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$\mathcal{PT}$-symmetric Darboux Transformation

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Table 1: Table of Notation

$\mathbb{R}$	Real axis.
$\mathbb{C}$	Complex plane.
$\mathbb{C}^+$	Upper half complex plane.
$y_x(x)$	$\frac{d}{dx}y(x)$, differentiation only applies to the real variable x.
$\mathcal{PT}$-symmetric	Functions invariant under simultaneous involution and complex conjugation, $u(x) = \overline{u(-x)}$.
Hermitian, Self-adjoint	Operators invariant under simultaneous transposition and complex conjugation, $A = \overline{A}^T$.
L^2	Space of equivalence classes of functions with respect to the norm: $\|\|f\|\|_{L^2} = \left(\int_R f(x)\overline{f(x)}dx \right)^{\frac{1}{2}}$. Viewed as a Hilbert space with the inner product given by $(f, g) = \|\|fg\|\|_{L^2}^2$
$\sim$	Asymptoticly equal, $f(x) \sim g(x) \iff \frac{f(x)}{g(x)} \to 1, \ \|x\| \to \infty$

Contents

Abstract

In this thesis we study Darboux transformation of the zero potential Schrödinger equation. By considering complex-valued solutions and eigenvalues we obtain two distinct families of regular, complex-valued and $\mathcal{PT}$-symmetric potentials not found in the literature. Formulas for the bound state eigenfunctions are provided and the associated scattering problems are solved. Both families of potentials are shown to decay exponentially fast and admit a finite number of bound states associated to the poles of the transmission coefficients, analogous to real-valued potentials in the Faddeev class.

1 Introduction

Sometimes mathematics is more art than mathematics, such is the case with Darboux transformation. In this thesis our goal is to obtain exactly solvable examples of $\mathcal{PT}$-symmetric Schrödinger operators by means of Darboux transformation. Historically, Darboux transformation was introduced as a covariance transformation [9] of the following equation:

$$-y_{xx}(x) + u(x)y(x) = \lambda y(x), \quad \lambda, x \in \mathbb{R}. \tag{1}$$

More precisely, given any equation of type (1), the time-independent Schrödinger equation, and all of its solutions, using Darboux transformation of rank N it is possible to construct a new Schrödinger equation from N solutions of the old Schrödinger equation in such a way that the N solutions provides an explicit formula for all solutions to the new equation. However, more than a given formula, the choice of the N solutions does not, a priori, reveal any properties of the new equation or any of its solutions. The art is to choose the starting solutions in such a way to obtain a desired outcome. Since one has, in this sense, complete control over the new equation and its solutions, Darboux transformation is indeed a ubiquitous tool in the study of exactly solvable systems and in soliton theory. For an introduction to the many applications of Darboux transformation, see for instance [6] and [9].

It is of interest to mention a few applications of Darboux transformation related to this thesis. For instance, in the paper [8] by V. Matveev, Darboux transformation is used to construct the so-called positon potential, which in turn is a singular, slowly decaying solution to the Korteweg-de Vries equation. Furthermore, P. Kurasov and F. Packalén studied the scattering problem for the positon potential using Darboux transformation in [7], where it is shown that the inverse scattering problem can not be solved uniquely. In [11], A L Sakhnovich uses a generalized matrix Darboux transformation to study a non self-adjoint matrix Schrödinger equation of type (1) and obtains a $\mathcal{PT}$-symmetric potential $\tilde{u}(x)$ and an explicit formula for the unique bound state solution of the corresponding Schrödinger equation. In this thesis, we obtain the same potential $\tilde{u}$, as a limit case of a more general potential.

Self-adjoint Schrödinger operators of the form (1) are a classical area of research in conventional quantum mechanics and mathematics. The requirement of being self-adjoint ensures that the spectrum of equation (1) is real and the time evolution, determined by the equation $iy_t(x,t) = -y_{xx}(x,t) + u(x)y(x,t)$, is unitary with respect to the Hilbert space L^2. However, in 1999, C.Bender, S. Boettcher and P. Meisinger introduced the seminal paper [1] in which they studied Schrödinger operators with potentials of the form $u(ix) = x^2(ix)^\epsilon,\ \epsilon > 0$. Schrödinger operators with potentials $u(ix) = x^2(ix)^\epsilon$ are not self-adjoint, but $\mathcal{PT}$-symmetric. It was proven by Dorey et al. in [5] that the spectrum of Schrödinger operators (1) with potentials $u(ix) = (ix)^N$ is in fact real and positive for all $N \geq 2$. Hence the notion of $\mathcal{PT}$-symmetric quantum mechanics was introduced as a form of alternative to conventional quantum mechanics. For a concise introduction to the still active research area of $\mathcal{PT}$-symmetric quantum mechanics, see [2] and the additional references therein. It is of interest to mention that A. Mostafazadeh showed in [10], that $\mathcal{PT}$-symmetric quantum mechanics and conventional quantum mechanics are in a sense equivalent as physical theories.

In this thesis we show that when using Darboux transformation, no assumption on the reality of the eigenvalues λ, or on the solutions $y(x)$ of equation (1) is required. We exploit this and obtain two distinct families of complex-valued, regular and exponentially decaying $\mathcal{PT}$-symmetric potentials. Furthermore, the scattering problems of the associated Schrödinger equations are solved and the potentials are shown to exhibit properties similar to real-valued potentials in the Faddeev class, i.e. potentials satisfying the estimate:

$$\int_{\mathbb{R}} (1 + |x|)|u(x)|dx < \infty.$$

For instance, it is well known [4], that potentials in the Faddeev class admit a class of solutions called Jost solutions, which characterize the scattering properties of the potential. Moreover, potentials in the Faddeev class can be shown to have a finite number of bound states associated to the poles of the transmission coefficient $T(k)$. These poles can be shown to lie on distinct points on the imaginary axis in $\mathbb{C}^+$.

This paper is organized as follows, in Section 2 we introduce the Darboux transformation and show that it is justified to consider complex solutions and eigenvalues. Next, we consider Darboux transformation of rank $N = 2$ and characterize two sets of solutions to the zero potential Schrödinger equation from which we are able to obtain the sought complex-valued, regular and $\mathcal{PT}$-symmetric potentials. In section 3 we study the properties of corresponding Schrödinger operators. We solve the associated scattering problems and provide explicit formulas for the bound state solutions. This is achieved by studying the asymptotic properties of a particular class of solutions, which are shown to have the same properties as the Jost solutions for ral-valued potentials in the Faddeev class.

2 Darboux Transformation and new Potentials

In this section we first define the Darboux Transformation (DT) originally introduced by Gaston Darboux in 1882, presented in a modern fashion by V. Matveev and M.Salle in [9]. Originally considered for Schrödinger equations with real-valued potentials we will show that the ideas extend to the case of complex-valued potentials. Following the introduction of Darboux transformation, in theorems 1 and 2 we characterize two distinct sets of solutions from which we are able to obtain two complex-valued, regular and $\mathcal{PT}$-symmetric potentials. The section is concluded by providing explicit formulas for the new potentials.

2.1 The Darboux Transformation

To introduce the Daroux transformation, suppose we have the following one-dimensional Schrödinger equation:

$$-y_{xx}(x) + u(x)y(x) = \lambda y(x), \quad x \in \mathbb{R}, \tag{2}$$

and suppose we know all the solutions $y(x, \lambda)$. The Darboux transformation using N fixed solutions $y_1, ..., y_N$, of the above equation is the function given by:

$$y[N](x) = \frac{\mathrm{W}(y_1, \dots, y_N, y)}{\mathrm{W}(y_1, \dots, y_N)}, \tag{3}$$

where the function y in the above definition is any solution of equation (2). We have denoted by $\mathrm{W}(y_1, ..., y_N)$ the Wronskian determinant of the N solutions $y_1, ..., y_N$ which is given by:

$$\mathrm{W}(y_1, ..., y_N) = \begin{vmatrix} y_1 & y_2 & \cdots & y_N \\ y_{1x} & y_{2x} & \cdots & y_{Nx} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{d^{N-1}}{dx^{N-1}} y_1 & \frac{d^{N-1}}{dx^{N-1}} y_2 & \cdots & \frac{d^{N-1}}{dx^{N-1}} y_N \end{vmatrix}.$$

It is required that the $N+1$ solutions $y_1, ..., y_N$ and y, are linearly independent, otherwise the Wronskian determinant vanishes everywhere and the transformation does not result in an interesting function. The theorem of Darboux then states that the function defined by formula (3) satisfies the following equation:

$$-y_{xx}[N] + u[N]y[N] = \lambda y[N]. \tag{4}$$

Where the potential $u[N](x)$ is determined by the N solutions $y_1, ..., y_N$ of equation (2) and is given by:

$$u[N](x) = u(x) - 2\frac{d^2}{dx^2}\ln \mathrm{W}(y_1, \dots, y_N). \tag{5}$$

It should be noted that equation (4) only makes sense whenever the Wronskian of the N solutions $y_1, ..., y_N$, is non-zero. Using Darboux transformation it is then possible to obtain new solvable Schrödinger equations, starting from any known Schrödinger equation. The proof that the function $y[N](x)$ is in fact a solution to equation (4) is due to M. Crum and can be found in [9]. Here we will prove the original case $N = 1$:

Let y_1 be a fixed solution and y be an arbitrary solution to equation (2). If we introduce σ as the logarithmic derivative of y_1:

$$\sigma = \frac{d}{dx}\ln y_1 = \frac{y_{1x}}{y_1},$$

then we have the following:

$$y[1] = \frac{\mathrm{W}(y_1, y)}{y_1} = y_x - \sigma y,$$
$$u[1] = u(x) - 2\sigma_x$$

We will consider each term in equation (4) separately. The first term is given by:

$$-y_{xx}[1] = -y_{xxx} + 2\sigma_x y_x + \sigma_{xx} y + \sigma y_{xx},$$

since y is a solution to equation (2), we have $-y_{xx} = (\lambda - u)y$ and hence:

$$-y_{xx}[1] = -u_x y - u y_x + \lambda y_x + 2\sigma_x y_x + \sigma_{xx} y + \sigma(u - \lambda)y \tag{6}$$
$$= (2\sigma_x + \lambda - u)y_x + (-u_x + \sigma_{xx} + u\sigma - \sigma\lambda)y \tag{7}$$

Next, we compute the second term:

$$u[1]y[1] = (u - 2\sigma_x)(y_x - \sigma y) = (u - 2\sigma_x)y_x - (u\sigma - 2\sigma\sigma_x)y \tag{8}$$

If we combine equations (7) and (8) above, we obtain:

$$-y_{xx}[1] + u[1]y[1] = \lambda y_x + (-u_x + \sigma_{xx} + 2\sigma\sigma_x - \sigma\lambda)y, \tag{9}$$

and by the definition of σ we have:

$$\sigma_{xx} + 2\sigma\sigma_x = \frac{d}{dx}(\sigma_x + \sigma^2) = \frac{d}{dx}(\frac{y_{1xx}}{y_1} - \frac{y_{1x}^2}{y_1^2} + \frac{y_{1x}^2}{y_1^2}) = \frac{d}{dx}(u - \lambda) = u_x.$$

Hence we may write equation (9) as:

$$-y_{xx}[1] + u[1]y[1] = \lambda y_x - \sigma\lambda y = \lambda y[1].$$

Indeed we have shown the function $y[1](x)$ defined by formula (3) is a solution to equation (4).

The theorem was originally proven for real-valued potentials and eigenvalues λ. However, we see that the preceding calculation hold without any assumption of λ or $u(x)$ being real-valued. The proof for general $N \geq 2$ essentially follows by repeated application of the same procedure as above. In this sense, Darboux transformation determines a set of potentials for which the Schrödinger equation is exactly solvable and provides explicit formulas for the solutions. This is precisely what we will use to obtain families of $\mathcal{PT}$-symmetric potentials in the following section.

2.2 New Potentials

In this section our goal is to construct two families of complex-valued, regular and $\mathcal{PT}$-symmetric potentials using Darboux transformation, starting with the zero potential Schrödinger equation:

$$-y_{xx} = \lambda y. \tag{10}$$

This goal is achieved by characterizing solutions to (10) which have non-vanishing and $\mathcal{PT}$-symmetric Wronskian determinants, theorems 1 and 2 show that there are essentially two such families of solutions. Following this characterization, we provide explicit formulas for the potentials by performing the calculation according to formula (5) and study their properties. We note that since $u(x) = 0$ the properties of the Wronskian of the two solutions y_1 and y_2 of equation (10) is what determines the properties of the potential. In this thesis we will consider Darboux transformation of rank $N = 2$, specifically we will consider solutions, y_1 and y_2, of equation (10) such that the Wronskian determinant satisfies the following conditions:

$$\begin{aligned} (i) \quad & W(x) \neq 0, \quad x \in \mathbb{R} \\ (ii) \quad & W(x) = \overline{W(-x)}, \quad x \in \mathbb{R}. \end{aligned}$$

Where condition (i) guarantees that the potential exists and is non-singular on the entire real axis and as we shall see, condition (ii) assures that the potential obtained from formula (5) is indeed $\mathcal{PT}$-symmetric. Hence consider the general solutions to equation (10) given by:

$$y_1(x) = \alpha e^{ik_1 x} + \beta e^{-ik_1 x}, \quad y_2(x) = \gamma e^{ik_2 x} + \delta e^{-ik_2 x}, \tag{11}$$

with $\alpha, \beta, \gamma, \delta, k_1, k_2 \in \mathbb{C}$, then derive constraints on the parameters such that the Wronskian determinant of the solutions y_1 and y_2 satisfy conditions (i) and (ii). If any two of $\alpha, \beta, \gamma, \delta$ are equal to zero then the potential vanish identically. Indeed, if either both α, β or γ, δ are equal to zero, then either y_1 or y_2 is identically equal to zero and there is nothing to prove. Suppose β and δ

are equal to zero, a similar calculation shows the other cases. We compute the derivatives of the Wronskian $\mathrm{W}(e^{ik_1x}, e^{ik_2x})$ and we find:

$$\mathrm{W}_x(e^{ik_1x}, e^{ik_2x}) = -(k_2 + k_1)(k_2 - k_1)e^{i(k_1+k_2)x}$$
$$\mathrm{W}_{xx}(e^{ik_1x}, e^{ik_2x}) = -i(k_2 + k_1)^2(k_2 - k_1)e^{i(k_1+k_2)x},$$

hence indeed, the numerator in the potential given by formula (5) vanishes identically. As the following theorems prove, we can essentially obtain two distinct families of Wronskian determinants able to satisfy conditions (i) and (ii).

Theorem 1. *Consider the solutions (11) and assume that $\alpha\beta\gamma\delta \neq 0$. Then, if k_1, k_2 are non-real the Wronskian $\mathrm{W}(y_1, y_2)$ satisfy condition (ii) if and only if:*

$$\alpha\delta = \overline{\beta\gamma}, \quad \alpha\gamma \in \mathbb{R}, \quad \beta\delta \in \mathbb{R}, \quad k_2 = \pm\overline{k}_1.$$

Furthermore, if $k_1, k_2 \in \mathbb{R}$ then the Wronskian $\mathrm{W}(y_1, y_2)$ satisfy condition (ii) if and only if:

$$arg(\beta) = arg(\alpha) + n2\pi, \quad arg(\delta) = arg(\gamma) + n2\pi \quad n \in \mathbb{Z}.$$

Proof. The Wronskian of the two solutions is given by:

$$W(x) = i\alpha\gamma\big(k_2 - k_1\big)e^{i(k_1+k_2)x} + i\alpha\delta\big(k_1 + k_2\big)e^{i(k_1-k_2)x}$$
$$+ i\beta\gamma\big(k_1 + k_2\big)e^{-i(k_1-k_2)x} + i\beta\delta\big(k_1 - k_2\big)e^{-i(k_1+k_2)x}.$$

To satisfy condition (ii) we search for Wronskian determinants able to satisfy:

$$\begin{aligned}\overline{W(-x)} = {} & -i\overline{\alpha\gamma}(\overline{k_2} - \overline{k}_1)e^{i(\overline{k}_1+\overline{k_2})x} - i\overline{\alpha\delta}(\overline{k}_1 + \overline{k_2})e^{i(\overline{k}_1-\overline{k_2})x} \\ & - i\overline{\beta\gamma}(\overline{k}_1 + \overline{k_2})e^{-i(\overline{k}_1-\overline{k_2})x} - i\overline{\beta\delta}(\overline{k}_1 - \overline{k_2})e^{-i(\overline{k}_1+\overline{k_2})x} = W(x).\end{aligned} \tag{12}$$

Since $\overline{\mathrm{W}(-x)}$ and $\mathrm{W}(x)$ are given by finite sums of exponential functions, for condition (ii) to hold it is necessary that the same set of frequencies appear in the exponential functions. There are four different cases to check when matching the frequencies. We check each case separately:

Case 1:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 + \overline{k_2}, \\ k_1 - k_2 = \overline{k}_1 - \overline{k_2}, \end{cases} \Longrightarrow k_1, k_2 \in \mathbb{R},$$

Case 2:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 + \overline{k_2}, \\ k_1 - k_2 = -\overline{k}_1 + \overline{k_2}, \end{cases} \Longrightarrow k_1 = \overline{k_2}.$$

Case 3:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 - \overline{k_2}, \\ k_1 - k_2 = \overline{k}_1 + \overline{k_2}, \end{cases} \Longrightarrow k_1, k_2 \in \mathbb{R}$$

Case 4:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 - \overline{k_2}, \\ k_1 - k_2 = -\overline{k}_1 - \overline{k_2}, \end{cases} \Longrightarrow k_1 = -\overline{k_2}.$$

Hence indeed, $k_1 = \pm\overline{k}_2$ or $k_1, k_2 \in \mathbb{R}$. We study first the case when $k_1 = a + ib,\ k_2 = a - ib,\ ab \neq 0$. The Wronskian determinant is given by:

$$\begin{aligned} \mathrm{W}(x) &= \alpha\gamma 2be^{2iax} - i\alpha\delta 2ae^{-2bx} + i\beta\gamma 2ae^{2bx} - \beta\delta 2be^{-2iax} \\ &= 2b\big(\alpha\gamma e^{2iax} - \beta\delta e^{-2iax}\big) + 2ia\big(\beta\gamma e^{2bx} - \alpha\delta e^{-2bx}\big). \end{aligned} \tag{13}$$

If we again study condition (ii), we have:

$$\begin{aligned} \mathrm{W}(x) &= 2b\big(\alpha\gamma e^{2iax} - \beta\delta e^{-2iax}\big) + 2ia\big(\beta\gamma e^{2bx} - \alpha\delta e^{-2bx}\big) = \\ \overline{\mathrm{W}(-x)} &= 2b\big(\overline{\alpha\gamma} e^{2iax} - \overline{\beta\delta} e^{-2iax}\big) - 2ia\big(\overline{\beta\gamma} e^{-2bx} - \overline{\alpha\delta} e^{2bx}\big), \end{aligned} \tag{14}$$

in particular we require the coefficients of the last two terms of both lines in the above equation to match, since $\mathrm{W}(x)$ and $\overline{W(-x)}$ must agree in the limits $x \to \pm\infty$. This immediately implies that the coefficients of the first two terms must match as well. Hence we indeed obtain the following conditions on the parameters $\alpha, \beta, \gamma, \delta$:

$$\alpha\delta = \overline{\beta\gamma}, \quad \alpha\gamma,\ \beta\delta \in \mathbb{R}.$$

Suppose instead that $k_1, k_2 \in \mathbb{R}$, if we study formula (12) we find that:

$$\alpha\gamma = -\overline{\alpha\gamma}, \quad \alpha\delta = -\overline{\alpha\delta}, \quad \beta\gamma = -\overline{\beta\gamma}, \quad \beta\delta = -\overline{\beta\delta},$$

from which we can indeed deduce that:

$$arg(\beta) = arg(\alpha) + n2\pi, \quad arg(\delta) = arg(\gamma) + n2\pi.$$

□

In what follows, will restrict our attention to the case when $k_1 = a + ib,\ k_2 = a - ib,\ ab \neq 0$. We note that from the conditions on the parameters $\alpha, \beta, \gamma, \delta$ we may introduce the single complex number $C = \frac{\alpha}{\beta} = \overline{(\frac{\gamma}{\delta})}$, and study the solutions:

$$y_1(x) = \frac{1}{\beta}\big(\ Ce^{ik_1x} + e^{-ik_1x}\big), \quad y_2(x) = \frac{1}{\delta}\big(\ \overline{C}e^{i\overline{k}_1x} + e^{-i\overline{k}_1x}\big)$$

. From formula (13) we can obtain a Wronskian determinant given by:

$$\mathrm{W}_1(x) = \frac{\mathrm{W}(x)}{\beta\delta} = 2b\big(\frac{\alpha\gamma}{\beta\delta}e^{2iax} - e^{-2iax}\big) + 2ia\big(\frac{\gamma}{\delta}e^{2bx} - \frac{\alpha}{\beta}e^{-2bx}\big)$$

Which we may write as:

$$\begin{aligned} \mathrm{W}_1(x) =& 2b\big(C\overline{C} - 1\big)\cos(2ax) + 4a\Im(C)\cosh(2bx) \\ &+ i\Big(2b\big(C\overline{C} + 1)\sin(2ax) + 4a\Re(C)\sinh(2bx)\Big). \end{aligned} \tag{15}$$

In the following section we will calculate the potential obtained from formula (5) using the Wronskian $W_1(x)$ with varying complex parameter C. We have not yet, however, assured that the above defined Wronskian satisfies condition (i). It is nontrivial to completely describe how the parameter C affects the zero-set of the equation:

$$W_1(x) = 0.$$

We note that the Wronskian is equal to zero if and only if the real and imaginary parts vanish simultaneously, hence for a fixed C we have to look for solutions, $x \in \mathbb{R}$, to the following system of equations:

$$\begin{cases} b(|C|^2 - 1)\cos(2ax) + 2a\Im(C)\cosh(2bx) = 0, \\ b(|C|^2 + 1)\sin(2ax) + 2a\Re(C)\sinh(2bx) = 0. \end{cases} \tag{16}$$

If there are no real x such that the system (16) is satisfied, then the Wronskian does not vanish on the real axis and hence satisfies condition (i). We will however restrict ourselves here to providing some examples where the system (16) has real solutions, and consequently condition (i) is not satisfied, and some examples where the system (16) has no real solutions and condition (i) is indeed satisfied.

Example 1:
Suppose $k = a + ib$ with $|\frac{a}{b}| > \frac{1}{2}$. If we pick the complex number $C = \frac{1}{2}(1 + i)$, then from the first line in (16) we obtain the following:

$$\begin{aligned} &-\frac{b}{2}\cos(2ax) + a\cosh(2bx) = 0, \\ &\Longrightarrow \cos(2ax) = 2\frac{a}{b}\cosh(2bx) > 1, \end{aligned}$$

from the last line above we find that there can be no real solutions. Hence in this case condition (i) is indeed satisfied since the real part of the Wronskian does not vanish for $x \in \mathbb{R}$.

Example 2:
If $C = \pm 1$ then the first line in (16) vanishes identically, and $x = 0$ is a zero of the Wronskian and condition (i) is not satisfied.

Example 3:
Our main example will be if we pick $C = \pm i$, then for any $k = a + ib \in \mathbb{C}$ the first line in (16) is equivalent to $\cosh(2bx) = 0$, which has no real solutions. In which case the Wronskian indeed satisfies conditions (i) and (ii). We will return to this example in the following section, where we explicitly calculate the potential in this case.

Before we proceed with calculating the potentials, we characterizes a second family of Wronskian determinants able to satisfy conditions (i) and (ii)

Theorem 2. *Consider again the solutions (11) and assume that α or β is equal to zero, then, for k_1, k_2 not both real the Wronskian $\mathrm{W}(y_1, y_2)$ satisfies conditions (i) and (ii) if and only if:*

$$k_1 \in \mathbb{R}, k_2 \in i\mathbb{R} \quad \text{and} \quad \delta = -\overline{\gamma}.$$

Similarly, if γ or δ is equal to zero and k_1, k_2 not both real then the Wronskian satisfies conditions (i) and (ii) if and only if:

$$k_2 \in \mathbb{R}, k_1 \in i\mathbb{R} \quad \text{and} \quad \beta = -\overline{\alpha}.$$

If both k_1 and k_2 are real, then the Wronskian satisfies condition (ii) if and only if:

$$\alpha\gamma = -\overline{\alpha\gamma}, \quad \alpha\delta = -\overline{\alpha\delta}.$$

Proof. Let $\beta = 0$, the proof extends word by word to the other case. The Wronskian to study is given by:

$$W(x) = i\alpha\gamma\big(k_2 - k_1\big)e^{i(k_1+k_2)x} + i\alpha\delta\big(k_1 + k_2\big)e^{i(k_1-k_2)x}.$$

From condition (ii) we require that:

$$\overline{W(-x)} = -i\overline{\alpha\gamma}(\overline{k}_2 - \overline{k_1})e^{i(\overline{k}_1+\overline{k_2})x} - i\overline{\alpha\delta}(\overline{k}_1 + \overline{k_2})e^{i(\overline{k}_1-\overline{k_2})x} = W(x).$$

By similar argument as in the case $\beta \neq 0$ it is necessary to match the frequencies of the exponential functions, hence we have to check the following cases:

Case 1:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 + \overline{k_2}, \\ k_1 - k_2 = \overline{k}_1 - \overline{k}_2, \end{cases} \Longrightarrow k_1, k_2 \in \mathbb{R}$$

Case 2:

$$\begin{cases} k_1 + k_2 = \overline{k}_1 - \overline{k_2}, \\ k_1 - k_2 = \overline{k}_1 + \overline{k}_2. \end{cases} \Longrightarrow k_1 \in \mathbb{R}, k_2 \in i\mathbb{R}.$$

Indeed, $k_1 \in \mathbb{R}$ and $k_2 \in i\mathbb{R}$ or $k_1, k_2 \in \mathbb{R}$. If we first let $k_1 = k \in \mathbb{R}$ and $k_2 = -i\kappa, \kappa \in \mathbb{R}$, then the solutions are on the form:

$$y_1(x) = \alpha e^{ikx}, \quad y_2(x) = \gamma e^{\kappa x} + \delta e^{-\kappa x}.$$

The Wronskian to study is then given by:

$$\begin{aligned} \mathrm{W}(x) &= \kappa\alpha e^{ikx}(\gamma e^{\kappa x} - \delta e^{-\kappa x}) - ik\alpha e^{ikx}(\gamma e^{\kappa x} + \delta e^{-\kappa x}) \\ &= \alpha e^{ikx}\bigg(\gamma\big(\kappa - ik\big)e^{\kappa x} - \delta\big(\kappa + ik\big)e^{-\kappa x}\bigg). \end{aligned}$$

If we again consider condition (ii), we require that:

$$\mathrm{W}(x) = \alpha e^{ikx}\Big(\gamma(\kappa - ik)e^{\kappa x} - \delta(\kappa + ik)e^{-\kappa x}\Big) =$$

$$\overline{\mathrm{W}(-x)} = \overline{\alpha} e^{ikx}\Big(\overline{\gamma}(\kappa + ik)e^{-\kappa x} - \overline{\delta}(\kappa - ik)e^{\kappa x}\Big),$$

in particular, by considering the limits $x \to \pm\infty$ we obtain $\delta = -\overline{\gamma}$, from which we immediately obtain that $\alpha \in \mathbb{R}$. Hence the Wronskian is given by:

$$\mathrm{W}_2(x) = \alpha e^{ikx}\Big(\gamma(\kappa - ik)e^{\kappa x} + \overline{\gamma}(\kappa + ik)e^{-\kappa x}\Big). \tag{17}$$

Which indeed satsifies condition (ii). If we consider condition (i), we have that the above Wronskian vanishes if and only if:

$$\gamma(\kappa - ik)e^{\kappa x} + \overline{\gamma}(\kappa + ik)e^{-\kappa x} = 0,$$

if we let $\gamma(\kappa - ik) = A$, we may write the above equation as:

$$\begin{aligned} Ae^{\kappa x} + \overline{A}e^{-\kappa x} &= (A + \overline{A})\cosh(\kappa x) + (A - \overline{A})\sinh(\kappa x) \\ &= 2\Re(A)\cosh(\kappa x) + i2\Im(A)\sinh(\kappa x) = 0 \end{aligned}$$

which evidently has no solutions for real x. Hence for the Wronskian given by formula (17) conditions (i) and (ii) are indeed satisfied.

If k_1 and k_2 are both real, then the solutions are given by:

$$y_1(x) = \alpha e^{ik_1 x}, \quad y_2(x) = \gamma e^{ik_2 x} + \delta e^{-ik_2 x},$$

and the Wronskian of the two solutions is given by:

$$\begin{aligned} \mathrm{W}(x) &= ik_2\alpha e^{ik_1 x}(\gamma e^{ik_2 x} - \delta e^{-ik_2 x}) - ik_1\alpha e^{ikx}(\gamma e^{ik_2 x} + \delta e^{-ik_2 x}) \\ &= i\alpha e^{ik_1 x}\Big(\gamma\big(k_2 - k_1\big)e^{ik_2 x} - \delta\big(k_2 + k_1\big)e^{-ik_2 x}\Big). \end{aligned}$$

If we study condition (ii) we require that:

$$\mathrm{W}(x) = i\alpha e^{ik_1 x}\Big(\gamma\big(k_2 - k_1\big)e^{ik_2 x} - \delta\big(k_2 + k_1\big)e^{-ik_2 x}\Big) =$$

$$\overline{\mathrm{W}(-x)} = -i\overline{\alpha} e^{ik_1 x}\Big(\overline{\gamma}\big(k_2 - k_1\big)e^{ik_2 x} - \overline{\delta}\big(k_2 + k_1\big)e^{-ik_2 x}\Big),$$

if we consider $\mathrm{W}(x) - \overline{\mathrm{W}(-x)} = 0$, we indeed have that:

$$\alpha\gamma = -\overline{\alpha\gamma}, \quad \alpha\delta = \overline{\alpha\delta}.$$

□

In what follows we will restrict our attention to the case when $k_1 \in \mathbb{R}$ and $k_2 \in i\mathbb{R}$. The potentials obtained from formula (5) using the Wronskian given by formula (17) constitute the second family of potentials. In the next section, we will provide explicit formulas for both families of potentials.

2.2.1 Constructing the Potentials

In the previous section we obtained two distinct families of $\mathcal{PT}$-symmetric Wronskian determinants from Darboux transformation of rank $N = 2$ starting with complex-valued solutions to the zero potential Schrödinger equation. We showed that both families of Wronskian determinants could be chosen to be $\mathcal{PT}$-symmetric and non-vanishing on the real axis. In what follows we use formula (5) to obtain two distinct families of complex-valued, regular and $\mathcal{PT}$-symmetric potentials $u_1(x)$ and $u_2(x)$ from the corresponding Wronskian determinants $\mathrm{W}_1(x)$ and $\mathrm{W}_2(x)$. The following Lemma shows that in the case of the free Schrödinger equation (10), it is indeed sufficient for the Wronskian determinant to satisfy condition (*ii*) in order for the potential obtained from formula (5) to be $\mathcal{PT}$-symmetric.

Lemma 1. *Assume that the initial potential $u(x)$ is $\mathcal{PT}$-symmetric:*

$$\overline{u(-x)} = u(x).$$

If the solutions $y_1, \dots, y_N$ are chosen such that the Wronskian is $\mathcal{PT}$-symmetric:

$$\overline{\mathrm{W}(-x)} = \mathrm{W}(x).$$

Then the potential obtained by Darboux transformation of order N is $\mathcal{PT}$-symmetric.

Proof. We have:

$$\begin{aligned}
\frac{d}{dx}\left(\overline{\mathrm{W}(-x)}\right) &= -\overline{\mathrm{W}_x(-x)} = -\mathrm{W}_x(x), \\
\frac{d^2}{dx^2}\left(\overline{\mathrm{W}(-x)}\right) &= -\frac{d}{dx}\overline{\mathrm{W}_x(-x)} = \overline{\mathrm{W}_{xx}(-x)} = \mathrm{W}_{xx}(x),
\end{aligned}$$

hence:

$$\begin{aligned}
\frac{d^2}{dx^2}\left(\ln \overline{\mathrm{W}(-x)}\right) &= \frac{\overline{\mathrm{W}_{xx}(-x)\mathrm{W}(-x)} - \overline{\mathrm{W}_x(-x)}^2}{\overline{\mathrm{W}(-x)}^2} \\
&= \frac{\mathrm{W}_{xx}(x)\mathrm{W}(x) - \mathrm{W}_x(x)^2}{\mathrm{W}(x)^2} = \frac{d^2}{dx^2}\left(\ln \mathrm{W}(x)\right),
\end{aligned}$$

and indeed we have:

$$\begin{aligned}
\overline{u[N](-x)} &= \overline{u(-x)} - 2\frac{d^2}{dx^2}\left(\ln \overline{\mathrm{W}(-x)}\right) \\
&= u(x) - 2\frac{d^2}{dx^2}\left(\ln \mathrm{W}(x)\right) = u[N](x).
\end{aligned}$$

□

In particular, since the initial potential $u(x)$ was chosen equal to zero and both $\mathrm{W}_1(x)$ and $\mathrm{W}_2(x)$ satisfy the assumptions of the above Lemma, we have that the corresponding families of potentials $u_1(x)$ and $u_2(x)$ are indeed $\mathcal{PT}$-symmetric. In what follows, we will calculate explicitly the potentials given by formula (5) for each of the Wronskian determinants given by formulas (15) and (17).

2.2.2 The Potential $u_1(x)$

We recall the Wronskian $\mathrm{W}_1(x)$:

$$\mathrm{W}_1(x) = C_1\cos(2ax) + C_2\cosh(2bx) + i\big(C_3\sin(2ax) + C_4\sinh(2bx)\big).$$

For each fixed $C \in \mathbb{C}$ the coefficients are given by:

$$C_1 = 2b(|C|^2 - 1), \quad C_2 = 4a\Im(C),$$
$$C_3 = 2b(|C|^2 + 1), \quad C_4 = 4a\Re(C),$$

In order to calculate the corresponding potential we need to compute the first and second derivative of the above defined Wronskian, they are given by:

$$\mathrm{W}_{1x}(x) = -2aC_1\sin(2ax) + 2bC_2\sinh(2bx) + i\big(2aC_3\cos(2ax) + 2bC_4\cosh(2bx)\big),$$
$$\mathrm{W}_{1xx}(x) = -4a^2C_1\cos(2ax) + 4b^2C_2\cosh(2bx) + i\big(-4a^2C_3\sin(2ax) + 4b^2C_4\sinh(2bx)\big).$$

To finnish the construction of the potential, we need to calculate the terms $\mathrm{W}_{1xx}(x)\mathrm{W}(x)$ and $\mathrm{W}_{1x}(x)^2$. If we calculate the term $\mathrm{W}_{1xx}(x)\mathrm{W}(x)$ find it is given by:

$$\begin{aligned}
&\mathrm{W}_{1xx}(x)\mathrm{W}(x) = \\
&-4a^2C_1^2\cos^2(2ax) - 4a^2C_1C_2\cos(2ax)\cosh(2bx) - i2a^2C_1C_3\sin(4ax) - i4a^2C_1C_4\cos(2ax)\sinh(2bx) \\
&+ 4b^2C_1C_2\cos(2ax)\cosh(2bx) + 4b^2C_2\cosh^2(2bx) + i4b^2C_2C_3\cosh(2bx)\sin(2ax) + i2b^2C_2C_4\sinh(4bx) \\
&- 2ia^2C_1C_3\sin(4ax) - 4ia^2C_2C_3\cosh(2bx)\sin(2ax) + 4a^2C_3^2\sin^2(2ax) + 4a^2C_3C_4\sin(2ax)\sinh(2bx) \\
&+ i4b^2C_4C_1\sinh(2bx)\cos(2ax) + i2b^2C_2C_4\sinh(4bx) - 4b^2C_4C_3\sinh(2bx)\sin(2ax) - 4b^2C_4\sinh^2(2bx).
\end{aligned}$$

Next, we calculate the term $\mathrm{W}_{1x}(x)^2$ and find that it is given by:

$$\begin{aligned}
&\mathrm{W}_{1x}(x)^2 = \\
&4a^2C_1^2\sin^2(2ax) - 4abC_1C_2\sinh(2bx)\sin(2ax) - i2a^2C_1C_3\sin(4ax) - i4abC_1C_4\sin(2ax)\sinh(2bx) \\
&- 4aC_2\sinh(2bx)\sin(2ax) + 4b^2C_2^2\sinh^2(2bx) + i4abC_2C_3\sinh(2bx)\cos(2ax) + i2b^2C_2C_4\sinh(4bx) \\
&- 2ia^2C_1C_3\sin(4ax) + i4abC_2C_3\sinh(2bx)\cos(2ax) - 4a^2C_3^2\cos^2(2ax) - 4abC_3C_4\cos(2ax)\cosh(2bx) \\
&- i4abC_1C_4\sin(2ax)\cosh(2bx) + 2ib^2C_2C_4\sinh(4bx) - 4abC_3C_4\cos(2ax)\cosh(2bx) - 4b^2C_4^2\cosh^2(2bx).
\end{aligned}$$

If we consider the similarities of the two expressions above, we find, for example, that the difference of the terms on the diagonals of the two expressions is equal to a constant. Proceeding similarly when we compute the factor $\mathrm{W}_{xx}(x)\mathrm{W}(x)-\mathrm{W}_x(x)^2$ appearing in formula (5), we collect the terms with the same functions and obtain the potential:

$$u_1(x)=\frac{D_1+\cosh(2bx)\Big(D_2\cos(2ax)+iD_3\sin(2ax)\Big)+\sinh(2bx)\Big(iD_4\cos(2ax)-D_5\sin(2ax)\Big)}{\Big(C_1\cos(2ax)+C_2\cosh(2bx)+i\big(C_3\sin(2ax)+C_4\sinh(2bx)\big)\Big)^2}, \tag{18}$$

With the coefficients D_i given by:

$$\begin{aligned}
D_1 &= -8\big(a^2(C_3^2-C_1^2)+b^2(C_2^2-C_4^2)\big),\\
D_2 &= -8C_1C_2(b^2-a^2)-16abC_3C_4,\\
D_3 &= -8C_2C_3(b^2-a^2)-16abC_1C_4,\\
D_4 &= -8C_1C_4(b^2-a^2)+16abC_2C_3,\\
D_5 &= -8C_2C_4(b^2-a^2)+16abC_1C_2.
\end{aligned}$$

If the parameter C is chosen such that $\mathrm{W}_1(x)$ does not vanish on the real axis, as was the case in Example 2, then formula (18) describes a regular, complex-valued, $\mathcal{PT}$-symmetric potential with the additional property that in both limits $x\to\pm\infty$, $u_1(x)$ decays exponentially. We can see that it is indeed $\mathcal{PT}$-symmetric by checking that simultaneous involution and conjugation acts trivially on all terms. Moreover, we can see that it decays exponentially if we note that the denominator contains an exponentially increasing function of a higher power than in the numerator.

It is of interest to again consider Example 2 from the previous section; If we take $C=-i$, we obtain:

$$C_1=0,\quad C_2=-4a,\quad C_3=4b,\quad C_4=0.$$

Furthermore if we let $b=\frac{1}{2}$, we have the coefficients:

$$D_1=-64a^2,\ D_2=0,\ D_3=64a(\frac{1}{4}-a^2),\ D_4=-64a^2,\ D_5=0,$$

hence we obtain the potential:

$$u(x)=\frac{-64a^2+i64a(\frac{1}{4}-a^2)\cosh(x)\sin(2ax)-i64a^2\cos(2ax)\sinh(x)}{\big(-4a\cosh(x)+i2\sin(2ax)\big)^2}.$$

If we let $a\to 0$ we obtain the potential:

$$\tilde{u}(x)=\frac{-4+2ix\cosh(x)-4i\sinh(x)}{\big(\cosh(x)-ix\big)^2}=2\bigg(\frac{(\sinh(x)+i)^2}{(\cosh(x)-ix)^2}-\frac{\cosh(x)}{(\cosh(x)-ix)}\bigg),$$

which is precisely the potential obtained by A.L. Sakhnovich in [11] using some generalized matrix Darboux transformation of the zero potential matrix Schrödinger equation. However, we note that we obtain this potential in the limit as $a \to 0$, in this sense the potential $u(x)$ is slight a generalization of $\tilde{u}(x)$.

2.2.3 The Potential $u_2(x)$

Following theorem 2, we choose $k_1 = k$ and $k_2 = -i\kappa,\ \kappa \in \mathbb{R}$ such that the Wronskian to study is given by:

$$\mathrm{W}_2(x) = e^{ikx}\left(\gamma(k-i\kappa)e^{\kappa x} + \overline{\gamma}(k+i\kappa)e^{-\kappa x}\right), \quad \gamma \in \mathbb{C}.$$

To complete the calculation of the potential given by formula (5), we compute the derivatives:

$$\mathrm{W}_{2x}(x) = e^{ikx}\left(\gamma(k-i\kappa)(\kappa+ik)e^{\kappa x} - \overline{\gamma}(k+i\kappa)(\kappa-ik)e^{-\kappa x}\right),$$
$$\mathrm{W}_{2xx}(x) = e^{ikx}\left(\gamma(k-i\kappa)(\kappa+ik)^2e^{\kappa x} + \overline{\gamma}(k+i\kappa)(\kappa-ik)^2e^{-\kappa x}\right).$$

With the above formulas for the derivatives of the Wronskian, we calculate first the term $\mathrm{W}_{2xx}(x)\mathrm{W}_2(x)$:

$$\mathrm{W}_{2xx}(x)\mathrm{W}_2(x) =$$
$$e^{2ikx}\left(\gamma^2(k-i\kappa)^2(k+i\kappa)^2e^{2\kappa x} + |\gamma(k+i\kappa)|^2\big((k+i\kappa)^2+(k-i\kappa)^2\big) + \overline{\gamma}^2(k+i\kappa)^2(k-i\kappa)^2e^{-2\kappa x}\right).$$

Next, we calculate the term $\mathrm{W}_{2x}(x)^2$:

$$\mathrm{W}_{2x}(x)^2 =$$
$$e^{2ikx}\left(\gamma^2(k-i\kappa)^2(k+i\kappa)^2e^{2\kappa x} - 2|\gamma(k+i\kappa)|^2(k^2+\kappa^2) + \overline{\gamma}^2(k+i\kappa)^2(k-i\kappa)^2e^{-2\kappa x}\right),$$

after some simplification we find from formula (5) the potential of type 2 is given by:

$$u_2(x) = \frac{-8|\gamma(k-i\kappa)|^2k^2}{\left(\gamma(k-i\kappa)e^{\kappa x} + \overline{\gamma}(k+i\kappa)e^{-\kappa x}\right)^2}. \tag{19}$$

The constructed potential is indeed $\mathcal{PT}$-symmetric. We can see this by noting that simultaneous involution and conjugation act trivially on both the numerator and denominator. Moreover we also have that in both limits $x \to \pm\infty$ the potential decays exponentially.

In the preceding sections started with the free Schrödinger equation (10) and constructed two families of potentials, $u_1(x)$ and $u_2(x)$. From these potentials we may obtain new Schrödinger equations and by the properties of Darboux transformation, equation (3) provides an explicit formula for all the solutions. In what follows we will study further the Schrödinger equations:

$$-y_{xx} + u_i(x)y = \lambda y,$$

with potential $u_1(x)$, given by formula (18) and potential $u_2(x)$ given by formula (19).

3 The Scattering Problem

In this section we study the scattering problem for the Schrödinger equations:

$$-y_{xx} + u_i(x)y = \lambda y, \tag{20}$$

with potential $u_i(x)$ given by formulas (18) and (19). We will determine the bound state eigenvalues, calculate the corresponding eigenfunctions and compute the associated scattering matrix for each Schrödinger equation. In theorem 3 we introduce a class of solutions, $y(k, x)$. By studying this class of solutions we obtain the bound state solutions and scattering matrix of the Schrödinger operator with potential (18) and (19) respectively. Before we proceed, we prove two auxiliary results regarding the Wronskian determinant of arbitrary solutions $y_1(k_1, x),\ y_2(k_2, x)$ to the free Schrödinger equation (10):

Lemma 2.

$$\begin{vmatrix} y_1 & y_2 \\ y_{1xx} & y_{2xx} \end{vmatrix} = W_x(x)$$

Proof. We compute:

$$\begin{aligned} W_x(x) &= \frac{d}{dx}\Big(y_1 y_{2x} - y_{1x} y_2\Big) \\ &= y_{1x}y_{2x} + y_1 y_{2xx} - y_{1xx}y_2 - y_{1x}y_{2x} \\ &= y_1 y_{2xx} - y_{1xx}y_2 = \begin{vmatrix} y_1 & y_2 \\ y_{1xx} & y_{2xx} \end{vmatrix} \end{aligned}$$

□

Lemma 3.

$$\begin{vmatrix} y_{1x} & y_{2x} \\ y_{1xx} & y_{2xx} \end{vmatrix} = -k_1 k_2 \mathrm{W}(x)$$

Proof. We compute:

$$\begin{aligned} \begin{vmatrix} y_{1x} & y_{2x} \\ y_{1xx} & y_{2xx} \end{vmatrix} &= y_{1x}y_{2xx} - y_{1xx}y_{2x} = -y_{1x}k_2^2 y_2 + k_1^2 y_1 y_{2x} \\ &= ik_2k_1^2 y_1 y_2 - ik_1k_2^2 y_1 y_2 = -k_1k_2\big(ik_2 y_1 y_2 - ik_1 y_1 y_2\big) \\ &= -k_1k_2\big(y_1 y_{2x} - y_{1x} y_2\big) = -k_1 k_2 \mathrm{W}(x) \end{aligned}$$

□

Theorem 3. *Suppose $y_1(k_1,x)$ and $y_2(k_2,x)$ are solutions of the zero potential Schrödinger equation (10), then solutions to the Schrödinger equation with potential $u[2](x)$ determined by formula (5) are given by:*

$$y(k,x) = \Big(-k^2 - k_1k_2 - ik\frac{\mathrm{W}_x(x)}{\mathrm{W}(x)}\Big)e^{ikx}. \tag{21}$$

We note that the Wronskian determinant appearing in the above defined function $y(k,x)$ is the Wronskian determinant $\mathrm{W}(y_1,y_2)$.

Proof. For each $k \in \mathbb{C}$, $y(x) = e^{ikx}$ is a solution to the free Schrödinger equation (10). For any such solution the properties of Darboux transformation guarantee that solutions to the Schrödinger equation with potential $u(x)$ are given by:

$$y[2](x) = \frac{\mathrm{W}(y_1,y_2,y)}{\mathrm{W}(y_1,y_2)}.$$

Hence using Lemmas 2 and 3 we may compute:

$$\begin{aligned}\mathrm{W}(y_1,y_2,y) &= \begin{vmatrix} y_1 & y_2 & y \\ y_{1x} & y_{2x} & y_x \\ y_{1xx} & y_{2xx} & y_{xx}\end{vmatrix} = y(x)\begin{vmatrix} y_{1x} & y_{2x} \\ y_{1xx} & y_{2xx}\end{vmatrix} - y_x(x)\begin{vmatrix} y_1 & y_2 \\ y_{1xx} & y_{2xx}\end{vmatrix} + y_{xx}(x)\begin{vmatrix} y_1 & y_2 \\ y_{1x} & y_{2x}\end{vmatrix} \\ &= -y(x)k_1k_2\mathrm{W}(x) - y_x(x)\mathrm{W}_x(x) + y_{xx}(x)\mathrm{W}(x),\end{aligned}$$

if we substitute $y(x) = e^{ikx}$ and divide by $\mathrm{W}(y_1,y_2)$ we immediately obtain the desired solutions:

$$y(k,x) = \frac{\mathrm{W}(y_1,y_2,e^{ikx})}{\mathrm{W}(y_1,y_2)} = \Big(-k^2 - k_1k_2 - ik\frac{\mathrm{W}_x(x)}{\mathrm{W}(x)}\Big)e^{ikx}.$$

□

In particular, we may substitute $\mathrm{W}(x)$ for any of the Wronskian determinants $\mathrm{W}_1(x)$ or $\mathrm{W}_2(x)$ and obtain solutions to the corresponding Schrödinger equations with potentials $u_1(x)$ and $u_2(x)$.

As we shall see, the properties of the above defined solutions allow us to solve the corresponding scattering problems.

3.1 Scattering Problem for $u_1(x)$

To solve the scattering problem for the Schrödinger equation (20) with potential $u_1(x)$ given by formula (18) we will study the solutions derived in Theorem 3. To this end, we first compute:

$$\begin{aligned}\lim_{x\to\pm\infty}\frac{\mathrm{W}_{1x}(x)}{\mathrm{W}_1(x)} &= \lim_{x\to\pm\infty}\frac{-2aC_1\sin(2ax) + 2bC_2\sinh(2bx) + i\big(2aC_3\cos(2ax) + 2bC_4\cosh(2bx)\big)}{C_1 cos(2ax) + C_2\cosh(2bx) + i\big(C_3\sin(2ax) + C_4\sinh(2bx)\big)} \\ &= \frac{\pm 2bC_2 + i2bC_4}{C_2 \pm iC_4} = \pm 2b,\end{aligned}$$

From the calculation above we obtain the following asymptotic behavior of the solutions $y(k,x)$:

$$\begin{aligned} y(k,x) &\sim \Big(-k^2-|k_1|^2-2ikb\Big)e^{ikx} \\ &= \Big(-k^2-|k_1|^2-k(k_1-\overline{k}_1)\Big)e^{ikx} \\ &= -\big(k+k_1\big)\big(k-\overline{k}_1\big)e^{ikx}, \quad x\to\infty. \end{aligned}$$

We may obtain the Jost solution from the left $f_l(k,x)$, characterized by the following properties:

$$\begin{aligned} f_l(k,x) &\sim e^{ikx}+o(1), \quad \Im(k)>0, \quad x\to\infty, \\ f_l(k,x) &\sim \frac{1}{T_l(k)}e^{ikx}+\frac{R_l(k)}{T_l(k)}e^{-ikx}+o(1), \quad x\to\infty, \end{aligned}$$

by considering the function:

$$\hat{y}(k,x)=\frac{y(k,x)}{-\big(k+k_1\big)\big(k-\overline{k}_1\big)},$$

indeed:

$$\hat{y}(k,x)=\frac{y(k,x)}{-\big(k+k_1\big)\big(k-\overline{k}_1\big)}\sim e^{ikx}, \quad x\to\infty.$$

Moreover, we have that:

$$\begin{aligned} y(k,x) &\sim \Big(-k^2-|k_1|^2+2ikb\Big)e^{ikx} \\ &= \Big(-k^2-|k_1|^2+k(k_1-\overline{k}_1)\Big)e^{ikx} \\ &= -\big(k-k_1\big)\big(k+\overline{k}_1\big)e^{ikx}, \quad x\to-\infty. \end{aligned}$$

Hence, from the above calculation we have that:

$$\begin{aligned} \hat{y}(k,x) &= \frac{y(k,x)}{-(k+k_1)(k-\overline{k}_1)}\sim e^{ikx}, \quad x\to\infty \\ \hat{y}(k,x) &\sim \frac{\big(k-k_1\big)\big(k+\overline{k}_1\big)}{\big(k+k_1\big)\big(k-\overline{k}_1\big)}e^{ikx}=\frac{1}{T_l(k)}e^{ikx}, \quad x\to-\infty. \end{aligned}$$

We also note that if we calculate the asymptotic behavior of $y(-k,x)$ we obtain the Jost solution from the right $f_r(k,x)$ characterized similarly by:

$$\begin{aligned} f_r(k,x) &\sim e^{-ikx}+o(1), \quad \Im(k)>0, \quad x\to-\infty \\ f_r(k,x) &\sim \frac{1}{T_r(k)}e^{-ikx}+\frac{R_r(k)}{T_r(k)}e^{ikx}+o(1), \quad x\to\infty, \end{aligned}$$

indeed, if we compute:

$$\tilde{y}(-k,x) = \frac{y(-k,x)}{-(k+k_1)(k-\overline{k}_1)} \sim e^{-ikx}, \quad x \to -\infty$$
$$\tilde{y}(-k,x) \sim \frac{(k-k_1)(k+\overline{k}_1)}{(k+k_1)(k-\overline{k}_1)} e^{-ikx} = \frac{1}{T_r(k)} e^{-ikx}, \quad x \to \infty.$$

Hence we find that the left and right transmission coefficients agree: $T_l(k) = T_r(k) = T(k)$ and the transmission coefficient of the potential $u_1(x)$ is given by:

$$T(k) = \frac{\big(k+k_1\big)\big(k-\overline{k}_1\big)}{\big(k-k_1\big)\big(k+\overline{k}_1\big)}.$$

We also find that $u_1(x)$ is in fact reflectionless since $R_r(k) = R_l(k) = 0$, hence we obtain the following:

Theorem 4. *The scattering matrix for the Schrödinger equation with potential $u_1(x)$ given by formula (18) is given by:*

$$S(k) = \begin{pmatrix} T(k) & 0 \\ 0 & T(k) \end{pmatrix},$$

where:

$$T(k) = \frac{(k+k_1)(k-\overline{k}_1)}{(k-k_1)(k+\overline{k}_1)}.$$

We find that if we take $\Im(k_1) = b > 0$ the poles of $T(k)$ also lie in $\mathbb{C}^+$ analogous to the case of real-valued potentials in the Faddeev class. Furthermore, the solutions given by:

$$y(k_1,x) = \bigg(-k_1^2 - |k_1|^2 - ik_1 \frac{\mathrm{W}_{1x}(x)}{\mathrm{W}_1(x)} \bigg) e^{ik_1 x}, \tag{22}$$
$$y(-\overline{k}_1,x) = \bigg(-\overline{k}_1^2 - |k_1|^2 + i\overline{k}_1 \frac{\mathrm{W}_{1x}(x)}{\mathrm{W}_1(x)} \bigg) e^{-i\overline{k}_1 x},$$

correspond to bound state eigenfunctions with eigenvalues $-k_1^2$ and $-\overline{k}_1^2$. We can see that they are indeed bounded on the entire real axis if we note that when $x \to \infty$ the exponential functions decay since $\Im(k_1) = -\Im(\overline{k}_1) = b > 0$. Moreover, the asymptotic analysis of the function $y(k,x)$ showed that in the limit $x \to -\infty$ we have that $y(k,x) \sim -(k-k_1)(k+\overline{k}_1)e^{ikx}$. Hence indeed $k = k_1$ and $k = -\overline{k}_1$ are the unique values of k such that the function $y(k,x)$ remain bounded on the entire real axis.

3.2 Scattering Problem for $u_2(x)$

Following the method of solving the scattering problem for the preceding potential we first compute:

$$\lim_{x\to\pm\infty} \frac{\mathrm{W}_{2x}(x)}{\mathrm{W}_2(x)} = \lim_{x\to\pm\infty} \frac{\gamma(\kappa-ik)(\kappa+ik_1)e^{\kappa x} - \overline{\gamma}(\kappa+ik)(\kappa-ik_1)e^{-\kappa x}}{\gamma(\kappa-ik)e^{\kappa x} + \overline{\gamma}(\kappa+ik)e^{-\kappa x}} = \begin{cases} (\kappa+ik_1), & x \to \infty, \\ -(\kappa-ik_1), & x \to -\infty. \end{cases}$$

We note that the number k_1k_2 appearing Lemma 3 is given by $i\kappa k_1$. Hence with the Wronskian $\mathrm{W}_2(x)$ the asymptotic behaviors of the solution $y(k,x)$ constructed in Theorem 3 are given by:

$$
\begin{aligned}
y(k,x) &\sim \Big(-k^2 - i\kappa k_1 - ik(\kappa + ik_1)\Big)e^{ikx} \\
&= -\big(k+i\kappa\big)\big(k-k_1\big)e^{ikx}, \quad x \to \infty \\
y(k,x) &\sim \Big(-k^2 - i\kappa k_1 + ik(\kappa - ik_1)\Big)e^{ikx} \\
&= -\big(k-i\kappa\big)\big(k-k_1\big)e^{ikx}, \quad x \to -\infty,
\end{aligned}
$$

from which we can obtain the left and right Jost solutions by considering the functions:

$$
\begin{aligned}
\hat{y}(k,x) &= \frac{y(k,x)}{-\big(k+i\kappa\big)\big(k-k_1\big)}, \\
\tilde{y}(-k,x) &= \frac{y(-k,x)}{-\big(k+i\kappa\big)\big(k-k_1\big)}.
\end{aligned}
$$

If we compute the asymptotic behavior of the functions above we indeed find that:

$$
\begin{aligned}
\hat{y}(k,x) &\sim e^{ikx}, \quad x \to +\infty \\
\hat{y}(k,x) &\sim \frac{k-i\kappa}{k+i\kappa}e^{ikx} = \frac{1}{T_r(k)}e^{ikx}, \quad x \to -\infty
\end{aligned}
$$

$$
\begin{aligned}
\tilde{y}(-k,x) &\sim e^{-ikx}, \quad x \to -\infty \\
\tilde{y}(-k,x) &\sim \frac{k-i\kappa}{k+i\kappa}e^{-ikx} = \frac{1}{T_l(k)}e^{-ikx}, \quad x \to \infty.
\end{aligned}
$$

From which we immediately obtain:

Theorem 5. *The scattering matrix for the Schrödinger equation with potential $u_2(x)$ given by formula (19) is given by:*

$$
S(k) = \begin{pmatrix} T(k) & 0 \\ 0 & T(k) \end{pmatrix},
$$

where:

$$
T(k) = \frac{k+i\kappa}{k-i\kappa}.
$$

We find that the solution:

$$
y(i\kappa,x) = \left(\kappa^2 - \kappa k_1 + \kappa\frac{\mathrm{W}_{2x}(x)}{\mathrm{W}_2(x)}\right)e^{-\kappa x} \tag{23}
$$

correspond to the solution with eigenvalue $-\kappa^2$, associated to the pole $i\kappa$ of the transmission coefficient $T(k)$, which may be chosen to lie in $\mathbb{C}^+$ if we take $\kappa > 0$, analogous to the case of real-valued potentials in the Faddeev class. Furthermore, by similar argument as for the solutions to the potential in the previous section, we indeed have that $k = i\kappa$ is the unique value of k such that the solution $y(k, x)$ remain bounded on the entire real axis.

4 Conclusion

In this thesis we studied $\mathcal{PT}$-symmetric Darboux transformation were able to obtain two distinct families of complex-valued, regular and $\mathcal{PT}$-symmetric potentials given by formulas (18) and (19), not found in the literature. Moreover, we obtained the bound state eigenfunctions of the corresponding Schrödinger operators, they are given by formulas (22) and (23). Furthermore, in Theorems 4 and 5 we obtained the associated scattering matrix of each Schrödinger operator, hence we were able to characterize the scattering properties of the new potentials. Moreover, we showed that both Schrödinger operators admit solutions characterized by the same asymptotic properties as the Jost solutions. By studying these solutions we found that each Schrödinger operator only admits a finite number of bound states, associated to the poles of the respective transmission coefficient. These properties are analogous to those of Schrödinger operators with real-valued potentials in the Faddeev class.

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