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New algorithms for computing exponential moments of hitting times and accumulated rewards of hitting type for semi-Markov processes are presented. The algorithms are based on special techniques of sequential phase space reduction and recurrence relations connecting exponential moments of rewards. Applications are discussed as well as possible generalizations of presented results and examples.

Key words: Semi-Markov process, Hitting time, Accumulated reward, Exponential moment, Phase space reduction, Recurrent algorithm.

AMS Mathematical Subject Classification 2010. Primary: 60J10, 60J22, 60J27, 60K15; Secondary: 65C40.

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1. Introduction

In this paper, we study recurrent relations for exponential moments of hitting times and accumulated rewards of hitting type for semi-Markov processes and present effective algorithms for computing these moments. These algorithms are based on procedures of sequential of phase space reduction for semi-Markov processes.

The results presented in this paper supplement results given in the paper Silvestrov and Manca (2015), where analogous results have been obtained for power moments of hitting times and accumulated rewards of hitting type for semi-Markov process. In order to make the present paper self-readable, we repeat some parts from the above paper, in particular, the definition of semi-Markov process given in Subsection 2.1, the description of reduced semi-Markov processes given in Subsection 3.1 and the slightly extended survey of literature given below.

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Hitting times are often interpreted as transition times for different stochastic systems describing by Markov-type processes, for example, occupation times or waiting times in queuing systems, life times in reliability models, extinction times in population dynamic models, etc. We refer to works by Korolyuk, Brodi and Turbin (1974), Kovalenko (1975), Korolyuk and Turbin (1976, 1978), Courtois (1977), Silvestrov (1980b), Anisimov, Zakusilo and Donchenko (1987), Ciardo, Raymonf, Sericola and Trivedi (1990), Kovalenko, Kuznetsov and Pegg (1997), Korolyuk, V.S. and Korolyuk, V.V. (1999), Limnios and Oprişan (2001, 2003), Barbu, Boussemart and Limnios (2004), Yin and Zhang (2005, 2013), Janssen and Manca (2006, 2007), Anisimov (2008), Gyllenberg and Silvestrov (2008), D'Amico, Petroni and Prattico (2013), and Papadopoulou (2013).

In financial and insurance applications, the hitting times for semi-Markov processes can be also interpreted as rewards accumulated up to some hitting terminating time for a financial or insurance contract. We refer here to works by D'Amico, Janssen and Manca (2005), Janssen and Manca (2006, 2007), Stenberg, Manca and Silvestrov (2006, 2007), Biffi, D'Amigo, Di Biase, Janssen, Manca and Silvestrov (2008), Silvestrov, Silvestrova and Manca (2008), D'Amico and Petroni (2012), Papadopoulou, Tsaklidis, McClean and Garg (2012), D'Amico, Guillen and Manca (2013), and D'Amico, Petroni and Prattico (2015).

Moments of hitting times also play an important role in limit and ergodic theorems for Markov type processes. As a rule, the first and second order moments are used in conditions of limit theorems theorems, higher order power and exponential moments in large deviation theorems, theorems on rates of convergence and asymptotical expansions. We refer here to works by Silvestrov (1974, 1980b, 1994, 1996), Korolyuk and Turbin (1976, 1978), Korolyuk, V. S. and Korolyuk, V. V. (1999), Hunter (2005), Koroliuk and Limnios (2005), Yin and Zhang (2005, 2013), Silvestrov and Drozdenko (2006), Anisimov (2008), Gyllenberg and Silvestrov (2008), and Silvestrov, D. and Silvestrov, S. (2015).

Recurrent relations, which link power moments of hitting times for Markov chains have been first obtained for Markov chains by Hodges and Rosenblatt (1953) and Chung (1954, 1960). Further development have been achieved by Kemeny and Snell (1961a, 1961b), Lamperty (1963), and Pitman (1974a, 1974b, 1977), Silvestrov (1980a, 1980b). Similar relations as well as description of these moments as minimal solutions of some algebraic or integral equations were considered for Markov chains and semi-Markov processes with discrete and arbitrary phase spaces by Cogburg (1975), Silvestrov (1980b, 1983a, 1983b, 1996), Nummelin and Tuominen (1983), Tweedie (1983), Nummelin (1984), and Silvestrov, Manca and Silvestrova (2014). Analogous results for exponential and mixed power exponential moments of first hitting times for semi-Markov processes have been obtained in Silvestrov (2004) and Gyllenberg and Silvestrov (2008).

The paper includes five sections. In Section 2, we introduce Markov renewal processes, semi-Markov processes and define hitting times and accumulated rewards of hitting type. We also present basic stochastic relations and systems of linear equations for exponential moments of these random functionals. In Section 3, we describe a procedure of phase space reduction for semi-Markov processes and formulas for computing transition characteristics for reduced semi-Markov processes. We also prove invariance of hitting times and their exponential moments with respect to the above procedure of phase space reduction. In Section 4, we describe a procedure of sequential phase space reduction for semi-Markov process and derive recurrent formulas for computing exponential moments of hitting times for semi-Markov processes. In Section 5, we present useful generalizations of the above results to real-valued and vector accumulated rewards of hitting type, general hitting times with hitting state indicators, place-dependent and time-dependent hitting times and accumulated rewards of hitting type and give a numerical example for the corresponding recurrent algorithm for computing exponential moments of hitting times for semi-Markov processes.

2. Semi-Markov processes and hitting times

In this section, we introduce Markov renewal processes and semi-Markov processes. We define also hitting times and accumulated rewards of hitting times, and give basic recurrent system of linear equations for their exponential moments, which are the main objects of our studies.

2.1. Markov renewal processes and semi-Markov processes. Let $\mathbb{X} = \{0, \ldots, m\}$ and $(J_n, X_n), n = 0, 1, \ldots$ be a Markov renewal process, i.e., a homogeneous Markov chain with the phase space $\mathbb{X} \times [0, \infty)$, an initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J_0 = i, X_0 = 0\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$Q_{ij}(t) = \mathsf{P}\{J_1 = j, X_1 \leq t / J_0 = i, X_0 = s\},\ (i, s), (j, t) \in \mathbb{X} \times [0, \infty). \quad (1)$$

In this case, the random sequence η_n is also a homogeneous (embedded) Markov chain with the phase space $\mathbb{X}$ and the transition probabilities,

$$p_{ij} = \mathsf{P}\{J_1 = j / J_0 = i\} = Q_{ij}(\infty),\ i, j \in \mathbb{X}. \quad (2)$$

As far as random variable X_n is concerned, it can be interpreted as sojourn time in state J_{n-1} or as a transition time from state J_{n-1} to state J_n, for $n = 1, 2, \ldots$.

We assume that the following communication conditions hold:

A: $\mathbb{X}$ is a communicative class of states for the embedded Markov chain J_n.

We also assume that the following condition excluding instant transitions holds:

B: $Q_{ij}(0) = 0, i, j \in \mathbb{X}$.

Let us now introduce a semi-Markov process,

$$J(t) = J_{N(t)},\ t \geq 0, \tag{3}$$

where $N(t) = \max(n \geq 0 : T_n \leq t)$ is a number of jumps in the time interval $[0, t]$, for $t \geq 0$, and $T_n = X_1 + \cdots + X_n,\ n = 0, 1, \ldots$, are sequential moments of jumps, for the semi-Markov process $J(t)$.

This process has the phase space $\mathbb{X}$, the initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J(0) = i\}, i \in \mathbb{X}\rangle$ and semi-Markov transition probabilities $Q_{ij}(t),\ t \geq 0$, $i, j \in \mathbb{X}$.

2.2. Hitting times and accumulated rewards of hitting type. Let us also introduce moments of sojourn times, for $\rho \geq 0$,

$$\phi_{ij}(\rho) = \mathsf{E}_i e^{\rho X_1} I(J_1 = j) = \int_0^\infty e^{\rho t} Q_{ij}^{(\varepsilon)}(dt),\ i, j \in \mathbb{X}. \tag{4}$$

Here and henceforth, notations P_i and E_i are used for conditional probabilities and expectations under condition $J(0) = i$.

Note that, $\phi_{ij}(0) = p_{ij},\ i, j \in \mathbb{X}$.

We assume that the following condition holds, for some integer $\rho > 0$:

$\mathbf{C}_\rho$: $\phi_{ij}(\rho) < \infty,\ i, j \in \mathbb{X}$.

Note that conditions **B** imply that $\phi_{ij}(\rho) > 0$, if $p_{ij} > 0$, while $\phi_{ij}(\rho) = 0$, if $p_{ij} = 0$.

The first hitting time to state 0 for the semi-Markov process $J(t)$ can be defined as,

$$W_0 = \inf(t \geq X_1 : J(t) = 0) = \sum_{n=1}^{U_0} X_n, \tag{5}$$

where $U_0 = \min(n \geq 1 : J_n = 0)$ is the first hitting time to state 0 for the Markov chain J_n.

The random variable W_0 can also be interpreted as a reward accumulated on trajectories of Markov chain J_n up to its first hitting to state 0.

The main object of our studies are power moments for the first hitting times,

$$\Phi_{i0}(\rho) = \mathsf{E}_i e^{\rho W_0}, \ \rho \geq 0, \ i \in \mathbb{X}. \tag{6}$$

Note that, $\Phi_{i0}(0) = 1$, $i \in \mathbb{X}$.

Conditions **A** and $\mathbf{C}_\rho$ imply that there exists $\rho_0 \in (0, \rho]$ such that, for $\rho' \in [0, \rho_0]$ and $i \in \mathbb{X}$,

$$\Phi_{i0}(\rho') < \infty. \tag{7}$$

Indeed, let us introduce conditional exponential moments, for $i, j \in \mathbb{X}$,

$$\psi_{ij}(\rho) = \begin{cases} \phi_{ij}(\rho)/p_{ij} & \text{if } p_{ij} > 0, \\ 1 & \text{if } p_{ij} = 0. \end{cases} \tag{8}$$

and define, for $\rho' \in [0, \rho]$, the following function,

$$\psi(\rho') = \max_{i,j \in \mathbb{X}} \phi_{ij}(\rho') \tag{9}$$

If conditions **A** and $\mathbf{C}_\rho$ holds, then function $\psi(\rho') \in [1, \infty)$, for $\rho' \in [0, \rho]$ and it is continuous nondecreasing function in this interval such that $\psi(\rho') \to 1$ as $\rho' \to 0$.

The following relation takes place, for $\rho' \in [0, \rho]$ and $i \in \mathbb{X}$,

$$\begin{aligned} \Phi_{i0}(\rho') &= \sum_{n=1}^{\infty} \sum_{i_0 = i, i_1, \ldots, i_{n-1} \neq 0, i_n = 0} \prod_{k=1}^{n} \psi_{i_{k-1}, i_k}(\rho') p_{i_{k-1}, i_k} \\ &\leq \sum_{n=1}^{\infty} \psi(\rho')^n \sum_{i_0 = i, i_1, \ldots, i_{n-1} \neq 0, i_n = 0} \prod_{k=1}^{n} p_{i_{k-1}, i_k} \\ &= \sum_{n=0}^{\infty} \psi(\rho')^n \mathsf{P}_i\{U_0 = n\}. \end{aligned} \tag{10}$$

Condition **A** implies that $\mathsf{P}_i\{U_0 \geq n\} \to 0$ as $n \to \infty$, for $i \in \mathbb{X}$. Thus, for any $0 < \theta < 1$, there exist integer $n_\theta \geq 1$ such that, for $i \in \mathbb{X}$,

$$\mathsf{P}_i\{U_0 \geq n_\theta\} \leq \theta. \tag{11}$$

This inequality implies that, for every $i \in \mathbb{X}$ and $k \geq 1$,

$$\begin{aligned} \mathsf{P}_i\{U_0 \geq k n_\theta\} &= \sum_{j \neq 0} \mathsf{P}_i\{U_0 \geq (k-1) n_\theta, J_{(k-1)n_\theta} = j\} \mathsf{P}_j\{U_0 \geq n_\theta\} \\ &\leq \theta \mathsf{P}_i\{U_0 \geq (k-1) n_\theta\} \leq \cdots \leq \theta^k. \end{aligned} \tag{12}$$

Inequalities (13) imply in an obvious way that, for for every $i \in \mathbb{X}$ and $n \geq 1$,

$$\mathsf{P}_i\{U_0 \geq n\} \leq \theta^{[\frac{n}{n_\theta}]} \leq L_\theta \theta^{\frac{n}{n_\theta}} \tag{13}$$

where $L_\theta = \theta^{-n_\theta}$.

Finally, relations (10) and (13) imply that the following inequality holds, for $\rho_0 \in [0, \rho]$ such that $\psi(\rho_0)\theta^{\frac{1}{n_\theta}} < 1$, and $i \in \mathbb{X}$,

$$\Phi_{i0}(\rho_0) \leq \sum_{n=0}^{\infty} L_\theta(\psi(\rho_0)\theta^{\frac{1}{n_\theta}})^n < \infty. \tag{14}$$

However, it should be noted that conditions **A** and $\mathbf{C}_\rho$ do not guarantee that the exponential moments $\Phi_{i0}(\rho) < \infty$. The corresponding example is given below.

In what follows, symbol $Y \stackrel{d}{=} Z$ is used to denote that random variables or vectors Y and Z have the same distribution.

The Markov property of the Markov renewal process (J_n, X_n) implies that following system of stochastic equalities takes place for hitting times,

$$\left\{ \begin{array}{l} W_{i,0} \stackrel{d}{=} X_{i,1} I(J_{i,1} = 0) + \sum_{j \neq 0}(X_{i,1} + W_{j,0}) I(J_{i,1} = j), \\ i \in \mathbb{X}, \end{array} \right. \tag{15}$$

where: (a) $W_{i,0}$ is a random variable which has distribution $\mathsf{P}\{W_{i,0} \leq t\} = \mathsf{P}_i\{W_0 \leq t\}, t \geq 0$, for every $i \in \mathbb{X}$; (b) $(J_{i,1}, X_{i,1})$ is a random vector, which takes values in space $\mathbb{X} \times [0, \infty)$ and has the distribution $P\{J_{i,1} = j, X_{i,1} \leq t\} = Q_{ij}(t), j \in \mathbb{X}, t \geq 0$, for every $i \in \mathbb{X}$; (c) the random variables $W_{i,0}$ and the random vector $(J_{i,1}, X_{i,1})$ are independent, for every $i \in \mathbb{X}$.

By computing exponential moments in stochastic relations (15) we get the following system of linear equations for moments $\Phi_{i0}(\rho), i \in \mathbb{X}$,

$$\left\{ \begin{array}{l} \Phi_{i0}(\rho) = \phi_{i0}(\rho) + \sum_{j \in \mathbb{X}, j \neq 0} \phi_{ij}(\rho)\Phi_{j0}(\rho), \\ i \in \mathbb{X}. \end{array} \right. \tag{16}$$

Note that it is possible that the moment $\phi_{ij}(\rho)$ equals to 0, while the moment $\Phi_{j0}(\rho)$ equal to $+\infty$ in relation (16). In such cases, one should set the product $0 \cdot \infty$ to be 0 when calculating the products at the right-hand side of equality (16).

Let consider the simplest semi-Markov process with the two-point phase space $\mathbb{X} = \{0, 1\}$ and, also, assume that all probabilities $p_{ij} > 0, i, j = 0, 1$, and, the exponential moments $\phi_{ij}(\rho) \in (0, \infty), i, j = 0, 1$.

In this case system of equations (16) takes the form,

$$\left\{ \begin{array}{l} \Phi_{00}(\rho) = \phi_{00}(\rho) + \phi_{01}(\rho)\Phi_{10}(\rho), \\ \Phi_{10}(\rho) = \phi_{10}(\rho) + \phi_{11}(\rho)\Phi_{10}(\rho). \end{array} \right. \tag{17}$$

If $\phi_{11}(\rho) \geq 1$, then $\Phi_{00}(\rho), \Phi_{10}(\rho) = \infty$ as follows from relations (17).

If $\phi_{11}(\rho) < 1$, then $\Phi_{00}(\rho), \Phi_{10}(\rho) < \infty$ and these moments are given by formulas,

$$\Phi_{10}(\rho) = \frac{\phi_{10}(\rho)}{1 - \phi_{11}(\rho)}, \ \Phi_{00}(\rho) = \phi_{00}(\rho) + \frac{\phi_{01}(\rho)\phi_{10}(\rho)}{1 - \phi_{11}(\rho)}. \tag{18}$$

It should be noted that the finiteness of the exponential moment for return time $\Phi_{00}(\rho)$ does not guarantee the finiteness of the exponential moment $\Phi_{11}(\rho)$. Indeed, according the above remarks, the exponential moments $\Phi_{11}(\rho), \Phi_{01}(\rho) = \infty$ if $\phi_{00}(\rho) \geq 1$.

Necessary and sufficient conditions of finiteness for exponential moments of hitting times are given in terms of so-called test-functions in Silvestrov (2004) and Gyllenberg and Silvestrov (2008).

We refer to functions $v(i), i \in \mathbb{X}$ defined on the space $\mathbb{X}$ and taking value in the interval $[0, \infty)$ as test-functions.

Let us introduce condition:

$\mathbf{D}_\rho$: There exists a test-function $v_\rho(i), i \in \mathbb{X}$ such that the following test inequalities hold,

$$v_\rho(i) \geq \phi_{i0}(\rho) + \sum_{j \in \mathbb{X}, j \neq 0} \phi_{ij}(\rho) v_\rho(j), \ i \in \mathbb{X}.$$

The following lemma gives the pointed above conditions finiteness for exponential moments of hitting times.

Lemma 1. *Let conditions* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}_\rho$, *for some* $\rho > 0$, *hold. Then, exponential moments* $\Phi_{i0}(\rho) < \infty, i \in \mathbb{X}$ *if and only if condition* $\mathbf{D}_\rho$ *holds. In this case, inequalities* $\Phi_{i0}(\rho) \leq v_\rho(i), \ i \in \mathbb{X}$ *hold and the exponential moments* $\Phi_{i0}(\rho), i \in \mathbb{X}$ *are the unique solution of the system of linear equations* (16).

It is useful to note that, in the above example with two-state semi-Markov process, this is impossible to find a test function $v_\rho(i), i = 0, 1$ such that the test inequalities penetrating condition $\mathbf{D}_\rho$ holds, if $\phi_{11}(\rho) \geq 1$.

Indeed, the second test inequality, which takes the form, $v_\rho(1) \geq \phi_{10}(\rho) + \phi_{11}(\rho) v_\rho(1)$, can not hold in this case, since $\phi_{10}(\rho) > 0$.

Condition $\mathbf{D}_\rho$, however, holds if $\phi_{11}(\rho) < 1$.

Indeed, the test inequality penetrating this condition holds for test-functions $v_\rho(1) = \frac{\phi_{10}(\rho)}{1 - \phi_{11}(\rho)}$ and $v_\rho(0) = \phi_{00}(\rho) + \phi_{01}(\rho) v_\rho(1)$ in the form of equalities.

The system of linear equation given in (16) has the matrix of coefficients $\mathbf{I} - {}_0\mathbf{P}(\rho)$, where $\mathbf{I} = \|I(i = j)\|$ is the unit matrix and matrix ${}_0\mathbf{P}(\rho) = \|\phi_{ij}(\rho) I(j \neq 0)\|$. Under conditions of Lemma 1, there exists the inverse matrix,

$$[\mathbf{I} - {}_0\mathbf{P}(\rho)]^{-1} = \|g_{i0j}(\rho)\|. \tag{19}$$

The elements of this matrix have the following probabilistic sense,

$$g_{i0j}(\rho) = \sum_{n=1}^{\infty} \mathsf{E}_i e^{\rho T_{n-1}} I(U_0 > n-1, J_{n-1} = j),\ i, j \in \mathbb{X}. \tag{20}$$

Thus, the formula for moments $\Phi_{i0}(\rho), i \in \mathbb{X}$ has the following form,

$$\Phi_{i0}(\rho) = \sum_{j \in \mathbb{X}} g_{i0j}(\rho)\phi_{j0}(\rho),\ i \in \mathbb{X}. \tag{21}$$

This is useful to note that the above remarks imply that condition **A** can be replaced by simpler hitting condition:

$\mathbf{A_0}$: $\mathsf{P}_i\{U_0 < \infty\} = 1,\ i \in \mathbb{X}$.

In this paper, we propose an alternative method, which can be considered as a stochastic analogue of Gauss elimination method for finding exponential moments $\Phi_{i0}(\rho), i \in \mathbb{X}$.

3. Semi-Markov processes with reduced phase spaces

In this section, we describe an one-step algorithm for reduction of a phase space for a semi-Markov process. We also give recurrent systems of linear equations for power moments of hitting times for a reduced semi-Markov process.

3.1. Reduced semi-Markov processes. Let us choose some state $k \in \mathbb{X}$ and consider the reduced phase space ${}_k\mathbb{X} = \mathbb{X} \setminus \{k\}$, with the state k excluded from the phase space $\mathbb{X}$.

Let us define the sequential moments of hitting the reduced space ${}_k\mathbb{X}$ by the embedded Markov chain J_n,

$${}_kV_n = \min(r > {}_kV_{n-1},\ J_r \in {}_k\mathbb{X}),\ n = 1, 2, \ldots,\ {}_kV_0 = 0. \tag{22}$$

Now, let us define the random sequence,

$$({}_kJ_n,\ {}_kX_n) = \begin{cases} (J_0, 0) & \text{for } n = 0, \\ (J_{{}_kV_n}, \sum_{r={}_kV_{n-1}+1}^{{}_kV_n} X_r) & \text{for } n = 1, 2, \ldots. \end{cases} \tag{23}$$

This sequence is also a Markov renewal process with phase space $\mathbb{X} \times [0, \infty)$, the initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J_0 = i, X_0 = 0\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$\begin{aligned} {}_kQ_{ij}(t) &= \mathsf{P}\{{}_kJ_1 = j,\ {}_kX_1 \leq t / {}_kJ_0 = i,\ {}_kX_0 = s\} \\ &= Q_{ij}(t) + \sum_{n=0}^{\infty} Q_{ik}(t) * Q_{kk}^{(*n)}(t) * Q_{kj}(t),\ t \geq 0,\ i, j \in \mathbb{X}. \end{aligned} \tag{24}$$

Here, symbol $*$ is used to denote the convolution of distribution functions (possibly improper), and $Q^{(*n)}_{kk}(t)$ is the n times convolution of the distribution function $Q_{kk}(t)$.

In this case, the Markov chain ${}_kJ_n$ has the transition probabilities,

$$\begin{aligned} {}_kp_{ij} = {}_kQ_{ij}(\infty) &= \mathsf{P}\{ {}_kJ_1 = j, / {}_kJ_0 = i\} \\ &= p_{ij} + \sum_{n=0}^{\infty} p_{ik}p^n_{kk}p_{kj} = p_{ij} + p_{ik}\frac{p_{kj}}{1 - p_{kk}},\ i, j \in \mathbb{X}. \end{aligned} \tag{25}$$

Note that condition **A** implies that probabilities $p_{kk} \in [0, 1),\ k \in \mathbb{X}$.

The transition distributions for the Markov chain ${}_kJ_n$ are concentrated on the reduced phase space ${}_k\mathbb{X}$, i.e., for every $i \in \mathbb{X}$,

$$\begin{aligned} \sum_{j\in {}_k\mathbb{X}} {}_kp_{ij} &= \sum_{j\in {}_k\mathbb{X}} p_{ij} + p_{ik} \sum_{j\in {}_k\mathbb{X}} \frac{p_{kj}}{1 - p_{kk}} \\ &= \sum_{j\in {}_k\mathbb{X}} p_{ij} + p_{ik} = 1. \end{aligned} \tag{26}$$

If the initial distribution $\bar{p}$ is concentrated on the phase space ${}_k\mathbb{X}$, i.e., $p_k = 0$, then the random sequence $({}_kJ_n,\ {}_kX_n), n = 0, 1, \ldots$ can be considered as a Markov renewal process with the reduced phase ${}_k\mathbb{X} \times [0, \infty)$, the initial distribution ${}_k\bar{p} = \langle p_i = \mathsf{P}\{{}_kJ_0 = i,\ {}_kX_0 = 0\} = \mathsf{P}\{{}_kJ_0 = i\}, i \in\ {}_k\mathbb{X}\rangle$ and transition probabilities ${}_kQ_{ij}(t), t \geq 0, i, j \in\ {}_k\mathbb{X}$.

If the initial distribution $\bar{p}$ is not concentrated on the phase space ${}_k\mathbb{X}$, i.e., $p_k > 0$, then the random sequence $({}_kJ_n,\ {}_kX_n), n = 0, 1, \ldots$ can be interpreted as a Markov renewal process with so-called transition period.

Let us now introduce the semi-Markov process,

$${}_kJ(t) = {}_kJ_{{}_kN(t)},\ t \geq 0, \tag{27}$$

where ${}_kN(t) = \max(n \geq 0 :\ {}_kT_n \leq t)$ is a number of jumps at time interval $[0, t]$, for $t \geq 0$, and ${}_kT_n = {}_kX_1 + \cdots + {}_kX_n,\ n = 0, 1, \ldots$ are sequential moments of jumps, for the semi-Markov process ${}_kJ(t)$.

As follows from the above remarks, the semi-Markov process ${}_kJ(t), t \geq 0$ has transition probabilities ${}_kQ_{ij}(t), t \geq 0, i, j \in \mathbb{X}$ concentrated on the reduced phase space ${}_k\mathbb{X}$, which can be interpreted as the actual "reduced" phase space of this semi-Markov process ${}_kJ(t)$.

If the initial distribution $\bar{p}$ is concentrated on the phase space ${}_k\mathbb{X}$, then process ${}_kJ(t), t \geq 0$ can be considered as the semi-Markov process with the reduced phase ${}_k\mathbb{X}$, the initial distribution ${}_k\bar{p} = \langle {}_kp_i = \mathsf{P}\{{}_kJ_1(0) = i\}, i \in\ {}_k\mathbb{X}\rangle$ and transition probabilities ${}_kQ_{ij}(t), t \geq 0, i, j \in\ {}_k\mathbb{X}$.

According to the above remarks, we can refer to the process ${}_kJ(t)$ as a reduced semi-Markov process.

If the initial distribution $\bar{p}$ is not concentrated on the phase space ${}_k\mathbb{X}$, then the process ${}_kJ(t), t \geq 0$ can be interpreted as a reduced semi-Markov process with transition period.

3.2. Transition characteristics for reduced semi-Markov processes. Relation (25) implies the following formulas, for probabilities ${}_kp_{kj}$ and ${}_kp_{ij}$, $i, j \in {}_k\mathbb{X}$,

$$\begin{cases} {}_kp_{kj} & = \frac{p_{kj}}{1-p_{kk}}, \\ {}_kp_{ij} & = p_{ij} + p_{ik}\,{}_kp_{kj} = p_{ij} + \frac{p_{ik}p_{kj}}{1-p_{kk}}. \end{cases} \tag{28}$$

It is useful to note that the second formula in relation (28) reduces to the first one, if to assign $i = k$ in this formula.

Taking into account that ${}_kV_1$ is Markov time for the Markov renewal process (J_n, X_n), we can write down the following system of stochastic equalities, for every $i, j \in {}_k\mathbb{X}$,

$$\begin{cases} {}_kX_{i,1}I({}_kJ_{i,1} = j) & \stackrel{d}{=} X_{i,1}I(J_{i,1} = j) \\ & \quad + (X_{i,1} + {}_kX_{k,1})I(J_{i,1} = k)I({}_kJ_{k,1} = j), \\ {}_kX_{k,1}I({}_kJ_{k,1} = j) & \stackrel{d}{=} X_{k,1}I(J_{k,1} = j) \\ & \quad + (X_{k,1} + {}_kX_{k,1})I(J_{k,1} = k)I({}_kJ_{k,1} = j), \end{cases} \tag{29}$$

where: (a) $(J_{i,1}, X_{i,1})$ is a random vector, which takes values in space $\mathbb{X} \times [0, \infty)$ and has the distribution $P\{J_{i,1} = j, X_{i,1} \leq t\} = Q_{ij}(t)$, $j \in \mathbb{X}$, $t \geq 0$, for every $i \in \mathbb{X}$; (b) $({}_kJ_{i,1}, {}_kX_{i,1})$ is a random vector which takes values in the space ${}_k\mathbb{X} \times [0, \infty)$ and has distribution $\mathsf{P}\{{}_kJ_{i,1} = j, {}_kX_{i,1} \leq t\} = \mathsf{P}_i\{{}_kJ_1 = j, {}_kX_1 \leq t\} = {}_kQ_{ij}(t)$, $j \in {}_k\mathbb{X}$, $t \geq 0$, for every $i \in \mathbb{X}$; (c) $(J_{i,1}, X_{i,1})$ and $({}_kJ_{k,1}, {}_kX_{k,1})$ are independent random vectors, for every $i \in \mathbb{X}$.

Let us denote,

$$\begin{aligned} {}_k\phi_{ij}(\rho) &= \mathsf{E}_i\, e^{\rho\,{}_kX_1} I({}_kJ_1 = j) \\ &= \int_0^\infty e^{\rho t}\,{}_kQ_{ij}(dt),\ \rho \geq 0,\ i, j \in {}_k\mathbb{X}. \end{aligned} \tag{30}$$

Note that ${}_k\phi_{ij}(0) = {}_kp_{ij},\ i \in \mathbb{X}, j \in {}_k\mathbb{X}$.

By computing exponential moments in stochastic relations (29) we get, for every $i, j \in {}_k\mathbb{X}$, the following system of linear equations for the moments ${}_k\phi_{kj}(\rho)$, ${}_k\phi_{ij}(\rho)$,

$$\begin{cases} {}_k\phi_{kj}(\rho) = \phi_{kj}(\rho) + \phi_{kk}(\rho)\,{}_k\phi_{kj}(\rho), \\ {}_k\phi_{ij}(\rho) = \phi_{ij}(\rho) + \phi_{ik}(\rho)\,{}_k\phi_{kj}(\rho). \end{cases} \tag{31}$$

Relation (31) yields the following formulas for moments ${}_k\phi_{kj}(\rho)$ and ${}_k\phi_{ij}(\rho)$, which should be used, for every $i, j \in {}_k\mathbb{X}$,

$$\begin{cases} {}_k\phi_{kj}(\rho) & = \frac{\phi_{kj}(\rho)}{1-\phi_{kk}(\rho)}, \\ {}_k\phi_{ij}(\rho) & = \phi_{ij}(\rho) + \frac{\phi_{ik}(\rho)\phi_{kj}(\rho)}{1-\phi_{kk}(\rho)}. \end{cases} \tag{32}$$

It is useful to note that the second formula in relation (32) reduces to the first one, if to assign $i = k$ in this formula.

Relation (32) imply that, under conditions **A**, **B** and $\mathbf{C}_\rho$, the following condition is necessary and sufficient for finiteness of exponential moments ${}_k\phi_{kj}(\rho)$, ${}_k\phi_{ij}(\rho), i, j \in {}_k\mathbb{X}$,

$\mathbf{E}^{(1)}_{k,\rho}$: $\phi_{kk}(\rho) < 1$.

3.3. Exponential moments for hitting times of reduced semi-Markov processes. Let us assume that $k \neq 0$ and introduce the first hitting time to state 0 for the reduced semi-Markov process ${}_kJ(t)$,

$${}_kW_0 = \inf(t \geq {}_kX_1 : {}_kJ(t) = 0) = \sum_{n=1}^{{}_kU_0} {}_kX_n, \tag{33}$$

where ${}_kU_0 = \min(n \geq 1 : {}_kJ_n = 0)$ is the first hitting time to state 0 by the reduced Markov chain ${}_kJ_n$.

Let also introduce moments,

$${}_k\Phi_{i0}(\rho) = \mathsf{E}_i\, e^{\rho\, {}_kW_0}, \ \rho \geq 0, \, i \in \mathbb{X}. \tag{34}$$

Note that, ${}_k\Phi_{i0}(0) = 1, i \in \mathbb{X}$.

The following theorem plays the key role in what follows.

Theorem 2. *The hitting times W_0 and ${}_kW_0$ to the state 0, respectively, for semi-Markov processes $J(t)$ and ${}_kJ(t)$, coincide, for every $k \neq 0$ and, thus, for every $i \in \mathbb{X}, k \neq 0$,*

$$\Phi_{i0}(\rho) = \mathsf{E}_i e^{\rho W_0} = {}_k\Phi_{i0}(\rho) = \mathsf{E}_i\, e^{\rho\, {}_kW_0}. \tag{35}$$

Proof. The first hitting times to a state 0 are connected for Markov chains J_n and ${}_kJ_n$ by the following relation,

$$U_0 = \min(n \geq 1 : J_n = 0) = \min({}_kV_n \geq 1 : {}_kJ_n = j) = {}_kV_{{}_kU_0}, \tag{36}$$

where ${}_kU_0 = \min(n \geq 1 : {}_kJ_n = 0)$.

The above relations imply that the following relation holds for the first hitting times to state 0, for the semi-Markov processes $J(t)$ and ${}_kJ(t)$,

$$W_0 = \sum_{n=1}^{U_0} X_n = \sum_{n=1}^{{}_kV_{{}_kU_0}} X_n = \sum_{n=1}^{{}_kU_0} {}_kX_n = {}_kW_0. \tag{37}$$

The equality for exponential moments of hitting times is an obvious corollary of relation (37). □

Lemma 2. *Let $\rho \geq 0$ and conditions* **A**, **B**, $\mathbf{C}_\rho$ *and* $\mathbf{D}_\rho$ *hold for the semi-Markov process $J(t)$. Then, these conditions also hold for the reduced semi-Markov process ${}_kJ(t)$, for any state $k \neq 0$.*

Proof. Holding of conditions **A** and **B** for the semi-Markov process ${}_kJ(t)$ is obvious. Holding of condition $\mathbf{C}_\rho$ for the semi-Markov process ${}_kJ(t)$ follows from relation (32). Holding of condition $\mathbf{D}_\rho$ for the semi-Markov process ${}_kJ(t)$ follows from Lemma 1. □

We can write down the recurrent systems of linear equations (16) for moments ${}_k\Phi_{k0}(\rho)$ and ${}_k\Phi_{i0}(\rho), i \in {}_k\mathbb{X}$ of the reduced semi-Markov process ${}_kJ(t)$,

$$\begin{cases} {}_k\Phi_{k0}(\rho) & = {}_k\phi_{k0}(\rho) + \sum_{j \in {}_k\mathbb{X}, j \neq 0} {}_k\phi_{kj}(\rho)\, {}_k\Phi_{j0}(\rho), \\ {}_k\Phi_{i0}(\rho) & = {}_k\phi_{i0}(\rho) + \sum_{j \in {}_k\mathbb{X}, j \neq 0} {}_k\phi_{ij}(\rho)\, {}_k\Phi_{j0}(\rho), \\ i \in {}_k\mathbb{X}. \end{cases} \tag{38}$$

Theorem 1 makes it possible to compute exponential moments $\Phi_{i0}(\rho) = {}_k\Phi_{i0}(\rho), i \in \mathbb{X}$ in the way alternative to solving recurrent systems of linear equations (16).

Instead of this, we can, first, compute exponential moments of transition times for the reduced semi-Markov process ${}_kJ(t)$ using, respectively, relation (32), and, then, by solving the systems of linear equations (38).

Note that the system of linear equations given in (16) has m equations for exponential moments $\Phi_{i0}(\rho), i \in \mathbb{X}, i \neq 0$ plus the explicit formula for computing exponential moment $\Phi_{00}(\rho)$ as function of exponential moments $\Phi_{i0}(\rho), i \in \mathbb{X}, i \neq 0$.

While, the system of linear equations given in (38) has, in fact, $m - 1$ equations for exponential moments ${}_k\Phi_{i0}(\rho), i \in {}_k\mathbb{X}, i \neq 0$, plus two explicit formulas for computing exponential moment ${}_k\Phi_{00}(\rho)$ and ${}_k\Phi_{k0}(\rho)$ as functions of exponential moments ${}_k\Phi_{i0}(\rho), i \in {}_k\mathbb{X}, i \neq 0$.

4. Algorithms of sequential phase space reduction

In this section, we present a multi-step algorithm for sequential reduction of phase space for semi-Markov processes. We also present the recurrent algorithm for computing exponential moments of hitting times for semi-Markov processes, which is based on the above algorithm of sequential reduction of the phase space.

4.1. Sequential reduction of phases space for semi-Markov processes. In what follows, let $i \in \{1, \ldots, m\}$ and let $\bar{k}_{i,m} = \langle k_{i,1}, \ldots, k_{i,m} \rangle = \langle k_{i,1}, \ldots, k_{i,m-1}, i \rangle$ be a permutation of the sequence $\langle 1, \ldots, m \rangle$ such that $k_{i,m} = i$, and let $\bar{k}_{i,n} = \langle k_{i,1}, \ldots, k_{i,n} \rangle$, $n = 1, \ldots, m$ be the corresponding chain of growing sequences of states from space $\mathbb{X}$.

Let us assume that $p_0 + p_i = 1$. Denote as ${}_{\bar{k}_{i,0}}J(t) = J(t)$, the initial semi-Markov process. Let us exclude state $k_{i,1}$ from the phase space ${}_{\bar{k}_{i,0}}\mathbb{X} = \mathbb{X}$ of semi-Markov process ${}_{\bar{k}_{i,0}}J(t)$ using the time-space screening procedure described in Section 3. Let ${}_{\bar{k}_{i,1}}J(t)$ be the corresponding reduced semi-Markov process. The above procedure can be repeated. The state $k_{i,2}$ can be excluded from the phase space of the semi-Markov process ${}_{\bar{k}_{i,1}}J(t)$. Let ${}_{\bar{k}_{i,2}}J(t)$ be the corresponding reduced semi-Markov process. By continuing the above procedure for states $k_{i,3}, \ldots, k_{i,n}$, we construct the reduced semi-Markov process ${}_{\bar{k}_{i,n}}J(t)$.

The process ${}_{\bar{k}_{i,n}}J(t)$ has, for every $n = 1, \ldots, m$, the actual "reduced" phase space,

$$
{}_{\bar{k}_{i,n}}\mathbb{X} = {}_{\bar{k}_{i,n-1}}\mathbb{X} \setminus \{k_{i,n}\} = \mathbb{X} \setminus \{k_{i,1}, k_{i,2}, \ldots, k_{i,n}\}. \tag{39}
$$

The transition probabilities ${}_{\bar{k}_{i,n}}p_{k_{i,n},j'}$, ${}_{\bar{k}_{i,n}}p_{i'j'}$, $i', j' \in {}_{\bar{k}_n}\mathbb{X}$, and the exponential moments ${}_{\bar{k}_{i,n}}\phi_{k_{i,n},j'}(\rho)$, ${}_{\bar{k}_{i,n}}e^{(r)}_{i'j'}$, $i', j' \in {}_{\bar{k}_{i,n}}\mathbb{X}$, $r = 1, \ldots, d$ are determined for the semi-Markov process ${}_{\bar{k}_{i,n}}J(t)$ by the transition probabilities and the expectations of sojourn times for the semi-Markov process ${}_{\bar{k}_{i,n-1}}J(t)$, respectively, via relations (28) and (32), which take the following recurrent forms, for $i', j' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ and $n = 1, \ldots, m$,

$$
\begin{cases}
{}_{\bar{k}_{i,n}}p_{k_{i,n},j'} & = \frac{{}_{\bar{k}_{i,n-1}}p_{k_{i,n},j'}}{1 - {}_{\bar{k}_{i,n-1}}p_{k_{i,n},k_{i,n}}}, \\
{}_{\bar{k}_{i,n}}p_{i'j'} & = {}_{\bar{k}_{i,n-1}}p_{i'j'} \\
& \quad + \frac{{}_{\bar{k}_{i,n-1}}p_{i'k_{i,n}} \, {}_{\bar{k}_{i,n-1}}p_{k_{i,n},j'}}{1 - {}_{\bar{k}_{i,n-1}}p_{k_{i,n},k_{i,n}}},
\end{cases} \tag{40}
$$

and

$$
\begin{cases}
{}_{\bar{k}_{i,n}}\phi_{k_{i,n},j'}(\rho) & = \frac{{}_{\bar{k}_{i,n-1}}\phi_{k_{i,n},j'}(\rho)}{1 - {}_{\bar{k}_{i,n-1}}\phi_{k_{i,n},k_{i,n}}(\rho)}, \\
{}_{\bar{k}_{i,n}}\phi_{i'j'}(\rho) & = {}_{\bar{k}_{i,n-1}}\phi_{i'j'}(\rho) \\
& \quad + \frac{{}_{\bar{k}_{i,n-1}}\phi_{i'k_{i,n}}(\rho) \, {}_{\bar{k}_{i,n-1}}\phi_{k_{i,n},j'}(\rho)}{1 - {}_{\bar{k}_{i,n-1}}\phi_{k_{i,n},k_{i,n}}(\rho)},
\end{cases} \tag{41}
$$

Relation (41) imply that, under conditions $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}_\rho$, the following condition is necessary and sufficient for finiteness of exponential moments ${}_{\bar{k}_{i,n'}}\phi_{k_{i,n'},j'}(\rho)$, ${}_{\bar{k}_{i,n'}}\phi_{i'j'}(\rho)$, $i', j' \in {}_{\bar{k}_{i,n'}}\mathbb{X}$, $n' = 1, \ldots, n$:

$\mathbf{E}^{(n)}_{\bar{k}_{i,n},\rho}$: ${}_{\bar{k}_{i,n'-1}}\phi_{k_{i,n'},k_{i,n'}}(\rho) < 1, n' = 1, \ldots, n.$

4.2. Recurrent algorithms for computing of moments of hitting times. Let us ${}_{\bar{k}_{i,n}}W_0$ be the first hitting time to state 0 for the reduced semi-Markov process ${}_{\bar{k}_{i,n}}J(t)$ and ${}_{\bar{k}_{i,n}}\Phi_{i'0}(\rho) = \mathsf{E}_{i'}e^{\rho\, {}_{\bar{k}_{i,n}}W_0}, i' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ be the exponential moments for these random variables.

By Theorem 1, the above exponential moments of hitting time coincide for the semi-Markov processes ${}_{\bar{k}_{i,0}}J(t), {}_{\bar{k}_{i,1}}J(t), \ldots, {}_{\bar{k}_{i,n}}J(t)$, i.e., for $n' = 0, \ldots, n$,

$$ {}_{\bar{k}_{j,n'}}\Phi_{k_{i,n'}0}(\rho) = \Phi_{k_{i,n'}0}(\rho), \ {}_{\bar{k}_{j,n'}}\Phi_{i'0}(\rho) = \Phi_{i'0}(\rho), \ i' \in {}_{\bar{k}_{i,n}}\mathbb{X}. \quad (42) $$

Moreover, exponential moments of hitting times ${}_{\bar{k}_{j,n}}\Phi_{k_{i,n}0}(\rho)$, ${}_{\bar{k}_{i,n}}\Phi_{i'0}(\rho)$, $i' \in {}_{\bar{k}_{i,n}}\mathbb{X}$, resulted by the recurrent algorithm of sequential phase space reduction described above, are invariant with respect to any permutation $\bar{k}'_{i,n} = \langle k'_{i,1}, \ldots, k'_{i,n}\rangle$ of sequence $\bar{k}_{i,n} = \langle k_{i,1}, \ldots, k_{i,n}\rangle$.

Indeed, for every permutation $\bar{k}'_{i,n}$ of sequence $\bar{k}_{i,n}$, the corresponding reduced semi-Markov process ${}_{\bar{k}'_{i,n}}J(t)$ is constructed as the sequence of states for the initial semi-Markov process $J(t)$ at sequential moment of its hitting into the same reduced phase space ${}_{\bar{k}'_{i,n}}\mathbb{X} = \mathbb{X} \setminus \{k'_{i,1}, \ldots, k'_{i,n}\} = {}_{\bar{k}_{i,n}}\mathbb{X} = \mathbb{X} \setminus \{k_{i,1}, \ldots, k_{i,n}\}$. The times between sequential jumps of the reduced semi-Markov process ${}_{\bar{k}'_{i,n}}J(t)$ are the times between sequential hitting of the above reduced phase space by the initial semi-Markov process $J(t)$.

This implies that the transition probabilities ${}_{\bar{k}_{i,n}}p_{k_{i,n}j'}$, ${}_{\bar{k}_{i,n}}p_{i'j'}, i', j' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ and the exponential moments ${}_{\bar{k}_{i,n}}\phi_{k_{i,n}j'}(\rho)$, ${}_{\bar{k}_{i,n}}\phi_{i'j'}(\rho), i', j' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ and, in sequel, exponential moments ${}_{\bar{k}_{i,n}}\Phi_{k_{i,n}0}(\rho)$, ${}_{\bar{k}_{i,n}}\Phi_{i'0}(\rho), i' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ are, for every $n = 1, \ldots, m$, invariant with respect to any permutation $\bar{k}'_{i,n}$ of the sequence $\bar{k}_{i,n}$.

Let us now choose $n = m$. In this case, the reduced semi-Markov process ${}_{\bar{k}_{i,m}}J(t)$ has the one-state phase space ${}_{\bar{k}_{i,m}}\mathbb{X} = \{0\}$ and state $k_{i,m} = i$.

In this case, the reduced semi-Markov process ${}_{\bar{k}_{i,m}}J(t)$ return to state 0 after every jump and hitting time to state 0 coincides with the sojourn time in state ${}_{\bar{k}_{i,m}}J(0)$.

Thus, the transition probabilities,

$$ {}_{\bar{k}_{i,m}}p_{i0} = {}_{\bar{k}_{i,m}}p_{00} = 1. \quad (43) $$

Also, by Theorem 1, moments,

$$ \Phi_{i0}(\rho) = {}_{\bar{k}_{i,m}}\Phi_{i0}(\rho) = {}_{\bar{k}_{i,m}}\phi_{i0}(\rho), \quad (44) $$

and

$$\Phi_{00}(\rho) = {}_{\bar{k}_{i,m}}\Phi_{00}(\rho) = {}_{\bar{k}_{i,m}}\phi_{00}(\rho). \quad (45)$$

Relations (44) and (45) imply that, under conditions **A**, **B** and $\mathbf{C}_\rho$, the following condition is necessary and sufficient for finiteness of exponential moments $\Phi_{i0}(\rho), i \neq 0, \Phi_{00}(\rho)$:

$\mathbf{E}^{(m)}_{\bar{k}_{i,m},\rho}$: ${}_{\bar{k}_{i,n-1}}\phi_{k_{i,n},k_{i,n}}(\rho) < 1, n = 1, \ldots, m.$

In fact, if condition $\mathbf{E}^{(m)}_{\bar{k}_{i,m},\rho}$ holds for some permutation $\bar{k}_{i,m} = \langle k_{i,1}, \ldots, k_{i,m-1}, i\rangle$ of the sequence $\langle 1, \ldots, m\rangle$, it also holds for any other permutation $\bar{k}'_{i',m} = \langle k'_{i',1}, \ldots, k'_{i',m-1}, i'\rangle$ of the sequence $\langle 1, \ldots, m\rangle$.

Thus, condition $\mathbf{E}^{(m)}_{\bar{k}_{i,m},\rho}$ is an alternative to condition $\mathbf{D}_\rho$.

The above remarks can be summarized in the following theorem, which presents the recurrent algorithm for computing of power moments for hitting times.

Theorem 2. *Let $\rho \geq 0$ and conditions* **A**, **B**, $\mathbf{C}_\rho$ *and* $\mathbf{D}_\rho$ *hold for the semi-Markov process $J(t)$. Exponential moments $\Phi_{i0}(\rho)$ and $\Phi_{00}(\rho)$ are given, for every $i = 1, \ldots, m$, by formulas* (44) *and* (45), *where the exponential moments ${}_{\bar{k}_{i,n}}\phi_{k_{i,n},j'}(\rho)$, ${}_{\bar{k}_{i,n}}\phi_{i'j'}(\rho)$, $i', j' \in {}_{\bar{k}_{i,n}}\mathbb{X}$ are determined, for $n = 1, \ldots, m$, by recurrent formulas* (41). *The moments $\Phi_{i0}(\rho)$ and $\Phi_{00}(\rho)$ are invariant with respect to any permutation $\bar{k}_{i,m}$ of sequence $\langle 1, \ldots, m\rangle$ used in the above recurrent algorithm.*

5. Generalizations and examples

In this section, we describe several variants for generalization of the results concerned recurrent algorithms for computing exponential moments of hitting times and accumulated rewards of hitting type.

5.1. Real-valued accumulated rewards of hitting type. First, we would like to mention that Theorems 1 and 2 can be generalized on the model, where of the Markov renewal process $(J_n, X_n), n = 0, 1, \ldots$ has the phase space $\mathbb{X} \times \mathbb{R}_1$, an initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J_0 = i, X_0 = 0\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$Q_{ij}(t) = \mathsf{P}\{J_1 = j, X_1 \leq t/J_0 = i, X_0 = s\},\ (i, s), (j, t) \in \mathbb{X} \times \mathbb{R}_1. \quad (46)$$

In this case, the random variable,

$$W_0 = \sum_{n=1}^{U_0} X_n \quad (47)$$

can be be interpreted as a reward accumulated on trajectories of Markov chain J_n up to its first hitting time $U_0 = \min(n \geq 1, J_n = 0)$ of this Markov chain to the state 0.

Conditions **A**, **B**, $\mathbf{C}_\rho$ and $\mathbf{D}_\rho$ do not change their formulations.

All recurrent relations for moments $\mathsf{E}_{i0}(\rho) = \mathsf{E}_i e^{\rho W_0}, i \in \mathbb{X}$, given in Sections 3 – 4, as well as Theorems 1 and 2 take the same forms as in the case of nonnegative rewards.

5.2. Vector accumulated rewards of hitting type. Second, we would like to show, how the above results can be generalized on the case of vector accumulated rewards.

Let us consider the model, where the Markov renewal process $(J_n, \bar{X}_n) = (J_n, (X_{1,n}, \ldots, X_{l,n})) = 0, 1, \ldots$ has the phase space $\mathbb{X} \times \mathbb{R}_k$, an initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J_0 = i, \vec{X}_0 = (0, \ldots, 0)\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$Q_{ij}(\bar{t}) = \mathsf{P}\{J_1 = j, \bar{X}_1 \leq \bar{t}/J_0 = i, \bar{X}_0 = s\}, \ (i, \bar{s}), (j, \bar{t}) \in \mathbb{X} \times \mathbb{R}_l. \quad (48)$$

Here and henceforth symbol $\bar{u} \leq \bar{v}$ for vectors $\bar{u} = (u_1, \ldots, u_l), \bar{v} = (v_1, \ldots, v_l) \in \mathbb{R}_l$ means that $u_1 \leq v_1, \ldots, u_l \leq v_l$.

The vector accumulated reward $\bar{W}_0 = (W_{1,0}, \ldots, W_{l,0})$ is defined as a l-dimensional random vector with components,

$$W_{l',0} = \sum_{n=1}^{U_0} X_{,n}, \ l' = 1, \ldots, l. \quad (49)$$

Let also $\bar{\rho} = (\rho_1, \ldots, \rho_l)$ be a vector with non-negative components.

Condition $\dot{\mathbf{C}}_\rho$ should be replaced by condition:

$\dot{\mathbf{C}}_{\bar{\rho}}$: $\mathsf{E}_i e^{\rho_1 X_{1,1} + \cdots \rho_l X_{l,1}} < \infty, \ i \in \mathbb{X}$.

Let us define random variables $W_0(\bar{\rho}) = \rho_1 W_{1,0} + \cdots + \rho_l W_{l,0}$ and mixed exponential moments $\Phi_{i0}(\bar{\rho}) = \mathsf{E}_i e^{W_0(\bar{\rho})} = \prod_{l'=1}^{l} e^{\rho_{l'} W_{l',0}}, i \in \mathbb{X}$.

By definition, $W_0(\bar{\rho}) = \sum_{n=1}^{U_0} (\rho_1 X_{1,n} + \cdots + \rho_l X_{l,n})$ is a scalar accumulated reward for the corresponding local rewards $X_n(\bar{\rho}) = \rho_1 X_{1,n} + \cdots + \rho_l X_{l,n}, n = 1, 2, \ldots$.

The exponential moments $\Phi_{i0}(\bar{\rho}) = \mathsf{E}_i e^{W_0(\bar{\rho})}, i \in \mathbb{X}$ are exponential moments for the above scalar accumulated reward, for parameter $\rho = 1$.

All recurrent relations for exponential moments of hotting times, given in Sections 3 – 4, as well as Theorems 1 and 2 can be reformulated in an obvious way for the above mixed exponential moments.

5.3. General hitting times with hitting state indicators. Third, the above results can be generalized on the case of more general hitting times,

$$W_{\mathbb{D}} = \sum_{n=1}^{U_{\mathbb{D}}} X_n, \tag{50}$$

where $U_{\mathbb{D}} = \min(n \geq 1, J_n \in \mathbb{D})$, for some nonempty set $\mathbb{D} \subset \mathbb{X}$.

In this case main object of studies are exponential moments for the hitting times with hitting state indicators,

$$\Phi_{\mathbb{D},ij}(\rho) = \mathsf{E}_i e^{\rho W_{\mathbb{D}}} I(J_{U_{\mathbb{D}}} = j), \ \rho \geq 0, \ j \in \mathbb{D}, i \in \mathbb{X}. \tag{51}$$

Note that $\Phi_{\mathbb{D},ij}(0) = \mathsf{P}_i\{J_{U_{\mathbb{D}}} = j\}, i \in \mathbb{X}, j \in \mathbb{D}$.

Condition $\mathbf{D}_\rho$ takes in this case the following form:

$\mathbf{D}_{\mathbb{D},\rho}$: There exists a test-function $v_\rho(i), i \in \mathbb{X}$ such that the following test inequalities hold,

$$v_\rho(i) \geq \phi_{i0}(\rho) + \sum_{j \in \mathbb{X},\, j \notin \mathbb{D}} \phi_{ij}(\rho) v_\rho(j), \ i \in \mathbb{X}.$$

Note also that condition $\mathbf{A}$ can, in fact, be replaced by a simpler condition:

$\mathbf{A}_{\mathbb{D}}$: $\mathsf{P}_i\{U_{\mathbb{D}} < \infty\} = 1, \ i \in \mathbb{X}$.

In this case, lemmas analogous to Lemma 1 and 2 and theorems analogous to Theorems 1 and 2 take place, and recurrent systems of linear equations and recurrent formulas analogous to those given in Sections 2 – 4 can be written down.

For example, let ${}_k\Phi_{\mathbb{D},ij}(\rho), \ i \in \mathbb{X}, j \in \mathbb{D}$ be the moments $\Phi_{\mathbb{D},ij}(\rho) < \infty, \ i \in \mathbb{X}, j \in \mathbb{D}$ computed for the reduced semi-Markov process ${}_kJ(t)$, for some $k \notin \mathbb{D}$.

The key recurrent systems of linear equations analogous to (38) take, for every $j \in \mathbb{D}$, nonempty set $\mathbb{D} \subset \mathbb{X}$ and $k \notin \mathbb{D}$, the following form,

$$\begin{cases} {}_k\Phi_{\mathbb{D},k0}(\rho) &= {}_k\phi_{k0}(\rho) + \sum_{j \in {}_k\mathbb{X},\, j \notin \mathbb{D}} {}_k\phi_{kj}(\rho)\, {}_k\Phi_{\mathbb{D},j0}(\rho), \\ {}_k\Phi_{\mathbb{D},i0}(\rho) &= {}_k\phi_{i0}(\rho) + \sum_{j \in {}_k\mathbb{X},\, j \notin \mathbb{D}} {}_k\phi_{ij}(\rho)\, {}_k\Phi_{\mathbb{D},j0}(\rho), \\ i \in {}_k\mathbb{X}. & \end{cases} \tag{52}$$

The corresponding changes caused by replacement of the hitting state 0 by state $j \in \mathbb{D}$ and set ${}_k\mathbb{X} \setminus \{0\}$ by set ${}_k\mathbb{X} \setminus \mathbb{D}$ sould be taken into account when writing down systems of linear equations (52) instead of systems of linear equations (38).

5.4. Place-dependent hitting times. Fourth, the above results can be generalized on so-called place-dependent hitting times,

$$Y_{\mathbb{G}} = \sum_{n=1}^{U_{\mathbb{G}}} X_n, \tag{53}$$

where $U_{\mathbb{G}} = \min(n \geq 1 : (J_{n-1}, J_n) \in \mathbb{G})$, for some nonempty set $\mathbb{G} \subset \mathbb{X} \times \mathbb{X}$.

Note that set $\mathbb{G}$ can be represented in the form $\mathbb{G} = \cup_{i \in \mathbb{X}} \{i\} \times \mathbb{G}_i$, where $\mathbb{G}_i = \{j \in \mathbb{X} : (i,j) \in \mathbb{G}\}$. Respectively, the first hitting time $U_{\mathbb{G}}$ can be represented as $U_{\mathbb{G}} = \min(n \geq 1 : J_n \in \mathbb{G}_{J_{n-1}})$. This representation explains using of the term "place-dependent hitting time".

In fact, the above model can be embedded in the previous one, if to consider the new Markov renewal process $(\bar{J}_n, X_n) = ((J_{n-1}, J_n), X_n), n = 0, 1, \ldots$ constructed from the initial Markov renewal process $(J_n, X_n), n = 0, 1, \ldots$ by aggregating sequential states for the initial embedded Markov chain J_n.

The Markov renewal process $(\bar{J}_n, X_n)$ has the phase space $(\mathbb{X} \times \mathbb{X}) \times [0, \infty)$. For simplicity, we can take the initial state $\bar{J}_0 = (J_{-1}, J_0)$, where J_{-1} is a random variable taking values in space $\mathbb{X}$ and independent on the Markov renewal process (J_n, X_n).

Note that the simpler condition **A** can, in fact, be replaced by a simpler condition:

$\mathbf{A}'_{\mathbb{G}}$: $\mathsf{P}_i\{U_{\mathbb{G}} < \infty\} = 1$, $i \in \mathbb{X}$.

The above assumption, that domain $\mathbb{G}$ is hittable, is implied by condition **A**, for any domain $\mathbb{G}$ containing a pair of states (i, j) such that $p_{ij} > 0$.

The results concerned exponential moments of usual accumulated rewards $W_{\mathbb{D}}$ can be expanded to the place-depended accumulated rewards $\mathbb{Y}_{\mathbb{G}}$ for hittable domains, using the above embedding procedure.

5.5. Time-dependent hitting times. Let $(J_n, X_n), n = 0, 1, \ldots$ be an inhomogeneous in time Markov renewal process, i.e., an inhomogeneous in time Markov chain with phase space with the phase space $\mathbb{X} \times [0, \infty)$, an initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J_0 = i, X_0 = 0\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities, defined for $(i, s), (j, t) \in \mathbb{X} \times [0, \infty)$ and $n = 0, 1, 2, \ldots$,

$$Q_{ij}^{(n+1)}(t) = \mathsf{P}\{J_{n+1} = j, X_{n+1} \leq t / J_n = i, X_n = s\}. \tag{54}$$

As in homogeneous in time case, we exclude instant jumps and assume that the following condition holds;

$\mathbf{B}'$: $Q_{ij}^{(n)}(0) = 0$, $i, j \in \mathbb{X}$, $n \geq 1$.

Process (J_n, X_n) can be transformed in a homogeneous in time Markov renewal process by adding to this process an additional counting time component $J'_n = n, n = 0, 1, \ldots$. Indeed, process $(\bar{J}_n, X_n) = ((J'_n, J_n), X_n), n = 0, 1, \ldots$ is a homogeneous in time Markov renewal process. This process has the phase space $(\mathbb{N} \times \mathbb{X}) \times [0, \infty)$, where $\mathbb{N} = \{0, 1, \ldots\}$. It has the initial distribution $\bar{p} = \langle p_i = \mathsf{P}\{J'_0 = 0, J_0 = i, X_0 = 0\} = \mathsf{P}\{J_0 = i\}, i \in \mathbb{X}\rangle$ and transition probabilities,

$$Q_{(n,i),(k,j)}(t) = \begin{cases} Q^{(n+1)}_{ij}(t) & \text{for } t \geq 0, k = n+1, n = 0, 1, \ldots, i, j \in \mathbb{X}, \\ 0 & \text{for } t \geq 0, k \neq n+1, n = 0, 1, \ldots, i, j \in \mathbb{X}. \end{cases} \tag{55}$$

The phase space of the process $(\bar{J}_n, X_n)$ is countable.

Let now define a time-truncated version of process $(\bar{J}_n, X_n)$ as the process $(\bar{J}^{(h)}_n, X^{(h)}_n) = ((J'_{n\wedge h}, J_{n\wedge h}), X_{n\wedge h}), n = 0, 1, \ldots$, for some integer $h \geq 1$.

The process $(\bar{J}^{(h)}_n, X^{(h)}_n), n = 0, 1, \ldots$ is also a homogeneous in time Markov renewal process. It has the finite phase space $(\mathbb{H} \times \mathbb{X}) \times [0, \infty)$, where $\mathbb{H} = \{0, 1, \ldots, h\}$.

Let $\langle \mathbb{D}_1, \ldots, \mathbb{D}_h \rangle$ be some sequence of subsets of space $\mathbb{X}$ such that $\mathbb{D}_h = \mathbb{X}$ and let $U_{\tilde{\mathbb{D}}_h} = \min(n \geq 1 : \bar{J}^{(h)}_n \in \{n\} \times \mathbb{D}_n) = \min(n \geq 1 : J_n \in \mathbb{D}_n)$ is the first hitting time to the domain $\tilde{\mathbb{D}}_h = \cup_{n=1}^h \{n\} \times \mathbb{D}_n$ for the Markov chain $\bar{J}^{(h)}_n$.

Obviously, $\mathsf{P}_i\{U_{\tilde{\mathbb{D}}_h} \leq h\} = 1, i \in \mathbb{X}$, i.e., domain $\tilde{\mathbb{D}}$ is hittable for the Markov chain $\bar{J}^{(h)}_n$.

Thus, all results presented in Sections 2 – 4 can be applied to the time-dependent accumulated rewards of hitting type,

$$Z_{\tilde{\mathbb{D}}_h} = \sum_{n=1}^{U_{\tilde{\mathbb{D}}_h}} X_n. \tag{56}$$

Note only hat condition $\mathbf{C}_\rho$ should be, in this case, replaced by condition:

$\mathbf{C}_{h,\rho}$: $\mathsf{E}\{e^{\rho X_n} I(J_n = j)/J_{n-1} = i\} < \infty, n = 1, \ldots, h, i, j \in \mathbb{X}$.

In conclusion, we would like also to note that it is possible to combine all five listed above generalization aspects in the frame of one semi-Markov model.

5.6. An example. Let us consider a numerical example illustrating the recurrent algorithm for computing power moment of hitting times and accumulated rewards of hitting times for semi-Markov processes, based on sequential reduction of their phase spaces.

Let $J(t)$ be a semi-Markov process with the phase space $\mathbb{X} = \{0, 1, 2, 3\}$, and the 4×4 matrix of transition probabilities, $\|Q_{ij}(t)\|$, which has the following form, for $t \geq 0$,

$$\left\| \begin{array}{llll} \frac{1}{2}I(t \geq \ln \frac{10}{9}) & 0 & 0 & \frac{1}{2}I(t \geq \ln \frac{10}{9}) \\ \frac{1}{2}(1 - e^{-9t}) & \frac{1}{6}(1 - e^{-9t}) & \frac{1}{6}(1 - e^{-9t}) & \frac{1}{6}(1 - e^{-9t}) \\ 0 & \frac{1}{2}(1 - e^{-10t}) & \frac{1}{4}(1 - e^{-10t}) & \frac{1}{4}(1 - e^{-10t}) \\ 0 & \frac{1}{2}I(t \geq \ln \frac{9}{8}) & \frac{1}{4}I(t \geq \ln \frac{9}{8}) & \frac{1}{4}I(t \geq \ln \frac{9}{8}) \end{array} \right\|. \quad (57)$$

Let us compute the exponential moments of hitting times $\Phi_{00}(\rho)$ and $\Phi_{10}(\rho)$, for $\rho = 1$, using the recurrent algorithm described in Sections 3 – 5.

Note that we chosed parameters of semi-Markov transition probabilities and the value of ρ in the way simplifying the corresponding numerical calculations.

The 4×4 matrices of transition probabilities $\|p_{ij}\|$, for the embedded Markov chain J_n and exponential moments $\|\phi_{ij}(1)\|$ of transition times, for the semi-Markov process $J(t)$, have the following forms,

$$\left\| \begin{array}{cccc} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{array} \right\| \text{ and } \left\| \begin{array}{cccc} \frac{5}{9} & 0 & 0 & \frac{5}{9} \\ \frac{9}{16} & \frac{3}{16} & \frac{3}{16} & \frac{3}{16} \\ 0 & \frac{5}{9} & \frac{5}{18} & \frac{5}{18} \\ 0 & \frac{9}{16} & \frac{9}{32} & \frac{9}{32} \end{array} \right\|. \quad (58)$$

Let us first exclude state 3 from the phase space $\mathbb{X} = \{0, 1, 2, 3\}$ of the semi-Markov process $J(t)$. The corresponding reduced semi-Markov process ${}_{\langle 3 \rangle}J(t)$ has the phase space ${}_{\langle 3 \rangle}\mathbb{X} = \{0, 1, 2\}$.

The recurrent formulas (40), for transition probabilities of the embedded Markov chain ${}_{\langle 3 \rangle}J_n$, and (41), for exponential moments ${}_{\langle 3 \rangle}\phi_{ij}(1)$ of sojourn times for the semi-Markov process ${}_{\langle 3 \rangle}J(t)$, have the following forms, respectively, ${}_{\langle 3 \rangle}p_{ij} = p_{ij} + \frac{p_{i3}p_{3j}}{1-p_{33}}$ and ${}_{\langle 3 \rangle}\phi_{ij}(1) = \phi_{ij}(1) + \frac{\phi_{i3}(1)\phi_{3j}(1)}{1-\phi_{33}(1)}$, for $i = 0, 1, 2, 3$, $j = 0, 1, 2$.

The 4×3 matrices of transition probabilities $\|{}_{\langle 3 \rangle}p_{ij}\|$ exponential moments $\|{}_{\langle 3 \rangle}\phi_{ij}(1)\|$, computed according the above recurrent formulas, take the following forms,

$$\left\| \begin{array}{ccc} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & \frac{5}{18} & \frac{2}{9} \\ 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{2}{3} & \frac{1}{3} \end{array} \right\| \text{ and } \left\| \begin{array}{ccc} \frac{5}{9} & \frac{10}{23} & \frac{5}{23} \\ \frac{9}{16} & \frac{123}{368} & \frac{6}{23} \\ 0 & \frac{160}{207} & \frac{15}{46} \\ 0 & \frac{160}{207} & \frac{15}{46} \end{array} \right\|. \quad (59)$$

Let us now exclude state 2 from the phase space ${}_{\langle 3 \rangle}\mathbb{X} = \{0, 1, 2\}$ of the semi-Markov process ${}_{\langle 3 \rangle}J(t)$. The corresponding reduced semi-Markov process ${}_{\langle 3,2 \rangle}J(t)$ has the phase space ${}_{\langle 3,2 \rangle}\mathbb{X} = \{0, 1\}$.

The recurrent formulas (40), for transition probabilities ${}_{\langle 3,2 \rangle}p_{ij}$ of the embedded Markov chain ${}_{\langle 3,2 \rangle}J_n$, and (41), for exponential moments ${}_{\langle 3,2 \rangle}\phi_{ij}(1)$ of sojourn times for the semi-Markov process ${}_{\langle 3,2 \rangle}J(t)$, have the following forms, respectively, ${}_{\langle 3,2 \rangle}p_{ij} = {}_{\langle 3 \rangle}p_{ij} + \frac{{}_{\langle 3 \rangle}p_{i2}\, {}_{\langle 3 \rangle}p_{2j}}{1 - {}_{\langle 3 \rangle}p_{22}}$ and ${}_{\langle 3,2 \rangle}\phi_{ij}(1) = {}_{\langle 3 \rangle}\phi_{ij}(1) + \frac{{}_{\langle 3 \rangle}\phi_{i2}(1)\, {}_{\langle 3 \rangle}\phi_{2j}(1)}{1 - {}_{\langle 3 \rangle}\phi_{22}(1)}$, for $i = 0, 1, 2$, $j = 0, 1$.

The 3×2 matrices of transition probabilities $\|{}_{\langle 3,2 \rangle}p_{ij}\|$ and exponential moments $\|{}_{\langle 3,2 \rangle}\phi_{ij}(\rho)\|$, computed according the above recurrent formulas, take the following forms,

$$\left\| \begin{array}{cc} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{array} \right\| \text{ and } \left\| \begin{array}{cc} \frac{5}{9} & \frac{3490}{4347} \\ \frac{9}{16} & \frac{65037}{102672} \\ 0 & \frac{320}{189} \end{array} \right\|. \tag{60}$$

Finally, let us exclude state 1 from the phase space ${}_{\langle 3,2 \rangle}\mathbb{X} = \{0, 1\}$ of the semi-Markov process ${}_{\langle 3,2 \rangle}J(t)$. The corresponding reduced semi-Markov process ${}_{\langle 3,2,1 \rangle}J(t)$ has the phase space ${}_{\langle 3,2,1 \rangle}\mathbb{X} = \{0\}$.

The recurrent formulas (40) and (41) for transition probabilities of the embedded Markov chain ${}_{\langle 3,2,1 \rangle}J_n$, expectations of sojourn times and second moments of sojourn times for the semi-Markov process ${}_{\langle 3,2,1 \rangle}J(t)$ have the following forms, respectively, ${}_{\langle 3,2,1 \rangle}p_{i0} = {}_{\langle 3,2 \rangle}p_{i0} + \frac{{}_{\langle 3,2 \rangle}p_{i1}\, {}_{\langle 3,2 \rangle}p_{10}}{1 - {}_{\langle 3,2 \rangle}p_{11}} = 1$ and ${}_{\langle 3,2,1 \rangle}\phi_{i0}(1) = {}_{\langle 3,2 \rangle}\phi_{i0}(1) + \frac{{}_{\langle 3,2 \rangle}\phi_{i1}(1)\, {}_{\langle 3,2 \rangle}\phi_{10}(1)}{1 - {}_{\langle 3,2 \rangle}\phi_{11}(1)}$, for $i = 0, 1$.

The 2×1 matrix of exponential moments $\|\Phi_{i0}(1)\| = \|{}_{\langle 3,2,1 \rangle}\phi_{i0}(1)\|$ computed according the above recurrent formulas, take the following forms,

$$\|\Phi_{i0}(1)\| = \left\| \begin{array}{c} \frac{282557}{158067} \\ \frac{57753}{37635} \end{array} \right\|. \tag{61}$$

In conclusion, we would like to note that recurrent algorithms presented in the paper are subjects of effective program realization. These programs let one compute power moments for hitting times and accumulated rewards of hitting times for semi-Markov processes with very large numbers of states. We are going to present such programs and results of large scale experimental studies in future publications.

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