Models of linear dependent type theory

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Abstract

In this paper, we construct a type theory which deals with non-linear, ordinary dependent types (which we will call *cartesian*), and *linear types*, where both constructs may depend on terms of cartesian types. In the interplay between the cartesian and linear types we introduce the new type formers $\sqcap_{x:A}B$ and $\sqsubset_{x:A} B$, akin to Π and Σ, but where the dependent type B, (and therefore the resulting construct) is a linear type. These can be seen as internalizing quantification over linear propositions. We also introduce the modalities M and L, transforming linear types into cartesian types and vice versa.

We interpret the theory in a split comprehension category $\pi : \mathcal{T} \to \mathcal{C}^{\to}$ [Jac93], accompanied by a split monoidal fibration (Definition 4.3), $q : \mathcal{L} \to \mathcal{C}$. The intuition is that $\mathcal{C}$ models a category of contexts, so that for any $\Gamma \in \mathcal{C}$, the fiber $\mathcal{T}_\Gamma$ is the category containing the cartesian types which can be formed in the context Γ, while the fiber $\mathcal{L}_\Gamma$ is a monoidal category of linear types in Γ. In this setting, the type formers $\sqcap_{x:A}$ and $\sqsubset_{x:A}$ are understood as right and left adjoints of the monoidal reindexing functor $\pi_A^* : \mathcal{L}_\Gamma \to \mathcal{L}_{\Gamma.A}$ corresponding to the weakening projection $\pi_A : \Gamma.A \to \Gamma$ in $\mathcal{C}$. The operators M and L give rise to a fiberwise adjunction $L \dashv M$ between $\mathcal{L}$ and $\mathcal{T}$, where we understand the traditional exponential modality as the comonad $! = LM$.

We provide two concrete examples of models, the *set-indexed families model* and the *diagrams model*. In the former, cartesian types are interpreted in the familiar way, as sets indexed by their context set Γ, and linear types are interpreted as Γ-indexed family of objects of a symmetric monoidal category $\mathcal{V}$. The latter model extends the groupoid model of dependent type theory [HS98] to accomodate linear types. Here, cartesian types over a context Γ are interpreted as a family of groupoids indexed over the groupoid Γ, while linear types are interpreted as diagrams over groupoids, $A : \Gamma \to \mathcal{V}$ in any symmetric monoidal category $\mathcal{V}$. We show that the *diagrams model* can under certain conditions model a linear analogue of the univalence axiom, and provide some discussion on the higher-dimensional nature of linear dependent types.