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An Explicit Formula for Optimal Portfolios in Complete Wiener Driven Markets: a Functional Itô Calculus Approach

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Abstract

The optimal investment problem is one of the most important problems in mathematical finance. The main contribution of the present paper is an explicit formula for the optimal portfolio process. Our optimal investment problem is that of maximizing the expected value of a standard general utility function of terminal wealth in a standard complete Wiener driven financial market. In order to derive the formula for the optimal portfolio we use the recently developed functional Itô calculus and more specifically an explicit martingale representation theorem. A main component in the formula for the optimal portfolio is a vertical derivative with respect to the driving Wiener process. The vertical derivative is an important component of functional Itô calculus.

Keywords: Functional Itô calculus, Martingale representation, Optimal investment, Optimal portfolios, Portfolio theory, Utility maximization, Vertical derivative.

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1 Introduction

Optimal investment and consumption problems are among the most important problems in mathematical finance. Problems of this type were first studied using Itô calculus by Merton (1969) and Merton (1971), where standard stochastic control methods were used (cf. *dynamic programming* and the *Hamilton-Jacobi-Bellman equation*). These methods typically relies on asset prices being Markovian. The *martingale methodology* was later developed by e.g. Pliska (1986) and Karatzas, Lehoczky & Shreve (1987). The martingale methodology typically relies on the financial market being complete, which allows for the optimization to be performed trajectory by trajectory.

In the present paper we consider the optimal investment problem of maximizing the expected value of a standard general utility function of terminal wealth. Our market is the standard complete Wiener driven financial market found in e.g. Karatzas & Shreve (1998). Optimal investment problems in this type of market have to a considerable extent been studied for several decades. For a survey of the literature on optimal investment (and consumption) problems and markets similar to that of the present paper see Karatzas & Shreve (1998) Chapters 1 and 3.

The following explicit formula for the optimal wealth process X^* is, in the setting of the present paper, a standard result:

$$X^*(t) = E_{\mathcal{F}_t}\left[\frac{H(T)}{H(t)} I(\mathcal{Y}(x_0)H(T))\right], \quad 0 \le t \le T,$$

where H is the *state price density* process, $I(\cdot)$ is the inverse of the derivative of the utility function for terminal wealth $U(\cdot)$ and $\mathcal{Y}(x_0)$ is a constant depending on the initial wealth x_0 (see Lemma 4.2 below). Using the standard martingale representation theorem and that HX^* is a martingale it is possible to implicitly characterize the optimal portfolio process π^*, and therefore to ascertain its existence.[1]

However, in order to derive an explicit formula for the optimal portfolio π^* the literature has resorted to Malliavin calculus and in particular the Clark-Ocone formula. One of the first papers in this direction was Ocone & Karatzas (1991). The *Malliavin approach* to optimal investment and consumption problems has since become popular, see e.g. Lakner (1998), Pham & Quenez (2001), Benth, Di Nunno, Lökka, Øksendal & Proske (2003), Detemple & Rindisbacher (2005), Lakner & Nygren (2006), Putschögl & Sass (2008) and Di Nunno & Øksendal (2009).

[1] For details see Karatzas (1989) or Karatzas & Shreve (1998) Chapter 3.

Functional Itô calculus was initially proposed by Dupire (2009). It has since been developed into a rigorous and coherent theory, see e.g. Cont & Fournié (2010a), Cont & Fournié (2010b), Cont & Fournié (2013) and Bally, Caramellino, Cont, Utzet & Vives (2016) (the latter contains extensive lecture notes covering most of the present theory).

In the present paper, we suggest a *functional Itô calculus approach* to the optimal investment problem. Specifically, using an explicit martingale representation result and the *vertical derivative* (see Section (2)) we derive an explicit formula for the optimal portfolio π^*.

We remark that Pang & Hussain (2015) study optimal investment using functional Itô calculus, in a way substantially different from that of the present paper: they study an optimal investment and consumption problem in a particular financial model with *bounded memory*, in which the dynamics of the *delay variables* are investigated by means of the *functional Itô formula*.

The main contribution of the present paper can be summarized as follows: we derive an explicit formula for the optimal portfolio process π^* for a standard general optimal investment problem, see Theorem 4.3. We remark that our formula may be more suitable for numerical studies of optimal portfolios compared to formulas based on the Malliavin approach, see Remark 5.2. We illustrate our findings by studying two well-known examples, see Section 4.1.

2 Preliminaries - explicit martingale representation

In order to prove our main result, Theorem 4.3, we need an explicit martingale representation result from functional Itô calculus. Functional Itô calculus is still not very well known and we therefore present this result and some of the related theory in this section (without proofs). For a comprehensive account we refer to Bally et al. (2016) (and also to Cont & Fournié (2013)).

Let $\Omega = D([0,T],\mathbb{R}^n)$ be the space of càdlàg paths on $[0,T]$, where $T<\infty$. The value of a path $\omega \in \Omega$ at a fixed time $t \in [0,T]$ is denoted by $\omega(t)$. For a fixed $t \in [0,T]$, a *stopped path* ω_t is obtained by fixing the value of the path ω on $(t,T]$ at $\omega(t)$, i.e.

$$\omega_t(s) = \omega(t \wedge s) = \begin{cases} \omega(s) & \text{for } s \in [0,t] \\ \omega(t) & \text{for } s \in (t,T]. \end{cases}$$

The *space of stopped paths* is defined as

$$\Lambda_T = \{(t,\omega_t) : (t,\omega) \in [0,T] \times D([0,T],\mathbb{R}^n)\}.$$

A stopped path ω_t can be identified with the pair (ω, t) and we therefore, without any risk of confusion, from now on let (t, ω) and (t, ω_t) denote the same object and use them interchangeably. Define the distance between two different paths, ω and $\tilde{\omega}$, stopped at two different times, t and $\tilde{t}$, by

$$\begin{aligned} d_\infty((t,\omega),(\tilde{t},\tilde{\omega})) &= \sup_{s\in[0,T]} ||\omega(s\wedge t) - \tilde{\omega}(s\wedge \tilde{t})|| + |t-\tilde{t}| \\ &= ||\omega(s\wedge t) - \tilde{\omega}(s\wedge \tilde{t})||_\infty + |t-\tilde{t}|. \end{aligned}$$

It can be shown that (Λ_T, d_∞) is a complete metric space.

A *non-anticipative functional* is a real-valued measurable map from the space of stopped paths $F : (\Lambda_T, d_\infty) \to \mathbb{R}$. Since a stopped path ω_t is identified with (t, ω) we may write the value of a functional at time t as either $F(t, \omega)$ or as $F(t, \omega_t)$.

A non-anticipative functional F is said to be *continuous at fixed times* if, for fixed $t \in [0, T)$, the functional $F(t, \cdot) : D([0, T], \mathbb{R}^n) \to \mathbb{R}$ is continuous (where the metric for the path is d_∞).

A non-anticipative functional F is said to be *horizontally differentiable* at $(t, \omega) \in \Lambda_T$ if the following limit (called the *horizontal derivative*) exists

$$\mathcal{D}F(t,\omega) = \lim_{h\searrow 0} \frac{F(t+h,\omega_t) - F(t,\omega_t)}{h}.$$

If F is horizontally differentiable at each $(t, \omega) \in \Lambda_T$ then the map $\mathcal{D}F : (t, \omega) \to \mathcal{D}F(t, \omega)$ defines a non-anticipative functional, called the *horizontal derivative* of F.

A non-anticipative functional F is *vertically differentiable* at $(t, \omega) \in \Lambda_T$ if the following limits exist

$$\partial_i F(t,\omega) = \lim_{h\to 0} \frac{F(t,\omega_t + e_i h 1_{[t,T]}) - F(t,\omega_t)}{h}, \quad i = 1, ..., n.^2$$

The vertical derivative at $(t, \omega) \in \Lambda_T$ is then defined as

$$\nabla_\omega F(t,\omega) = (\partial_i F(t,\omega), i = 1, ..., n)'.^3$$

If F is vertically differentiable at each (t, ω), then $\nabla_\omega F(t, \omega)$ defines a non-anticipative functional called the *vertical derivative* of F.[4] Higher order vertical derivatives are defined by vertically differentiating the vertical derivative

[2] $e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_n = (0, ...0, 1)$ and $1_.$ is the indicator function.

[3] $'$ denotes transposition.

[4] The vertical derivative is more precisely an $\mathbb{R}^n$-valued non-anticipative functional (a similar comment applies for higher order derivatives). The definition of such a functional is the natural extension of the $\mathbb{R}$-valued non-anticipative functional. All functionals which are not derivatives should be understood to be $\mathbb{R}$-valued unless other information is provided.

(whenever such derivatives exist). We then, for example, write

$$\nabla_\omega^2 F(t,\omega) = (\partial_j(\partial_i F(t,\omega)), i,j = 1,...,n).$$

Let $\mathbb{C}_l^{0,0}(\Lambda_T)$ be the set of *left-continuous* functionals, which are defined as non-anticipative functionals that are continuous at fixed times and satisfy: $\forall (t,\omega) \in \Lambda_T, \forall \epsilon > 0, \exists \eta$ such that $\forall (\tilde{t},\tilde{\omega}) \in \Lambda_T$,

$$\tilde{t} < t \quad and \quad d_\infty((t,\omega),(\tilde{t},\tilde{\omega})) < \eta \Rightarrow |F(t,\omega) - F(\tilde{t},\tilde{\omega})| < \epsilon.$$

Let $\mathbb{B}(\Lambda_T)$ be the set of *boundedness preserving* functionals, which are defined as non-anticipative functionals satisfying: for any compact set $K \subset \mathbb{R}^n$ and $t_0 \in [0,T), \exists C_{K,t_0} > 0$ such that $\forall t \leq t_0, \forall \omega \in D([0,T],\mathbb{R}^n)$,

$$\omega([0,t]) \subset K \Rightarrow |F(t,\omega)| \leq C_{K,t_0}.$$ [5]

Let $\mathbb{C}_b^{1,2}(\Lambda_T)$ denote the set of left-continuous non-anticipative functionals (i.e. in $\mathbb{C}_l^{0,0}(\Lambda_T)$) which are also horizontally differentiable and twice vertically differentiable such that

1. $\mathcal{D}F(t,\cdot) : D([0,T],\mathbb{R}^n) \to \mathbb{R}$ is continuous for each $t \in [0,T)$,
2. $\nabla_\omega F(t,\omega), \nabla_\omega^2 F(t,\omega) \in \mathbb{C}_l^{0,0}(\Lambda_T)$,
3. $\mathcal{D}F, \nabla_\omega F(t,\omega), \nabla_\omega^2 F(t,\omega) \in \mathbb{B}(\Lambda_T)$.

Remark 2.1. *Let f be a real-valued function with $f \in C^{1,2}([0,T] \times \mathbb{R}^n)$ and $F(t,\omega) = f(t,\omega(t))$, then the horizontal and vertical derivatives reduce to the standard partial derivatives.*

Now consider a stochastic basis $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ where $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is the P-augmented filtration generated by an n-dimensional Wiener process W and $T < \infty$. Using the functional Itô formula[6] it is not difficult to prove the following explicit martingale representation theorem.

Proposition 2.2. (Martingale representation I) *If a martingale Y satisfies*

$$Y(t) = F(t, W_t) \quad a.s. \tag{1}$$

for some non-anticipative functional $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$, then, for any $t \in [0,T]$,

$$Y(t) = Y(0) + \int_0^t \nabla_\omega F(s, W_s)' dW(s) \quad a.s.^7 \tag{2}$$

[5] $\omega([0,t]) \subset K$ means that the path ω restricted to $[0,t]$ takes its values in the compact set $K \subset \mathbb{R}^n$. C_{K,t_0} is a constant which may depend on K and t_0.

[6] See e.g. Cont & Fournié (2013) Theorem 4.1. or Bally et al. (2016) Theorem 6.2.3.

[7] This is a version of e.g. Bally et al. (2016) Proposition 7.1.3 or Cont & Fournié (2013) Theorem 5.2. A more general setting where the second argument of F (see (1)) and the integrator (see (2)) are both on the form $X = X(0) + \int_0^\cdot \sigma(s)dW(s)$ is studied in Cont & Fournié (2013) and Bally et al. (2016) (this last remark applies also to the rest of this section).

We now define the *vertical derivative of the process* Y (satisfying (1)) with respect to W as

$$\nabla_W Y(t) = \nabla_\omega F(t, W_t). \tag{3}$$

Using this definition and (2) we see that our martingale Y (which satisfies (1)) can be written as

$$Y(t) = Y(0) + \int_0^t \nabla_W Y(s)' dW(s) \quad a.s. \tag{4}$$

We will now extend the definition of the vertical derivative $\nabla_W Y$. Using this extended definition we will see that the martingale representation in (4) is valid for any square integrable martingale. First we need som more definitions.

Let $\mathcal{L}^2(W)$ be the space of progressively measurable processes ϕ with $E[\int_0^T \phi(s)'\phi(s)ds] < \infty$. Let $\mathcal{M}^2(W)$ be the space of square integrable martingales starting at zero, with the norm $\sqrt{E(|Y(T)|^2)}$ (where in general $|\cdot|^2 = \sum_{i,j}(\cdot)^2$). Let $\mathcal{C}_b^{1,2}(W)$ be the set of adapted processes Y which can be represented as $Y(t) = F(t, W_t)$ a.s. with $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$. Now denote the set of "smooth square integrable martingales starting at zero" by $D(W) = \mathcal{C}_b^{1,2}(W) \cap \mathcal{M}^2(W)$. The following result characterizes the vertical derivative for martingales in $D(W)$.

Lemma 2.3. *Let* $Y \in D(W)$*. The vertical derivative* $\nabla_W Y$ *(defined in (3) as* $\nabla_W Y(t) = \nabla_\omega F(t, W_t)$ *whenever* $Y(t) = F(t, W_t)$ *a.s. with* $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$*) is then the unique element of* $\mathcal{L}^2(W)$ *satisfying*

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_W Y(t)' \nabla_W Z(t) dt\right]$$

for every $Z \in D(W)$*.*[8]

We will now use this lemma to extend the current definition of the vertical derivative $\nabla_W Y$ in (3) to a weak derivative which can be applied to any process in $\mathcal{M}^2(W)$.

Theorem 2.4 (The vertical derivative)**.** $\nabla_W : D(W) \to \mathcal{L}^2(W)$ *has a continuous extension* $\nabla_W : \mathcal{M}^2(W) \to \mathcal{L}^2(W)$ *satisfying*

$$\nabla_W \left[\int_0^\cdot \phi(s)' dW(s)\right] = \phi.$$

[8]This is a simplified version of e.g. Bally et al. (2016) Lemma 7.3.2, see also Cont & Fournié (2013) Proposition 5.5.

Let $Y \in \mathcal{M}^2(W)$, the (weak) vertical derivative $\nabla_W Y$ is then the unique element in $\mathcal{L}^2(W)$ satisfying

$$E[Y(T)Z(T)] = E\left[\int_0^T \nabla_W Y(t)' \nabla_W Z(t) dt\right]$$

for every $Z \in D(W)$.[9]

We further extend the definition of the vertical derivative $\nabla_W Y$ to the space of square integrable martingales by noting that if Y is a square integrable martingale then $Y - Y(0) \in \mathcal{M}^2(W)$ and defining $\nabla_W Y = \nabla_W (Y - Y(0))$. We are now ready to present the main result of this section.

Theorem 2.5 (Martingale representation II)**.** *Let Y be a square integrable martingale. Then, for any $t \in [0, T]$,*

$$Y(t) = Y(0) + \int_0^t \nabla_W Y(s)' dW(s) \quad a.s.$$ [10]

3 A standard complete financial market and the optimal investment problem

This section describes our optimization problem and our market, which is the Wiener driven complete market of Karatzas & Shreve (1998).[11]

Consider a continuous time financial market corresponding to the stochastic basis $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$, where $\underline{\mathcal{F}} = \{\mathcal{F}_t\}_{0 \le t \le T}$ is the P-augmented filtration generated by an n-dimensional Wiener process W. The time horizon is a positive constant $T < \infty$. The market is endowed with the following objects.

- A money market process B given by $B(t) = e^{\int_0^t r(s) ds}, 0 \le t \le T$, where r is a progressively measurable instantaneous risk-free rate process, satisfying $\int_0^T |r(t)| dt < \infty$ a.s.
- n stock price processes $S_i, i = 1, ..., n$ which are continuous, strictly positive and satisfy the SDEs

$$dS_i(t) = S_i(t)\alpha_i(t) dt + S_i(t)\sigma_i(t) dW(t), \quad S_i(0) > 0, \quad 0 \le t \le T,$$

[9] These are simplified versions of the results in e.g. Bally et al. (2016) Lemma 7.3.3, see also Cont & Fournié (2013) Theorem 5.8.

[10] This is a simplified version of e.g. Bally et al. (2016) Lemma 7.3.4, see also Cont & Fournié (2013) Proposition 5.9.

[11] We simplify the financial market of Karatzas & Shreve (1998) by considering an absolutely continuous money market process B and a volatility process σ which is non-singular (and not only non-singular almost everywhere $dt \times dP$), see Chapter 1 (ibid.).

- α_i is the i:th element of an n-dimensional progressively measurable mean rate of return process α satisfying $\int_0^T |\alpha(t)|dt < \infty$ a.s.
- σ_i is the i:th row of an $n \times n$-dimensional progressively measurable volatility process σ, where $\sigma(t)$ is non-singular for all $t \in [0, T]$ and all $\omega \in \Omega$, and $\int_0^T |\sigma(t)|^2 dt < \infty$ a.s.

Definition 3.1. *A* **portfolio process** (π, π^0) *consists of an n-dimensional progressively measurable process π (corresponding to the amount of capital invested in each stock S_i) and a 1-dimensional progressively measurable process π^0 (corresponding to the amount of capital invested in the money market B) for which $\int_0^T |\pi^0(t) + \pi(t)'\mathbf{1}||r(t)|dt < \infty$, $\int_0^T |\pi_t'(\alpha(t) - r(t)\mathbf{1})|dt < \infty$ and $\int_0^T |\pi(t)'\sigma(t)|^2 dt < \infty$ a.s.*[12] *The corresponding* **wealth process** *X is given by*

$$\begin{aligned} X(t) \quad &= x_0 + \int_0^t (\pi^0(s) + \pi(s)'\mathbf{1})r(s)ds + \int_0^t \pi(t)'(\alpha(s) - r(s)\mathbf{1})ds \\ &+ \int_0^t \pi(s)'\sigma(s)dW(s), \quad 0 \le t \le T \end{aligned} \tag{5}$$

where x_0 is initial wealth. We call the portfolio process **self-financing** *if $X(t) = \pi^0(t) + \pi(t)'\mathbf{1}, 0 \le t \le T$, and* **tame** *if the process $X(t)B(t)^{-1}, 0 \le t \le T$ is a.s. bounded from below by a real constant which does not depend on t.*

Remark 3.2. *All portfolio processes in the present paper are self-financing. For any given portfolio process (π, π^0) it is therefore enough to know π to find X (use the self-financing condition and (5) to see this). We can then use X and the self-financing condition to find π^0. Therefore, we will from now on refer to π alone as the portfolio process. We denote the wealth process corresponding to π by X^π.*

Definition 3.3. *For any given initial wealth $x_0 \ge 0$, we say that a portfolio process π is* **admissible** *if the corresponding wealth process is self-financing and satisfies $X^\pi(t) \ge 0, 0 \le t \le T$ a.s.*

We need the following objects and the related Assumption 3.5.

Definition 3.4. *The* **market price** *of risk process θ is defined by*

$$\theta(t) = \sigma(t)^{-1}(\alpha(t) - r(t)\mathbf{1}), \quad 0 \le t \le T.$$

The **likelihood process** *Z is defined by*

$$Z(t) = e^{-\int_0^t \theta(s)'dW(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds}, \quad 0 \le t \le T.$$

[12] $\mathbf{1}$ is an n-dimensional vector with each element equal to 1.

The **state price density process** H *is defined by*

$$H(t) = B(t)^{-1}Z(t) = e^{-\int_0^t r(s)ds - \int_0^t \theta(s)'dW(s) - \frac{1}{2}\int_0^t |\theta(s)|^2 ds}, \quad 0 \le t \le T. \quad (6)$$

Assumption 3.5. $\int_0^T |\theta(t)|^2 dt < \infty$ *a.s. The local martingale* Z *is in fact a martingale.* $E[H(T)] < \infty$.[13]

Definition 3.6. *The market is said to be* **complete** *if every* $\mathcal{F}_T$*-measurable random variable* ξ *(called a contingent claim), with* $\xi B(T)^{-1}$ *bounded from below and* $E_0[B(T)^{-1}\xi] < \infty$*, can be replicated by an admissible tame portfolio process* π *in the sense that*

$$\frac{\xi}{B(T)} = x_0 + \int_0^T B(t)^{-1}\pi(t)'\sigma(t)dW_0(t) \quad a.s.$$

for some x_0*, where* E_0 *is the expected value under the probability measure* P_0 *induced by* Z *(so that e.g.* $E_0[B(T)^{-1}\xi] = E\left[Z(T)B(T)^{-1}\xi\right]$*) and where* $W_0 = W + \int_0^{\cdot} \theta(s)ds$ *(so that* W_0 *is a Wiener process under* P_0*, by the Girsanov theorem).*

It is well-known that our market is arbitrage free and complete.[14] Let us now introduce a standard general terminal wealth utility function U and the inverse of its derivative I.

Definition 3.7 (Terminal wealth utility function)**.** $U : \mathbb{R} \to [-\infty, \infty)$ *is concave, non-decreasing and upper semicontinuous. The half-line dom*$(U) = \{x \in \mathbb{R} : U(x) > -\infty\}$ *is a nonempty subset of* $[0, \infty)$*. The derivative* U' *is continuous, positive and strictly decreasing on the interior of dom*(U)*, and* $\lim_{x\to\infty} U'(x) = 0$*. Define* $\overline{x} = \inf\{x \in \mathbb{R} : U(x) > -\infty\}$ *and the (generalized) inverse* $I : (0, \infty] \to [\overline{x}, \infty)$ *by*

$$I(y) = \begin{cases} (U')^{-1}(y), & y \in (0, \lim_{x\searrow\overline{x}} U'(x)) \\ \overline{x}, & y \in [\lim_{x\searrow\overline{x}} U'(x), \infty]. \end{cases}$$

The inverse I is finite, continuous and defined on $(0, \infty]$ and strictly decreasing for $y \in (0, \lim_{x\searrow\overline{x}} U'(x))$. Moreover, $I(U'(x)) = x$ for $x \in (\overline{x}, \infty)$.[15]

We are now ready to formulate our optimization problem.

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[13] A sufficient condition for the first and second assumption is for example that θ is bounded. A sufficient condition for the third assumption is that B is bounded away from zero.

[14] See Karatzas & Shreve (1998) Chapter 1 Theorem 4.2 and Theorem 6.6.

[15] See Karatzas & Shreve (1998) Chapter 3. Our notation is that $(f)^{-1}$ denotes the standard inverse function of an invertible function f.

Problem 1. (The optimal investment problem) *Given a fixed initial wealth* $x_0 > 0$ *and a utility function* U *we consider the following maximization problem*

$$sup_{\pi \in \mathcal{A}(x_0)} E\left[U(X^{\pi}(T))\right]$$

where $\mathcal{A}(x_0)$ *is the set of admissible portfolio processes satisfying*

$$E[min[U(X^{\pi}(T), 0)]] > -\infty.$$

Our main objective is now to find an explicit formula for the optimal portfolio process, which we denote by π^*.

4 The optimal portfolio process π^*

This section contains our main result, Theorem 4.3. The results in this section rely on the following integrability assumption (which therefore needs to be verified for any particular problem that is studied).[16]

Assumption 4.1. $E\left[(H(T)I(yH(T)))^2\right] < \infty, \forall y \in (0, \infty)$.

We need the following well-known result.

Lemma 4.2. *Consider initial wealth* $x_0 \in (\lim_{y \to \infty} E\left[H(T)I(yH(T))\right], \infty)$. *The optimal wealth process* X^* *is then given by*

$$X^*(t) = E_{\mathcal{F}_t}\left[\frac{H(T)}{H(t)} I(\mathcal{Y}(x_0)H(T))\right], \quad 0 \le t \le T \tag{7}$$

where $\mathcal{Y}(x_0) > 0$ *is a constant determined by*

$$E[H(T)I(\mathcal{Y}(x_0)H(T))] = x_0. \tag{8}$$

Moreover, an optimal portfolio process π^* *exists uniquely, up to almost everywhere equivalence* $dt \times dP$.[17]

We are now ready to prove our main result, which is an explicit formula for the optimal portfolio process.

[16] This assumption is stronger than the corresponding assumption in Karatzas & Shreve (1998). We demand square integrability and not only integrability. See Section 3.6 and 3.7 (ibid.) for remarks concerning conditions implying the validity of this assumption.

[17] See Karatzas & Shreve (1998) Chapter 3, mainly Theorem 7.6 (where π^* is also implicitly characterized). See also Theorem 3.5, Corollary 6.5, Remark 6.4 and p. 102 (ibid.).

Theorem 4.3 (The optimal portfolio process π^*)**.** *Consider initial wealth $x_0 \in (\lim_{y\to\infty} E\left[H(T)I(yH(T))\right], \infty)$. The optimal portfolio process π^* can then be represented as*

$$\pi^*(t) = \sigma(t)'^{-1} \frac{\nabla_W E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right] + \theta(t) E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right]}{H(t)}, \tag{9}$$

$0 \leq t \leq T$, where ∇_W is the vertical derivative (operator) with respect to the driving Wiener process W.

Remark 4.4. *Use (7) and (9) to see that the optimal portfolio process π^* can also be represented as*

$$\pi^*(t) = \sigma(t)'^{-1}\left[H(t)^{-1}\nabla_W[H(t)X^*(t)] + \theta(t)X^*(t)\right], \quad 0 \leq t \leq T. \tag{10}$$

Proof. Consider (7) and define the martingale M by

$$M(t) = H(t)X^*(t) = E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right]. \tag{11}$$

Now, use Jensen's inequality, the tower property, Assumption 4.1 and $\mathcal{Y}(x_0) > 0$ (see Lemma 4.2) to see that, for any t,

$$\begin{aligned}
E[M(t)^2] \quad &= E\left[E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right]^2\right] \\
&\leq E\left[E_{\mathcal{F}_t}\left[(H(T)I(\mathcal{Y}(x_0)H(T)))^2\right]\right] \\
&= E\left[(H(T)I(\mathcal{Y}(x_0)H(T)))^2\right] < \infty.
\end{aligned}$$

Thus, M is in fact a square integrable martingale.

Now use (5), (6), the standard Itô formula, the self-financing condition (Definition 3.1) and the definition of θ (Definition 3.4) to see that

$$\begin{aligned}
dM(t) \quad &= H(t)dX^*(t) + X^*(t)dH(t) + dX^*(t)dH(t) \\
&= H(t)[X^*(t)r(t)dt + \pi^*(t)'(\alpha(t) - r(t)\mathbf{1})dt + \pi^*(t)'\sigma(t)dW(t)] \\
&+X^*(t)[-r(t)H(t)dt - \theta(t)'H(t)dW(t)] \\
&+\pi^*(t)'\sigma(t)(-\theta(t)H(t))dt \\
&= H(t)\pi^*(t)'\sigma(t)dW(t) - X^*(t)\theta(t)'H(t)dW(t).
\end{aligned}$$

This implies that π^* satisfies, for any t,

$$M(t) = M(0) + \int_0^t H(s)(\pi^{*\prime}(s)\sigma(s) - X^*(s)\theta(s)')dW(s).$$

Since M is a square integrable martingale we may now use the explicit martingale representation of Theorem 2.5 which says that, for any t,

$$M(t) = M(0) + \int_0^t \nabla_W M(s)' dW(s) \quad a.s.$$

where $\nabla_W M$ is unique in $\mathcal{L}^2(W)$ (cf. Theorem (2.4)). Using the two last equations we see that we may represent π^* by

$$\nabla_W M(t)' = H(t)(\pi^{*\prime}(t)\sigma(t) - X^*(t)\theta(t)'), \quad 0 \le t \le T.$$

We recall that H given by (6) is an exponential process and we may therefore write

$$\pi^*(t)'\sigma(t) = \left[\frac{\nabla_W M(t)'}{H(t)} + X^*(s)\theta(t)'\right], \quad 0 \le t \le T$$

which implies that

$$\pi^*(t) = \sigma(t)'^{-1}\left[\frac{\nabla_W M(t)}{H(t)} + X^*(t)\theta(t)\right], \quad 0 \le t \le T.$$

Now substitute the right hand side of (7) for $X^*(t)$ and the right hand side of (11) for $M(t)$. The result follows. ∎

4.1 Examples

Let us use our explicit formula for the optimal portfolio π^* in two well-known examples.

4.1.1 Logarithmic utility

Let $U(x) = ln(x)$ for $x \in (0, \infty)$, so that $I(y) = \frac{1}{y}$ for $y \in (0, \infty)$. We easily obtain that Assumption 4.1 is satisfied and that $\lim_{y\to\infty} E[H(T)I(yH(T))] = 0$ (use calculations similar to those in (12) below). We may therefore use Lemma 4.2 and Theorem 4.3 for any initial wealth $x_0 > 0$.

Let us start by studying the components in (9). Using $I(y) = \frac{1}{y}$ we obtain that, for all t,

$$E_{\mathcal{F}_t}[H(T)I(\mathcal{Y}(x_0)H(T))] = E_{\mathcal{F}_t}\left[H(T)\frac{1}{\mathcal{Y}(x_0)H(T)}\right] = \frac{1}{\mathcal{Y}(x_0)}. \tag{12}$$

Lemma 4.2 says that $\mathcal{Y}(x_0)$ is a positive constant and (12) therefore implies that

$$\nabla_W E_{\mathcal{F}_t}[H(T)I(\mathcal{Y}(x_0)H(T))] = \nabla_W\left[\frac{1}{\mathcal{Y}(x_0)}\right] = 0,$$

where we used that the vertical derivative reduces to the standard derivative in this case. The above implies that the formula in (9) in Theorem 4.3 gives us

$$\pi^*(t) = \sigma(t)'^{-1}\frac{0+\theta(t)\frac{1}{\mathcal{Y}(x_0)}}{H(t)} = \frac{(\sigma(t)\sigma(t)')^{-1}(\alpha(t)-r(t)\mathbf{1})}{\mathcal{Y}(x_0)H(t)}.$$

Now use Lemma 4.2 and (12) to see that $x_0 = E\left[H(T)I(\mathcal{Y}(x_0)H(T))\right] = \frac{1}{\mathcal{Y}(x_0)}$, which implies that the optimal portfolio can be expressed as

$$\pi^*(t) = (\sigma(t)\sigma(t)')^{-1}(\alpha(t)-r(t)\mathbf{1})\frac{x_0}{H(t)}.$$

For completeness sake let us also find an expression for the optimal wealth process X^*. Use Lemma 4.2 and the above to see that

$$X^*(t) = E_{\mathcal{F}_t}\left[\frac{H(T)}{H(t)}I(\mathcal{Y}(x_0)H(T))\right] = \frac{1}{\mathcal{Y}(x_0)H(t)} = \frac{x_0}{H(t)}.$$

4.1.2 Power utility with deterministic coefficients

Let the coefficient processes r, α and σ be deterministic functions of time. This implies that also θ is a deterministic function of time. Let $U(x) = \frac{x^\gamma}{\gamma}$ for $x \in (0,\infty)$ with $\gamma < 1, \gamma \neq 0$. It follows that $I(y) = y^{\frac{1}{\gamma-1}}$ for $y \in (0,\infty)$. This implies that

$$E\left[(H(T)I(yH(T)))\right] = y^{\frac{1}{\gamma-1}}E\left[H(T)H(T)^{\frac{1}{\gamma-1}}\right] = y^{\frac{1}{\gamma-1}}E\left[H(T)I(H(T))\right] \tag{13}$$

which means that $\lim_{y\to\infty} E\left[H(T)I(yH(T))\right] = 0$ under Assumption 4.1 (to see this use that Assumption 4.1 implies that $E\left[H(T)I(H(T))\right] < \infty$). We may therefore use Lemma 4.2 and Theorem 4.3 for any initial wealth $x_0 > 0$, under Assumption 4.1. A sufficient condition for Assumption 4.1 is in this case that θ and r are bounded.

Recall from the proof of Theorem 4.3 that the process HX^* given by

$$H(t)X^*(t) = E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right]$$

is a square integrable martingale under Assumption 4.1. Now use $I(y) =$

$y^{\frac{1}{\gamma-1}}$, (6) and (7) to perform the following calculations.[18]

$$\begin{aligned}
H(t)X^*(t) &= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} E_{\mathcal{F}_t}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right] \qquad (14)\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} E_{\mathcal{F}_t}\left[e^{-\int_0^T \frac{\gamma}{\gamma-1}\theta(s)'dW(s)+\int_0^T(\ldots)ds}\right]\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} e^{\int_0^T(\ldots)ds} E_{\mathcal{F}_t}\left[e^{\int_0^T \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)}\right]\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} e^{\int_0^T(\ldots)ds} e^{\int_0^t \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)} E_{\mathcal{F}_t}\left[e^{\int_t^T \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)}\right]\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} e^{\int_0^T(\ldots)ds} e^{\int_0^t \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)} e^{\frac{1}{2}\int_t^T |\frac{\gamma\theta(s)}{\gamma-1}|^2 ds}\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} e^{\int_0^T(\ldots)ds} e^{\int_0^t \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)} e^{\frac{1}{2}\int_0^T |\frac{\gamma\theta(s)}{\gamma-1}|^2 ds-\frac{1}{2}\int_0^t |\frac{\gamma\theta(s)}{\gamma-1}|^2 ds}\\
&= \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} e^{\int_0^T(\ldots)ds} e^{\int_0^t \frac{-\gamma}{\gamma-1}\theta(s)'dW(s)} e^{-\frac{1}{2}\int_0^t |\frac{\gamma\theta(s)}{\gamma-1}|^2 ds}.
\end{aligned}$$

Thus, HX^* is in fact a square integrable *exponential* martingale. Using Itô's formula we may therefore write

$$H(t)X^*(t) = H(0)X^*(0) + \int_0^t H(s)X^*(s)\frac{-\gamma}{\gamma-1}\theta(s)'dW(s)$$

which together with Theorem 2.5 implies that we may represent the vertical derivative of HX^* with respect to W as

$$\nabla_W[H(t)X^*(t)]' = H(t)X^*(t)\frac{-\gamma}{\gamma-1}\theta(t)'. \qquad (15)$$

Now use Theorem 4.3 (specifically (10)) and (15) to see that the optimal portfolio can be represented as

$$\begin{aligned}
\pi^*(t) &= \sigma(t)'^{-1}\left[H(t)^{-1}\nabla_W[H(t)X^*(t)] + \theta(t)X^*(t)\right]\\
&= \sigma(t)'^{-1}\left[X^*(t)\frac{-\gamma}{\gamma-1}\theta(t) + \theta(t)X^*(t)\right]\\
&= \sigma(t)'^{-1}\left[\frac{1}{1-\gamma}\theta(t)X^*(t)\right]\\
&= (\sigma(t)\sigma(t)')^{-1}(\alpha(t) - r(t)\mathbf{1})\frac{X^*(t)}{1-\gamma}.
\end{aligned}$$

Let us also find an expression for the optimal wealth process X^*. Use Lemma 4.2 and $I(y) = y^{\frac{1}{\gamma-1}}$ to see that

$$x_0 = E\left[H(T)I(\mathcal{Y}(x_0)H(T))\right] = \mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} E\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]$$

[18] $(\ldots)$ denotes a deterministic function of time which is comprised of r, θ and γ, and which can easily be calculated at each instance this notation is used.

which implies that

$$\mathcal{Y}(x_0)^{\frac{1}{\gamma-1}} = \frac{x_0}{E\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]}.$$

Using this and (14) we see that optimal wealth process is given by

$$X^*(t) = \frac{1}{H(t)} \frac{x_0}{E\left[H(T)^{\frac{\gamma}{\gamma-1}}\right]} E_{\mathcal{F}_t}\left[H(T)^{\frac{\gamma}{\gamma-1}}\right].$$

5 Summary and remarks

We consider a standard complete Wiener driven market and a general optimal investment problem for terminal wealth. It is well-known that the optimal wealth process X^* is then given by the formula

$$X^*(t) = E_{\mathcal{F}_t}\left[\frac{H(T)}{H(t)} I(\mathcal{Y}(x_0)H(T))\right],$$

where H is the state price density, $I(\cdot)$ is the inverse of the derivative of the utility function and $\mathcal{Y}(x_0)$ is a constant depending on the initial wealth x_0. It is also well-known that the optimal portfolio process π^* can be implicitly characterized using standard martingale representation and that there, moreover, exist explicit formulas for the optimal portfolio π^* involving the Malliavin derivative.

Using the recently developed functional Itô calculus we derived the following explicit formula for the optimal portfolio π^*

$$\pi^*(t) = \sigma(t)'^{-1} \frac{\nabla_W E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right] + \theta(t) E_{\mathcal{F}_t}\left[H(T)I(\mathcal{Y}(x_0)H(T))\right]}{H(t)}$$

where ∇_W is the vertical derivative (operator) with respect to the driving Wiener process, θ is the market price of risk and σ is the asset price volatility. We showed that the formula can also be expressed as

$$\pi^*(t) = \sigma(t)'^{-1}\left[H(t)^{-1}\nabla_W[H(t)X^*(t)] + \theta(t)X^*(t)\right].$$

We also used the formula to derive the optimal portfolio π^* in two specific examples (logarithmic utility and power utility with deterministic coefficients).

Remark 5.1. *The relationship between the vertical derivative and the Malliavin derivative is studied in Cont & Fournié (2013) and Bally et al. (2016).*

Remark 5.2. *Our explicit formula for optimal portfolios may have computational advantages compared to explicit formulas based on the Malliavin approach: in the words of Bally et al. (2016)* "From a computational viewpoint, unlike the Clark-Haussmann-Ocone representation, which requires to simulate the anticipative process $\mathbb{D}_t H$ *[i.e. the Malliavin derivative]* and compute conditional expectations, $\nabla_X Y$ *[i.e. the vertical derivative]* only involves non-anticipative quantities which can be computed path by path. It is thus more amenable to numerical computations"*.*

We leave for future research how the functional Itô calculus approach to optimal investment suggested in the present paper compares to the Malliavin approach, and other approaches, when it comes to the performance in numerical studies of particular optimal investment problems.

A method for the computation of explicit approximations to functional Itô calculus martingale representation is presented in Cont & Lu (2016). Numerical studies of optimal portfolios using the Malliavin approach can be found in Detemple, Garcia & Rindisbacher (2003) and Takahashi & Yoshida (2004). Short surveys of other numerical approaches to optimal portfolios are included in Detemple et al. (2003) and Cvitanić, Goukasian & Zapatero (2003).

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