}^n$.

Definition 16. Let M be a differentiable manifold, p a point in M and $\varphi : U \to V$ a coordinate patch covering p. A function $\omega : M \to \mathbb{R}$ is of *class* $C^\infty(M)$ if for all $p \in M$ there exist a coordinate patch φ (covering p) such that $w \circ \varphi \in C^\infty(U)$.

[6] In some literature, an *atlas*.

Definition 17. Let M be a differentiable manifold and p a point in M. A linear map $w : C^\infty(M) \to \mathbb{R}$ which satisfy the product rule

$$w(fg) = (wf)g(p) + f(p)(wg) \tag{80}$$

for $f, g \in C^\infty(M)$ is a *derivation at p*. The set of all such derivations at p is denoted $T_p(M)$ and it is a vector space called the *tangent space to M at p*. The dimension is $\dim(T_p(M)) = \dim(M)$. Elements of $T_p(M)$ are called *tangent vectors at p*.

For an open subset $V \subset M$ and $p \in V$, the definitions of a *k-tensor field* and a *differential k-form* at p are identical to those in Definition 3 but with $T_p(M)$ instead of $T_x(\mathbb{R}^n)$. This leads to

Definition 18. Let M be a differentiable manifold. The set $\mathcal{A}^k(T(M))$ is a vector space denoted $\bigwedge^k(M)$ or $\bigwedge^k$ and referred to the *linear space of k-forms* on M.

A coordinate basis is created analogously to the case in $\mathbb{R}^n$. The following proposition is a result based on a coordinate basis in $\mathbb{R}^n$ and the use of coordinate patches, for a detailed proof see [Lee, 2013].

Proposition 2. Let M be a differentiable manifold and $p \in M$. Then $T_p(M)$ is a k-dimensional vector space. For any smooth coordinate patch φ whose image contains p and gives p a local coordinate representation $\{x^i\}$, the set of vectors $\{\frac{\partial}{\partial x^i}\}$ forms a basis for $T_p(M)$ and this basis is called a *coordinate basis* for $T_p(M)$.

The dual basis $\{dx^i\}$ or $\{dx^I\}$ [which *is* possible to define without an inner product] is the corresponding *coordinate basis* for 1- or k-forms at p.

The properties of the exterior differential d relies on the concept of a pullback, which is defined below.

Definition 19. Let M and N be two differentiable manifolds and $F : M \to N$ a smooth map between these. The *pullback* of F is a function

$$F^* : \bigwedge^k(N) \to \bigwedge^k(M). \tag{81}$$

If $\alpha \in \bigwedge^k(N)$, the coordinate basis representation of α is $\alpha = \alpha_I \, dy^I$ for any set of locally smooth coordinates $\{y^i\}$. The pullback form in $\bigwedge^k(M)$ is then given by

$$F^*\alpha = (\alpha_I \circ F) \; d\left(y^I \circ F\right). \tag{82}$$

Existence, uniqueness and the properties of the exterior differential $d : \bigwedge^k(M) \to \bigwedge^{k+1}(M)$

are almost identically defined as in Theorem 1. The only differences are (i) M is not necessarily in $\mathbb{R}^n$, but rather a differentiable manifold, and (ii) if $\alpha \in \bigwedge^k(M)$, for every coordinate patch φ on M, $d\alpha$ is the unique $k+1$ form to satisfy

$$\varphi^*(d\alpha) = d(\varphi^*\alpha). \tag{83}$$

The above structure on a differentiable manifold is fundamental for doing calculus. However, since orthogonality remains undefined, this approach is limited and obviously not enough to compute curvature. It is necessary to introduce an inner product.

Riemannian manifolds

An inner product on M enables the definition of length and angle, which is necessary for orthonormality on a vector space. There are a few conditions to consider. First, an inner product has two arguments and in this case each argument is a vector in $T_p(M)$. Consequently, the inner product must be a 2-tensor. Second, the properties of an inner product (linearity, symmetry, positive definiteness and non-degeneracy) need to be considered and this gives rise to the Riemannian metric. However, not all of these properties need to be fulfilled in order to define orthogonality. Linearity follows trivially from assuming it to be a tensor and symmetry is clearly necessary for uniqueness. But it turns out that positive definiteness cannot be assumed to hold in GR. Relaxing this condition gives rise to the pseudo-Riemannian metric.

Definition 20. Let M be a differentiable manifold and p a point in M.
A *Riemannian metric* on M, denote it g, is a 2-tensor $g \in \mathcal{L}^2(T_p(M))$ which is

(i) symmetric: $g(v, w) = g(w, v)$ for all $v, w \in T_p(M)$,

(ii) non-degenerate: if $g(v, w) = 0$ for all $w \in T_p(M)$, then it implies $v = 0$, and

(iii) positive definite: $g(v, v) > 0$ for all $v \in T_p(M) \setminus \{0\}$,

for all $p \in M$. From the previous conventions made, it follows that g is of class C^∞.

With a Riemannian metric all the separate tangent spaces $T_p(M)$ are equipped with an inner product, which may vary smoothly with p. Due to this dependence on p there exist different notations for g. In this thesis the following notations are used.

Convention. g denotes a Riemannian metric and $\overline{g}$ denotes the usual dot product in $\mathbb{R}^n$.

Definition 21. A *Riemannian manifold* is a differentiable manifold M equipped with a Riemannian metric g, denoted (M, g).

Proposition 3. A differentiable manifold has a Riemannian metric.

Proof. Let M be a differentiable manifold, U_i an open subset in $\mathbb{R}^k$ and V_i an open subset in M, then M is covered by the coordinate patches $\{\varphi_i\}$, $\varphi_i : U_i \to V_i$.

For submanifolds in $\mathbb{R}^k$, the Riemannian metric is given by the dot product $\overline{g}$ (this is rigorously proven in Corollary 1). Using this, and the pullback of the coordinate patches, each set V_i has it is own Riemannian metric $g_i = (\varphi_i^{-1})^*\overline{g}$.

It is also clear from [Munkres, 1997] (Theorem 41.1) that there exist a partition of unity, $\{\phi_i\}$, where $\phi_i : M \to \mathbb{R}$ is C^∞ and where $\{\phi_i\}$ is dominated by $\{\varphi_i\}$. With g_i and φ_i, it is possible to (pointwise) define a new metric on M

$$g = \sum_i \phi_i g_i. \tag{84}$$

Since supp ϕ_i is locally finite, terms outside of this are zero. Moreover, g_i is pointwise defined for a finite subset V_i. This implies that $\{\phi_i g_i\}$ is locally finite and, thus, only finitely many non-zero points in a neighbourhood around each point. Hence, the sum g is well defined and C^∞ for all points p in M. Moreover, for $v, w \in T_p(M)$

$$g(v,w) = \sum_i \phi_i(p) g_i(v,w) = \sum_i \phi_i(p) g_i(w,v) = g(w,v) \tag{85}$$

so symmetry holds. Next, consider the case

$$g(v,w) = \sum_i \phi_i(p) g_i(v,w) = 0 \tag{86}$$

for all $w \in T_p(M)$. The dot product is obviously non-degenerate, so the only other possibility for this to hold, except non-degeneracy, is if $\phi_i(p) = 0$ for all i. But this is a direct contradiction of a partition of unity. Consequently, g must also be non-degenerate. The last property is positive definiteness. For a non-zero vector $v \in T_p(M)$, $g_i(v,v) > 0$ clearly holds. Also, at least one term $\phi_i(p)$ must be positive not to violate the properties of a partition of unity. Thus, $g(v,v) > 0$ and all properties are verified. This g is the sought Riemannian metric. □

Example 5. The metric space $M = \mathbb{R}^n$ together with the collection of $\{\varphi_i = \mathrm{id} : U_i \to U_i\}$ where U_i is an open subset of $\mathbb{R}^n$ is a differentiable n-manifold. When adding the Riemannian metric $g = \overline{g}$ the pair $(\mathbb{R}^n, \overline{g})$ is a Riemannian manifold.

Another example is the Poincaré upper half-space which in one sense is an embedding in $\mathbb{R}^2$, but with another Riemannian metric than the usual dot product. The differentiable manifold $M = \mathbb{H} = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ together with

$$g = \frac{\overline{g}}{y^2} \tag{87}$$

is a Riemannian manifold. //

Orthogonality and the norm of a tangent vector can now be defined.

Definition 22. Let $v \in T_p(M)$ be a tangent vector. The *norm* of v is defined

$$|v|_g = g(v,v)^{\frac{1}{2}}. \tag{88}$$

Definition 23. For two non-zero vectors $v, w \in T_p(M)$, the angle between them is defined as the unique value $\theta \in [0, \pi]$ where

$$\cos\theta = \frac{g(v,w)}{|v|_g |w|_g}. \tag{89}$$

Two vectors are *orthogonal* if $g(v,w) = 0$.

Submanifolds and the Riemannian metric

A useful property of Riemannian manifolds is that submanifolds inherit the Riemannian metric. This is a natural thing to assume for the dot product and submanifolds in $\mathbb{R}^n$ and it can be proven more generally for Riemannian manifolds. In order to do so the concepts of an immersion and a pushforward between manifolds needs to be defined.

Consider two differentiable manifolds M, N and a smooth function $F : M \to N$. Any point $p \in M$ will map to $F(p) \in N$, but how will F act on a tangent vector $v \in T_p(M)$? This is given by the pushforward of F.

The pushforward can be identified with a linear map, i.e. a matrix, between tangent spaces. When M and N are Euclidean spaces this matrix is the Jacobian of F. When M and N are not Euclidean spaces, use the fact that both are differentiable manifolds and can be mapped to Euclidean spaces via coordinate patches. For open subsets $U \subset M$ and $V \subset N$, if $\varphi : U \to \mathbb{R}^k$ and $\psi : V \to \mathbb{R}^k$ are two such patches covering p and $F(p)$, by defining

$$\widetilde{p} = \varphi(p) \tag{90}$$
$$\widetilde{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U \cap F^{-1}(V)) \to \psi(V), \tag{91}$$

the linear map of interest is the Jacobian of $\widetilde{F}$ at $\widetilde{p}$ ([Lee, 2013]). See Figure 2 for the complete picture. For simplicity, both the Euclidean and the non-Euclidean case are referred to the *Jacobian of F at p* or the *matrix F_* at p*.

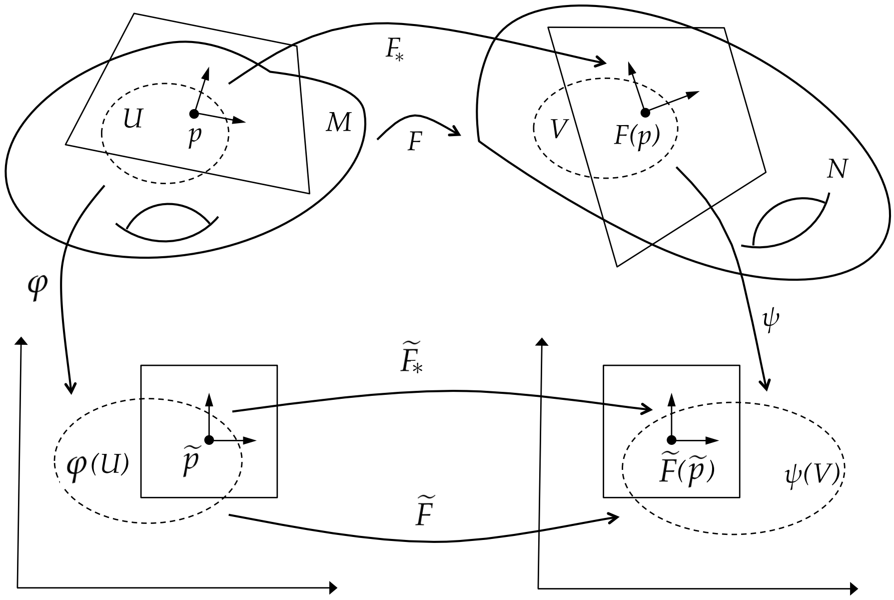


Figure 2: *Illustration of the functions involved to construct the matrix F_* at p.*

Proposition 4. Let M, N be two differentiable manifolds and $F : M \to N$ a smooth function. If $\{x^i\}$ and $\{y^i\}$ are local coordinates in $\varphi(U)$ and $\psi(V)$, the *pushforward of* F at $p \in M$ is defined

$$F_* : T_p(M) \to T_{F(p)}(N). \tag{92}$$

Where $F_*(v)$ for $v \in T_p(M)$ is given by

$$F_*(v) = F_* \left(v^i \frac{\partial}{\partial x^i}\bigg|_p \right) = v^i \frac{\partial \widetilde{F}^j}{\partial x^i}(\widetilde{p}) \, \frac{\partial}{\partial y^j}\bigg|_{F(p)}. \tag{93}$$

The below figure illustrates this geometrically. The proof can be found in [Lee, 2013].

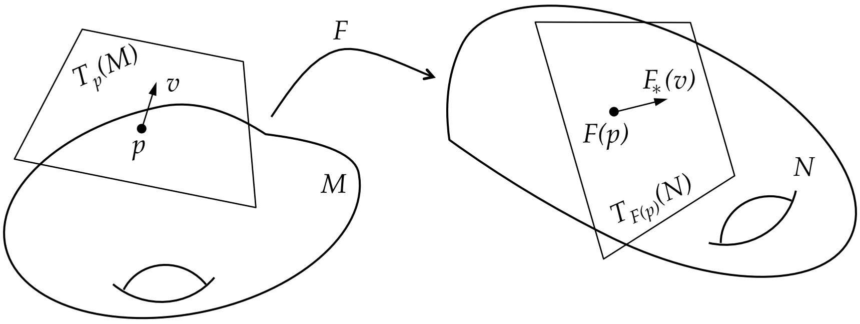


Figure 3: *Geometrical illustration of the pushforward F_* of a smooth function $F : M \to N$ acting on a tangent vector $v \in T_p(M)$. M, N are differentiable manifolds.*

Example 6. Let $M = \mathbb{R}^2$, $N = \mathbb{R}^3$ and $F : \mathbb{R}^2 \to \mathbb{R}^3$,

$$F(x,y) = (x,\, y,\, f(x,y)), \tag{94}$$

for a real-valued smooth function f. The matrix F_* at $p \in M$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial f}{\partial x}(p) & \frac{\partial f}{\partial y}(p) \end{pmatrix}. \tag{95}$$

Consider now a tangent vector $v = v^1 \frac{\partial}{\partial x} + v^2 \frac{\partial}{\partial y}$ in $T_p(M)$. By proposition 4, the pushforward $F_*(v) \in T_{F(p)}(N)$ is given by

$$\begin{pmatrix} v^1 & 0 \\ 0 & v^2 \\ v^1 \frac{\partial f}{\partial x}(p) & v^2 \frac{\partial f}{\partial y}(p) \end{pmatrix}, \tag{96}$$

in the coordinate basis $\{\frac{\partial}{\partial x}|_{F(p)},\, \frac{\partial}{\partial y}|_{F(p)},\, \frac{\partial}{\partial z}|_{F(p)}\}$. //

The matrix F_* can be analysed using linear algebra and this is the rationale behind an immersion.

Definition 24. A smooth function $F : M \to N$ between differentiable manifolds is called a *smooth immersion* if the rank of the matrix F_* fulfils $\text{rank}(F_*) = \dim(M)$ at each point of M.

Proposition 5. Let (N, g) be a Riemannian manifold and suppose $F : M \to N$ is a smooth immersion. Then F^*g is a Riemannian metric on M.

Proof. Verify that F^*g satisfies the properties of a Riemannian metric. First, let $p \in M$ and consider the tangent vectors $v_1, v_2, v_3 \in T_p(M)$. Then,

$$(F^*g)(v_1,\, v_2) = g(F_*(v_1), F_*(v_2)) = g(F_*(v_2), F_*(v_1)) = (F^*g)(v_2,\, v_1), \tag{97}$$
$$(F^*g)(v_3,\, v_3) = g(F_*(v_3), F_*(v_3)) > 0 \tag{98}$$

from the symmetric and positive definite properties of g, assuming v_3 is non-zero. Second, let

$$(F^*g)(v,\, w) = g(F_*(v), F_*(w)) = 0 \tag{99}$$

for all $w \in T_p(M)$. Since g is non-degenerate this implies $F_*(v) = 0$. By assuming F is a smooth immersion, the matrix F_* is non-singular for all p and consequently the only possibility is that $v = 0$. Hence, F^*g is a Riemannian metric. □

Corollary 1. Submanifolds in $\mathbb{R}^n$ inherit the dot product and are therefore Riemannian manifolds.

Proof. From Example 5, $(\mathbb{R}^n, \overline{g})$ is a Riemannian manifold. Consider a submanifold M in $\mathbb{R}^n$ and the trivial construction of $\overline{g}$ restricted to M, $\overline{g}|_M$. In this case F is just the identity matrix for all $p \in M$ and clearly a smooth immersion. Applying the above proposition, $(M, \overline{g}|_M)$ is a Riemannian manifold and M trivially inherit the dot product. □

Pseudo-Riemannian manifolds

By relaxing the condition of positive definiteness in Definition 20 the sign of g can either be positive, negative or zero.

Definition 25. Let M be a differentiable manifold and p a point in M.
A *pseudo-Riemannian metric* on M, also denoted by g, is a C^∞ 2-tensor $g \in \mathcal{L}^2(T_p(M))$ which is (i) symmetric and (ii) non-degenerate for all $p \in M$.

A *pseudo-Riemannian manifold* is a differentiable manifold M equipped with a pseudo-Riemannian metric g: (M, g).

This is an important property because some pseudo-Riemannian manifolds turns out to solve the Einstein field equations. Hence, these manifolds admits a good way for modelling spacetime itself and other physical phenomena therein.

The components of g, in a basis, constitutes a matrix. Since g is symmetric, so is the matrix. Moreover, every symmetric matrix is orthogonally diagonalisable, so when acting on an orthonormal basis, g is similar to a diagonal matrix with entries ± 1 ([Holst & Ufnarovski, 2014]). More on the components of g is given later on. For different pseudo-Riemannian metrics the relaxation of positive definiteness creates a variable in the number of positive and negative values. When acting on an arbitrary basis the variable is the composition of signs of the eigenvalues, which for a diagonal matrix are the diagonal values. The formal definition is

Definition 26. Let g be a pseudo-Riemannian metric represented by a real symmetric matrix. The *signature* of g is the pair (p, q) of the number of positive, p, and negative, q, eigenvalues counted with multiplicity. [7]

[7]Sometimes the definition of the signature also includes the number of zero values, denoted r. But, assuming non-degeneracy excludes zero as an eigenvalue, i.e. for all pseudo-Riemannian metrics the signature in this notation is $(p, q, r = 0)$.

The signature is an important invariant property for g specified by Sylvester's law of inertia.

Theorem 5 (*Sylvester's law of inertia*)**.** Let g be a symmetric 2-tensor. Then the number of positive diagonal entries and the number of negative diagonal entries in any diagonal matrix representation of g are each independent of the diagonal representation.

The original theorem is more general than for 2-tensors, but considering the purpose of this thesis this formulation is enough. For the original theorem and the proof see [Friedberg et al., 2003].

Remark. Sylvester's law of inertia states that (p, q) is an invariant of g. However, different sign conventions are used in different areas of physics. For example, in GR the same g is used but applying the *timelike* convention comes with the signature $(1, 3)$ while the *spacelike* convention has signature $(3, 1)$, both equivalent ([Hartle, 2003]).

Remark. Neither Proposition 3 nor 5 holds in general for pseudo-Riemannian manifolds. Due to the relaxation of positive definiteness a few extra conditions need to be satisfied in order to guarantee the existence of a metric. This is not a problem in GR where these conditions are satisfied, but for other mathematical applications one need to be aware of this.

While introducing an inner product on tangent vectors, it is reasonable to also transfer the idea of an inner product on k-forms from $\mathbb{R}^n$ to pseudo-Riemannian manifolds.

Definition 27. Let $(M, \tilde{g})$ be a pseudo-Riemannian or Riemannian manifold. The function $g : \bigwedge^k \times \bigwedge^k \to \mathbb{R}$, $k \geq 1$, defined similarly as Definition 8 or 10 is an *inner product on k-forms*.

An *ON-basis* for k-forms is a basis $\{\sigma^i\}$ for $k = 1$ or $\{\sigma^I\}$ for $k > 1$ that is orthonormal under an inner product on k-forms.

Convention. The letter g is used for both pseudo-Riemannian metrics and for inner products on k-forms. This is no coincidence since they behave exactly the same on tangent vectors and forms and one can think of them as "duals of inner products". Hence, from this point onwards, the name pseudo-Riemannian metric will also include inner products on k-forms and both will be denoted with g.

Vector-valued differential forms

The full theory and structure of vector-valued forms can be found in [Taubes, 2011]. In this context, the starting point is

Definition 28. Let M be a differentiable manifold, p a point in M and E a smooth vector bundle over M. Assume that for each $p \in M$ there exist an orthonormal and finite basis $\{\hat{\mathbf{e}}_i\}$ that locally span E. The linear space of E-valued k-forms is defined

$$\bigwedge^k (M, E) = E \otimes \mathcal{A}^k(T(M)), \tag{100}$$

where $\beta \in \bigwedge^k (M, E)$ is written

$$\beta = \alpha^i \, \hat{\mathbf{e}}_i = \alpha^i{}_J \, \sigma^J \, \hat{\mathbf{e}}_i. \tag{101}$$

The following properties holds and the proofs can be found in [Taubes, 2011].

Proposition 6. Let M be a differentiable manifold.

Wedge product. If E_1 and E_2 are two smooth vector bundles over M, there exist a wedge product that takes

$$\wedge \, : \, \bigwedge^p (M, E_1) \times \bigwedge^q (M, E_2) \to \bigwedge^{p+q} (M, E_1 \otimes E_2) \, . \tag{102}$$

All properties of the usual wedge product also holds for vector-valued k-forms.

Connection. Let E be a smooth vector bundle over M. It is possible to define a *connection* on $\bigwedge^0 (M, E)$ which connects sections of E, take this to be the local frames $\{\hat{\mathbf{e}}_i\}$, to the corresponding E-valued 1-forms $\bigwedge^1 (M, E)$. Formally, a connection on E is a map

$$\triangledown \, : \, \bigwedge^0 (M, E) \to \bigwedge^1 (M, E) \tag{103}$$

that (i) respects the vector space structure

$$\triangledown(c \, \hat{\mathbf{e}}_i) = c \, \triangledown \, \hat{\mathbf{e}}_i \tag{104}$$

$$\triangledown(\hat{\mathbf{e}}_i + \hat{\mathbf{e}}_j) = \triangledown\hat{\mathbf{e}}_i + \triangledown\hat{\mathbf{e}}_j, \tag{105}$$

for a real constant c, and (ii) obeys the analogue of Leibnitz's rule

$$\triangledown(F \, \hat{\mathbf{e}}_i) = F \, \triangledown \, \hat{\mathbf{e}}_i + \hat{\mathbf{e}}_i \otimes dF, \tag{106}$$

for all $F \in C^\infty(M)$.

Exterior differential. An analogue to the usual exterior differential d is the *exterior differential of vector-valued forms* which is an extension of a connection. If E is equipped with the connection $\triangledown$, there exist a unique operator $d_\triangledown$ for all $p \in M$ where

$$d_\triangledown \, : \, \bigwedge^k (M, E) \to \bigwedge^{k+1} (M, E) \, , \tag{107}$$

for each $k \geq 0$. The differential $d_{\bigtriangledown}$ is defined by the rules

(i) If α is a k-form and $\hat{\mathbf{e}}_i$ an element of E, then

$$d_{\bigtriangledown}(\alpha\,\hat{\mathbf{e}}_i) = d\alpha\,\hat{\mathbf{e}}_i + (-1)^k \alpha \wedge \bigtriangledown\hat{\mathbf{e}}_i. \tag{108}$$

(ii) If β_1 and β_2 are two vector-valued k-forms, then

$$d_{\bigtriangledown}(\beta_1 + \beta_2) = d_{\bigtriangledown}\beta_1 + d_{\bigtriangledown}\beta_2. \tag{109}$$

Additionally, and in big contrast to the ordinary exterior differential d, the general case for a vector-valued k-form β is

$$d^2_{\bigtriangledown}\beta \neq 0. \tag{110}$$

The property $d^2_{\bigtriangledown}\beta \neq 0$ will turn out to be related to curvature.

The metric tensor and its components

In theoretical physics and GR, the pseudo-Riemannian metric g is almost exclusively referred to as the *metric tensor* or the *metric*. From here on these will be used synonymous. The metric tensor is perhaps the single most important tool for understanding GR which is why understanding its components are equally important. Consider first g acting on forms.

Definition 29. If g is a metric tensor, then the *components* of g expressed in an ON-basis are given by

$$g_{IJ} = g(\sigma^I, \sigma^J). \tag{111}$$

In a general coordinate basis

$$g_{IJ} = g(dx^I, dx^J). \tag{112}$$

Trivially, in an ON-basis $g_{IJ} = \pm\delta_{IJ}$ whereas in a coordinate basis g_{IJ} can be real-valued functions. It follows that g, summed over all its components, is a matrix which can be expressed in an ON-basis

$$g = \pm\delta_{IJ}\,\sigma^I \otimes \sigma^J = \pm\delta_{IJ}\,\sigma^I\,\sigma^J, \tag{113}$$

or in a coordinate basis

$$g = g_{IJ}\,dx^I \otimes dx^J = g_{IJ}\,dx^I\,dx^J. \tag{114}$$

Remark. First, make a distinction between the symmetry properties of g itself and the symmetry properties of the arguments σ^I or dx^I. While the arguments are forms and therefore antisymmetric tensors, the metric tensor as a function of forms *is* symmetric, which follows from the definition. Second, in equation (113) and (114), since g is symmetric the tensor product can be written as a symmetric product, i.e $\sigma^I \otimes \sigma^J = \sigma^I \sigma^J$ and $dx^I \otimes dx^J = dx^I dx^J$ (for more details see [Lee, 2013]). The symmetric product leads to a natural way of expressing the diagonal basis elements as squared, i.e.

$$\mathrm{Diag}(g) = \pm 1\, \sigma^I \otimes \sigma^I = \pm 1 \left(\sigma^I\right)^2 \quad \text{or} \tag{115}$$

$$\mathrm{Diag}(g) = g_{II}\, dx^I \otimes dx^I = g_{II} \left(dx^I\right)^2. \tag{116}$$

The symmetric and non-degeneracy criteria from Definition 25 implies that g is a real symmetric and invertible matrix. The components of the inverse matrix are denoted g^{IJ}, with the indices superscripted. This gives the identity matrix

$$g^{IJ} g_{JK} = \delta^I_K. \tag{117}$$

Example 7. Consider the Riemannian manifold $(\mathbb{R}^3, \overline{g})$ with the coordinate basis $\{dx, dy, dz\}$. The metric tensor is a function $\overline{g} : \bigwedge^1 \times \bigwedge^1 \to \mathbb{R}$ given by

$$\overline{g} = \delta_{ij}\, dx^i dx^j = dx^2 + dy^2 + dz^2. \tag{118}$$

Clearly, representing g as a matrix is trivial

$$\overline{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{119}$$

The matrix is real symmetric and invertible. The inverse is $\overline{g}$ itself. Moreover, the eigenvalues are $+1$ with multiplicity three which gives a metric signature of $(3, 0)$.

Consider now a cylinder, C, as a submanifold in $\mathbb{R}^3$. The smooth function

$$F(r, \theta, z) = (r\cos\theta, r\sin\theta, z), \tag{120}$$

where $r > 0$ and θ is limited to a full rotation, maps between the cylindrical and Cartesian coordinates injectively. Both manifolds are differentiable and in Euclidean space, so investigating the rank of the Jacobian of F will tell if F is a smooth immersion. The Jacobian of F,

$$\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{121}$$

has rank 3 at each point. Thus, F is a smooth immersion. Since $(\mathbb{R}^3, \overline{g})$ is a Riemannian manifold, it follows from Proposition 5 that $(C, F^*\overline{g})$ is a Riemannian manifold. The metric tensor on C can be calculated explicitly by the pullback

$$
\begin{aligned}
F^*\overline{g} &= d(r\cos\theta)^2 + d(r\sin\theta)^2 + dz^2 && (122)\\
&= \left(\cos^2\theta + \sin^2\theta\right)\, dr^2 + r^2\left(\cos^2\theta + \sin^2\theta\right)\, d\theta^2 + dz^2 && (123)\\
&= dr^2 + r^2\, d\theta^2 + dz^2. && (124)
\end{aligned}
$$

In the coordinate basis $\{dr,\, d\theta,\, dz\}$, the matrix $F^*\overline{g}$ is

$$F^*\overline{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{125}$$

and in the ON-basis $\{dr,\, r\, d\theta,\, dz\}$ the matrix is

$$F^*\overline{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{126}$$

Both matrices are real symmetric, invertible and have signature (3, 0). //

Example 8. Let $(\mathbb{R}^4, g)$ be a pseudo-Riemannian manifold where g is given by

$$g = -dt^2 + dx^2 + dy^2 + dz^2. \tag{127}$$

The matrix

$$\overline{g} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{128}$$

is real symmetric, invertible and has signature (3, 1). //

In Example 7 it is not a coincidence that both $\overline{g}$ and $F^*\overline{g}$ have the same signature. In the next section it will become clear that the metric signature is connected to an isometric relation between metrics.

Definition 30. If g is a metric tensor, then the *components* of g expressed in a basis to $E|_p$ are

$$g_{ij} = g(\hat{\mathbf{e}}_i,\, \hat{\mathbf{e}}_j). \tag{129}$$

IV. Curvature in Riemannian geometry

With the structure of Riemannian geometry and vector-valued k-forms in place, curvature of Riemannian manifolds can now be investigated.

Connection forms

In light of Proposition 6, the previous geometric introduction to connection 1-forms in section II can now be justified.

Definition 31. A *Lie algebra* over $\mathbb{R}$ is a real vector space, denoted $\mathfrak{g}$, together with a map called the *Lie bracket* that takes $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Remark. The structure of Lie algebras and Lie brackets are left to the reader to investigate (see e.g. [Lee, 2013], [Taubes, 2011]). There is no explicit use of Lie brackets, but most of the results derived here can also be derived using this.

Consider now a Lie algebra-valued k-form, $\beta \in \bigwedge^k (M, \mathfrak{g})$, over a differentiable manifold M. As usual, β is written in its component form $\beta = \alpha^i \,\hat{\mathbf{e}}_i$. Let $\mathfrak{g}$ be equipped with the connection $\bigtriangledown$. The exterior differential of β is

$$d_{\bigtriangledown}(\beta) = d_{\bigtriangledown}(\alpha^i \,\hat{\mathbf{e}}_i) = d\alpha^i \,\hat{\mathbf{e}}_i + (-1)^k \alpha^i \wedge \bigtriangledown \hat{\mathbf{e}}_i, \tag{130}$$

where $d_{\bigtriangledown}\beta \in \bigwedge^{k+1} (M, \mathfrak{g})$. The interesting part in the above calculation is $\bigtriangledown \hat{\mathbf{e}}_i$. The vector space $\bigwedge^0 (M, \mathfrak{g})$ is just a section of $\mathfrak{g}$ that locally can be treated as a real vector space. That is, the elements $\hat{\mathbf{e}}_i$ are vectors. The action of $\bigtriangledown$ on $\hat{\mathbf{e}}_i$ transforms this vector to a Lie algebra-valued 1-form, $\bigwedge^1 (M, \mathfrak{g})$, and based on the importance of these, the components are given a special name.

Definition 32. The 1-form components of the Lie algebra-valued 1-forms $\bigtriangledown \hat{\mathbf{e}}_i \in \bigwedge^1 (M, \mathfrak{g})$ are called *connection 1-forms* and are denoted $\omega^i{}_j$. In component form

$$\bigtriangledown \hat{\mathbf{e}}_j = \omega^i{}_j \,\hat{\mathbf{e}}_i. \tag{131}$$

Naturally, the next step is to derive a way to calculate these connection 1-forms. Let the connections ω_{ij}, with both indices downstairs, be set to $\omega_{ij} = g(\hat{\mathbf{e}}_i, \bigtriangledown \hat{\mathbf{e}}_j)$. Then,

$$\omega_{ij} = g(\hat{\mathbf{e}}_i, \bigtriangledown \hat{\mathbf{e}}_j) = g(\hat{\mathbf{e}}_i, \omega^k{}_j \,\hat{\mathbf{e}}_k) = g(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_k)\omega^k{}_j = g_{ik}\omega^k{}_j. \tag{132}$$

Using the fact that g is invertible it is possible to deduce that

$$\omega^i{}_j = g^{ik}\omega_{kj}. \tag{133}$$

This shows that raising or lowering an index of the connection will alter it and this alteration depends on the metric g. If $g = \overline{g}$ raising or lowering the index will not make any difference since all components are 1. But if the metric from Example 8 is used, raising the index in the t-coordinate will change sign, i.e. $\omega_{tt} = g_{tk}\omega^k{}_t = -\omega^t{}_t$. This alteration can also be a real function. The importance of the connection 1-forms motivates a special notation for its components.

Definition 33. Let $\{\sigma^k\}$ be an ON-basis for 1-forms. The connection 1-forms ω_{ij} are written

$$\omega_{ij} = \Gamma_{ijk}\,\sigma^k, \tag{134}$$

where Γ_{ijk} are called *Christoffel symbols of the first kind*.

The connection 1-forms $\omega^i{}_j$ are written

$$\omega^i{}_j = \Gamma^i{}_{jk}\,\sigma^k, \tag{135}$$

where $\Gamma^i{}_{jk}$ are called *Christoffel symbols of the second kind*. From

$$\omega_{ij} = \Gamma_{ijk}\,\sigma^k = g_{il}\omega^l{}_j = g_{il}\Gamma^l{}_{jk}\,\sigma^k, \tag{136}$$

it is clear that these two components are related through

$$\Gamma_{ijk} = g_{il}\Gamma^l{}_{jk}. \tag{137}$$

In order to derive an expression for the Christoffel symbols, called *Koszul's formula*, a couple of relations need to be investigated further. Start by defining a lower index for the basis elements σ^k.

Definition 34. For an ON-basis of 1-forms, $\{\sigma^k\}$, the lower index components are defined

$$\sigma_i = g(\hat{\mathbf{e}}_i, \sigma^j\,\hat{\mathbf{e}}_j) = g(\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_j)\sigma^j = g_{ij}\sigma^j. \tag{138}$$

It follows that

$$g(\sigma^i, \sigma_j) = g(\sigma^i, g_{jk}\sigma^k) = g(\sigma^i, \sigma^k)g_{jk} = g^{ik}g_{jk} = g^{ik}g_{kj} = \delta^i{}_j, \tag{139}$$

$$g(\sigma^i \wedge \sigma^j, \sigma_p \wedge \sigma_q) = \delta^i{}_p\delta^j{}_q - \delta^i{}_q\delta^j{}_p. \tag{140}$$

The sum of two connection 1-forms with their index flipped is another 1-form, denoted $\Phi = \omega_{ij} + \omega_{ji}$. In the basis $\{\sigma^k\}$

$$\Phi_k\sigma^k = \Gamma_{ijk}\sigma^k + \Gamma_{jik}\sigma^k = (\Gamma_{ijk} + \Gamma_{jik})\sigma^k \quad \Rightarrow \quad \Gamma_{ijk} = \Phi_k - \Gamma_{jik}. \tag{141}$$

Next, consider g acting on $\hat{\mathbf{e}}_k$ and $d_\bigtriangledown(\sigma^j\,\hat{\mathbf{e}}_j)$, where σ^j is an ON-basis for 1-forms.

$$
\begin{aligned}
g(\hat{\mathbf{e}}_k,\, d_\bigtriangledown(\sigma^j\,\hat{\mathbf{e}}_j)) &= g(\hat{\mathbf{e}}_k,\, d\sigma^j\,\hat{\mathbf{e}}_j - \sigma^j \wedge \bigtriangledown\hat{\mathbf{e}}_j) && (142)\\
&= g(\hat{\mathbf{e}}_k,\, d\sigma^j\,\hat{\mathbf{e}}_j) - g(\hat{\mathbf{e}}_k,\, \sigma^j \wedge \bigtriangledown\hat{\mathbf{e}}_j) && (143)\\
&= g(\hat{\mathbf{e}}_k,\, \hat{\mathbf{e}}_j)d\sigma^j - g(\hat{\mathbf{e}}_k,\, \sigma^j \wedge \omega^i{}_j\,\hat{\mathbf{e}}_i) && (144)\\
&= g(\hat{\mathbf{e}}_k,\, \hat{\mathbf{e}}_j)d\sigma^j - g(\hat{\mathbf{e}}_k,\, \hat{\mathbf{e}}_i)\sigma^j \wedge \omega^i{}_j && (145)\\
&= g_{kj}d\sigma^j - g_{ki}\sigma^j \wedge \omega^i{}_j = g_{kj}d\sigma^j - \sigma^j \wedge g_{ki}\omega^i{}_j && (146)\\
&= g_{kj}d\sigma^j - \sigma^j \wedge \omega_{kj} = g_{kj}d\sigma^j + \omega_{kj} \wedge \sigma^j && (147)\\
&= dg_{kj}\sigma^j + \omega_{kj} \wedge \sigma^j = d\sigma_k + \omega_{kj} \wedge \sigma^j && (148)\\
&= d\sigma_k + \Gamma_{kjl}\sigma^l \wedge \sigma^j. && (149)
\end{aligned}
$$

The result is a 2-form, which is denoted $\Psi_k = d\sigma_k + \Gamma_{kjl}\sigma^l \wedge \sigma^j$. Using the expression for Ψ_k and (140), the last relation is given by g acting on the 2-forms $d\sigma_i$ and $\sigma_p \wedge \sigma_q$.

$$
\begin{aligned}
g(d\sigma_i,\, \sigma_p \wedge \sigma_q) &= g(\Psi_i - \Gamma_{ijk}\sigma^k \wedge \sigma^j,\, \sigma_p \wedge \sigma_q) && (150)\\
&= g(\Psi_i,\, \sigma_p \wedge \sigma_q) - \Gamma_{ijk}g(\sigma^k \wedge \sigma^j,\, \sigma_p \wedge \sigma_q) && (151)\\
&= g(\Psi_i,\, \sigma_p \wedge \sigma_q) - \Gamma_{ijk}(\delta^k{}_p\delta^j{}_q - \delta^k{}_q\delta^j{}_p) && (152)\\
&= g(\Psi_i,\, \sigma_p \wedge \sigma_q) + \Gamma_{ipq} - \Gamma_{iqp}. && (153)
\end{aligned}
$$

To summarise, the two fundamental relations for the derivation of Koszul's formula are

$$
\begin{aligned}
\Gamma_{ijk} &= \Phi_k - \Gamma_{jik}, && (154)\\
\Gamma_{ipq} - \Gamma_{iqp} &= g(d\sigma_i,\, \sigma_p \wedge \sigma_q) - g(\Psi_i,\, \sigma_p \wedge \sigma_q). && (155)
\end{aligned}
$$

Proposition 7 (*Koszul's formula*)**.** The Christoffel symbols of the first kind are given by the formula

$$
\begin{aligned}
\Gamma_{ijk} = \frac{1}{2}\Big(& g(d\sigma_i,\, \sigma_j \wedge \sigma_k) - g(d\sigma_j,\, \sigma_i \wedge \sigma_k) - g(d\sigma_k,\, \sigma_i \wedge \sigma_j) && (156)\\
& - g(\Psi_i,\, \sigma_j \wedge \sigma_k) + g(\Psi_j,\, \sigma_i \wedge \sigma_k) + g(\Psi_k,\, \sigma_i \wedge \sigma_j) - \Phi_i + \Phi_j + \Phi_k\Big).
\end{aligned}
$$

If all Ψ and Φ are known, Γ_{ijk} are uniquely determined.

Proof. Using (154) and (155),

$$
\begin{aligned}
2\Gamma_{ijk} &= \Gamma_{ijk} + \Gamma_{ijk} = \Gamma_{ijk} + (\Phi_k - \Gamma_{jik}) = \Gamma_{ijk} - \Gamma_{jik} + \Phi_k && (157)\\
&= \Gamma_{ijk} + (\Gamma_{ikj} - \Gamma_{ikj}) + (\Gamma_{jki} - \Gamma_{jki}) - \Gamma_{jik} + \Phi_k && (158)\\
&= (\Gamma_{ijk} - \Gamma_{ikj}) + \Gamma_{ikj} - \Gamma_{jki} + (\Gamma_{jki} - \Gamma_{jik}) + \Phi_k && (159)\\
&= (\Gamma_{ijk} - \Gamma_{ikj}) + (\Phi_j - \Gamma_{kij}) - (\Phi_i - \Gamma_{kji}) + (\Gamma_{jki} - \Gamma_{jik}) + \Phi_k && (160)\\
&= (\Gamma_{ijk} - \Gamma_{ikj}) + (\Gamma_{jki} - \Gamma_{jik}) + (\Gamma_{kji} - \Gamma_{kij}) + \Phi_j - \Phi_i + \Phi_k && (161)\\
&= g(d\sigma_i, \sigma_j \wedge \sigma_k) - g(\Psi_i, \sigma_j \wedge \sigma_k) + g(d\sigma_j, \sigma_k \wedge \sigma_i) - g(\Psi_j, \sigma_k \wedge \sigma_i) && (162)\\
&\quad + g(d\sigma_k, \sigma_j \wedge \sigma_i) - g(\Psi_k, \sigma_j \wedge \sigma_i) + \Phi_j - \Phi_i + \Phi_k\\
&= g(d\sigma_i, \sigma_j \wedge \sigma_k) - g(d\sigma_j, \sigma_i \wedge \sigma_k) - g(d\sigma_k, \sigma_i \wedge \sigma_j) - g(\Psi_i, \sigma_j \wedge \sigma_k) && (163)\\
&\quad + g(\Psi_j, \sigma_i \wedge \sigma_k) + g(\Psi_k, \sigma_i \wedge \sigma_j) - \Phi_i + \Phi_j + \Phi_k.
\end{aligned}
$$

Dividing each side by 2 gives the final formula. Moreover, since the basis $\{\sigma^k\}$ and the metric g are known and well defined, if all Ψ and Φ are known the values of Γ_{ijk} can be uniquely determined. □

There is an analogous formula for the Christoffel symbols of the second kind that can be derived from (137) and the inverse metric, however, this is not shown here.

The Koszul formula shows that there are some freedom in choosing the connection forms. The most obvious choice, and also the most useful, is to set both variables to zero. This choice of connection is unique and comes with a name.

Definition 35. A *Levi-Civita connection* is a connection 1-form where $\Psi = \Phi = 0$. This is equivalent to

$$\omega_{ij} = -\omega_{ji} \quad \Leftrightarrow \quad \Gamma_{ijk} = -\Gamma_{jik} \quad \text{or} \quad \Gamma^i_{jk} = \Gamma^i_{kj}, \tag{164}$$

a relation called *metric compatible*. Moreover,

$$0 = d\sigma_k + \omega_{kj} \wedge \sigma^j, \tag{165}$$

a relation called *torsion free*.

One advantage with a Levi-Civita connection is that the Koszul formula for the Christoffel symbols reduces considerably.

Corollary 2. Given a Levi-Civita connection, the Koszul's formulas for both kind of Christoffel symbols are

$$\Gamma_{ijk} = \frac{1}{2}\Big(g(d\sigma_i, \sigma_j \wedge \sigma_k) - g(d\sigma_j, \sigma_i \wedge \sigma_k) - g(d\sigma_k, \sigma_i \wedge \sigma_j)\Big), \tag{166}$$

$$\Gamma^i{}_{jk} = \frac{1}{2}\Big(g(d\sigma^i, \sigma_j \wedge \sigma_k) - g(d\sigma_j, \sigma^i \wedge \sigma_k) - g(d\sigma_k, \sigma^i \wedge \sigma_j)\Big). \tag{167}$$

Proof. The formula for Γ_{ijk} follows trivially from Proposition 7 and $\Psi = \Phi = 0$. The formula for $\Gamma^i{}_{jk}$ follows from

$$\Gamma_{ijk} = g_{il}\Gamma^l{}_{jk} \quad \Leftrightarrow \quad \Gamma^l{}_{jk} = g^{il}\Gamma_{ijk} \tag{168}$$

where the below derivation and a subsequent variable change $l \to i$ gives the final result.

$$\Gamma^l{}_{jk} = \frac{1}{2}g^{il}\Big(g(d\sigma_i, \sigma_j \wedge \sigma_k) - g(d\sigma_j, \sigma_i \wedge \sigma_k) - g(d\sigma_k, \sigma_i \wedge \sigma_j)\Big) \tag{169}$$

$$= \frac{1}{2}\Big(g(dg^{il}\sigma_i, \sigma_j \wedge \sigma_k) - g(d\sigma_j, g^{il}\sigma_i \wedge \sigma_k) - g(d\sigma_k, g^{il}\sigma_i \wedge \sigma_j)\Big) \tag{170}$$

$$= \frac{1}{2}\Big(g(d\sigma^l, \sigma_j \wedge \sigma_k) - g(d\sigma_j, \sigma^l \wedge \sigma_k) - g(d\sigma_k, \sigma^l \wedge \sigma_j)\Big). \square \tag{171}$$

Theorem 6 (*Fundamental theorem of Riemannian geometry*)**.** Let (M, g) be a pseudo-Riemannian manifold and p a point in M. Then there exist a unique Levi-Civita connection at p.

The proof can be found in [Godinho & Natário, 2014]. The Levi-Civita connection is the preferred connection and several intrinsic geometric properties are built around this.

Example 9. A good illustration and a useful result for later purposes is to calculate the Christoffel symbols for the Levi-Civita connection in Euclidean space $(\mathbb{R}^n, \overline{g})$. The Christoffel symbols are of course depending on the choice of basis, so this task implies choosing a basis first.

Consider the Cartesian coordinate basis $\{\sigma^k\} = \{dx, dy, dz\}$. Since the metric is the usual dot product there is no difference between up or down index, i.e. $\sigma^k = \sigma_k$ for all k. From (166), it is clear that all Γ_{ijk} components is calculated by taking the exterior derivative of the basis. But

$$d(dx) = d(dy) = d(dz) = 0 \tag{172}$$

for all components. That is, $\Gamma_{ijk} = 0$ for all i, j, k.

Changing basis to the ON-basis $\{dr, r\,d\theta, r\sin\theta\,d\phi\}$ makes it a bit more complex. Still, there is no difference between up or down index due to the metric. However, without accounting for symmetries, there are 27 different Christoffel symbols to calculate. The exterior derivative does not vanish for all basis elements and they need to be calculated manually. A couple of chosen ones are calculated below.

Denote the index $i, j, k = r, \theta, \phi$ and set $i = \theta$, then

$$\Gamma_{\theta rr} = \frac{1}{2}\Big(g(d(r\,d\theta),\, dr \wedge dr) - g(d(dr),\, r\,d\theta \wedge dr) - g(d(dr),\, r\,d\theta \wedge dr)\Big) = 0, \tag{173}$$

$$\Gamma_{\theta r\theta} = \frac{1}{2}\Big(g(d(r\,d\theta),\, dr \wedge r\,d\theta) - g(d(r\,d\theta),\, r\,d\theta \wedge dr)\Big) = g(d(r\,d\theta),\, dr \wedge r\,d\theta) \tag{174}$$

$$= g(dr \wedge d\theta,\, dr \wedge r\,d\theta) = \frac{1}{r} g(dr \wedge r\,d\theta,\, dr \wedge r\,d\theta) = \frac{1}{r}, \tag{175}$$

$$\Gamma_{\theta\phi\phi} = \frac{1}{2}\Big(- g(d(r\sin\theta\,d\phi),\, r\,d\theta \wedge r\sin\theta\,d\phi) - g(d(r\sin\theta\,d\phi),\, r\,d\theta \wedge r\sin\theta\,d\phi)\Big) \tag{176}$$

$$= -g((\sin\theta\,dr \wedge d\phi + r\cos\theta\,d\theta \wedge d\phi),\, r\,d\theta \wedge r\sin\theta\,d\phi) \tag{177}$$

$$= -g(r\cos\theta\,d\theta \wedge d\phi,\, r\,d\theta \wedge r\sin\theta\,d\phi) \tag{178}$$

$$= -\frac{cos\theta}{r\sin\theta} g(r\,d\theta \wedge r\sin\theta d\phi,\, r\,d\theta \wedge r\sin\theta\,d\phi) = -\frac{\cot\theta}{r}. \tag{179}$$

The rest turns out to be zero, i.e.

$$\Gamma_{\theta jk} = \begin{pmatrix} 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{\cot\theta}{r} \end{pmatrix}. \tag{180}$$

The complete description of the Levi-Civita connection are given by the matrices $\Gamma_{\theta jk}$, Γ_{rjk} and $\Gamma_{\phi jk}$. //

Cartan's structural equations

It is now possible to show the calculations leading up to the more general version of Cartan's structural equations, previously Theorem 3. First, torsion and curvature are defined.

Definition 36. Let $\{\sigma^j\}$ be an ON-basis and $\{\hat{\mathbf{e}}_k\}$ a local frame to a Lie algebra-valued 1-form on a pseudo-Riemannian manifold (M, g). The 2-forms Θ^i, given by

$$\Theta^i = g^{ki} g(\hat{\mathbf{e}}_k,\, d_\triangledown(\sigma^j\,\hat{\mathbf{e}}_j)), \tag{181}$$

are called *torsion* 2-*forms*.

The 2-forms $\Omega^i{}_j$, given by

$$\Omega^i{}_j\,\hat{\mathbf{e}}_i = d^2_\triangledown(\hat{\mathbf{e}}_j), \tag{182}$$

are called *curvature* 2-*forms*.

Cartan's structural equations gives a simpler and a more practical way to calculate torsion and curvature, only in terms of the ON-basis and the connection 1-forms.

Theorem 7 (*Cartan's structural equations*)**.** Let $\{\sigma^j\}$ be an ON-basis to a Lie algebra-valued 1-form and ${\omega^i}_j$ its corresponding connection 1-forms, all on a pseudo-Riemannian manifold (M, g). Then,

(1) the first structure equation (*torsion*) states that

$$\Theta^i = d\sigma^i + {\omega^i}_j \wedge \sigma^j, \tag{183}$$

(2) the second structure equation (*curvature*) states that

$${\Omega^i}_j = d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j. \tag{184}$$

Proof. The torsion 2-form is related to $g(\hat{\mathbf{e}}_k, d_\nabla(\sigma^j \hat{\mathbf{e}}_j)) = \Psi_k$, previously derived in (143) - (149).

$$\begin{aligned}
\Psi_k &= g_{kj} d\sigma^j - g_{ki}\sigma^j \wedge {\omega^i}_j \quad [\text{relabel } g_{kj}d\sigma^j \to g_{ki}d\sigma^i] &(185)\\
&= g_{ki}d\sigma^i - g_{ki}\sigma^j \wedge {\omega^i}_j = g_{ki}\left(d\sigma^i - \sigma^j \wedge {\omega^i}_j\right) = g_{ki}\left(d\sigma^i + {\omega^i}_j \wedge \sigma^j\right), &(186)
\end{aligned}$$

taking the inverse of g gives the final result

$$g^{ki}\Psi_k = \Theta^i = d\sigma^i + {\omega^i}_j \wedge \sigma^j. \tag{187}$$

For curvature, remember in Proposition 6 that in general $d^2_\nabla \beta \neq 0$ for a vector-valued k-form. The calculations for a basis element $\hat{\mathbf{e}}_j$ are easy to show. The composition d^2_∇ will act on $\hat{\mathbf{e}}_j$ by taking it into $\bigwedge^2(M, \mathfrak{g})$, i.e. a Lie algebra-valued 2-form is sought.

$$\begin{aligned}
d^2_\nabla(\hat{\mathbf{e}}_j) &= d^2_\nabla(1 \cdot \hat{\mathbf{e}}_j) = d_\nabla(d_\nabla(1 \cdot \hat{\mathbf{e}}_j)) = d_\nabla(1 \wedge \nabla\hat{\mathbf{e}}_j) = d_\nabla(\nabla\hat{\mathbf{e}}_j) &(188)\\
&= d_\nabla({\omega^i}_j\, \hat{\mathbf{e}}_i) = d{\omega^i}_j\, \hat{\mathbf{e}}_i - {\omega^i}_j \wedge \nabla\hat{\mathbf{e}}_i = d{\omega^i}_j\, \hat{\mathbf{e}}_i - {\omega^i}_j \wedge \omega^k_i\, \hat{\mathbf{e}}_k &(189)\\
&\quad [\text{relabel } {\omega^i}_j \wedge \omega^k_i\, \hat{\mathbf{e}}_k \to {\omega^k}_j \wedge {\omega^i}_k\, \hat{\mathbf{e}}_i] &(190)\\
&= d{\omega^i}_j\, \hat{\mathbf{e}}_i - {\omega^k}_j \wedge {\omega^i}_k\, \hat{\mathbf{e}}_i = (d{\omega^i}_j - {\omega^k}_j \wedge {\omega^i}_k)\, \hat{\mathbf{e}}_i &(191)\\
&= (d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j)\, \hat{\mathbf{e}}_i. &(192)
\end{aligned}$$

Fixing i and j gives the curvature 2-forms ${\Omega^i}_j = d{\omega^i}_j + {\omega^i}_k \wedge {\omega^k}_j$. $\square$

With the Levi-Civita connection the torsion is always zero. This reduces the first structure equation to the one given in Theorem 3. Since the Levi-Civita connection is always in use, there is no need to derive an expression for the components of a torsion 2-form. However, this is not the case for a curvature 2-form.

Definition 37. Consider a curvature 2-form in the following component form

$$\Omega^i{}_j = \frac{1}{2} R^i{}_{jkl}\, \sigma^k \wedge \sigma^l. \tag{193}$$

The elements $R^i{}_{jkl}$ are called components of the *Riemann curvature tensor*.

Proposition 8. The components of the Riemann curvature tensor are given by[8]:

$$R^i{}_{jkl} = \partial_k \Gamma^i{}_{jl} - \partial_l \Gamma^i{}_{jk} + \Gamma^i{}_{ks}\Gamma^s{}_{jl} - \Gamma^i{}_{ls}\Gamma^s{}_{jk}. \tag{194}$$

The only non-trivial trace of the Riemann curvature tensor is called the *Ricci curvature tensor* and is given by

$$R^m_{imj}\, \sigma^j = R_{ij}\, \sigma^j \tag{195}$$

where

$$R_{ij} = \partial_l \Gamma^l{}_{ij} - \partial_j \Gamma^l{}_{il} + \Gamma^m_{ij}\Gamma^l{}_{lm} - \Gamma^m_{il}\Gamma^l{}_{jm}. \tag{196}$$

Last, the trace of the Ricci curvature tensor is called the *Ricci curvature scalar* and is denoted

$$R = R_{ij} g^{ij}. \tag{197}$$

In an ON-basis, the curvature 2-forms satisfies the symmetries

$$\Omega_{ij} = -\Omega_{ji}, \tag{198}$$

$$R^i{}_{jlk} = -R^i{}_{jkl}. \tag{199}$$

The proof is omitted due to being more or less straight forward, although tedious, calculations using Cartan's structural equations, a Levi-Civita connection, the definition of the Christoffel symbols and the *Bianchi identities* which are two identities for $R^i{}_{jkl}$. For more details see e.g. [Dray, 2015].

An important result for defining flatness is the following Corollary.

Corollary 3. In Euclidean space $(\mathbb{R}^n, \bar{g})$, all the curvature 2-forms vanishes, i.e. $\Omega^i{}_j = 0$ for all i, j.

Proof. As previously shown, the Christoffel symbols vanishes using a Cartesian coordinate basis. From (194), it is clear that in this basis $R^i{}_{jkl} = 0$ for all i, j, k, l and consequently, $\Omega^i{}_j = 0$ for all i, j.

[8] Since Γ is a 0-form, the exterior derivative $d\Gamma$ is a 1-form. Hence, the following notation is used: $d\Gamma^i{}_{jl} \wedge \sigma^l = \partial_k \Gamma^i{}_{jl}\, \sigma^k \wedge \sigma^l$.

Assume a coordinate transformation from $\{x^i\}$ to $\{y^i\}$. In terms of tensor calculus, a general tensor will be transform

$$\overline{T}^{I'}_{J'}(y^1, \cdots y^n) = \frac{\partial y^{I'}}{\partial x^I} \frac{\partial x^J}{\partial y^{J'}} T^I_J(x^1, \cdots x^n), \tag{200}$$

where I', J' are index set for the new variable y and I, J index set for the old variable x. Using this, the Riemann curvature tensor will transform accordingly

$$\overline{R}^i_{\ jkl}(y^1, \cdots y^n) = \frac{\partial y^i}{\partial x^m} \frac{\partial x^n}{\partial y^j} \frac{\partial x^s}{\partial y^k} \frac{\partial x^t}{\partial y^l} R^m_{\ nst}(x^1, \cdots x^n). \tag{201}$$

Since $R^m_{\ nst} = 0$, the transformed curvature tensor also vanish. Hence, $\overline{\Omega}^i_{\ j} = 0$ for all i, j in any coordinate system. □

This corollary together with a Levi-Civita connection proves the Cartan's structural equations in $\mathbb{R}^n$, Theorem 3. Moreover, in a coordinate transformation from Cartesian to a general set of coordinates, the Christoffel symbols can be shown to transform

$$\overline{\Gamma}^k_{ij}(y) = \frac{\partial x^s}{\partial y^i} \frac{\partial x^t}{\partial y^j} \frac{\partial y^k}{\partial x^m} \Gamma^m_{st}(x) + \frac{\partial y^k}{\partial x^m} \frac{\partial^2 x^m}{\partial y^i \partial y^j} = \frac{\partial y^k}{\partial x^m} \frac{\partial^2 x^m}{\partial y^i \partial y^j}, \tag{202}$$

where the first term vanish since $\Gamma^m_{st}(x) = 0$. The fact that $\overline{\Gamma}^k_{ij}$ not necessarily vanishes can be interpret as a measure of how much the new coordinate basis "curve". This is not a measure of curvature in the way previously defined, but rather a consequence of the coordinates in use. Since curvature is an invariant property of the geometry, the curvature tensor must vanish regardless of the choice of coordinates. A common way to separate coordinate systems based on whether or not the Christoffel symbols vanishes is given by

Definition 38. A coordinate system where the Christoffel symbols vanishes completely is called a *flat coordinate system*. It is possible to do a change of variables from this flat coordinate system to another (in the same geometry). If this new coordinate system has non-zero Christoffel symbols it is called a *curvilinear coordinate system*.

As seen in Example 9, spherical coordinates in $\mathbb{R}^3$ is a curvilinear coordinate system.

Flatness

The Euclidean geometry was previously referred to as flat. A flat geometry can now be accurately defined.

Definition 39. A pseudo-Riemannian manifold (M, g) is called *flat* if all the curvature 2-forms vanishes. That is, $\Omega^i_{\ j} = 0$ for all i, j.

A common misunderstanding with this definition is to assume that if a geometry (i.e. a pseudo-Riemannian manifold) is flat, the curvature of embedded shapes in this geometry are also flat. This is not correct and the classical example of the sphere with non-flat curvature embedded in Euclidean space proves this. Rather, if a geometry is flat the curvature 2-forms in Cartan's second structure equation vanishes. This implies that the intrinsic curvature of an embedded shape in this geometry can be calculated by $d\omega^i{}_j = -\omega^i{}_k \wedge \omega^k{}_j$. Shapes in a flat geometry can be flat or curved. Likewise, shapes in a curved geometry, where the curvature 2-forms are non-zero, can also be flat or curved. In this case the calculations usually becomes more complex due to a non-zero Riemann curvature tensor.

The method for proving Corollary 3 was to find one set of coordinates for which the Riemann curvature tensor vanish and then prove that regardless of the basis, the curvature will remain zero. An alternative way to show that a general Riemannian manifold is flat is to use the fact that Euclidean space is flat and find an isometry between the two manifolds. The same holds for a general pseudo-Riemannian manifold. In this case, the analogy of a flat geometry is a *Minkowski manifold*, $(\mathbb{R}^n, g_{Mink})$, where the metric is called a *Minkowski metric*. This metric is given by

$$g_{Mink} = -(dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2 \tag{203}$$

and has the signature $(n-1, 1)$.

Definition 40. Let (M, g) and $(\widetilde{M}, \widetilde{g})$ be pseudo-Riemannian manifolds. A function $F : M \to \widetilde{M}$ is called a *Riemannian isometry* if F is a diffeomorphism and $F^*\widetilde{g} = g$ holds. In that case (M, g) and $(\widetilde{M}, \widetilde{g})$ are *isometric*.

A function $F : M \to \widetilde{M}$ is called a *local Riemannian isometry* if for all points $p \in M$, F is a local diffeomorphism where $F^*\widetilde{g} = g$ holds. Then (M, g) and $(\widetilde{M}, \widetilde{g})$ are *locally isometric*.

Proposition 9. Consider a Riemannian manifold (M, g). If (M, g) is (locally) isometric to $(\mathbb{R}^n, \overline{g})$, then (M, g) is (locally) flat. In this case one also refers to g as a (locally) *flat metric*.

Identically, a pseudo-Riemannian manifold (locally) isometric to $(\mathbb{R}^n, g_{Mink})$ is (locally) flat.

Proof. Let (M, g) be locally isometric to $(\mathbb{R}^n, \overline{g})$. There exist a neighbourhood V around each $p \in M$ where $F^*\overline{g} = g$ holds and $F : V \to \mathbb{R}^n$ is a diffeomorphism. Also, let $F(p) = x_0$. For each point p and x_0 let $\{dy^i\}$ and $\{dx^i\}$ be the local coordinate bases in M and $\mathbb{R}^n$. It is possible, if necessary, to shrink V in order for the two coordinate bases to be simultaneously valid over V and $F(V)$.

The relation between the metric components are

$$g_{ij}(p)\, dy^i dy^j = F^*(\overline{g}_{ij}(p)\, dy^i dy^j) \tag{204}$$
$$= \overline{g}_{ij}(x_0)\, \frac{\partial F_i}{\partial x^s}(x_0)\, dx^s\, \frac{\partial F_j}{\partial x^t}(x_0)\, dx^t \tag{205}$$
$$= \overline{g}_{ij}(x_0)\, \frac{\partial F_i}{\partial x^s}(x_0) \frac{\partial F_j}{\partial x^t}(x_0)\, dx^s dx^t \tag{206}$$

Based on $\overline{g}$ being a diagonal matrix where $\overline{g}_{ij} = \delta_{ij}$

$$g_{ij} = 0, \qquad \text{if } i \neq j, \tag{207}$$
$$g_{ii} = \sum_{s=1}^{n} \left(\frac{\partial F_i}{\partial x^s} \right)^2, \quad \text{if } i = j. \tag{208}$$

Assume a Levi-Civita connection, then it is possible to derive an alternative Koszul's formula where

$$2\Gamma_{ijk} = \partial_k g_{ij} + \partial_j g_{ik} - \partial_i g_{jk}. \tag{209}$$

The Christoffel symbols can now be calculated.

$$\Gamma_{ijk} = 0 \qquad \text{if } i \neq j, \tag{210}$$
$$\Rightarrow \quad \Gamma^i{}_{jk} = 0 \qquad \text{if } i \neq j, \tag{211}$$
$$\Gamma_{iik} = \partial_k g_{ii} = \sum_{s=1}^{n} \frac{\partial F_i}{\partial x^s} \frac{\partial^2 F_i}{\partial x^s \partial x^k}, \qquad \text{if } i = j, \tag{212}$$
$$\Rightarrow \quad \Gamma^i{}_{ik} = g^{ii}\Gamma_{iik} = \frac{1}{\sum_{m=1}^{n} \left(\frac{\partial F_i}{\partial x^m} \right)^2} \sum_{s=1}^{n} \frac{\partial F_i}{\partial x^s} \frac{\partial^2 F_i}{\partial x^s \partial x^k}, \quad \text{if } i = j. \tag{213}$$

The Riemann curvature tensor is given by (194) and there exist three different cases: (i) $i \neq j$, (ii) $i = j$ and $k = l$ and (iii) $i = j$ and $k \neq l$.

(i) If $i \neq j$ all Christoffel symbols vanishes and $R^i{}_{jkl} = 0$.

(ii) If $i = j$ and $k = l$

$$R^i{}_{ikk} = \partial_k \Gamma^i{}_{ik} - \partial_k \Gamma^i{}_{ik} + \Gamma^i{}_{ks}\Gamma^s{}_{ik} - \Gamma^i{}_{ks}\Gamma^s{}_{ik} = 0. \tag{214}$$

(iii) If $i = j$ and $k \neq l$, consider first the last two terms $\Gamma^i{}_{ks}\Gamma^s{}_{il} - \Gamma^i{}_{ls}\Gamma^s{}_{ik}$. They are only non-zero of $i = k$ or $i = l$. Thus, using the symmetry of Γ under a Levi-Civita connection (Definition 35), for $i = k$

$$\Gamma^i{}_{is}\Gamma^s{}_{il} - \Gamma^i{}_{ls}\Gamma^s{}_{ik} = \Gamma^i{}_{is}\Gamma^s{}_{il} = \Gamma^i{}_{ii}\Gamma^i{}_{il} = \Gamma^i{}_{ii}\Gamma^i{}_{li} = 0. \tag{215}$$

The same holds for $i = l$, so $\Gamma^i{}_{ks}\Gamma^s{}_{il} - \Gamma^i{}_{ls}\Gamma^s{}_{ik} = 0$. This reduces the components to

$$R^i{}_{ikl} = \partial_k \Gamma^i{}_{il} - \partial_l \Gamma^i{}_{ik} \tag{216}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{1}{\sum_{m=1}^n \left(\frac{\partial F_i}{\partial x^m}\right)^2}\sum_{s=1}^n \frac{\partial F_i}{\partial x^s}\frac{\partial^2 F_i}{\partial x^s \partial x^l}\right) \tag{217}$$

$$- \frac{\partial}{\partial x^l}\left(\frac{1}{\sum_{m=1}^n \left(\frac{\partial F_i}{\partial x^m}\right)^2}\sum_{s=1}^n \frac{\partial F_i}{\partial x^s}\frac{\partial^2 F_i}{\partial x^s \partial x^k}\right) = 0.$$

This expression vanish due to the commutative nature of partial derivatives.

Hence, taking all cases together, $R^i{}_{jkl} = 0$ and (M, g) is flat. The proof for a Minkowski manifold is identical up to a sign in the first coordinate and the same calculations will produce a vanishing Riemann curvature tensor. Also, the first part of the proposition is given if $V = M$ and the coordinate bases $\{dy^i\}$ and $\{dx^i\}$ are valid for all of M and $F(M)$. □

The following theorem can be useful for deciding whether or not a particular geometry is flat. The proof is not given here, but it relies on Sylvester's law of inertia and the existence of an orthonormal basis (See e.g. [Friedberg et al., 2003] and [Lee, 2013]).

Theorem 8. A necessary (but not sufficient) condition for two pseudo-Riemannian manifolds, (M, g) and $(\widetilde{M}, \widetilde{g})$, to be locally isometric is that the signatures of g and $\widetilde{g}$ are locally equal.

Thus, an arbitrary geometry (M, g) can be evaluated based on the eigenvalues of the matrix g. If the signature of g does not (locally) equal either the signature of $\overline{g}$ or g_{Mink}, the geometry is certain to be locally curved.

Example 10. The cylinder manifold $(C, F^*\overline{g})$ from Example 7, where

$$F^*\overline{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{218}$$

has signature $(3, 0)$. Moreover, the function

$$F(r, \theta, z) = (r\cos\theta, r\sin\theta, z) \tag{219}$$

is a diffeomorphism in the chosen domain. This implies that $(C, F^*\overline{g})$ is isometric to $(\mathbb{R}^3, \overline{g})$ and flat. //

Example 11. The geometry (M, g), where

$$g = -dt^2 + a^2 \left(\frac{1}{1-kr^2}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \right), \tag{220}$$

for a constant $a > 0$ and $k = -1, 0, 1$ is called the *Robertson-Walker geometry*. In the ON-basis $\{dt, a/\sqrt{1-kr^2}\, dr, ar\, d\theta, ar \sin\theta\, d\phi\}$ the signature $(p, q)(k)$, depends on k

$$(p, q)(k) = \begin{cases} (3,\,1), & k = -1, 0 \\ (2,\,2), & k = 1,\, r > 1 \end{cases}. \tag{221}$$

The signature for $k = 1$ and $r > 1$ is clearly not compatible with flat space, so this geometry is curved. For $k = -1$, despite having the same signature as g_{Mink} it turns out that there exist no isometry between the two, i.e. it is also curved. The geometry where $k = 0$ together with the diffeomorphism $F(t, r, \theta, \phi) = (t, ar, \theta, \phi)$ is isometric to $(\mathbb{R}^4, g_{Mink})$ and therefore flat. //

Example 12. The geometry (M, g), where

$$g = -\left(1 - \frac{a}{r}\right) dt^2 + \frac{1}{1-\frac{a}{r}}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \tag{222}$$

for $M = \mathbb{R} \times \mathbb{R}^+ \times S^2$ and a constant $a > 0$ is called the *Schwarzschild geometry*. In the ON-basis $\{\sqrt{1-a/r}\, dt, (\sqrt{1-a/r})^{-1}\, dr, r\, d\theta, r \sin\theta\, d\phi\}$ the signature $(p, q)(r)$, depends on r as $(p, q)(r) = (3,\,1)$ for $r \in (0, \infty) \setminus \{0, a\}$. The signature corresponds to g_{Mink} which makes this whole region a candidate for flat geometry. However, it turns out that due to the r dependence the geometry is only asymptotically flat, i.e. asymptotically isometric to $(\mathbb{R}^4, g_{Mink})$ as $r \to \infty$.

For the space enthusiast, this metric models the gravitational field outside of a point mass at $r = 0$. One application is a black hole where all the mass is concentrated at a singular point. At $r = a$, there exist a "fake" singularity due to the choice of coordinates. This *coordinate singularity* is the event-horizon of a black hole. Physically, it is not a true singularity since it can be removed by changing to *Eddington-Finkelstein* coordinates, where the only singularity occurs at $r = 0$. But strange things happens when $r < a$. In this domain time (dt) and space (dr) switch sign and place. Consequently, the singularity at $r = 0$ is *not* a position in space, but rather a moment in time. So, the unlucky person who finds himself inside the event horizon will not be able to escape the singularity since one cannot escape a moment in time. //

V. General relativity and the connection to Riemannian geometry

With Riemannian geometry it is now possible to introduce the basic ideas in general relativity. The first step is to mathematically define spacetime. Thereafter Einstein's field equations, and how to solve these, will be presented.

Most of the results in this section and the next requires additional concepts from physics, which is why no proofs are presented. If no explicit reference is given, proofs and detailed derivations can be found in [Hartle, 2003] or [Dray, 2015].

Lorentzian manifolds

An important subclass of pseudo-Riemannian manifolds is generated by a Lorentz metric.

> **Definition 41.** A pseudo-Riemannian metric with signature $(n-1, 1)$ is called a *Lorentz metric*.
>
> A *Lorentzian manifold* is a differentiable manifold equipped with a Lorentz metric.

The flat Minkowski manifold $(\mathbb{R}^n, g_{Mink})$ is a Lorentzian manifold, but a Lorentzian manifold need not to be flat. The convention is to use *Minkowski space* when one talks about flat space.

The geometry of spacetime is almost self explanatory. Space has three spatial dimensions and time one dimension. Hence, spacetime is modelled as a four dimensional manifold, either flat or curved.

> **Definition 42.** The Minkowski manifold $(\mathbb{R}^4, g_{Mink})$ is called *flat spacetime* or *Minkowski spacetime*.
>
> A Lorentzian manifold (M, g), where M has four dimensions and g is not a flat metric, is called *curved spacetime*.

The existence of a Riemannian metric on a differentiable manifold is ensured by Proposition 3. However, the same result does not hold for pseudo-Riemannian metrics and Lorentz metrics for which there are additional conditions. It turns out that the only reasonable model of spacetime is a non-compact manifold and, fortunately, a non-compact manifold satisfy these extra conditions to guarantee the existence of a Lorentz metric. For more details see [Hawking & Ellis, 1973].

Einstein's field equations

Consider a spacetime (M, g) where M is a Lorentzian manifold. The metric is either flat or curved and has signature $(3, 1)$. The *Einstein field equations* (EFE) are then given by

$$\mathbf{G} + \mathbf{\Lambda} = \frac{8\pi G}{c^2}\mathbf{T}, \tag{223}$$

or in component form

$$R_{ij} - \frac{1}{2}g_{ij}R + \Lambda g_{ij} = \frac{8\pi G}{c^2}T_{ij}. \tag{224}$$

The left side consist of the *Einstein tensor* $\mathbf{G}$ and the cosmological constant $\mathbf{\Lambda}$. The right side of the *stress-energy tensor* $\mathbf{T}$ scaled by the gravitational constant G and the speed of light in vacuum c.

The Einstein tensor can be expressed as a Lie algebra-valued 1-form which describes the curvature of spacetime.

$$\mathbf{G} = G^i\hat{\mathbf{e}}_i = G^i{}_j\sigma^j\hat{\mathbf{e}}_i, \tag{225}$$

where $\{\sigma^j\}$ is an ON-basis of 1-forms, $\{\hat{\mathbf{e}}_i\}$ an orthonormal vector basis and $G^i{}_j$ the components given by

$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_jR. \tag{226}$$

Or equivalently

$$G^i{}_j = R^i{}_j - \frac{1}{2}\delta^i{}_jR = g^{ik}R_{kj} - \frac{1}{2}g^{ik}g_{kj}R \quad \Leftrightarrow \tag{227}$$

$$g_{ik}G^i{}_j = g_{ik}g^{ik}R_{kj} - \frac{1}{2}g_{ik}g^{ik}g_{kj}R \quad \Leftrightarrow \tag{228}$$

$$G_{kj} = R_{kj} - \frac{1}{2}g_{kj}R \quad \Leftrightarrow \tag{229}$$

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R. \tag{230}$$

R_{ij} are the components of the Ricci curvature tensor and R the Ricci curvature scalar. The Einstein tensor is truly curvature of spacetime.

The cosmological constant is an attempt to model the acceleration of the expansion of the Universe and is related to the mysterious source of dark energy. The components simply consist of a constant real value Λ. The value of Λ is a matter of observational cosmology, but in most rough models one can assume it is zero.

If the left side of EFE represents curvature then the right side represents matter. By the famous $E = mc^2$, matter is simply a representation of energy. In the three spatial dimensions, $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, energy takes the form of forces acting along each axis. Three spatial dimensions creates a nine component tensor T^{ij} where $i, j = x, y, z$. In physics, forces acting along different axes $i \neq j$ are called *stress*. This is a quite natural wording if one thinks of applying asymmetric force to e.g. a piece of paper. The paper will experience stress and likely tear apart. In the temporal dimension, $\hat{\mathbf{t}}$, the corresponding tensor components are called energy and momentum density. For T^{tt}, think of a scientist sitting completely still in space, preferably in a spacesuit. This scientist would be able to measure the amount of energy in this resting frame of space, a measure called energy density. The $T^{it} = T^{tj}$ components ($i, j = x, y, z$) represents the flow of energy, also called momentum density, in the i^{th} or j^{th} direction, measured by a scientist sitting in different resting frames. The summarising picture is

$$T^{ij} = \left(\begin{array}{c|c} \begin{array}{c}\text{energy}\\ \text{density}\end{array} & \begin{array}{c}\text{momentum}\\ \text{density}\end{array} \\ \hline \rotatebox{90}{$\begin{array}{c}\text{momentum}\\ \text{density}\end{array}$} & \text{stress} \end{array}\right) \tag{231}$$

Energy and momentum densities are far from intuitive concepts and a good source for deeper understanding is [Hartle, 2003].

Similarly as the Einstein tensor, instead of using tensor notation the stress-energy tensor is expressed as a Lie algebra-valued 1-form.

$$\mathbf{T} = T^i \hat{\mathbf{e}}_i = T^i{}_j \sigma^j \hat{\mathbf{e}}_i, \tag{232}$$

where the components $T^i{}_j$, or $T_{ij} = g_{ik} T^k{}_j$, can be derived from T^{ij}.

Solutions to Einstein's field equations

A solution to EFE is a specific Minkowski or Lorentzian spacetime that match a certain matter distribution. This is more or less a way to say that the left side need to equal the right side. But finding a solution is hard and the complexity is hidden in the deceivingly simple looking equation (224). The left side of the equation consists of non-linear partial differential equations given by (196) and (197). The right side constitutes a dynamical system (potentially non-linear in itself) of how matter is distributed in spacetime. This dynamical system describes matter and energy based on physical laws of nature, their coupling with each other and in some cases the coupling with gravitational fields.

In practice, there exist two different methods to approach a solution. One way is to fix the right side, i.e. choose a matter distribution **T**, and then try to find a Lorentzian manifold, a spacetime, to match this. The other way is to fix the left side **G** by choosing a specific spacetime and then try to find a physical matter distribution. Due to the non-linearity of EFE exact solutions are rare. In order to handle this, there exist three common approximations: *vacuum*, *weak field* and *linearised weak field* solutions.

For vacuum, consider completely empty space[9] and a cosmological constant equal to zero, then $T_{ij} = 0$ and

$$R_{ij} = \frac{1}{2} g_{ij} R \quad \Rightarrow \tag{233}$$

$$g^{ij} R_{ij} = \frac{1}{2} g^{ij} g_{ij} R \quad \Rightarrow \tag{234}$$

$$R = \frac{1}{2} R \quad \Rightarrow \tag{235}$$

$$R = 0. \tag{236}$$

Definition 43. The reduced EFE $R_{ij} = 0$ are called the *vacuum equations*, and a solution is called a *vacuum solution*.

A vacuum solution is one of the few solutions that can be shown in detail without too lengthy calculations, the most trivial being flat spacetime.

Example 13. Let the matter distribution be vacuum, i.e. $T_{ij} = 0$ and $\Lambda = 0$. Consider Minkowski spacetime as a candidate solution to the EFE. The vacuum equations states that

$$R_{ij} = \partial_l \Gamma^l{}_{ij} - \partial_j \Gamma^l{}_{il} + \Gamma^m_{ij} \Gamma^l{}_{lm} - \Gamma^m_{il} \Gamma^l{}_{jm} = 0. \tag{237}$$

Since g_{Mink} is a flat metric all the Christoffel symbols vanishes and solves the vacuum equations. Clearly then, Minkowski spacetime is an exact vaccum solution. //

Assume now that one is situated in flat space, far away from a source of gravity. Over the distance, this source creates a weak field of gravity only with the strength of a small perturbation.

Definition 44. A *weak field solution* is a metric

$$g = g_{Mink} + h, \tag{238}$$

where h is small compared to g_{Mink} and solves the EFE. Applying perturbation theory, a solution can be expanded in terms of h and, if necessary, truncated to first order

[9]Completely empty space does not exist. Even in the darkest part of this vast emptiness the cosmic microwave background radiation makes a mark. So, this assumption is an approximation.

for a linearised solution. This linearisation is very common to make in weak field solutions and they are referred to *linearised weak field solutions*.

A rough sketch for common ways of solving EFE is given in Figure 4. If the method is to fix **T**, **T** might be coupled to **G** via some law of nature that binds a specific matter distribution to gravitational fields. Since the sought solution *is* gravity, this coupling might be problematic. One way to proceed is to solve for g and **T** simultaneously, most likely numerically. Another way is to approximate **T** with an uncoupled approximation, if this is possible. The uncoupled version of **T** is a system of linear or non-linear differential equations that need to be solved. Based on the complexity of this system, there exist either an analytic solution or one need to approximate **T** further to find a solution. Vacuum is one approximation, *perfect fluid* another. For perfect fluid

$$T_{ij} = (\rho + p)u_i u_j + p g_{ij}, \tag{239}$$

where ρ is energy density, p the pressure density and u the velocity vector. The uncoupled version is an approximation where $p = 0$ and this model is called *dust*. After an expression for **T** is finally given, this is plugged into EFE and then one proceeds by solving this system of equations in some manner.

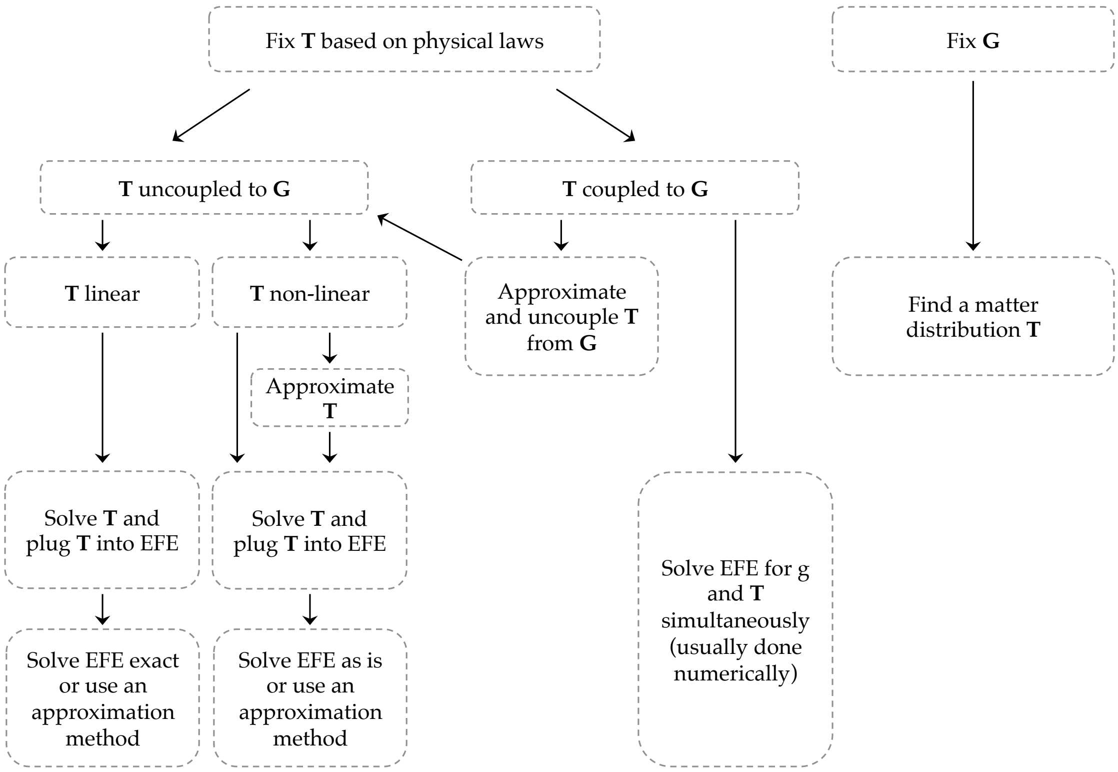


Figure 4: *Possible ways to approach a solution to EFE.*

VI. Predicting and detecting gravitational waves

General relativity is a theory from which predictions can be made. A prediction is a consequence of solving the EFE and analyse what the solution means physically. Theory must of course be verified by observations and this can be a matter of months, years or even centuries, and GR had it all. The prediction about the bending of light around stars was verified early. Also Mercury's orbit precession could be explained by GR. In recent years, as the observational technology has become more accurate, black holes have been verified to exist and the GPS is perhaps the most widely used application relying on GR. For long, gravitational waves were assumed impossible to detect. The prediction was that a passing gravitational wave, here on Earth, would curve the space around it by about $1/10000^{\text{th}}$ the diameter of a proton, or around 10^{-19} m. But with over 40 years of research the prediction was verified in 2016.

Weak gravitational waves

The physics behind predicting the existence of gravitational waves can be divided into two different regimes, strong and weak gravitational waves. The strong regime is close to the source. In terms of a solution to EFE, this regime is where any weak field approximation breaks down and the curvature from the source can no longer be handled as a small perturbation of flat spacetime. There are no known analytical solutions to these strong gravitational waves and the numerical models require supercomputers to work out a viable solution.

Weak gravitational waves origin from strong regime events. Over the distance travelled to Earth, the wave loses its strength and becomes, eventually, a small perturbation. The properties of these weak gravitational waves are (i) they propagate with the speed of light, (ii) they are weak enough that a weak field solution is possible and (iii) they satisfies the wave equation. A simplifying assumption is that they propagate through vacuum, i.e. they are a vacuum solution. The general solution can be derived from these properties and with the help of gauge theory. The full derivation can be found in [Hartle, 2003].

The *general solution for weak gravitational waves* is

$$g = g_{Mink} + h, \tag{240}$$
$$h_{ij}(p) = a_{ij} e^{\iota k^s p_s}, \tag{241}$$

where a_{ij} are constants, ι the imaginary unit, k the wave vector and p a point in spacetime. The temporal part of the wave vector k consist of the angular wavenumber $k^t = K$ which is related to the wavelength λ as $K = 2\pi/\lambda$. The spatial components describes the direction of the wave and the magnitude of this vector must equal K.

If the wavelength is fixed and the wave assumed to travel along a single axis, say the positive x-direction, the solution can be simplified to

$$h_{ij}(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & a_{34} & -a_{33} \end{pmatrix} e^{-\imath K(t-x)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & A & B \\ 0 & 0 & B & -A \end{pmatrix} e^{-\imath K(t-x)}, \tag{242}$$

where A, B are constants. The full solution is given by different waves superimposed on each other. Each wave has a unique K and the constants $A(K)$ and $B(K)$ will also depend on K. Still, assume the wave travels along the positive x-direction, then

$$h_{yy} = -h_{zz} = \int_K A(K) e^{-\imath K(t-x)} = f(t, x), \tag{243}$$

$$h_{yz} = h_{zy} = \int_K B(K) e^{-\imath K(t-x)} = u(t, x), \tag{244}$$

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 + f(t,x) & u(t,x) \\ 0 & 0 & u(t,x) & 1 - f(t,x) \end{pmatrix}. \tag{245}$$

Mathematically, solving the EFE when the candidate solution is more complex than Minkowski spacetime the true use of differential forms becomes obvious. Instead of using Koszul's formula to calculate an endless amount of Christoffel symbols, one can simply derive the connection 1-forms from Cartan's first structure equation and then derive the Riemann curvature tensor from Cartan's second structure equation. The following example illustrates this, as well as what gravitational waves might look like.

Example 14. Consider the Lorentz spacetime $(\mathbb{R}^4, g)$, where

$$g = -dt^2 + dx^2 + (1 + f(t,x))\, dy^2 + (1 - f(t,x))\, dz^2 \tag{246}$$

and f is small compared to 1. A linearised weak field solution, with the off-diagonal elements u set to zero, is sought. In spacetime coordinates, $\{\hat{\mathbf{e}}_i\} = \{\hat{\mathbf{t}}, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, the ON-basis $\{\sigma^t, \sigma^x, \sigma^y, \sigma^z\} = \{dt, dx, \sqrt{1+f}\, dy, \sqrt{1-f}\, dz\}$ is a natural choice of basis.

Assuming a Levi-Civita connection, Cartan's first structure equation gives the following system of equations (let f_t and f_x denote the partial derivatives of f):

$$0 = \omega^t{}_t \wedge \sigma^t + \omega^t{}_x \wedge \sigma^x + \omega^t{}_y \wedge \sigma^y + \omega^t{}_z \wedge \sigma^z, \tag{247}$$

$$0 = \omega^x{}_t \wedge \sigma^t + \omega^x{}_x \wedge \sigma^x + \omega^x{}_y \wedge \sigma^y + \omega^x{}_z \wedge \sigma^z, \tag{248}$$

$$\frac{1}{2(1+f)} \left(f_t\, \sigma^t \wedge \sigma^y + f_x\, \sigma^x \wedge \sigma^y \right) = \omega^y{}_t \wedge \sigma^t + \omega^y{}_x \wedge \sigma^x + \omega^y{}_y \wedge \sigma^y + \omega^y{}_z \wedge \sigma^z, \tag{249}$$

$$-\frac{1}{2(1-f)} \left(f_t\, \sigma^t \wedge \sigma^z + f_x\, \sigma^x \wedge \sigma^z \right) = \omega^z{}_t \wedge \sigma^t + \omega^z{}_x \wedge \sigma^x + \omega^z{}_y \wedge \sigma^y + \omega^z{}_z \wedge \sigma^z. \tag{250}$$

By Theorem 6 the sought solution is unique and there are two ways to proceed to find this. One way is to use the symmetry of the connection 1-forms to set up a system of 16 equations and solve for the 16 unique variables ${\omega^i}_j$. Another way is to guess a solution from the last two equations with a non-zero left-hand side and then use the symmetry of the connection forms to derive the corresponding forms with flipped index. If this guess solves the system, it is the correct solution. In most easy cases, such as this, this is the best way to proceed. From the last two equations, a possible solution is

$$
\begin{aligned}
{\omega^y}_t &= -\frac{f_t}{2(1+f)}\,\sigma^y, && (251)\\
{\omega^z}_t &= \frac{f_t}{2(1-f)}\,\sigma^z, && (252)\\
{\omega^y}_x &= -\frac{f_x}{2(1+f)}\,\sigma^y, && (253)\\
{\omega^z}_x &= \frac{f_x}{2(1-f)}\,\sigma^z, && (254)
\end{aligned}
$$

as well as putting all other connections in these two equations to zero. Using the symmetry $\omega_{ij} = -\omega_{ji}$ and $\omega_{ij} = g_{il}{\omega^l}_j$ the other non-zero connections are

$$
\begin{aligned}
& \omega_{ty} = g_{tl}{\omega^l}_y = -{\omega^t}_y = -\omega_{yt} = -(g_{yl}{\omega^l}_t) = -((1+f){\omega^y}_t) && (255)\\
\Rightarrow\quad & {\omega^t}_y = (1+f){\omega^y}_t = -\frac{f_t}{2}\,\sigma^y, && (256)\\
& {\omega^t}_z = (1-f){\omega^z}_t = \frac{f_t}{2}\,\sigma^z, && (257)\\
& {\omega^x}_y = -(1+f){\omega^y}_x = \frac{f_x}{2}\,\sigma^y, && (258)\\
& {\omega^x}_z = -(1-f){\omega^z}_x = -\frac{f_x}{2}\,\sigma^z. && (259)
\end{aligned}
$$

It is easy to check that these eight connection forms solves the system. Using Cartan's second structure equation the curvature 2-forms can be calculated. The ten non-zero

2-forms are

$$\Omega^t{}_y = -\frac{1}{4(1+f)}\left(f_t^2 + 2(1+f)f_{tt}\,\sigma^t \wedge \sigma^y + f_x f_t + 2(1+f)f_{tx}\,\sigma^x \wedge \sigma^y\right), \tag{260}$$

$$\Omega^t{}_z = -\frac{1}{4(1-f)}\left(f_t^2 + 2(f-1)f_{tt}\,\sigma^t \wedge \sigma^z + f_x f_t + 2(f-1)f_{tx}\,\sigma^x \wedge \sigma^z\right), \tag{261}$$

$$\Omega^x{}_y = \frac{1}{4(1+f)}\left(f_x f_t + 2(1+f)f_{tx}\,\sigma^t \wedge \sigma^y + f_x^2 + 2(1+f)f_{xx}\,\sigma^x \wedge \sigma^y\right), \tag{262}$$

$$\Omega^x{}_z = \frac{1}{4(1-f)}\left(f_x f_t + 2(f-1)f_{tx}\,\sigma^t \wedge \sigma^z + f_x^2 + 2(f-1)f_{xx}\,\sigma^x \wedge \sigma^z\right), \tag{263}$$

$$\Omega^y{}_t = \frac{1}{4(1+f)^2}\left(f_t^2 - 2(1+f)f_{tt}\,\sigma^t \wedge \sigma^y + f_x f_t - 2(1+f)f_{tx}\,\sigma^x \wedge \sigma^y\right), \tag{264}$$

$$\Omega^y{}_x = \frac{1}{4(1+f)^2}\left(f_x f_t - 2(1+f)f_{tx}\,\sigma^t \wedge \sigma^y + f_x^2 - 2(1+f)f_{xx}\,\sigma^x \wedge \sigma^y\right), \tag{265}$$

$$\Omega^y{}_z = \frac{f_x^2 - f_t^2}{4(1+f)}\,\sigma^y \wedge \sigma^z, \tag{266}$$

$$\Omega^z{}_t = \frac{1}{4(1-f)^2}\left(f_t^2 - 2(f-1)f_{tt}\,\sigma^t \wedge \sigma^z + f_x f_t - 2(f-1)f_{tx}\,\sigma^x \wedge \sigma^z\right), \tag{267}$$

$$\Omega^z{}_x = \frac{1}{4(1-f)^2}\left(f_x f_t - 2(f-1)f_{tx}\,\sigma^t \wedge \sigma^z + f_x^2 - 2(f-1)f_{xx}\,\sigma^x \wedge \sigma^z\right), \tag{268}$$

$$\Omega^z{}_y = \frac{f_t^2 - f_x^2}{4(1-f)}\,\sigma^y \wedge \sigma^z. \tag{269}$$

The non-zero curvature 2-forms imply that the chosen Lorentz spacetime is indeed curved. Moreover, from Proposition 8 the components of the Riemann curvature tensor can be derived.

$$R^t{}_{yty} = -\frac{1}{2(1+f)}(f_t^2 + 2(1+f)f_{tt}), \qquad R^t{}_{yxy} = -\frac{1}{2(1+f)}(f_x f_t + 2(1+f)f_{tx}), \tag{270}$$

$$R^t{}_{ztz} = -\frac{1}{2(1-f)}(f_t^2 + 2(f-1)f_{tt}), \qquad R^t{}_{zxz} = -\frac{1}{2(1-f)}(f_x f_t + 2(f-1)f_{tx}), \tag{271}$$

$$R^x{}_{yty} = \frac{1}{2(1+f)}(f_x f_t + 2(1+f)f_{tx}), \qquad R^x{}_{yxy} = \frac{1}{2(1+f)}(f_x^2 + 2(1+f)f_{xx}), \tag{272}$$

$$R^x{}_{ztz} = \frac{1}{2(1-f)}(f_x f_t + 2(f-1)f_{tx}), \qquad R^x{}_{zxz} = \frac{1}{2(1-f)}(f_x^2 + 2(f-1)f_{xx}), \tag{273}$$

$$R^y{}_{tty} = \frac{1}{2(1+f)^2}(f_t^2 - 2(1+f)f_{tt}), \qquad R^y{}_{txy} = \frac{1}{2(1+f)^2}(f_x f_t - 2(1+f)f_{tx}), \tag{274}$$

$$R^y{}_{xty} = \frac{1}{2(1+f)^2}(f_x f_t - 2(1+f)f_{tx}), \qquad R^y{}_{xxy} = \frac{1}{2(1+f)^2}(f_x^2 - 2(1+f)f_{xx}), \tag{275}$$

$$R^z{}_{ttz} = \frac{1}{2(1-f)^2}(f_t^2 - 2(f-1)f_{tt}), \qquad R^z{}_{txz} = \frac{1}{2(1-f)^2}(f_x f_t - 2(f-1)f_{tx}), \tag{276}$$

$$R^z{}_{xtz} = \frac{1}{2(1-f)^2}(f_x f_t - 2(f-1)f_{tx}), \qquad R^z{}_{xxz} = \frac{1}{2(1-f)^2}(f_x^2 - 2(f-1)f_{xx}), \tag{277}$$

$$R^y{}_{zyz} = \frac{1}{2(1+f)}(f_x^2 - f_t^2), \qquad R^z{}_{yyz} = \frac{1}{2(1-f)}(f_t^2 - f_x^2). \tag{278}$$

These are the independent components and applying the symmetry $R^i{}_{jlk} = -R^i{}_{jkl}$ gives the additional ones. It is now a simple matter to calculate the Ricci curvature tensor $R_{ij} = R^m_{imj}$. The non-zero components are

$$R_{tt} = \frac{1}{(f^2-1)^2}(2f(f^2-1)f_{tt} - (1+f^2)f_t^2), \tag{279}$$

$$R_{tx} = -\frac{1}{(f^2-1)^2}(2f(1-f^2)f_{tx} + (1+f^2)f_x f_t), \tag{280}$$

$$R_{xt} = R_{tx}, \tag{281}$$

$$R_{xx} = -\frac{1}{(f^2-1)^2}(2f(f^2-1)f_{xx} - (1+f^2)f_x^2), \tag{282}$$

$$R_{yy} = \frac{1}{f^2-1}(f_{tt} - f_{xx} + f^2(f_{xx} - f_{tt}) + f_t^2 - f_x^2), \tag{283}$$

$$R_{zz} = \frac{1}{f^2-1}(f_{xx} - f_{tt} + f^2(f_{tt} - f_{xx}) + f_t^2 - f_x^2). \tag{284}$$

The vacuum equations $R_{ij} = 0$ can be reduced down to

$$(f^2-1)(f_{tt} - f_{xx}) = 0, \tag{285}$$

$$2f(f^2-1)(f_{tt} - f_{xx} + f_{tx}) + (f^2+1)(f_x^2 - f_t^2 - f_t f_x) = 0 \tag{286}$$

and clearly this is not easy to solve for the non-linear case. For the linearised solution, think of f as a function $f(t,x) = \epsilon X(t,x)$ where ϵ is the small perturbation variable. The second order factor ϵ^2 will be truncated, i.e. $\epsilon^2 \approx 0$. In case of real detections, this perturbation will be of the order $\epsilon = 10^{-19}$, so removing the second order $\epsilon^2 = 10^{-38}$ truly makes sense since there is no way to detect such small vibrations. Thus, the above vacuum equations becomes

$$(\epsilon^2 X^2 - 1)(\epsilon X_{tt} - \epsilon X_{xx}) \approx \epsilon X_{xx} - \epsilon X_{tt} = f_{xx} - f_{tt} = 0, \tag{287}$$

$$2\epsilon X(\epsilon^2 X^2 - 1)(\epsilon X_{tt} - \epsilon X_{xx} + \epsilon X_{tx}) + (\epsilon^2 X^2 + 1)(\epsilon^2 X_x^2 - \epsilon^2 X_t^2 - \epsilon X_t \epsilon X_x) \tag{288}$$

$$= 2\epsilon^2 X(\epsilon^2 X^2 - 1)(X_{tt} - X_{xx} + X_{tx}) + \epsilon^2(\epsilon^2 X^2 + 1)(X_x^2 - X_t^2 - X_t X_x) \approx 0. \tag{289}$$

The linearised weak field solution has only one condition and that is $f_{xx} - f_{tt} = 0$, i.e. f need to be a wave function.

In order to understand what these waves look like consider the wave function

$$f(t,x) = A\sin(\omega(t-x)), \tag{290}$$

with a small amplitude A and frequency ω. The metric g is

$$g = -dt^2 + dx^2 + (1 + A\sin(\omega(t-x)))\,dy^2 + (1 - A\sin(\omega(t-x)))\,dz^2 \tag{291}$$

and the wave only affects spacetime in the $y-z$ plane. For fixed x, A and ω, think of a stationary circle of small mass elements. When this gravitational wave passes the ring, the particles will remain stationary in terms of their coordinate positions because these coordinates move in symmetry with the wave and no forces are involved. That is, the circle stays as a perfect circle in the spacetime it is embedded in. Instead, in order to measure this wave it is necessary to measure the distance over time between two such mass elements. In Riemannian geometry, the distance is given by the Riemannian distance function.

Definition 45. Consider a parametrised curve $\gamma(t) : [a,b] \to M$. If γ is a smooth curve with $\gamma'(t) \neq 0$ for $t \in [a,b]$, it is called a *regular curve*.

Definition 46. If $\gamma(t) : [a,b] \to M$ is a regular curve, the *length* of γ is given by

$$L(\gamma) = \int_{[a,b]} |\gamma'(t)|_g. \tag{292}$$

For two points $p, q \in M$ (M need to be connected) the *Riemann distance*, or simply the *distance*, is then given by

$$d(p,q) = \inf\{L(\gamma) : \gamma \text{ is a regular curve joining } p \text{ and } q\}. \tag{293}$$

In terms of how much the distance changes it is possible to illustrate how this ring, in this case squeeze and stretch, due to the curvature of spacetime. Figure 5 shows a few instances in time t.

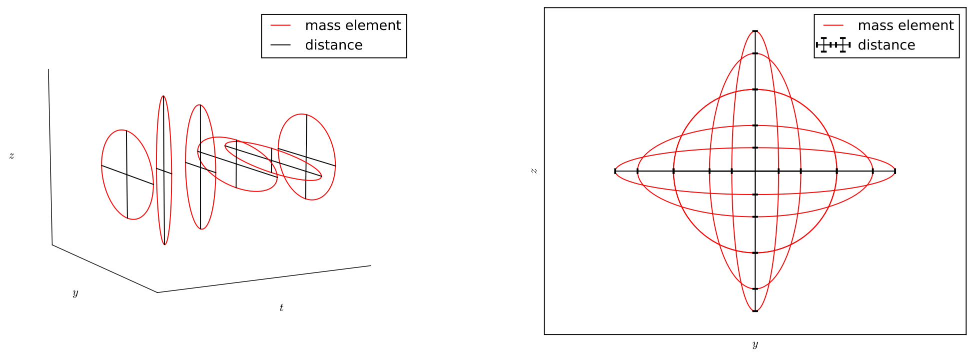


Figure 5: *The wave function $f(t, x) = A\sin(\omega(t - x))$ perturbing the distance between mass elements in a circle in the $y - z$ plane. This measured change in distance is not due to any forces, but rather curved spacetime.*

Consider instead the wave function

$$f(t, x) = Ae^{-(t-x)^2/\sigma^2}, \tag{294}$$

with a maximum height A and width σ. A similar illustration using the distance between the mass elements in a circle is given below.

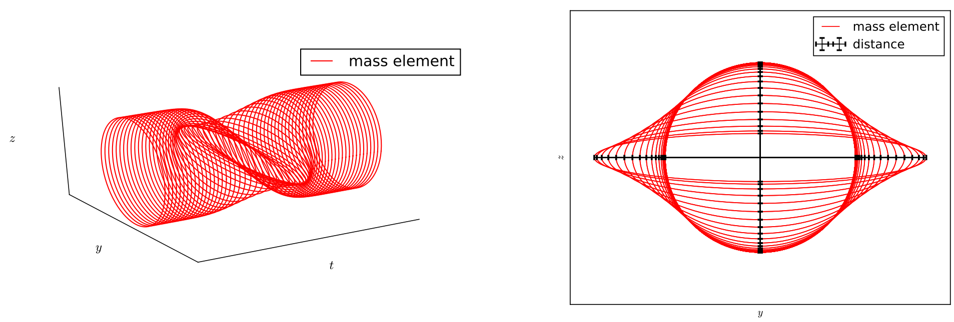


Figure 6: *The wave function $f(t, x) = Ae^{-(t-x)^2/\sigma^2}$ perturbing the distance between mass elements in a circle in the $y - z$ plane.*

This function models a single Gaussian wave packet, squeezing the distance in the z-direction and expanding it in the y-direction as it propagates through spacetime. //

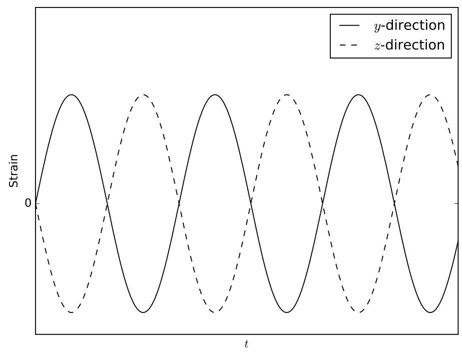
LIGO and the detection of gravitational waves

In 2016 the first detection of a gravitational wave was published and presented in [Abbott et al., 2016]. The detection happened in 2015 and the wave originated from the inspiral and merger of two black holes. The event was estimated to release about 3 solar masses of *pure* energy into gravitational waves, an amount incomprehensible to any source of energy we can relate to. The waves travelled about 1.3 billion light years until it reached earth as a tiny perturbation of order $\epsilon = 10^{-21}$.

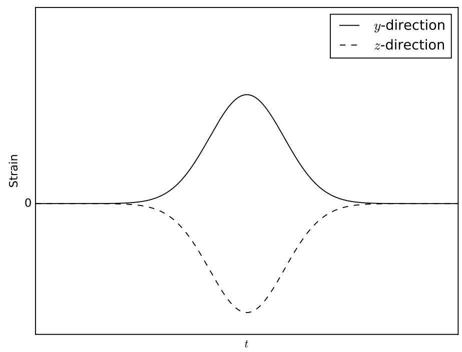
The discovery was made possible by the Laser Interferometer Gravitational-Wave Observatory (LIGO) and the use of a laser interferometer. This tool measures the deviation of distances between test particles very accurately by using constructive and destructive interference of a laser beam.

> **Definition 47.** Denote the distance between two test particles in flat spacetime by L_* and the perturbation of this distance due to curvature by δL. The relative deviation from flat spacetime is called *strain* and defined as the ratio $\delta L / L_*$.

The figure below shows the strain for the two wave functions in Example 14.



(a) *Perturbation factor* $A \sin(\omega(t - x))$.



(b) *Perturbation factor* $Ae^{-(t-x)^2/\sigma^2}$.

Figure 7: *Strain calculated for two wave functions.*

In order to predict how the event in 2015 would look like here on Earth, theorists needed to numerically work out the strong gravitational wave pattern at the event and then predict the decay in strength and pattern over the distance to Earth. Quite remarkably the predictions were almost spot on as can be seen in the two below figures.

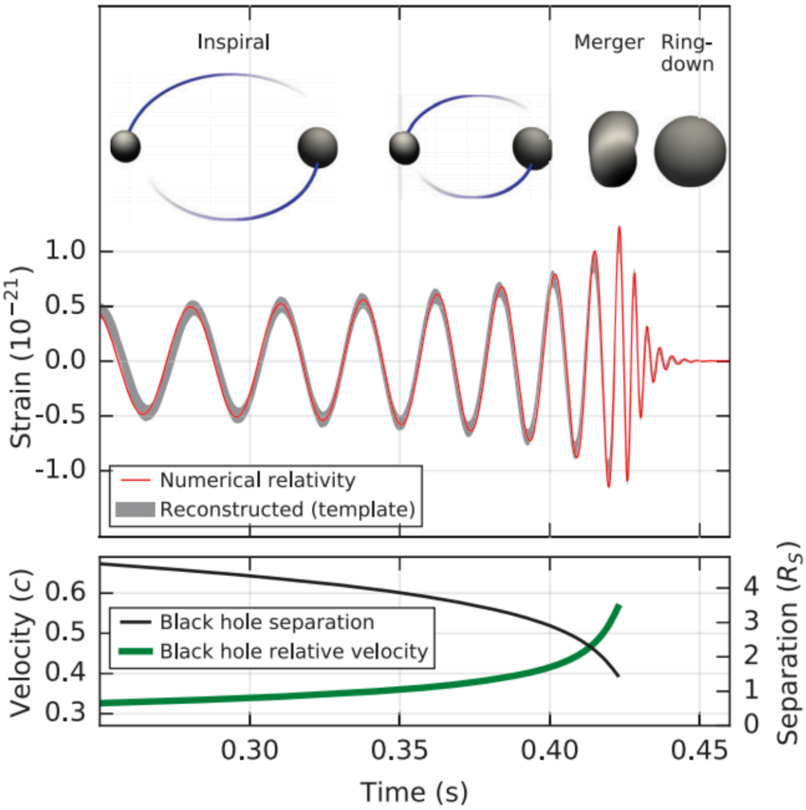


Figure 8: *Numerical predictions of the wave pattern on Earth following the event of two merging black holes. Source: [Abbott et al., 2016].*

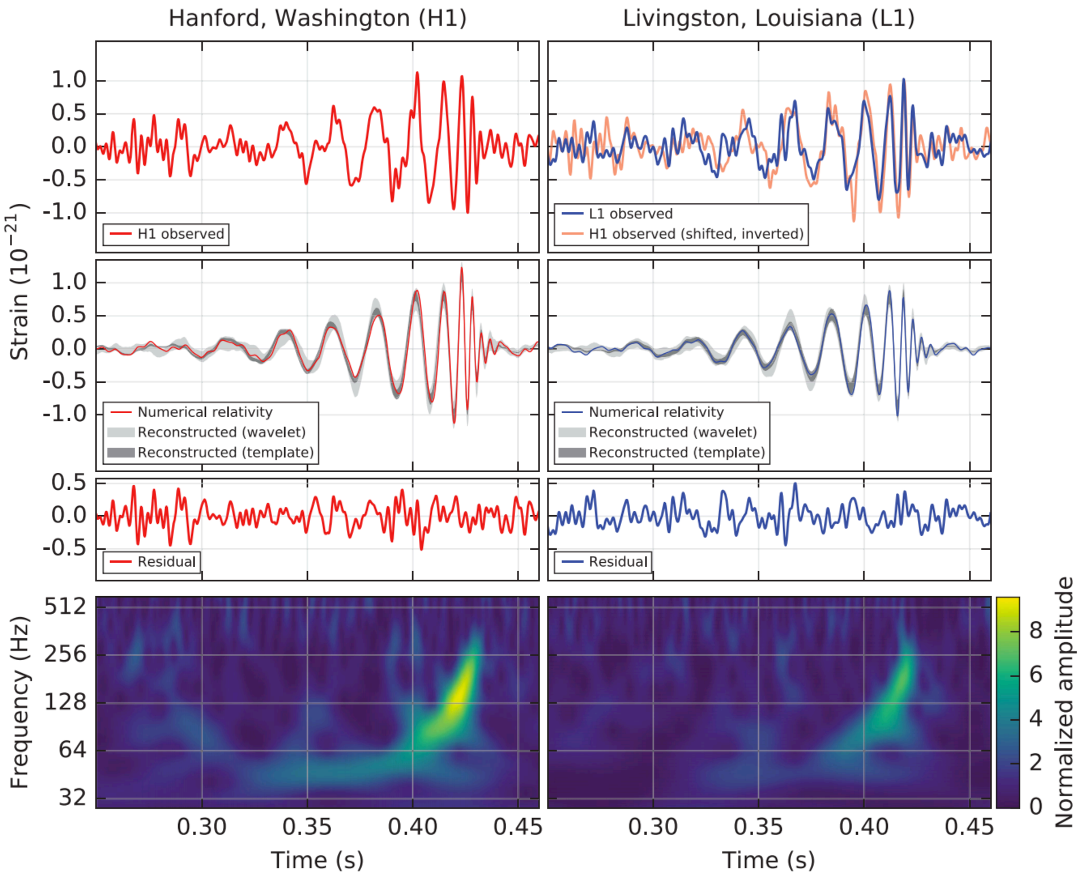


Figure 9: *Weak gravitational wave observations of strain and frequency at two separate observatories vs. the numerical predictions. The residual is sorted out background noise. Source: [Abbott et al., 2016].*

VII. Summary and concluding remark

The generalisation of Euclidean geometry leads to the extensive subject of Riemannian geometry. Introducing an inner product, also known as a metric tensor, together with a differentiable manifold creates the structure of a pseudo-Riemannian or a Riemannian manifold, differing only in the signature of the metric tensor. While this generalisation opens up a wide field of mathematics, it also redefine concepts in Calculus. Differential forms are a perfect tool for differential calculus in Riemannian geometry. On Riemannian manifolds, vector-valued differential forms and Lie-algebra valued 1-forms can be developed. This leads to the definition of a connection 1-form. The components of a connection 1-form are called Christoffel symbols and they can be uniquely determined with the use of a Levi-Civita connection.

Curvature is an important property of Riemannian manifolds, described by the Riemann curvature tensor and curvature 2-forms. Calculating these 2-forms relies on Cartan's structural equations. Curvature is essential for theoretical physics, especially in GR where curved spacetime is related to gravity via Einstein's field equations. Cartan's structural equations are also practical for reducing the number of calculations needed.

The Universe, as we know it today, consist of four dimensions: space and time. This is best modelled by a pseudo-Riemannian manifold with signature $(3, 1)$, also called a Lorentzian manifold. Solving Einstein's field equations means finding a Lorentzian manifold to satisfy the requirements of a specific matter and energy distribution. Usually, one has to approximate this distribution by e.g. perfect vacuum or a perfect fluid. A particularly interesting vacuum solution is weak gravitational waves, which are wave perturbations of flat spacetime. If flat spacetime is set to a magnitude of 1, then these perturbations have a magnitude of around 10^{-21}.

Final remark. This thesis puts focus on a particular path through Riemannian geometry leading to GR and gravitational waves. However, there are many related areas not included and the two most obvious are geodesics and global curvature. Geodesics are the generalisation of straight lines in Riemannian geometry and they can be used in measuring curvature, both locally for paths on a manifold and globally with Gauss-Bonnet theorem. In GR, geodesics are also a tool for doing analysis on particles in space and how their paths are effected by gravity. For example, photons (i.e. light) always travel along geodesics in spacetime. The global curvature of a 2-dimensional manifold is given by the Gauss-Bonnet theorem and higher $2n$-dimension manifolds can be analysed with the generalised Gauss-Bonnet theorem, where both theorems are related to the Euler characteristic of the manifold. Global curvature can be used in e.g. Einstein-Gauss-Bonnet gravity, which is a generalisation of spacetime gravity to higher dimension gravity. Even if these subjects are excluded here, they make up a perfect continuation for the study of differential forms and Riemannian geometry.

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