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Motives and the Tate Conjecture

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Abstract

This thesis is an overview of the theory of pure motives, as well as an introduction to the Tate conjecture. After going through some preliminaries, we introduce algebraic cycles and Weil cohomology. We then give Grothendieck's classical definition of pure motives and discuss some properties. After that, we move on to André's motivated cycles introduced in [And96], and the category of pure motives they give rise to. We briefly discuss the motivic Galois groups attached to these motives.

The rest of this thesis regards the Tate conjecture, which says that the Tate classes of ℓ-adic cohomology are algebraic. Moonen showed in [Moo18] that if this is the case over a field of characteristic zero, then the Galois representations given by ℓ-adic cohomology are semi-simple. We explain the proof in detail, taking the opportunity to use the theory of algebraic cycles and motives developed in the earlier chapters.

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Contents

Chapter 1

Introduction

This thesis is mainly concerned with the theory of motives. Coming from algebraic geometry, motives are objects with connections to many areas of mathematics such as number theory and representation theory. The purpose of the present text is two-fold, and this is reflected in the division of the last two chapters. We survey the basics of the theory of motives by giving some definitions and elementary properties, and discussing open questions. Then we give a brief introduction to the Tate conjecture, and go through an application of the theory discussed beforehand.

Algebraic geometry. Algebraic geometry was born as the study of geometrical objects, *varieties*, defined by systems of polynomial equations. It saw a huge explosion in the last century, notably with the scheme theory introduced by the school of Grothendieck. In the following decades, the impact of algebraic geometry on disparate mathematical disciplines has mounted considerably and algebraic geometry now plays a central role in modern research mathematics.

Classical algebraic geometry concerns varieties over the complex numbers and in this setting we have a lot of geometrical intuition. Modern tools have facilitated the study of varieties over arbitrary fields, even finite fields. Varieties over finite fields always have finitely many points, and one may try to count how many there are. The roots of this problem go back to Gauss, and Weil worked thought about it in the 1940's.

The Weil conjectures. In the late 1940's, Weil formulated some now famous (and resolved) conjectures regarding the number of points on varieties over finite fields. Investigating these questions, he saw connections to cohomological properties of complex varieties. A complex variety may be given the structure of a complex analytic space. As such, it may be given the complex analytic topology, which is much finer than the standard topology in algebraic geometry, the Zariski topology. Moreover, one may take the singular cohomology of the corresponding complex analytic space, yielding the *Betti cohomology* of the variety.

Unfortunately, there was no cohomology theory in positive characteristic with quite as many desirable properties as Betti cohomology had over the complex numbers. The existence of a cohomology theory with some of those properties over arbitrary base fields, a so-called *Weil cohomology*, then seemed to have the potential of making a lot of problems more approachable.

The substitute for Betti cohomology over arbitrary characteristic came when Grothendieck axiomatised coverings and defined Grothendieck topologies and in particular the étale topology on the category of schemes. This enabled ℓ-adic étale cohomology to be defined over arbitrary base fields. Unlike Betti cohomology though, the coefficients could not be taken as rational but rather as ℓ-adic.

Motives. Since the time of the Weil conjectures, plenty of Weil cohomologies have been defined, most of them connected in one way or another by comparison theorems, analogies, etc. A wish to unify all these different approaches and construct a "universal" cohomology theory led Grothendieck to conjecture the existence of motives. A definition of (pure) motives was proposed, but progress in showing that it had some desirable properties was swiftly halted by a lack of proof of Grothendieck's standard conjectures on algebraic cycles.

Conjecturally, motives also generalise Galois theory. The category of pure motives is expected to be equivalent to the category of finite-dimensional representations of a group scheme called the *motivic Galois group*. The properties of such a group again hinge largely on the standard conjectures.

The Tate conjectures. The standard conjectures are intimately connected with one of the biggest open problems in algebraic geometry, the Tate conjecture. Although rather analogous to the Hodge conjecture, the Tate conjecture has in many ways proven more difficult.

One can say that the Tate conjecture comes in two parts, one part about the semi-simplicity of ℓ-adic cohomology as a Galois representation and one part asserting the algebraicity of a certain aspect of ℓ-adic cohomology. Ben Moonen recently made an advance ([Moo18]) in the understanding of the relationship between these two parts. This thesis is dedicated largely to explaining this result.

1.1 Outline

After the introduction, the present text is divided into three main chapters. First, in chapter 2, we go through some theory needed for the subsequent chapters. We begin by giving a crash course in the theory of profinite groups. Then we go on to describe the basic theory of Galois extensions, and their Galois groups which are the reason we started with profinite groups. We take a brief detour into representation theory for a partial generalisation of Maschke's theorem to profinite groups. Finally, we round off the chapter with sections on graded vector spaces and category theory. These exist mainly to nail down notation and make sure we have solid footing when discussing those concepts in chapter 3.

With the preliminaries taken care of, we attempt a quick tour of the theory of pure motives in chapter 3. We start off with the fundamentals: algebraic cycles and the definition of a Weil cohomology. After that we give an introduction to the classical construction of pure motives. Lastly, we have a section dealing with André's modified category of pure motives. We start this section with some motivation for the constructions that follow and go on to give a sketchy account of how it works.

Chapter 4 is the final chapter, and in it all that came before is brought together. The chapter is intended as an instructive application of pure motives (we use André's construction), and is centered around a recent result by Moonen [Moo18].

Chapter 2

Preliminaries

In this chapter, we introduce some of the preliminary theory which we need in the subsequent chapters. We use these sections to fix notation, and to recall important results for easier referencing later. Perhaps the biggest omission from this chapter is any material on algebraic geometry. We therefore assume familiarity with basic scheme theory, as well as the basics of étale cohomology. For an introduction to the former, see [Har77]. For the latter, see [Del77] or [FK88].

2.1 Profinite groups

This section introduces some basics regarding profinite groups. Our motivating example of a profinite group is that of a Galois group. A concise introduction to profinite groups is given in chapter 1 of [Sza09]. A much more comprehensive treatment may be found in [RZ10]. We shall start with some prerequisites about projective and inductive limits, which are general concepts, ubiquitous in mathematics in general and in algebra in particular.

2.1.1 Limits

In what follows, let objects and morphisms be from some fixed category $\mathcal{C}$. A *projective system* is a directed poset $(I, \leq)$ with a family of objects A_i, for $i \in I$ and for every pair $i \leq j$ in I, a morphism $f_{ij} \colon A_j \to A_i$. Moreover, f_{ii} should be the identity on A_i and composition should be respected: $f_{ij} \circ f_{jk} = f_{ik}$. A *projective limit* (or *inverse limit*) is an object A with morphisms $p_i \colon A \to A_i$, for $i \in I$, called *(natural) projections*, which satisfy $p_i = f_{ij} \circ p_j$ for all $i \leq j$. Moreover, (A, p_i) should satisfy the following universal property. If (B, q_i) is any other such object, with its own projections q_i onto A_i, then there exists a unique morphism $\varphi \colon B \to A$ such that $p_i \circ \varphi = q_i$ for all i in I. If the projective limit exists then it is unique up to unique isomorphism.

When $\mathcal{C}$ is the category of e.g. groups, rings, or topological spaces we can construct a projective limit explicitly as follows, Let (A_i, f_{ij}) be a projective system of groups (or rings or topological spaces). Then consider the subgroup (or subring or subspace) of the direct product of the A_i

defined by

$$\varprojlim_{j\in I} A_j := \left\{ (a_i)_{i\in I} \in \prod_{i\in I} A_i : f_{ij}(a_j) = a_i \ \forall i \leq j \right\}.$$

This is the projective limit of the system (A_i, f_{ij}). The natural projections are then defined by restricting the projection morphisms $\prod_i A_i \to A_j$.

An *inductive system* is analogous to a projective system, but with the arrows reversed. The *inductive limit* (or *direct limit*) is defined in the same way as the projective limit, but again with all the arrows reversed. We can construct it explicitly in the case of groups (or rings or topological spaces). Consider the equivalence relation $\sim$ on the disjoint union $\coprod_{i\in I} A_i$, defined by

$$a_i \sim a_j \iff \exists k \in I \text{ such that } i \leq k,\ j \leq k, \text{ and } f_{ik}(a_i) = f_{jk}(a_j).$$

Then define the inductive limit as

$$\varinjlim_{j\in I} A_j := \left(\coprod_{i\in I} A_i \right) / \sim .$$

2.1.2 Profinite groups

Definition 2.1.1. A *profinite group* is a projective limit of finite groups.

A profinite group G carries a natural topology. Indeed, let (G_i) be a projective system of finite groups such that

$$G = \varprojlim_i G_i.$$

Then, we give each G_i the discrete topology and the direct product $\prod_i G_i$ the product topology. Since G is isomorphic to a subgroup of $\prod_i G_i$ we can give it the induced subspace topology. This turns G into a topological group, i.e. group multiplication and inversion are continuous maps. Profinite groups are compact: a product of finite groups is compact by Tychonoff's theorem, and a closed subset of a compact set is compact.

Being defined as a projective limit, G has a natural projection $p_i\colon G \to G_i$ for each G_i. These are continuous: the preimage of any subset of G_i under $\prod_i G_i \to G_i$ is open in the product topology, and intersecting the preimage with G yields an open set in the subspace topology. In particular, $\ker p_i$ is open for every i, and

$$G/\ker p_i \cong \operatorname{im} p_i \subseteq G_i,$$

so that $\ker p_i$ has finite index.

Lemma 2.1.2. *The* $\ker p_i$ *form a neighbourhood basis of* 1 *in* G.

Proof. Let V be a neighbourhood of 1 in G. We need to show that $\ker p_i \subset V$ for some i. That is, we need to show that there exists a p_i such that $p_i(g) = 1$ implies $g \in V$. Because the topology of G is inherited from the product topology there are finitely many indices $i(1), \dots, i(N)$ such that

$$\left[V_{i(1)} \times \cdots \times V_{i(N)} \times \prod_{j \neq i(1), \dots, i(N)} G_j \right] \cap G, \tag{2.1}$$

is an open subset of V, where the $V_{i(k)}$ are subsets of the $G_{i(k)}$. As an open subset of (2.1) we find

$$\left[\{1\}_{i(1)} \times \cdots \times \{1\}_{i(N)} \times \prod_{j \neq i(1), \dots, i(N)} G_j \right] \cap G. \tag{2.2}$$

Since the indexing set is assumed to be directed, there is an index $i(0)$ such that $i(0) \geq i(k)$ for $k = 1, \dots, N$. Then, since the projections p_i are compatible with the morphisms f_{ij}, we have that, for $g \in G$, if $p_{i(0)}(g) = 1$ then $p_{i(k)}(g) = f_{i(k),i(0)} \circ p_{i(0)}(g) = 1$. Thus, the following is an open subset of (2.2)

$$\left[\{1\}_{i(0)} \times \prod_{j \neq i(1), \dots, i(N)} G_j \right] \cap G.$$

But this is exactly $\ker p_{i(0)}$ so we are done. □

Corollary 2.1.3. *Any open subgroup of a profinite group has finite index.*

Proof. If H is an open subgroup of G then H is in particular an open neighbourhood of 1. Thus $\ker p_i \subseteq H$ for some p_i, and $[G : H] \leq [G : \ker p_i] < \infty$. □

In fact, we have the following result:

Lemma 2.1.4. *The open subgroups of a profinite group are exactly the closed subgroups of finite index.*

Proof. Let G be profinite. Suppose $H \subset G$ is an open subgroup. Then gH is homeomorphic to H, for all g in G. By the continuity of multiplication by g, and the fact that it has a continuous inverse: multiplication by g^{-1}. Thus $\{gH\}_{g \in G}$ is an open cover of G. Compactness implies existence of a finite subcover, but since the cosets are disjoint this actually means that the whole cover is finite. Thus $[G : H]$ is finite. Moreover, the complement $G \setminus H$ is $\bigcup_{g \notin H} gH$, which is open, and hence H is closed.

Conversely, suppose H is closed of finite index in G. Then again, $G \setminus H$ is $\bigcup_{g \notin H} gH$ and this is a finite union of closed sets and hence closed. Thus H is open. □

Example 2.1.5. Examples of profinite groups include finite groups, Galois groups (Lemma 2.2.9) and étale fundamental groups. In fact, every profinite group is the Galois group of some Galois extension: [Wat73].

Example 2.1.6 (p-adic numbers)**.** Let p be a prime. The finite rings $\mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$ form a projective system, with the obvious maps. The projective limit is $\mathbb{Z}_p$, the ring of *p-adic integers*. It is an integral domain and its field of fractions is the field $\mathbb{Q}_p$ of *p-adic numbers*.

2.2 Galois theory

Chapter 4 of this thesis is about the Tate conjecture, which is a statement about Galois representations, i.e. representations of Galois groups. The purpose of this section is to recall some basics of Galois theory, which we use in chapter 4. For us, the two main results are (1) that Galois groups are profinite (Lemma 2.2.9) and (2) the fundamental correspondence of Galois theory (Prop. 2.2.10). For proofs of all the results in this section, see e.g. [Lan05] or [Bou07a].

2.2.1 Field extensions

We begin with some definitions regarding field extensions. Let L/K be a field extension, i.e. an inclusion of fields $K \subseteq L$. Recall that an element α in L is *algebraic* over K if it is a zero of a polynomial with coefficients in K. If α is algebraic over K then there is a unique monic irreducible polynomial $m_{\alpha,K}$ with coefficients in K, such that $m_{\alpha,K}(\alpha) = 0$. This is the *minimal polynomial* of α over K. If $m_{\alpha,K} = m_{\beta,K}$ then α and β are *conjugate* over K. Say L/K is an *algebraic extension* if every $\alpha \in L$ is algebraic over K.

Definition 2.2.1. A field K is *algebraically closed* if the only algebraic extension of K is K itself.

Proposition 2.2.2. *Any field K has an* algebraic closure*, i.e. an algebraic extension $\bar{K}$ which is algebraically closed.*

An algebraic closure isn't unique or even canonical in any way. Its isomorphism class is however uniquely determined by K.

We say that an irreducible polynomial is *separable* if its formal derivative is non-zero. If α is algebraic over K, then it is *separable* over K if its minimal polynomial $m_{\alpha,K}$ is separable. An algebraic extension L/K is *separable* if every element α in L is separable over K. Note that every algebraic extension of characteristic zero is separable.

Definition 2.2.3. A field K is *separably closed* if the only separable extension of K is K itself.

Proposition 2.2.4. *Any field K has a* separable closure*, i.e. a separable extension K^s which is separably closed.*

An algebraic extension L/K is *normal* if for every irreducible polynomial $p(X)$ in $K[X]$ either has no roots in L or has all its roots in L. Equivalently, L/K is a normal extension if whenever α is in L, all its conjugates are too.

Proposition 2.2.5. *Let K^s be a separable closure of K. Then K^s is normal over K.*

The *degree* $[L : K]$ of an extension L/K is the dimension of L as a vector space over K.

2.2.2 Galois theory

Galois theory is about the relationship between an algebraic field extension L/K and the group of automorphisms of L that fix K, denoted $\mathrm{Aut}(L/K)$. A special role is played by Galois extensions.

Definition 2.2.6. An algebraic extension L/K is *Galois* if the subfield of L fixed under $\mathrm{Aut}(L/K)$ is K. Say that L is *Galois* over K.

Lemma 2.2.7. *Let L/K be an algebraic extension and let $\bar{K}$ be an algebraic closure of K.*

1. *If L is finite over K, then $\mathrm{Hom}_K(L, \bar{K}) \leq [L : K]$, with equality if and only if L is separable.*
2. *L is normal over K if and only if the image of every K-embedding of L into $\bar{K}$ lies in L.*
3. *An algebraic extension L/K is* Galois *if and only it is both normal and separable.*

Definition 2.2.8. When L/K is Galois, write $\mathrm{Gal}(L/K) = \mathrm{Aut}(L/K)$. It is the *Galois group* of L over K.

Note that for Galois extensions L/M, M/K and L/K we have natural projections

$$\mathrm{Gal}(L/K) \to \mathrm{Gal}(M/K)$$

obtained by restriction from L to M. The reason we can do this is part 2 of Lemma 2.2.7. These are in fact surjective, since any automorphism of M can be extended into one of L, when L/M is Galois. In particular we then have projections to $\mathrm{Gal}(M/K)$ for every finite Galois subextension M/K, and intuitively it might seem that the Galois extension L/K should be described as a limit of its finite subextensions. This leads us to the following result.

Lemma 2.2.9. *The Galois group* $\mathrm{Gal}(L/K)$ *of any Galois extension is a profinite group.*

Proof. We will define a projective system and show that its limit is $\mathrm{Gal}(L/K)$. As our indexing set I we will take all the subextensions M of L/K which are finite over K. We order I by inclusion. Now for each M in I, let $G_M = \mathrm{Gal}(M/K)$. Then G_M are finite groups of orders $[M : K]$. For $M \subset M'$ in I, let $f_{M,M'} \colon G_{M'} \to G_M$ be the morphism that restricts automorphisms of M' to automorphisms of M. Denote the limit of this projective system by G.

Now define a homomorphism

$$\varphi \colon Gal(L/K) \to \prod_{M \in I} G_M,$$

by sending an automorphism of L to the product of its restrictions to all the $M \in I$. We need to show that (1) φ is injective and (2) that $\operatorname{im} \varphi = G$.

(1) Suppose $\sigma \in \mathrm{Gal}(L/K)$ is sent to 1 under φ. This means that the restriction of σ to any finite extension of K is trivial. But then σ must fix every element α in L since otherwise the finite extension $K(\alpha)$ would be a counterexample. That $K(\alpha)/K$ is finite follows from the fact that α is algebraic over K.

(2) Let's start with $\operatorname{im} \varphi \subseteq G$. Let $M \subset M'$ be elements in I. Then, taking $\sigma \in \mathrm{Gal}(L/K)$,

$$\varphi(\sigma)_{M'}\big|_M = (\sigma\big|_{M'})\big|_M = \sigma\big|_M = \varphi(\sigma)_M,$$

so that $\varphi(\sigma)$ is indeed an element of G. For the reverse inclusion, $G \subseteq \operatorname{im} \varphi$, take an element τ in G. Then we can define an automorphism σ in $\mathrm{Gal}(L/K)$ as follows: for each α in L, let $\sigma(\alpha) = \tau_M(\alpha)$ for some M in I containing α. This is well-defined and independent of the choice of M since the G_M form a projective system. By definition, $\varphi(\sigma) = \tau$, and we are done. □

A Galois group $\mathrm{Gal}(L/K)$ thus has a natural topology, namely the profinite topology. In the context of Galois groups, it is called the *Krull topology.* With the knowledge that Galois groups are profinite, we may state the following result.

Proposition 2.2.10 (Fundamental Correspondence of Galois Theory). *Let L/K be a Galois extension. For an intermediate extension E, let $\varphi(E)$ be* $\mathrm{Gal}(L/E)$ *as a subgroup of* $\mathrm{Gal}(L/K)$. *Conversely, for a* ***closed*** *subgroup H of* $\mathrm{Gal}(L/K)$, *let $\psi(H) = L^H$ be the subfield of L fixed by H. Then ϕ and ψ are mutual inverses and set up a 1-1 correspondence*

$$\{E : L/E/K\} \longleftrightarrow \{H \leq G : H \text{ is closed}\}.$$

Moreover, letting E be an intermediate field and H the corresponding closed subgroup, the following properties hold:

1. *Inclusion-reversing: $H \subset H'$ if and only if $E \supset E'$.*
2. *E is Galois over K if and only if H is normal in* $\mathrm{Gal}(L/K)$. *In this case,*

$$\mathrm{Gal}(E/K) = \mathrm{Gal}(L/K)/H.$$

3. *H is open if and only if E/K is a finite extension. In this case,*

$$[\mathrm{Gal}(L/K) : H] = [E : K].$$

With the fundamental correspondence in mind, the following lemma might be of interest.

Lemma 2.2.11. *An extension L/K is algebraic if and only if every sub-K-algebra of L is a field.*

Example 2.2.12 (Absolute Galois groups). Let K^{s} be a separable closure of K. Then K^{s}/K is Galois by Prop 2.2.5. We say that $\mathrm{Gal}(K^{\mathrm{s}}/K)$ is the *absolute Galois group* of K. Every finite Galois group of K appears as a quotient of the absolute Galois group.

The case when $K = \mathbb{Q}$ will be of special interest for us. In this case a separable closure is an algebraic closure.

Example 2.2.13 (Cyclotomic extensions). Let μ_n be the n:th roots of unity over $\mathbb{Q}$. Then $\mathbb{Q}(\mu_n)/\mathbb{Q}$ is a Galois extension, and the Galois group is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$. This isomorphism is given by the action on a primitive n:th root of unity ζ

$$\begin{aligned} \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q}) &\xrightarrow{\sim} (\mathbb{Z}/n\mathbb{Z})^\times \\ (\zeta \mapsto \zeta^a) &\mapsto a \pmod{n} \end{aligned}$$

When $n = p$ is a prime, let μ_{p^∞} be the union of μ_{p^m} where m ranges over $\mathbb{N}$. Then we get

$$\mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \varprojlim_m (\mathbb{Z}/p^m\mathbb{Z})^\times = \mathbb{Z}_p^\times,$$

an infinite extension with the p-adic units as Galois group.

2.3 A result from representation theory

This section introduces a lemma regarding representations of profinite groups. We will use this result when dealing with Galois representations in chapter 4.

Lemma 2.3.1. *Let G be a profinite group and V a representation of G. Let $W \subseteq V$ be a subrepresentation. Suppose $V = W \oplus W^\perp$, for some $W^\perp$ which is stable under an open subgroup H of G. Then there exists a $\tilde{W}^\perp$ stable under all of G, such that $V = W \oplus \tilde{W}^\perp$.*

Proof. Let π be the projection onto W associated with the direct sum $V = W \oplus W^\perp$. That is, π is idempotent with $\operatorname{im} \pi = V$ and $\ker \pi = W$. Then π is H-equivariant.

Since H is open in G and G is profinite, H has finite index in G by Lemma 2.1.4. Let T be a left transversal of H in G, i.e. a set of one representative for every left coset of H in G. Then T is a finite set and we can define

$$\tilde{\pi} = \frac{1}{|T|} \sum_{t \in T} t\pi t^{-1}.$$

We first show that $\tilde{\pi}$ is independent of the choice of transversal. Indeed, let $s = th$ for some $t \in T$ and $h \in H$, i.e. s is a different choice of representative for the coset of t. Then $s\pi s^{-1} = th\pi h^{-1}t^{-1}$, and since π is H-equivariant, this is just $t\pi t^{-1}$.

Now we show that $\tilde{\pi}$ is a projection onto W. It is the identity on W:

$$\begin{aligned}
\tilde{\pi}(w) &= \frac{1}{|T|} \sum_{t \in T} t\pi t^{-1}(w) \\
&= \frac{1}{|T|} \sum_{t \in T} tt^{-1}(w) && (W \text{ is stable under } t^{-1}, \pi \text{ is identity on } W) \\
&= \frac{1}{|T|} \sum_{t \in T} w \\
&= w.
\end{aligned}$$

Now let $v \in V$. Then $\pi t^{-1}(v)$ is in W for every $t \in T$ since $\operatorname{im} \pi = W$. Moreover, W is stable under t, so $t\pi t^{-1}(v)$ is in W for every $t \in T$. Thus $\tilde{\pi}(v)$ is the sum of elements in W and hence an element of W, i.e. the image of $\tilde{\pi}$ is W.

Next, we show that $\tilde{\pi}$ is G-equivariant. Let g be in G. We have that

$$\begin{aligned}
g\tilde{\pi}g^{-1} &= \frac{1}{|T|} \sum_{t \in T} gt\pi t^{-1}g^{-1} \\
&= \frac{1}{|T|} \sum_{t \in T} gt\pi (gt)^{-1},
\end{aligned}$$

and this is just $\tilde{\pi}$ since left multiplication by g permutes the cosets of H, and since we have shown the definition of $\tilde{\pi}$ to not depend on the choice of representatives.

Finally, letting $\tilde{W}^\perp = \ker \tilde{\pi}$ we get the desired result. □

Corollary 2.3.2 (Maschke's theorem)**.** *Finite-dimensional representations of finite groups are semi-simple, i.e. decompose into irreducible subrepresentations.*

Proof. A finite group is profinite, and the trivial group is an open subgroup. □

2.4 Category theory

The purpose of this section is to fix some notation and naming conventions in category theory. We do not go into details, we only give rough definitions. For more details, see chapter 2 of [And04], as we use the same notation.

A $\otimes$-*category* is an F-linear category which is also monoidal, such that the product $\otimes$ is bilinear. Such a category is *rigid* if every object has a "dual". In a rigid $\otimes$-category, every endomorphism has a "trace", and the *rank* of an object M is the trace of id_M. A functor between $\otimes$-categories which respects the $\otimes$-structure is a $\otimes$-*functor.*

Now suppose $\mathcal{T}$ is a rigid abelian $\otimes$-category such that $\mathrm{End}\,\mathbf{1} = F$, where $\mathbf{1}$ is the (two-sided) unit with respect to $\otimes$. Then a *fibre functor* ω on $\mathcal{T}$ is a faithful exact $\otimes$-functor

$$\omega\colon \mathcal{T} \to \mathrm{Vec}_L,$$

to the category of finite-dimensional L-vector spaces, for some extension L/F. If such a fibre functor exists, $\mathcal{T}$ is *Tannakian.* If such a fibre functor exists with $L = F$, then $\mathcal{T}$ is *neutral* Tannakian.

For much more about Tannakian categories, see [Del90].

2.5 Graded vector spaces

Cohomology will play a key role in the subsequent chapters. Roughly, cohomology theories are nice ways of attaching graded vector spaces to different objects, in an attempt to linearise the study of them. Therefore, we will spend some time in this section dealing with graded vector spaces, so that we can work with them more easily in the sequel. In the literature, the reader may refer to [Bou07b].

Fix a field F and let $(M,+)$ be a monoid (which, for us, will always be $\mathbb{N}$ or $\mathbb{Z}$). An *(M-)graded vector space* over F is a collection $(V^i)_{i\in M}$ of F-vector spaces, indexed by M. We will usually denote such a graded vector space simply by V^*. We may view V^* as a vector space by taking the direct sum $\oplus_{i\in M} V^i$. If we want to forget the grading on V^*, we shall write V. The elements of V^i are *homogeneous* of *degree* $i \in M$.

An F-linear map $T\colon V^* \to W^*$ between graded vector spaces is *homogeneous* of *degree* i if

$$T(V^j) \subseteq W^{j+i}$$

for all $j \in M$. Write $\mathrm{Hom}_i(V^*, W^*)$ for the homogeneous maps of degree i from V^* to W^*. A homogeneous map of degree zero is simply *homogeneous.*

The M-graded vector spaces over F, together with homogeneous maps, form the *category of M-graded vector spaces*, which is denoted GVec_F^M. Note that $\mathrm{GVec}_F^{\mathbb{N}}$ sits naturally inside $\mathrm{GVec}_F^{\mathbb{Z}}$. We have that GVec_F^M is an abelian category with direct sum given by

$$V^* \oplus W^* = (V^i \oplus W^i)_{i\in M}$$

It is moreover a $\otimes$-category with tensor product given by

$$V^* \otimes W^* = \Big(\bigoplus_{j+k=i} V^j \otimes V^k\Big)_{i\in M}$$

The unit object with respect to $\otimes$ is F concentrated in degree 0.

Write $\mathrm{Hom}_F(V, W)$ for the F-linear maps from V to W. We may consider

$$\mathrm{Hom}_*(V^*, W^*) = (\mathrm{Hom}_i(V^*, W^*))_{i\in M}.$$

as a graded vector space, but the way to forget the grading here is through point-wise addition of maps, so we'll write $\sum_j \mathrm{Hom}_j(V^*, W^*)$ for the underlying vector space. Since we view V^* as a direct sum of its homogeneous parts, we get short exact sequences

$$0 \to V^i \xrightarrow{\iota^i} V^* \xrightarrow{\pi^i} V^i \to 0.$$

Using the ι^i and π^j for V^* and W^* we get can decompose a linear map $T\colon V \to W$ into linear maps $T_{ij}\colon V^i \to W^j$. Thus we get a homogeneous map of degree j

$$T^j := \bigoplus_{i \in M} (\iota^{i+j} \circ T_{i,i+j})\colon V^* \to W^*$$

for each j, where the ι^{i+j} here map into W^*. Then $T = \sum_j T^j$, and this gives us an isomorphism

$$\sum_j \mathrm{Hom}_j(V^*, W^*) \cong \mathrm{Hom}_F(V, W).$$

As a last note, we will in the sequel come across not only graded vector spaces but also e.g. graded algebras. Such objects are defined analogously.

Chapter 3

Motives

This chapter is greatly indebted to André's book [And04]. We try to simplify the exposition found there by leaving out many (often important) details, in an effort to get a view of the big picture as quickly as possible.

3.1 Algebraic cycles

The purpose of this section is to give a brief overview of the fundamental definitions regarding algebraic cycles. Apart from André's book, a short and sweet introduction to Chow groups and intersection theory is found in Appendix A of Hartshorne [Har77]. For a lot more details, the reader may refer to for example the chapter on intersection theory in the Stacks Project [Stacks].

Let K be a field and let X be a smooth projective variety over K. Write $\mathcal{Z}^*(X)$ for the free abelian group generated by the closed integral subschemes of X, graded by codimension. It is the *graded group of algebraic cycles* on X. The elements of $\mathcal{Z}^1(X)$ are called *(Weil) divisors* and the cycles of dimension zero (not codimension) are called *zero cycles* For any commutative ring F, write $\mathcal{Z}^*(X)_F$ for the tensor product $\mathcal{Z}^*(X) \otimes_{\mathbb{Z}} F$. This is the graded group of algebraic cycles on X with coefficients in F.

For Z a closed integral subscheme of X, write $[Z]$ for its image in $\mathcal{Z}^*(X)$, or in $\mathcal{Z}^*(X)_F$, or in any of their quotients which will be defined later.

3.1.1 Intersection product and adequate relations

We would like to make the algebraic cycles into a graded algebra, by defining a suitable product. The product in question is the intersection product from intersection theory.

If X is not of pure dimension, interpret $\dim X$ as a function which is locally constant. Say two closed integral subschemes Z and Z' *intersect properly* if the dimension of their intersection is $\dim Z + \dim Z' - \dim X$. In this case, their *intersection product* $Z \cdot Z'$ can be computed. We won't go into the exact definition here, but it roughly amounts to taking the intersection while keeping track of certain multiplicities. Please see [Stacks] or [Ful98] for details.

In order to make the intersection product defined for every pair of elements, we will introduce an equivalence relation $\sim$ on the algebraic cycles. By passing to the quotient

$$\mathcal{Z}^*_\sim(X)_F := \mathcal{Z}^*(X)_F/\sim$$

we'll get a graded algebra. A hypothetical equivalence relation $\sim$ should satisfy three conditions. The first condition (M) is meant to alleviate the need for a proper intersection and is often called the "moving lemma".

(M) For every $\alpha, \beta \in \mathcal{Z}^*(X)_F$, there exists $\alpha' \sim \alpha$ such that α' and β' intersect properly.

Secondly, when we mod out by $\sim$, the F-linear structure should be preserved:

(L) $\sim$ is compatible with the F-linear structure and the grading.

More explicitly, condition (L) means that the equivalence classes are homogeneous, and that if $\alpha \sim \alpha'$ and $\beta \sim \beta'$, then $a(\alpha+\beta) \sim a(\alpha'+\beta')$ for all $a \in F$.

The third and last condition is intended to ensure that the construction $\mathbb{Z}^*_\sim(X)_F$ a contravariant functor in X. Given two smooth projective K-schemes X and Y, we have a projection

$$p_X^{XY} \colon X \times Y \to X,$$

under which we take the may preimage of $\alpha \in \mathcal{Z}^*(X)_F$. Given any $\gamma \in \mathcal{Z}^*(X \times Y)_F$ which intersects $(p_X^{XY})^{-1}(\alpha)$ properly, we can define a cycle on Y by $\gamma_*(\alpha) := p_Y^{XY}(\gamma \cdot (p_X^{XY})^{-1}(\alpha))$. In this way, cycles in the product $X \times Y$ lets us go from cycles on X to cycles on Y. We would like the equivalence relation to satisfy:

(C) for every $\alpha \in \mathcal{Z}^*(X)_F$ and every $\gamma \in \mathcal{Z}^*(X \times Y)_F$ intersecting $(p_X^{XY})^{-1}(\alpha)$ properly, if $\alpha \sim 0$ on X then $\gamma_*(\alpha) \sim 0$ on Y.

To see what this has to do with $\mathcal{Z}^*_\sim(-)$ being contravariant, take a morphism $f \colon X \to Y$. Then let Γ_f be its graph in $X \times Y$ and consider its transpose ${}^t\Gamma_f$ as a cycle on $Y \times X$. Then if β is a cycle on Y intersecting ${}^t\Gamma_f$ properly, we get a cycle on X by $f^*(\beta) := ({}^t\Gamma_f)_*(\beta)$. One can check that this is functorial, and condition (C) ensures that this construction works well after passing to the quotient. A proper morphism f also induces a covariant map f_* on the Chow groups. It is a morphism of F-modules is not compatible with the intersection product. We skip the definition of f_* here.

Definition 3.1.1. An equivalence relation $\sim$ on $\mathcal{Z}^*(X)$ is *adequate* if it satisfies the three conditions (M), (L) and (C).

3.1.2 Rational and numerical equivalence, the Chow ring

The finest adequate relation is called *rational equivalence*. Under rational equivalence, we get the *Chow ring* $\mathcal{Z}^*_{\text{rat}}(X)$.

On the other side of the spectrum, the coarsest adequate relation is called *numerical equivalence*. In this case we write $\mathcal{Z}^*_{\text{num}}(X)$.

3.1.3 Correspondences

Let $\sim$ be an adequate relation. The *(algebraic) correspondences* of *degree* r from X to Y (with respect to $\sim$) are the elements of

$$C^r_\sim(X,Y) := \mathcal{Z}^{\dim X + r}_\sim(X \times Y).$$

Due to condition (C) on $\sim$, we may interpret the correspondences as homomorphisms from $\mathcal{Z}^*_\sim(X)$ to $\mathcal{Z}^*_\sim(Y)$. Moreover, due to the discussion after condition (C), we think of correspondences as generalisations of morphisms of schemes.

Define *composition* of correspondences $f \in C^r_\sim(X,Y)$ and $g \in C^s_\sim(Y,Z)$ by

$$g \circ f := (p^{XYZ}_{XZ})_* \big((p^{XYZ}_{XY})^*(f) \cdot (p^{XYZ}_{YZ})^*(g)\big) \in C^{r+s}_\sim(X,Z),$$

where again, we use p to denote projection on fibre product components.

3.2 Weil cohomology

When we have an embedding $\sigma\colon K \to \mathbb{C}$ of the base field into the complex numbers, we have Betti cohomology. Given a K-scheme X, we base change to get a complex variety $X_\sigma := X \times_\sigma \operatorname{Spec} \mathbb{C}$. We can then consider the corresponding complex analytic space, which has a much finer topology than the Zariski topology. Here, we have singular cohomology, which we call the *Betti cohomology* of X, and it has many interesting properties.

If K is for example of characterstic $p > 0$, then no such embedding σ exists. When formulating the Weil conjectures in [Wei49], Weil drew tentative connections between the number of points on varieties over finite fields, and cohomological properties of varieties over $\mathbb{C}$. If one could define a cohomology theory with characteristic zero coefficients, for when the base field is of positive characteristic, such that some of the properties enjoyed by Betti cohomology held, then this should provide a way to tackle the Weil conjectures. So was born the notion of a Weil cohomology theory.

In defining and discussing Weil cohomologies we follow section 3.3 in André's book [And04]. In particular we don't (yet) demand that the coefficient field is of characterstic zero. Let K be a field and denote the category of smooth projective schemes over K by $\mathcal{P}(K)$. The fibre product $\times_K$ makes this a monoidal category with unit $\operatorname{Spec} K$. Recall the category $\mathrm{GVec}^{\mathbb{N}}_F$ of $\mathbb{N}$-graded F-vector spaces, for some field F. As we discussed in section 2.5, this too is a monoidal category.

Definition 3.2.1. A *Weil cohomology* is a covariant functor

$$H^*\colon \mathcal{P}(K) \to \mathrm{GVec}^{\mathbb{N}}_F,$$

respecting the monoidal structure and satisfying

$$\dim_F H^2(\mathbb{P}^1) = 1,$$

together with the trace maps and cycle class maps defined in Definition 3.2.2 below.

Cup product. Any scheme X in $\mathcal{P}(K)$ has a canonical morphism $\Delta_X \colon X \to X \times X$ sending it to the diagonal. Under H^* this diagonal morphism is sent to a homogeneous map

$$H^*(\Delta_X)\colon H^*(X \times X) \to H^*(X).$$

Since H^* respects the monoidal structure we have a canonical isomorphism, called the *Künneth isomorphism*,

$$H^i(X \times X) \cong \bigoplus_{j+k=i} H^j(X) \otimes H^k(X).$$

And thus $H^*(\Delta_X)$ induces a product on $H^*(X)$, the *cup product*, which is additive in the degree of homogeneous elements. The cup product is denoted by $\cup$.

Tate twists. Denote the one-dimensional $H^2(\mathbb{P}^1)$ by $F(-1)$. Then $F(-1)$ is (non-canonically) isomorphic to F but in degree 2 as a graded vector space. Write $F(1)$ for its graded dual $\mathrm{Hom}_*(F(-1), F)$ in $\mathrm{GVec}_F^{\mathbb{Z}}$ (note the extension of the grading monoid), which is then concentrated in degree -2. This is the inverse of $F(-1)$ under $\otimes$ in $\mathrm{GVec}_F^{\mathbb{Z}}$. Also write $F(-1) = F(1)^{\otimes(-1)}$ and $F(r) = F(1)^{\otimes r}$ for $r \in \mathbb{Z}$. Then $F(r) \otimes F(s) = F(r+s)$. The operation of tensoring V^* in $\mathrm{GVec}_F^{\mathbb{Z}}$ by $F(r)$ is called *Tate twisting* and is denoted by

$$V^*(r) := V^* \otimes F(r).$$

If V^* is concentrated between degrees 0 and k, then $V^*(r)$ is shifted to degrees between $-2r$ and $k - 2r$.

Definition 3.2.2. We now come to the trace maps and cycle maps referenced in Definition 3.2.1 of Weil cohomologies.

1. (Trace map, Poincaré duality) For every equidimensional X in $\mathcal{P}(K)$ of dimension d, we should have a K-linear map

   $$\mathrm{tr}_X \colon H^{2d}(X)(d) \to K,$$

   called the *trace map*. It should satisfy $\mathrm{tr}_{X\times Y} = \mathrm{tr}_X \,\mathrm{tr}_Y$ and be an isomorphism when $X \times \operatorname{Spec} \bar{K}$ is connected. The trace map should lastly satisfy the *Poincaré duality*: the pairing

   $$\langle , \rangle \colon H^i(X) \otimes H^{2d-i}(X)(d) \xrightarrow{\cup} H^{2d}(X)(d) \xrightarrow{\mathrm{tr}_X} K$$

   is a perfect pairing.

2. (Cycle class map) For every X in $\mathcal{P}(X)$ we should have group homomorphisms

   $$\gamma_X^r \colon \mathrm{CH}^r(X) \to H^{2r}(X)(r),$$

   called *cycle maps*, such that γ is contravariant in X, and such that

   $$\gamma_{X\times Y}^{r+s}(\alpha \times \beta) = \gamma_X^r(\alpha) \otimes \gamma_Y^s(\beta),$$

   and finally, such that when X is equidimensional of dimension d, the composition of γ^d with tr_X is the same as the degree map on zero-cycles.

Homological equivalence. If L is a subring of F, and H^* is a Weil cohomology with coefficients in F, then a cycle $\alpha \in \mathcal{Z}^r(X)_L$ is *homologically equivalent* to 0 if $\gamma^r(\alpha) = 0$. Homological equivalence is an adequate equivalence relation. It is finer than numerical equivalence and coarser than rational equivalence. It is conjectured that numerical and homological equivalence coincide.

Homological correspondences. Let $X, Y \in \mathcal{P}(K)$, H^* a Weil cohomology and $d = \dim X$. By the Künneth isomorphism

$$H^i(X \times Y)(d) \cong \bigoplus_{j+k=i} H^j(X)(d) \otimes H^k(Y).$$

Note that the degrees on the right-hand side add up to $i - 2d$ as they should. Through Poincaré duality we have an isomorphism

$$H^j(X)(d) \cong \mathrm{Hom}_F(H^{2d-j}(X), K),$$

and thus

$$H^i(X \times Y)(d) \cong \bigoplus_{j+k=i} \mathrm{Hom}_F(H^{2d-j}(X), H^k(Y)) = \mathrm{Hom}_{i-2d}(H^*(X), H^*(Y))$$

Summing over all degrees yields an F-linear isomorphism

$$H(X \times Y)(d) \cong \mathrm{Hom}_F(H(X), H(Y)),$$

identifying $H(X \times Y)(d)$ with the linear maps from $H(X)$ to $H(Y)$. The elements of $H^*(X \times Y)(d)$ are the *homological correspondences* from X to Y with respect to H^*. Note that the correspondences in $H^{2d}(X \times Y)(d)$ are identified with the homogeneous maps $\mathrm{Hom}_0(H^*(X), H^*(Y))$. As a special case of this, note that the cycle maps

$$\gamma_X^r \colon \mathrm{CH}^r(X) \to H^{2r}(X)(r),$$

can be seen as mapping into correspondences. Namely,

$$H^{2r}(X)(r) \cong H^{2r}(\operatorname{Spec} K \times X)(r) \cong \mathrm{Hom}_{2r}(F, H^*(X)(r)).$$

3.3 Grothendieck's pure motives

We would like a functor $\mathfrak{h}$ from $\mathcal{P}(K)$ into some abelian $\otimes$-category of "motives", such that if H^* is a Weil cohomology, then H^* factors through $\mathfrak{h}$ via a faithful *realisation functor*:

$$\begin{array}{ccc} X & \longmapsto & \mathfrak{h}(X) \\ & \searrow & \vdots \\ & & H^*(X) \end{array} \tag{3.1}$$

We have seen that the algebraic cycles on X give rise to cohomology classes through cycle class maps. We have also seen that correspondences give homomorphisms between the algebraic cycle algebras, and that homological correspondences are the same as homomorphisms between the cohomology algebras. This gives some intuition for the following construction by Grothendieck.

Definition 3.3.1. The category of *pure motives* over K, with an adequate relation $\sim$ and coefficients in F, is denoted by $M_\sim(K)_F$ and has objects and morphisms as follows. The objects are triples (X, e, r) where $X \in \mathcal{P}(K)$ is a smooth projective scheme over K, $e \in C^0_\sim(X, X)_F$ is an idempotent correspondence from X to X, and r is an integer. The morphisms from (X, e, r) to (Y, f, s) are the correspondences in $C^{s-r}_\sim(X, Y)$ of the form

$$f \circ g \circ e, \quad \text{such that} \quad g \in C^{s-r}_\sim(X, Y),$$

with composition as correspondences.

We write $\mathfrak{h}\colon \mathcal{P}(K) \to M_\sim(K)_F$ for the contravariant functor sending X to $(X, \mathrm{id}, 0)$ and sending $f\colon X \to Y$ to the transpose ${}^t\Gamma_f = f^*$ of its graph. For a triple (X, e, r) we use the suggestive notation $e\mathfrak{h}(X)(r)$ and call it the motive *cut out* of X by e, *(Tate) twisted* r times.

The category of pure motives is an F-linear category with direct sum given by disjoint union of schemes. It is moreover a $\otimes$-category by

$$e\mathfrak{h}(X)(r) \otimes f\mathfrak{h}(Y)(s) := (e \otimes f)\mathfrak{h}(X \times Y)(r + s).$$

The unit object is $\mathbf{1} := \mathfrak{h}(\operatorname{Spec} K)$. Tate twists make $M_\sim(K)_F$ into a *rigid* $\otimes$-category, where duality on the objects is given by

$$[e\mathfrak{h}(X)(r)]^\vee := ({}^te)\mathfrak{h}(X)(d - r), \quad d := \dim X,$$

for equidimensional X (extended additively), and duality on the morphisms is given by transposition of correspondences.

Example 3.3.2 (The motive of $\mathbb{P}^1$.)**.** Let x be a point in $\mathbb{P}^1(K)$. Then the diagonal in $\mathbb{P}^1 \times \mathbb{P}^1$ decomposes modulo $\sim$ as the sum of the idempotent correspondences $[x] \times \mathbb{P}^1$ and $\mathbb{P}^1 \times [x]$, and these are independent of the choice of x ([And04, Exercise 3.2.2.2 1]). Thus $\mathfrak{h}(\mathbb{P}^1)$ decomposes as $\mathbf{1} \oplus \tilde{\mathfrak{h}}(\mathbb{P}^1)$, where $\mathbf{1}$ is the trivial motive and $\tilde{\mathfrak{h}}(\mathbb{P}^1)$ is the *reduced motive* of $\mathfrak{h}(\mathbb{P}^1)$ cut out by the idempotent $[x] \times \mathbb{P}^1$. Then $\tilde{\mathfrak{h}}(\mathbb{P}^1)$ is canonically identified with $\mathbf{1}(-1)$, since $[x] \times \mathbb{P}^1$ is the transpose of $\mathbb{P}^1 \times [x]$, so

$$\mathbf{1} = \mathbf{1}^\vee = \left[(\mathbb{P}^1 \times [x])\mathfrak{h}(\mathbb{P}^1)\right]^\vee = ([x] \times \mathbb{P}^1)\mathfrak{h}(\mathbb{P}^1)(1) = \tilde{\mathfrak{h}}(\mathbb{P}^1)(1),$$

Call $\mathbf{1}(-1)$ the *Lefschetz motive.* Its dual $\mathbf{1}(1)$ is the *Tate motive.*

When $\sim$ is rational equivalence, $M_\sim(K)_F$ is the category of *Chow motives*, and we'll denote it by $\mathrm{CHM}(K)_F$. With Chow motives, the universality outlined at the top of the section, in (3.1), is realised.

Proposition 3.3.3. *Defining a Weil cohomology over K with coefficients in F is equivalent to defining a $\otimes$-functor*

$$H^*\colon \mathrm{CHM}(K)_F \to \mathrm{GVec}^{\mathbb{Z}}_F,$$

such that $H^i(\mathbf{1}(-1)) = 0$ for every $i \neq 2$.

Proof. Proposition 4.2.5.1 in [And04]. □

The proof of Proposition 3.3.3 rests on a stronger universality property which is explained in section 4.2.4 of [And04]. The functors from $\mathrm{CHM}(K)_F$ corresponding to Weil cohomologies are called *realisations* of motives. The category of Chow motives is neither abelian nor semi-simple.

When $\sim$ is numerical equivalence, $M_\sim(K)_F$ is the category of *numerical motives*, or *Grothendieck motives*. It is denoted by $\mathrm{NM}(K)_F$. We have the following theorem.

Proposition 3.3.4 (Jannsen)**.** *The category* $\mathrm{NM}(K)_F$ *is abelian and semi-simple. Conversely, if* $M_\sim(K)_F$ *is abelian and semi-simple for some adequate relation* $\sim$*, then* $\sim$ *is numerical equivalence.*

Proof. Theorème 4.5.1.1 in [And04]. □

On the other hand, Weil cohomologies do not factor through Grothendieck motives unless certain conjectures hold.

An adequate relation which Weil cohomologies do factor through is $M_{\mathrm{hom}}(K)_F$. If numerical and homological equivalence coincide, then $M_{\mathrm{hom}}(K)_F = \mathrm{NM}(K)_F$ is an abelian semi-simple category of pure motives through which Weil cohomologies factor.

Artin motives Consider the subcategory $\mathcal{V}$ of $\mathcal{P}(K)$ consisting of schemes that are finite étale over K. Let $\mathrm{AM}(K)_F$ be the smallest full subcategory of $M_\sim(K)_F$ that contains $\mathfrak{h}(X)$ for all $X \in \mathcal{V}$, and that is stable under $\oplus$, $\otimes$, direct summands and duals. This is the category of *Artin motives* over K and is actually independent of the choice of adequate relation $\sim$. It is equivalent to the category $\mathrm{Rep}_F(\mathrm{Gal}(K^s/K))$.

3.4 André's pure motives

Apart from unifying the different Weil cohomologies, another source of motivation for motives comes from Galois theory. The Galois correspondence can be reframed in the language of schemes as an equivalence of categories

$$\text{(finite étale } K\text{-schemes)} \xrightarrow{\sim} \text{(finite sets with an action by } \mathrm{Gal}(K^s/K))$$

One may wish to find a *linearised* version of this equivalence. On the right-hand side this is easily done: if S is a finite set with an action by $\mathrm{Gal}(K^s/K)$ then the free F-vector space F^S on S is finite-dimensional and carries a natural action by $\mathrm{Gal}(K^s/K)$. This gives a functor into (continuous) finite-dimensional representations $\mathrm{Rep}_F(\mathrm{Gal}(K^s/K))$ of $\mathrm{Gal}(K^s/K)$. The finite étale K-schemes on the other hand are linearised by Artin motives, and this does yield a linearised equivalence.

$$\begin{array}{ccc} \text{(finite étale } K\text{-schemes)} & \xrightarrow{\sim} & \text{(finite sets with an action by } \mathrm{Gal}(K^s/K)) \\ \downarrow{\scriptstyle \mathfrak{h}} & & \downarrow \\ \mathrm{AM}(K)_F & \xrightarrow{\sim} & \mathrm{Rep}_F(\mathrm{Gal}(K^s/K)) \end{array}$$

The idea is to generalise this to arbitrary dimension by replacing $\mathrm{AM}(K)_F$ by the full category of Grothendieck motives $\mathrm{NM}(K)_F$ and $\mathrm{Gal}(K^s/K)$ by some group scheme, called the *motivic Galois group*.

If $\mathrm{NM}(K)_F$ was Tannakian, with a fibre functor given by a realisation, e.g. ℓ-adic cohomology, then the motivic Galois group would be the Tannakian group, i.e. the $\otimes$-automorphism group of the fibre functor.

There are several hurdles to $\mathrm{NM}(K)_F$ being Tannakian in this way. Most importantly, Grothendieck's *standard conjectures* on algebraic cycles are still open. We won't go into too much detail on this important topic. For details, see [And04] or [Kle68]. We will focus on conjecture D.

Conjecture 3.4.1 ($D(X)$)**.** *Homological and numerical equivalence on X coincide, that is*

$$\mathcal{Z}^*_{hom}(X)_{\mathbb{Q}} = \mathcal{Z}^*_{num}(X)_{\mathbb{Q}}, \qquad (D(X))$$

where X is a smooth projective K-scheme.

If conjecture $D(X)$ holds on all of $\mathcal{P}(K)$, then Weil cohomologies do factor through $\mathrm{NM}(K)_F$, giving us an abelian semi-simple category of pure motives, in the way we hoped for. Trying to use a Weil cohomology as fibre functor would still not make $\mathrm{NM}(K)_F$ Tannakian though, as is explained in section 6.1 of [And04]. The issue causing this set-back can be resolved either by modifying $\mathrm{NM}(K)_F$ into a slightly different category, or by considering *super representations* instead of representations.

3.4.1 Motivated cycles

When the base field K is of characteristic zero the standard conjecture of Lefschetz type implies the rest of the standard conjectures. We'll give a brief introduction to this conjecture. Let K be of characterstic zero and let H^* be a Weil cohomology. Given an ample Cartier divisor on X, one may define a *Lefschetz operator*

$$L\colon H^i(X)(r) \to H^{i+2}(X)(1+r).$$

We say that the *strong Lefschetz theorem* holds for H^* if

$$L^{d-i}\colon H^i(X)(r) \to H^{2d-i}(X)(d-i+r)$$

is an isomorphism for every i such that $i \leq d$ and every $r \in \mathbb{Z}$. The strong Lefschetz theorem holds for every classical Weil cohomology. There is also a *weak Lefschetz theorem*, which also holds for every classical Weil cohomology, but we won't say more than that about it here.

Now suppose the strong and weak Lefschetz theorems holds for H^*. Define the *Lefschetz involution* $*_L$ on $\oplus_{i,r} H^i(X)(r)$ as L^{d-i} when $i \leq d$ and as the inverse of L^{i-d} when $i > d$. We may now state the standard conjecture of Lefschetz type.

Conjecture 3.4.2 ($B(X)$: Lefschetz type)**.** *The Lefschetz involution is given by an algebraic correspondence (over $\mathbb{Q}$).*

André proposed a way around the Lefschetz type standard conjecture in [And96]. The idea is to formally adjoin the Lefschetz involution to the algebraic correspondences, thereby bypassing the question of its algebraicity.

Definition 3.4.3 ([And96, Déf. 1])**.** A *motivated cycle* of degree r on X is an element in $H^{2r}(X)(r)$ of the form $(p_X^{XY})_*(\alpha \cup *_L(\beta))$ where Y is arbitrary, α and β are in $\mathcal{Z}^*_{\mathrm{hom}}(X \times Y)_F$, and $*_L$ is the Lefschetz involution. Denote the set of motivated cycles on X by $\mathcal{Z}^*_{\mathrm{mot}}(X)_F$.

The cycles in $\mathscr{Z}_{\mathrm{mot}}^{\dim X+r}(X\times Y)_F$ are the *motivated correspondences* of degree r from X to Y. We mirror the notation for algebraic correspondences and write

$$C^r_{\mathrm{mot}}(X\times Y)_F := \mathscr{Z}_{\mathrm{mot}}^{\dim X+r}(X\times Y)_F.$$

André tells us in [And96, Prop. 2.1] that $\mathscr{Z}_{\mathrm{mot}}(X)_F$ is a sub-F-algebra of $\oplus_r H^*(X)(r)$, under cup product. Using motivated instead of algebraic correspondences, we get a new notion of pure motives.

Definition 3.4.4. The category of *André motives* has objects (X, e, r) where X is a smooth projective K-scheme, e is an idempotent motivated correspondence in $\mathscr{Z}_{\mathrm{mot}}^{\dim X}(X\times X)$, and r is an integer. The morphisms from (X, e, r) to (Y, f, s) are motivated correspondences of the form $f\circ g\circ e$ where g is in $\mathscr{Z}_{\mathrm{mot}}^{\dim X-r+s}(X\times Y)$. We denote this category by $\mathrm{Mot}(K; F)$.

Taking $F=\mathbb{Q}_\ell$, this is a semi-simple $\mathbb{Q}_\ell$-linear neutral Tannakian category with fibre functor given by ℓ-adic étale cohomology. By section 4 of [And96] we may conclude that the corresponding motivic Galois group, which we'll denote by $\mathscr{G}_K$, is a pro-reductive group and that $\mathrm{Mot}(K;\mathbb{Q}_\ell)$ is equivalent to $\mathrm{Rep}_{\mathbb{Q}_\ell}(\mathscr{G}_K)$. Weil cohomologies also factor through $\mathrm{Mot}(K;\mathbb{Q}_\ell)$, and we have somewhat succeeded in generalising the linearised Galois correspondence at the top of the section.

By abuse of notation, write H^* for the functor from $\mathrm{Mot}(K;\mathbb{Q}_\ell)$, through which the Weil cohomology H^* factors. Then a motive (X, e, r) is sent to $H^*(e)H^*(X)(r)$ under H^*, i.e. the subspace of $H^*(X)$ cut out by the projector $H^*(e)$, Tate twisted r times. A *motivated subspace* V of $H^*(X, e, r)$ is a linear subspace stable under the action of $\mathscr{G}_K$. This is equivalent to V being the realisation of a submotive M of (X, e, r). Since $\mathrm{Mot}(K;\mathbb{Q}_\ell)$ is semi-simple, V motivated implies that there is a submotive M' such that $(X, e, r) = M\oplus M'$.

Chapter 4

The Tate Conjecture

We begin by fixing some notation. Let K a field which is finitely generated over its prime subfield. Fix a prime number ℓ different from the characteristic of K and an algebraic closure $\bar{K}$ of K. Denote the absolute Galois group of K by $\Gamma_K = \mathrm{Gal}(\bar{K}/K)$.

Let X be a smooth projective scheme over K. Write $H^i(X)$ for the ℓ-adic étale cohomology $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell)$. Denote its Tate twists $H^i_{\text{ét}}(X_{\bar{K}}, \mathbb{Q}_\ell(n))$ by $H^i(X)(n)$.

Note that Γ_K acts on $\bar{K}$ by definition, and hence by functoriality on $\operatorname{Spec} \bar{K}$. Thus, Γ_K acts on the product $X_{\bar{K}} \times_{\operatorname{Spec} K} \operatorname{Spec} \bar{K}$. Finally, Γ_K then acts linearly on $H^i(X)$ by functoriality, turning the cohomology to an ℓ-adic Galois representation. The action of Γ_K on the cohomology of $X_{\bar{K}}$ should in some sense encode the K-structure on X which we forget when we extend scalars.

4.1 Statement

A cohomology class $\xi \in H^i(X)(n)$ is a *Tate class* if its stabiliser in Γ_K is an open subgroup. In other words, if it is fixed by $\mathrm{Gal}(\bar{K}/K')$ for some finite extension K'/K. If $\xi \neq 0$, then we must have $i = 2n$. With this in mind, define $\mathscr{T}^n(X) \subset H^{2n}(X)(n)$ to be the subspace of Tate classes.

Tate conjectured the following in [Tat65] and elsewhere.

Conjecture 4.1.1. *For every X/K as above and all integers $i \geq 0$ and n,*

(S) $H^i(X)(n)$ is a semi-simple representation of Γ_K, and

(T) when $n \geq 0$, the cycle class map $\mathrm{CH}^n(X_{\bar{K}}) \otimes \mathbb{Q}_\ell \to \mathscr{T}^n(X)$ is surjective.

4.2 Motivation and known cases

The Tate conjecture (T) is known for divisors (that is, $n = 1$) on abelian varieties. When K is a finite field, this is due to Tate [Tat66]. Faltings then showed it for K a number field in [Fal83] (English translation in [CS86]). Totaro writes in the beginning of section 4 of [Tot17] that Zarhin extended these results to finitely generated fields, but no reference is given. For an abelian variety A, we have $H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell A, \mathbb{Z}_\ell)$ as Galois representations, where $T_\ell A$ is the Tate module of A, i.e. the inverse limit of the ℓ^n torsion points $A[\ell^n]$ on A. Moreover,

$H^*_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell) \cong \bigwedge^* H^1_{\text{ét}}(A_{\bar{K}}, \mathbb{Z}_\ell)$ as graded algebras. See for example [CS86, Ch. V, Theorem 15.1]. This gives some feeling for why the Tate conjecture is a lot more approachable for abelian varieties than more general varieties.

More recently, the Tate conjecture has been confirmed for divisors on K3 surfaces. This is done through the Kuga-Satake construction, which relates the cohomology of K3 surfaces to that of abelian varieties. Note that for surfaces, divisors make up the only interesting case of the Tate conjecture, codimension 0 and 2 cycles map into the one-dimensional H^0 and H^4 respectively (since 4 is twice the dimension). For more of a discussion about this development, see [Tot17].

It is worth noting that the situation is very different for the Hodge conjecture, where the general case of divisors has been known since 1924, long before the Hodge conjecture was even formulated.

Most of Tate's original motivation for his conjecture was for divisors. Tate writes in [Tat94] about how he was lead to the divisor case of the Tate conjecture through the Tate-Shafarevich conjecture as well as the Birch and Swinnerton-Dyer conjecture. He writes also that the conjecture in higher codimension arose mainly as a generalisation of the divisor case.

4.3 Moonen's theorem

In 2017, Ben Moonen published the following result in [Moo18].

Proposition 4.3.1 (Moonen 2017)**.** *Suppose the characteristic of K is zero. If assertion (T) of Conjecture 4.1.1 is true, then so is (S). In other words, if the cycle class map is surjective onto the Tate classes, then the ℓ-adic cohomology groups are semi-simple as representations of Γ_K.*

What follows is a short sketch of the main ideas in the proof. From here on, K will always be of characteristic zero. We want to show that $H^i(X)$ is semi-simple, i.e. decomposable as a direct sum of irreducible subrepresentations of Γ_K. Write $H = H^i(X)$ and assume that H has a subrepresentation V. It is then enough to show that V has a complement that is also stable under the action of Γ_K. By Lemma 2.3.1 it is actually enough to show that V has a complement stable under the action of an **open subgroup** of Γ. The proof can then be seen as having three parts.

Step 0: reduction. Reduce to the case $K = \mathbb{Q}$. We shall skip this step and instead refer to Moonen's paper. Now let $\Gamma = \Gamma_{\mathbb{Q}}$.

Step 1: produce a Tate class. We want to use (T) in some way. That means we want to at some point produce a Tate class ξ in $H^{2k}(Y)(k)$ for some Y and k. Assume that we have such a Tate class ξ. Since ξ is fixed under the action of an open subgroup of Γ, the span of ξ is a subrepresentation (of that open subgroup) isomorphic to the trivial representation $\mathbb{Q}_\ell$. Tate twisting by $-k$ we get a subrepresentation, of the open subgroup, of $H^{2k}(Y)$ isomorphic to $\mathbb{Q}_\ell(-k)$. Thus, a reasonable strategy would be to try and produce such a subrepresentation for some Y.

We have one obvious representation to work with, namely V. The dimension $m := \dim V$ can be anything so it doesn't make sense to try and show that V is isomorphic to $\mathbb{Q}_\ell(-k)$, which is

1-dimensional. However, consider $\mathrm{Alt}^m(V)$. This is a one-dimensional representation of Γ and

$$\mathrm{Alt}^m(V) \subset \mathrm{Alt}^m(H) \subset H^{mi}(Y),$$

where Y is the m-fold fibre product of X with itself. We will show that $\mathrm{Alt}^m(V)$ is isomorphic to $\mathbb{Q}_\ell(-k)$ as representations of an open subgroup of Γ. Then $\mathrm{Alt}^m(V)(k)$ will be spanned by a Tate class.

Step 2: bring in the motives. Having $\mathrm{Alt}^m(V)(k)$ spanned by a Tate class, it is by (T) spanned by the cohomology class of an algebraic cycle. That is, it is spanned by the image of an algebraic cycle under the cycle class map. We will then get a decomposition $H = V \oplus V'$ as André motives. Then the ℓ-adic realisation functor respects direct sums, so $H = V \oplus V'$ as Galois representations, and we will be done.

4.4 The proof

By step zero, we set $K = \mathbb{Q}$ and let $\Gamma = \Gamma_{\mathbb{Q}}$. As in the previous discussion we write $H = H^i(X)$ and let $V \subset H$ be a sub-Γ-representation. We want to show that V has a complement invariant under the action of an open subgroup of Γ.

4.4.1 Step 1

Let m denote $\dim V$ and write

$$Y = X \times_{\mathrm{Spec}\, K} \cdots \times_{\mathrm{Spec}\, K} X$$

with m copies of X. By the Künneth isomorphism, $H^{\otimes m}$ occurs as a summand in $H^{mi}(Y)$. We then have a chain of inclusions

$$\mathrm{Alt}^m V \subset \mathrm{Alt}^m H \subset H^{\otimes m} \subset H^{mi}(Y).$$

As discussed in the proof sketch, $\mathrm{Alt}^m V$ is one-dimensional. Our present goal is to show that, for some open subgroup of Γ, the restricted representation on $\mathrm{Alt}^m V$ is isomorphic to $\mathbb{Q}(-k)$. By the fundamental Galois correspondence, Proposition 2.2.10, an open subgroup of Γ corresponds to a field F with finite degree over $\mathbb{Q}$, i.e. a number field. Open subgroups of Γ are thus of the form Γ_F, for F a number field. Being a one-dimensional representation, $\mathrm{Alt}^m V$ is given by a character

$$\psi\colon \Gamma \to \mathbb{Q}_\ell^\times.$$

Showing that $\mathrm{Alt}^m V$ and $\mathbb{Q}(-k)$ are isomorphic as Γ_F-representations, for some number field F, thus amounts to showing that $\psi = \chi^{-k}$ on Γ_F, where χ is the cyclotomic character

$$\chi\colon \Gamma \to \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \cong \mathbb{Z}_\ell^\times.$$

We start by showing that the image of ψ is actually in $\mathbb{Z}_\ell^\times$. For this we need the following lemma.

Lemma 4.4.1. *V contains a Γ-stable $\mathbb{Z}$-lattice.*

Proof. See [DS05, Prop. 9.3.5]. ☐

Lemma 4.4.2. *The image of ψ is contained in $\mathbb{Z}_\ell^\times$.*

Proof. Take a Γ-stable lattice $v_1, \ldots, v_m$ from Lemma 4.4.1 as a basis for V. Letting $v = v_1 \wedge \cdots \wedge v_m$ in $\bigwedge^m V$ we get an isomorphism

$$\mathrm{Alt}^m V \cong \textstyle\bigwedge^m V \cong \mathbb{Q}_\ell v.$$

The action of $g \in \Gamma$ is $g(v) = \psi(g)v$, but it's also

$$g(v) = g(v_1) \wedge \cdots \wedge g(v_m),$$

which is an $\mathbb{Z}_\ell$-multiple of v since the v_i make up a stable $\mathbb{Z}_\ell$-lattice. Combining the two yields that $\psi(g)$ is in $\mathbb{Z}_\ell^\times$, since it's invertible in $\mathbb{Z}_\ell$ (the inverse being $\psi(g^{-1})$). ☐

We arrive at the main claim of the section. Let χ be the ℓ-adic cyclotomic character.

Proposition 4.4.3. *There is an open subgroup Γ_F of Γ such that the restriction of ψ to Γ_F is an integral power of χ.*

Start with a simple case.

Lemma 4.4.4. *Proposition 4.4.3 is true if* $\operatorname{im}\psi$ *is finite.*

Proof. We have that $\operatorname{im}\psi \cong \Gamma/\ker\psi$ so $\ker\psi$ has finite index in Γ, and thus is open. Now the restriction of ψ to $\ker\psi$ is by definition trivial, so it is equal to the zeroth power, χ^0, of the cyclotomic character on $\ker\psi$. ☐

We may thus assume that $\operatorname{im}\psi$ is infinite, for the purposes of Proposition 4.4.3.

The subgroup $\ker\psi \subset \Gamma$ is normal and closed. By the fundamental theorem of Galois theory, it thus corresponds to an intermediate field extension, $\bar{\mathbb{Q}}/L/\mathbb{Q}$, Galois over $\mathbb{Q}$. Here, $L = \bar{\mathbb{Q}}^{\ker\psi}$ (the fixed field of $\ker\psi$). Let $\mathbb{Q}_\infty$ be a $\mathbb{Z}_\ell$-extension of $\mathbb{Q}$, i.e. an extension with Galois group $\mathbb{Z}_\ell$. It is unique up to choice of $\bar{\mathbb{Q}}$, for example by [Lan90, Thm. 5.2], since the Leopoldt conjecture is known for $\mathbb{Q}$ and since there are no imaginary embeddings of $\mathbb{Q}$ in $\mathbb{C}$.

Lemma 4.4.5. *L contains the $\mathbb{Z}_\ell$-extension $\mathbb{Q}_\infty$ as a subfield of finite index.*

Proof. We have an isomorphism

$$\mathbb{Z}_\ell^\times \cong \mu(\ell-1) \times (1 + \ell\mathbb{Z}_\ell),$$

and thus a quotient map

$$\mathbb{Z}_\ell^\times \twoheadrightarrow (1 + \ell\mathbb{Z}_\ell) \cong \mathbb{Z}_\ell.$$

Since $\operatorname{im}\psi$ is an infinite subgroup of $\mathbb{Z}_\ell^\times$, it contains an element of infinite order, which hence generates a copy of $\mathbb{Z}$. Moreover, since ψ is continuous it sends the compact $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ to a compact, and hence closed subset of $\mathbb{Z}_\ell^\times$. Thus, $\operatorname{im}\psi$ contains a copy of the closure of $\mathbb{Z}$ in $\mathbb{Z}_\ell^\times$, which is a copy of $\mathbb{Z}_\ell$. We get a quotient map

$$\operatorname{im}\psi \twoheadrightarrow \mathbb{Z}_\ell,$$

and by the fundamental Galois correspondence, this means that there is an intermediate field $L/M/\mathbb{Q}$ such that $\mathrm{Gal}(M/\mathbb{Q}) \cong \mathbb{Z}_\ell$. Thus, $M = \mathbb{Q}_\infty$, by uniqueness. The degree $[L : \mathbb{Q}_\infty]$ is at most size of $\mu(\ell-1)$, which is finite. ☐

The prime ℓ is totally ramified in $\mathbb{Q}_\infty$. For a prime λ of L above ℓ, its inertia group $I_\lambda \subset \mathrm{Gal}(L/\mathbb{Q})$ is thus an open subgroup.

Proof of Prop. 4.4.3. In order to prove Proposition 4.4.3, we will need some results from p-adic Hodge theory, a theory which we will use as a black box. The first claim is that H is a *Hodge-Tate representation* of $\Gamma_{\mathbb{Q}_\ell}$, and this is proven by Faltings in [Fal88]. Although we won't define this concept, we'll use it as a bridge. We have that V and $\mathrm{Alt}^m V$ are also Hodge-Tate. Then, Remark 3.9(iv) of [Fon94] says, roughly translated:

> A one-dimensional ℓ-adic representation is Hodge-Tate if and only if there is an integer $\nu \in \mathbb{Z}$, and an open subgroup $J \subset I_\lambda$ such that every g in J acts by multiplication of $\chi(g)^\nu$.

Thus, applying this to $\mathrm{Alt}^m V$ we get that $\psi = \chi^\nu$ on $J \subset I_\lambda$. Let J' be the preimage of J under the quotient map $\Gamma \twoheadrightarrow \mathrm{Gal}(L/\mathbb{Q})$. Then J' is open, by continuity, and $J = J'/\ker\psi$, so that ψ is the same on J and J' Thus, $J' \subset \Gamma$ is the open subgroup we need. □

We thus have that $\psi = \chi^\nu$ for an integer ν, on an open subgroup Γ_F of Γ. In other words, $\mathrm{Alt}^m V \cong \mathbb{Q}_\ell(\nu)$ as a Γ_F-representation. But then

$$\mathbb{Q}_\ell \cong (\mathrm{Alt}^m V)(-\nu) \subset H^{mi}(Y)(-\nu),$$

so that $(\mathrm{Alt}^m V)(-\nu)$ is fixed by Γ_F. But since it is nonzero we must have $mi = 2k$ and $-\nu = k$ for some $k \geq 0$. Thus, the line

$$(\mathrm{Alt}^m V)(k) \subset \mathscr{T}^k(Y) \subset H^{2k}(Y)(k)$$

is spanned by a Tate class.

4.4.2 Step 2

By part (T) of the Tate conjecture, we have that $(\mathrm{Alt}^m V)(k)$ is spanned by the cohomology class of an algebraic cycle in $\mathrm{CH}^k(Y_F)$. Recall the discussion regarding motivated subspaces in section 3.4. We have that $\mathrm{Alt}^m V$ is a motivated subspace of $\mathrm{Alt}^m H$. Letting $\mathscr{G}_F$ be the motivic Galois group of $\mathrm{Mot}(F;\mathbb{Q}_\ell)$, we thus have that $\mathrm{Alt}^m V$ is a $\mathscr{G}_F$-submodule of $\mathrm{Alt}^m H$.

Lemma 4.4.6. *If $\mathrm{Alt}^m V$ is a $\mathscr{G}_F$-submodule of $\mathrm{Alt}^m H$, then V is a $\mathscr{G}_F$-submodule of H*

Proof. This is a general representation-theoretic result. We have that Alt^m is a map from the set of m-dimensional subspaces in H to the set of one-dimensional subspaces of $\mathrm{Alt}^m H$. It is the *Plücker embedding* and is injective. To see this, let W and W' be m-dimensional subspaces of H. Then, by first choosing a basis of $W \cap W'$, then extending it to the rest of W and W' and finally the rest of H, we may construct a basis $e_1, \ldots, e_n$ of H such that $e_1, \ldots, e_m$ is a basis of W and $e_j, \ldots, e_{j+m-1}$ is a basis of W'. But a basis of $\bigwedge^m H \cong \mathrm{Alt}^m H$ is then given by

$$e_{i_1} \wedge \cdots \wedge e_{i_m}, \quad 1 \leq i_1 < \cdots < i_m \leq n.$$

Thus, $\mathrm{Alt}^m W = \mathrm{Alt}^m W'$ if an only if $j = 1$, i.e. $W = W'$.

Let $g \in \mathscr{G}_F$. We have that $g(\mathrm{Alt}^m V) = \mathrm{Alt}^m V$ by assumption and hence

$$\mathrm{Alt}^m(gV) = g(\mathrm{Alt}^m V) = \mathrm{Alt}^m V,$$

so, by the injectivity of the Plücker embedding, $gV = V$. This is true for all $g \in \mathscr{G}_F$ so we are done. □

Now $\mathrm{Mot}(F, \mathbb{Q}_\ell)$ is a semi-simple category, and thus H decomposes as $V \oplus V'$ as a motive over F. Applying the étale realisation functor, which respects sums, we thus get a decomposition $H = V \oplus V'$ as representations of Γ_F. This is exactly what we needed by Lemma 2.3.1, so we are done.

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