Mathematical Statistics
Stockholm University

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## Research Report 2016:20

ISSN 1650-0377

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Research Report **2016:20**,
http://www.math.su.se

# Time-inconsistent stochastic control: solving the extended HJB system is a necessary condition for regular equilibria

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November 2016

**Abstract**

*Time-inconsistent stochastic control* stochastic control is a game-theoretic generalization of standard stochastic control. An important result of standard stochastic control is the characterization of the optimal value function as the solution to the Hamilton-Jacobi-Bellman equation. Time-inconsistent stochastic control offers a similar possibility: Björk, Khapko and Murgoci (2016) [2] introduce a system of PDEs, the extended HJB system, and prove a verification theorem saying that *if* the extended HJB system has a solution then it is an equilibrium of a corresponding time-inconsistent stochastic control problem. In the present paper we show that a *regular* equilibrium is necessarily a solution to the extended HJB system.

*Keywords:* Dynamic inconsistency, Extended HJB system, Equilibrium, Hamilton-Jacobi-Bellman, Time inconsistent preferences, Time-inconsistent stochastic control.

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# 1 Introduction

Time-inconsistent stochastic control is a game theoretic generalization of standard stochastic control, based on the notion of Nash equilibrium. Time-inconsistent control problems are common in applications. They have, for example, been a part of the finance and economics literature since at least the 1950s. The general mathematical theory of time-inconsistent stochastic control is, however, still highly incomplete, as Björk and Murgoci (2014) [4] write "*What has been lacking in the literature...is a...general theory of time-inconsistent stochastic control*".

We will now describe a simple standard (time-consistent) stochastic control problem in a somewhat unusual fashion and, as a contrast, introduce the time-inconsistent stochastic control problem.

A simple but standard stochastic control problem amounts to choosing a (deterministic) function $\mathbf{u} : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ (known as a *Markovian feedback control law*) that maximizes

$$\tilde{J}(t,x,\mathbf{u}) = E_{t,x}[\tilde{F}(X_T^{\mathbf{u}})] \tag{1}$$

for each $(t,x)$ in $[0,T] \times \mathbb{R}^n$, where $\tilde{F}$ is some nice real-valued function, often called *terminal payoff (function)*, and $X^{\mathbf{u}}$ is a controlled Itô diffusion:

$$dX_s^{\mathbf{u}} = \mu(s, X_s^{\mathbf{u}}, \mathbf{u}(s, X_s^{\mathbf{u}}))ds + \sigma(s, X_s^{\mathbf{u}})dW_s$$

where $W$ is a Wiener process and $\sigma$ and $\mu$ are nice deterministic functions.

At first glance this optimization problem does not look well-posed since the optimal control (function) would apparently depend on the particular starting point $(t,x)$. Fortunately, the problem turns out to be well-posed due to the *dynamic programming principle* (DPP), which can be heuristically formulated as follows: *if an optimal control for (1) exists for every fixed* $(t,x)$ *then these controls do, in fact, coincide.* Thus, the problem of choosing a control (function) $\mathbf{u}$ such that (1) is maximal *for each* $(t,x)$ turns out to be well-posed.

The standard stochastic control theory cannot, however, satisfactorily handle the problem of choosing a control $\mathbf{u}$ such that it maximizes

$$J(t,x,\mathbf{u}) = E_{t,x}[F(x, X_T^{\mathbf{u}})] \tag{2}$$

for each $(t,x)$ in $[0,T] \times \mathbb{R}$, where $F$ is a nice deterministic function which we call a *time-inconsistent terminal payoff (function).* To see this, note that if $t$ is current time and $X_t = x$, and if a Markovian feedback control law $\mathbf{u}$ is optimal at $(t,x)$ then it will at any future point $(s, X_s^{\mathbf{u}})$ be optimal for the problem of maximizing $E_{s,X_s^{\mathbf{u}}}[F(x, X_T^{\mathbf{u}})]$, according to the DPP. However, it will not generally be optimal for the problem of maximizing $E_{s,X_s^{\mathbf{u}}}[F(X_s^{\mathbf{u}}, X_T^{\mathbf{u}})]$ which is our task, cf. (2). This optimization problem is therefore equivalent to us having to deal with different terminal payoff functions at different points in time: at $(t, x)$ we consider maximizing the conditional expected value (see (2)) based on the function $F(x, \cdot)$, but later, at for example $(s, X_s^{\mathbf{u}})$ we want to maximize the conditional expected value based on the currently unknown (random) function $F(X_s^{\mathbf{u}}, \cdot)$. In other words: the set of preferences we have today are not the same as the set of preferences we have tomorrow. The optimization problem is therefore *time-inconsistent*, and it is in fact unclear what optimality for (2) even means (the problem is not well-posed).

In order to deal with the issue of time-inconsistency the literature has (primarily) resorted to game theory and the notion of *Nash equilibrium* (not only for the setting above, but also more generally). Heuristically, the game-theoretic nature of the problem is due to the terminal payoff function changing over time, and we view the optimization problem as a game between agents (one agent for each time $t$) who optimize their own value function $E_{t,X_t^{\mathbf{u}}}[F(X_t^{\mathbf{u}}, X_T^{\mathbf{u}})]$, by choosing the control's (i.e. $\mathbf{u}$:s) value only at the time $t$. This idea is formalized by the introduction of an *equilibrium control* and a corresponding *equilibrium value function*, see Definition 3.2.

One of the main results of standard (time-consistent) stochastic control is the characterization of the optimal value function as the unique solution to a certain non-linear partial differential equation known as the Hamilton-Jacobi-Bellman equation (HJB). This type of characterization was originally often performed under ad hoc assumptions regarding mainly sufficient differentiability (see Remark 8.2 for further comments). Another approach, which demands less regularity, is to consider *viscosity solutions*, see Remark 8.4. The characterization is often presented as two companion theorems (see e.g. [16, 18, 19, 23]) which in simplified versions can be phrased as follows: 1. *A necessary condition for two functions to be an optimal control and the optimal value function is that they solve the HJB*, and 2. *Solving the HJB is a sufficient condition for two functions to be an optimal control and the optimal value function* (this type of result is known as a *verification theorem*).

Naturally, we hope that time-inconsistent stochastic control problems offer a similar possibility. Indeed, the *extended HJB system* was around 2010 proposed in the influential [2][1], where a time-inconsistent stochastic control problem similar to the one in the present paper is studied. The main result of [2] is a verification theorem which in a simplified version can be phrased as

[1]Different versions of this paper have been available as preprints since at least 2010. The most cited version is the older [3].

follows: *Solving the extended HJB system is a sufficient condition for being an equilibrium.* The type of solutions considered are classical solutions (i.e. sufficiently differentiable, in the usual sense).

The main contribution of the present paper can be summarized as follows.

- We show that solving the extended HJB system is a necessary condition for being a *regular* equilibrium, see Theorem 5.5 and the more general Theorem 6.3.
- As an example, we study a time-inconsistent version of the *regulator problem*, see Section 7.

**Remark 1.1.** *We consider only regular equilibria. This means that we implicitly only consider models which satisfy certain conditions, mainly regarding the existence of an equilibrium control and differentiability (see Assumption 5.2 and Assumption 5.3). These assumptions, therefore, need to be verified for each particular model that is studied. We remark that the example studied in Section 7 has a regular equilibrium.*

## 1.1 Previous literature

The game-theoretic approach to time-inconsistency was first used by Strotz (1955) [22] when studying utility maximization problems. Other influential papers following a similar line in economics and finance are [14, 17, 21, 25]. These and other papers in mainly economics and finance have sparked a recent interest in time-inconsistent stochastic control from a more mathematical perspective. Early financial mathematics papers in time-inconsistent stochastic control include Ekeland and Lazrak (2006) [10] and Ekeland and Pirvu (2008) [11], who study a classic time-inconsistent finance problem (optimal consumption and investment under hyperbolic discounting). The ideas developed in these papers served as an inspiration to the general framework for time-inconsistent stochastic control proposed in [2, 4], which also include several interesting applications. A particular type of time-inconsistent stochastic control problem is mean-variance optimization (studied in e.g. optimal investment theory). Different versions of this problem have recently been studied by [5, 7, 20]. A numerical technique for mean-variance optimization problems is studied in [8].

### 1.1.1 Alternative approaches to time-inconsistent stochastic control

Yong (2012) [28] (see also [27]) and Wang and Wu (2015) [26] study time-inconsistent stochastic control, but they use a slightly different equilibrium construction than that of the present paper. They start with $N$ agents each controlling the state process on a small time interval and then solve the equilibrium problem by using backward recursion, which is similar to how the problem is solved in discrete time in [4]. $N$ is then sent to infinity. Moreover, in [26] the theory of FBSDEs is used to characterize the equilibrium (rather than with a PDE system). Djehiche and Huang (2015) [9] study a problem similar to that of the present paper, for which they characterize an equilibrium control (if it exists) via a stochastic maximum principle for general nonlinear diffusion models (i.e. as the solution to a particular SDE). Their main focus is, however, an equilibrium approach to mean field games.

# 2 The model

Let there be a stochastic basis $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ with a fixed time horizon $T < \infty$, where $\underline{\mathcal{F}}$ is the augmented filtration generated by a $d$-dimensional Wiener process $W$. We consider $n$-dimensional controlled diffusion processes

$$dX_s^{\mathbf{u}} = \mu(s, X_s^{\mathbf{u}}, \mathbf{u}(s, X_s^{\mathbf{u}}))ds + \sigma(s, X_s^{\mathbf{u}})dW_s, \quad t \leq s \leq T, \quad X_t^{\mathbf{u}} = x \tag{3}$$

where,

- controls are measurable functions $\mathbf{u} : [0,T] \times \mathbb{R}^n \to U$ (known as Markovian feedback control laws), where $U \in \mathbb{R}^k$ is compact. We denote the set of such functions by $\mathbf{U}$,
- $\mu : [0,T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$ and $\sigma : [0,T] \times \mathbb{R}^n \to M(n,d)$ are bounded and continuous functions. Moreover, $\sigma(t,x)$ is Lipschitz in $x \in \mathbb{R}^n$ and $\sigma\sigma^T$ is uniformly elliptic.[2]

**Definition 2.1** (Admissible control). *A control* $\mathbf{u}$ *is* admissible *if for any starting point* $(t,x) \in [0,T) \times \mathbb{R}^n$ *there exists a unique strong solution* $X^{\mathbf{u}}$ *to (3).* $X^{\mathbf{u}}$ *has continuous sample paths (by the definition of a strong solution).*

From (2) in Section 1 we recall that the basis of our optimization problem is a function $F$ , which we call a time-inconsistent terminal payoff function.

**Definition 2.2** (Time-inconsistent terminal payoff function).
$F(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *is continuous and satisfies*

$$|F(x,y)| \leq C_0(1 + |x|^2) \tag{4}$$

*for some constant* $C_0$ *(which may depend on* $y$*).*

[2] $M(n,d)$ denotes the set of real-valued matrices with dimension $n \times d$. $\sigma\sigma^T$ uniformly elliptic means that there exists a constant $v > 0$ such that $\lambda^T\sigma(t,x)\sigma^T(t,x)\lambda \geq v|\lambda|^2$, for all for $\lambda, x \in \mathbb{R}^n$ and $t \in [0,T]$, where $|v|^2 = \sum_i v_i^2$ and $\sigma^T$ is the transpose of $\sigma$.

We will use of the following function indexed by a control.

**Definition 2.3** (Auxiliary function)**.** *For a fixed* $\mathbf{u} \in \mathbf{U}$ *we define the function* $f_{\mathbf{u}} : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ *by*

$$f_{\mathbf{u}}(t,x,y) = E_{t,x}\left[F(X_T^{\mathbf{u}}, y)\right],$$

*where* $E_{t,x}[\cdot]$ *denotes the expected value given the starting point* $(t,x)$ *in (3).*

**Lemma 2.4.**

1. *Each control is an admissible control.*
2. *For any* $\mathbf{u} \in \mathbf{U}$ *and any* $(t,x) \in [0,T] \times \mathbb{R}^n$, $E_{t,x}\left[(sup_{t \le s \le T}|X_s^{\mathbf{u}}|)^2\right] < C_1(1+|x|^2)$ *for some constant* $C_1$ *which is independent of* $t$ *and* $\mathbf{u}$.
3. *For any* $\mathbf{u} \in \mathbf{U}$ *and any* $(t,x,y) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$, $|f_{\mathbf{u}}(t,x,y)| \le E_{t,x}[|F(X_T^{\mathbf{u}}, y)|] \le C_2(1+|x|^2)$ *for some constant* $C_2$ *which is independent of* $t$ *and* $\mathbf{u}$ *(but which may depend on* $y$*).*

Note that item 1. above implies that the set of controls $\mathbf{U}$ is actually also the set of admissible controls.

**Proof.** For each function $\mathbf{u} \in \mathbf{U}$ and starting value $(t,x)$ the SDE (3) has a unique strong solution $X^{\mathbf{u}}$, by a result found in e.g. [24] (and clarified in [15]). Each control is therefore admissible.

The integrability condition for $sup_{t \le s \le T}|X_s^{\mathbf{u}}|$ is a consequence of $\mu$ and $\sigma$ being bounded (see e.g [16, ch. 5.3].). We remark that $C_1$ can be taken to depend only on $n$, $T$ and the least upper bounds of $|\mu|$ and $|\sigma|$ (where $|\cdot|^2 = \Sigma_{i,j}(\cdot)^2$).

Using (4) and item 2. it is easy to see that

$$\begin{aligned}&|f_{\mathbf{u}}(t,x,y)| \le E_{t,x}\left[|F(X_T^{\mathbf{u}}, y)|\right] \le E_{t,x}\left[C_0(1+|X_T^{\mathbf{u}}|^2)\right] \\ &\le C_0 + C_0 E_{t,x}\left[|X_T^{\mathbf{u}}|^2\right] \le C_0 + C_0(C_1 + |x|^2) \le C_2(1+|x|^2)\end{aligned}$$

for some $C_2$ depending only on $n, T$, the least upper bounds of $|\mu|$ and $|\sigma|$ and $C_0$ (and therefore on $y$). ∎

We need some more notation.

- $\mathcal{A}^{\mathbf{u}}$ denotes the following indexed (by a control $\mathbf{u}$) differential operator

$$\mathcal{A}^{\mathbf{u}} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} \mu_i(t,x,\mathbf{u}(t,x)) \frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma\sigma_{ij}^T(t,x) \frac{\partial^2}{\partial x_i x_j}$$

  which operates on sufficiently differentiable functions $[0,T] \times \mathbb{R}^n \to \mathbb{R}$.

- $\mathcal{A}^{\mathbf{u}}$ operates only on variables in parentheses. For any function $g(t,x,y) : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we allow for the possibility of placing the third variable as a superscript $g^y(t,x) = g(t,x,y)$. This implies, for example, that $\mathcal{A}^{\mathbf{u}} g^y(t,x)$ involves only derivatives with respect to the first and second variables $t$ and $x$. Moreover, $\mathcal{A}^{\mathbf{u}} g(t,x,x)$ is equal to $\mathcal{A}^{\mathbf{u}} h(t,x)$, with $h(t,x) := g(t,x,x)$.

- $\mathcal{A}^{\mathbf{u}} g^y(t,x)$ and $\mathcal{A}^{\mathbf{u}(t,x)} g^y(t,x)$ denote the same object.

**Remark 2.5.** *The expected values found throughout this paper are finite (to see this use arguments similar to those in the proof of Lemma 2.4).*

# 3 The time-inconsistent stochastic control equilibrium

Consider the time-inconsistent terminal payoff function $F$ in (4) and define the time-inconsistent value function as follows.

**Definition 3.1** (Value function)**.**

$$J(t,x,\mathbf{u}) = f_{\mathbf{u}}(t,x,x) = E_{t,x}\left[F(X_T^{\mathbf{u}}, x)\right]$$

Now, we wish to choose the control $\mathbf{u}$ so that $J(t,x,\mathbf{u})$ is as large as possible for each $(t,x)$. But, this is not (yet) a well-posed problem, as we saw in the introduction. The first objective of our theory of time-inconsistent stochastic control is to make this a well-posed problem.

To this end, we introduce a game-theoretic notion of equilibrium very similar to that in [2], which was inspired by [10, 11]. The definition is a continuous time version of a *subgame perfect Nash equilibrium.*

**Definition 3.2** (Equilibrium)**.**

- $\hat{\mathbf{u}}$ *is an* **equilibrium control** *if it is an admissible control such that*

$$\liminf_{h \searrow 0} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{t+h})}{h} \geq 0, \quad \forall (t,x) \in [0,T) \times \mathbb{R}^n, \qquad (5)$$

*for all* $\mathbf{u}_{t+h} = \begin{cases} \mathbf{u}, & \text{on } [t,t+h] \times B[x] \\ \hat{\mathbf{u}}, & \text{on } \{[t,t+h] \times B[x]\}^c, \end{cases}$ *with* $h > 0, t+h \leq T$,

*where* $\mathbf{u} \in \mathbf{U}$ *and* $B[x]$ *is some arbitrary* $n$*-dimensional ball centered at* $x$*.*

- *For a given equilibrium control* $\hat{\mathbf{u}}$ *the corresponding* **equilibrium value function** *is defined by* $V(t,x) = f_{\hat{\mathbf{u}}}(t,x,x)$.

- *An* **equilibrium** *is a triplet* $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,y))$.

**Remark 3.3.** $\mathbf{u}_{t+h}$ *is bounded and measurable (cf. pasting of measurable functions) and is therefore in* $\mathbf{U}$ *(this comment also applies to all controls we introduce later).* $\mathbf{u}_{t+h}$ *clearly depends on* $x$ *and* $|B[x]|$*, and also on* $t$ *and* $h$ *rather than* $t+h$*. The fact that our notation does not indicate this should not cause any confusion.*

Definition 3.1 and Definition 3.2 imply the following identities

$$V(t,x) = f_{\hat{\mathbf{u}}}(t,x,x) = J(t,x,\hat{\mathbf{u}}) = E_{t,x}\left[F(X_T^{\hat{\mathbf{u}}}, x)\right]. \tag{6}$$

**Remark 3.4** (Interpretation of $\mathbf{u}_{t+h}$)**.** *If we at* $(t,x)$ *consider* $\mathbf{u}_{t+h}$ *then we consider an alternative control* $\mathbf{u}$ *(compared to* $\hat{\mathbf{u}}$*) which is to be used until either 1. the short time period* $h>0$ *has passed, or 2. the controlled process* $X^{\mathbf{u}}$ *has left the ball* $B[x]$.

**Remark 3.5** (Interpretation of equilibrium)**.** *We can think of the agents in this game as future incarnations of the person that is doing the optimization. The interpretation of an equilibrium control is that it is a control which every agent wants to use if every agent after her (i.e. every future incarnation) uses that control. Heuristically, we can for small* $h$ *think of* $J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{t+h})$ *as the expected payoff an agent at time* $t$ *(the current "self") obtains when choosing the equilibrium control* $\hat{\mathbf{u}}$ *minus the expected payoff she would obtain if she chose the alternative control* $\mathbf{u}$*, provided that* $\hat{\mathbf{u}}$ *is used when she is no longer able to choose the control. Note that* $J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{t+h})$ *would approach zero when sending* $h$ *to zero, and we therefore normalize this difference in (5).*

# 4 The extended HJB system

Before we introduce the extended HJB system we remark that $\mathcal{A}^u f_{\hat{\mathbf{u}}}^y(t,x)$ involves, by definition, derivatives only with respect to the first and second variables $t$ and $x$ in $f_{\hat{\mathbf{u}}}(t,x,y)$. Consequently, $\mathcal{A}^u f_{\hat{\mathbf{u}}}^x(t,x)$ involves derivatives only with respect to $t$ and *the first* $x$ in $f_{\hat{\mathbf{u}}}(t,x,x)$.

**Definition 4.1** (The extended HJB system)**.** *For $(t,x,y) \in [0,T) \times \mathbb{R}^n \times \mathbb{R}^n$,*

$$\begin{aligned} \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(t,x) &= 0 \\ f^y_{\hat{\mathbf{u}}}(T,x) &= F(x,y) \\ \sup_{u\in U} \{\mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x)\} &= 0 \\ V(T,x) &= F(x,x) \end{aligned} \tag{7}$$

*where $\hat{\mathbf{u}}$ satisfies*

$$\hat{\mathbf{u}}(t,x) \in \arg sup_{u\in U} \{\mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x)\} \tag{8}$$

*and where any solution triplet $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,y))$ must satisfy*

- $\hat{\mathbf{u}} \in \mathbf{U}$ *(this requirement corresponds only to measurability, cf. the definition of* $\mathbf{U}$*),*
- $f^y_{\hat{\mathbf{u}}}(t,x) \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *for any fixed* $y \in \mathbb{R}^n$*,*
- $V(t,x)$ *and* $f_{\hat{\mathbf{u}}}(t,x,x)$ *have existing derivatives to the extent that* $\mathcal{A}^u V(t,x)$ *and* $\mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x)$ *exist, for any* $(t,x) \in [0,T) \times \mathbb{R}^n$*. In other words,* $\frac{\partial V(t,x)}{\partial t}, \frac{\partial V(t,x)}{\partial x_i}, \frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}, \frac{\partial h(t,x)}{\partial t}, \frac{\partial h(t,x)}{\partial x_i}, \frac{\partial^2 h(t,x)}{\partial x_i \partial x_j}$ *exist for each* $(t,x) \in [0,T) \times \mathbb{R}^n$ *and* $i,j = 1,...,n$ *with* $h(t,x) := f_{\hat{\mathbf{u}}}(t,x,x)$*.*

**Remark 4.2** (The maximization in (8))**.** *The maximization in (8) should be construed as follows. For a fixed point $(t,x)$ and **any** two fixed functions $V(t,x)$ and $f_{\hat{\mathbf{u}}}(t,x,y)$ (which are sufficiently differentiable for the derivatives in (9) to exist) the maximization procedure regards*

$$\mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x) \tag{9}$$

*with respect to $u \in U$.*

**Remark 4.3.** *The two first rows of the extended HJB system (7) correspond to a parametrized (by $y$ and $\hat{\mathbf{u}}$) Kolmogorov backward equation (we call these two rows the* **Kolmogorov part***). To be precise, the Kolmogorov part corresponds to a family of Kolmogorov backward equations, one for each $y \in \mathbb{R}^n$. The third and fourth row (we call these two rows the* **HJB part***) consist of a standard HJB equation, but with the non-standard feature that it includes derivatives of the solution to the Kolmogorov part.*

**Remark 4.4** (Interpretation of the extended HJB system)**.** *It may shed some light on the extended HJB system to consider the following schedule. Start by guessing a solution, i.e. any three fixed functions $\hat{\mathbf{u}}, V(t,x)$ and $f_{\hat{\mathbf{u}}}(t,x,y)$, and then*

1. *check that* $\hat{\mathbf{u}}, V(t,x)$ *and* $f_{\hat{\mathbf{u}}}(t,x,y)$ *satisfy the regularity conditions presented after (8),*

2. *check that* $V(t,x)$ *and* $f_{\hat{\mathbf{u}}}(t,x,y)$ *satisfy the two boundary conditions in (7),*

3. *check that* $f_{\hat{\mathbf{u}}}(t,x,y)$ *satisfies the Kolmogorov part,*

4. *for each* $(t,x)$, *check that at least one maximizer of (9) exists, produces the maximal value* $0$, *and is equal to the value* $\hat{\mathbf{u}}(t,x)$.

*If the guessed three functions* $\hat{\mathbf{u}}, V(t,x)$ *and* $f_{\hat{\mathbf{u}}}(t,x,y)$ *pass 1,2,3 and 4, then they are a solution to the extended HJB system.*

**Remark 4.5.** *The extended HJB system in Definition 4.1 does not include a component which is included in the extended HJB system in [2]. The reason for this is that they include in their time-inconsistent optimization problem a second payoff function, which is allowed to depend non-linearly on* $E_{t,x}[X_T^{\mathbf{u}}]$.

# 5 A regular equilibrium necessarily solves the extended HJB system

Let us first define what we mean by a regular equilibrium.

**Definition 5.1** (Regular equilibrium)**.** *An equilibrium* $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,y))$ *(see Definition 3.2) is said to be regular if Assumption 5.2 and Assumption 5.3 below are satisfied.*

**Assumption 5.2** (Right-continuous equilibrium control)**.** *There exists a right-continuous function* $\hat{\mathbf{u}} : [0,T] \times \mathbb{R}^n \to U$ *such that (5) is satisfied.*

**Assumption 5.3** (Differentiability)**.** *The auxiliary function indexed with* $\hat{\mathbf{u}}$ *and the equilibrium value function satisfy the following conditions.*

1. $f_{\hat{\mathbf{u}}}^y(t,x) \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *for each fixed* $y \in \mathbb{R}^n$.

2. $\mathcal{A}^u$ *can, for any* $(t,x,u) \in [0,T) \times \mathbb{R}^n \times U$, *operate on* $V(t,x)$, *i.e.* $\frac{\partial V(t,x)}{\partial t}, \frac{\partial V(t,x)}{\partial x_i}, \frac{\partial^2 V(t,x)}{\partial x_i \partial x_j}$ *exist for each* $(t,x) \in [0,T) \times \mathbb{R}^n$ *and* $i,j = 1,...,n$.

Note that (6) implies that item 2. above is equivalent to assuming that $\mathcal{A}^u$ can, for any $(t,x,u)$, operate on $f_{\hat{\mathbf{u}}}(t,x,x)$.

**Remark 5.4.** *We consider only regular equilibria. This means that we implicitly only consider models (i.e. $\sigma, \mu, U$ and $F$) which are such Assumption 5.2 and Assumption 5.3 are satisfied. Thus, these assumptions need to be verified for each particular model that is studied. See Section 7 for a concrete example with a regular equilibrium.*

We are now ready to present our main result.

**Theorem 5.5** (Main result)**.** *A regular equilibrium $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,y))$ necessarily solves the extended HJB system (Definition 4.1). In particular, if a regular equilibrium exists then the extended HJB system has a solution.*

The following result follows directly from Theorem 5.5 and (6).

**Corollary 5.6.** *An equilibrium control $\hat{\mathbf{u}}$ and the corresponding auxiliary function $f_{\hat{\mathbf{u}}}(t,x,y)$ necessarily solve the following simplified version of the extended HJB system,*

$$\begin{aligned} \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(t,x) = 0, \quad & f^y_{\hat{\mathbf{u}}}(T,x) = F(x,y) \\ \sup_{u\in U} \{\mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x)\} = 0, \quad & \hat{\mathbf{u}}(t,x) \in \arg sup_{u\in U} \{\mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x)\}, \end{aligned}$$

*under Assumption 5.2 and Assumption 5.3.*

**Remark 5.7.** *It can be argued that it would be more natural for the result in Corollary 5.6 to be our main result. We have, however, chosen to formulate our main result as in Theorem 5.5 in order for it to be in accordance with the existing literature.*

**Proof of Theorem 5.5.** The last assertion of the theorem follows directly from the first. The regularity conditions of the extended HJB system (presented directly after (8)) are satisfied as a direct consequence of the equilibrium being regular. The proof comprises 4 parts.

*Part 1.* From Definition 2.3 and Definition 3.2 it directly follows that $f_{\hat{\mathbf{u}}}(T,x,y) = F(x,y)$ and $V(T,x) = F(x,x)$. The boundary conditions of the extended HJB system are therefore satisfied.

*Part 2* (this part corresponds to a version of the Feynman-Kac formula). Here we will show that our auxiliary function

$$f^y_{\hat{\mathbf{u}}}(t,x) = E_{t,x}[F(X^{\hat{\mathbf{u}}}_T, y)] \tag{10}$$

solves the Kolmogorov part of the extended HJB system. Let $(t,x,y)$ be an arbitrary point in $[0,T) \times \mathbb{R}^n \times \mathbb{R}^n$. Consider the sequence of stopping times given by

$$\tau_n = \inf\left\{s > t : (s, X^{\hat{\mathbf{u}}}_s) \notin [t, t+h_n) \times B[x]\right\} \wedge T \tag{11}$$

where $B[x]$ is an arbitrary ball centered at $x$ and $\{h_n\}$ is an arbitrary positive and decreasing sequence satisfying $\lim_{n\to\infty} h_n = 0$. To see that the hitting time (11) is a stopping time, see e.g. [1] or [16, ch. 1.2,1.7,5.2]. It follows directly from (11) that $X_s^{\hat{\mathbf{u}}}$ is bounded on the stochastic interval $[t, \tau_n]$. Note also that for almost all outcomes $\omega \in \Omega$ there exists an $N(\omega)$ such that $\tau_n = t + h_n$ for $n \geq N(\omega)$, so that $\tau_n \to t$ in the same way as $t + h_n \to t$ as $n \to \infty$ a.s.

Now, view $f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})_{s\in[t,T]}$ as a stochastic process, for which Itô's formula (the necessary differentiability is provided by Assumption 5.3) implies that

$$\begin{aligned} f_{\hat{\mathbf{u}}}^y(t,x) \quad &= E_{t,x}\left[f_{\hat{\mathbf{u}}}^y(\tau_n, X_{\tau_n}^{\hat{\mathbf{u}}}) - \int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})ds\right] \\ &-E_{t,x}\left[\int_t^{\tau_n} \sum_{i=1}^n \sigma_i(s, X_s^{\hat{\mathbf{u}}}) \frac{\partial f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})}{\partial x_i} dW_s\right] \\ &= E_{t,x}\left[f_{\hat{\mathbf{u}}}^y(\tau_n, X_{\tau_n}^{\hat{\mathbf{u}}}) - \int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})ds\right] \end{aligned} \tag{12}$$

where $\sigma_i$ is the $i$:th row of $\sigma$ and where the expected value of the Itô integral is zero since its integrand is bounded (which follows from the facts that $X_s^{\hat{\mathbf{u}}}$ is, for each $n$, bounded on the stochastic interval $[t, \tau_n]$, $\sigma$ is bounded, and the derivatives $\frac{\partial f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})}{\partial x_i}$ satisfy Assumption 5.3).

Now, use (10) and then the tower property (Lemma 2.4 ensures that $F(y, X_T^{\hat{\mathbf{u}}})$ is integrable) to see that

$$E_{t,x}\left[f_{\hat{\mathbf{u}}}^y(\tau_n, X_{\tau_n}^{\hat{\mathbf{u}}})\right] = E_{t,x}\left[E_{\tau_n, X_{\tau_n}^{\hat{\mathbf{u}}}}[F(X_T^{\hat{\mathbf{u}}}, y)]\right] = E_{t,x}\left[F(X_T^{\hat{\mathbf{u}}}, y)\right] = f_{\hat{\mathbf{u}}}^y(t,x),$$

which with (12) implies, for each $n$, that

$$E_{t,x}\left[\frac{\int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})ds}{h_n}\right] = 0 \tag{13}$$

Consider now the (obviously related) sequence of random variables given by

$$\frac{\int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})ds}{h_n}. \tag{14}$$

Note that the integrand in (14) is bounded and has right-continuous trajectories on $[t, \tau_n]$, for each $n$.[3] It follows that we may, for each fixed outcome

[3]To see this use that the integrand of (14) is

$$\mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}}) = \frac{\partial f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})}{\partial s} + \sum_{i=1}^n \mu_i(s, X_s^{\hat{\mathbf{u}}}, \hat{\mathbf{u}}(s, X_s^{\hat{\mathbf{u}}})) \frac{\partial f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n \sigma\sigma_{ij}^T(s, X_s^{\hat{\mathbf{u}}}) \frac{\partial^2 f_{\hat{\mathbf{u}}}^y(s, X_s^{\hat{\mathbf{u}}})}{\partial x_i x_j},$$

where $\sigma$ is bounded and continuous, $\mu(s, x, \hat{\mathbf{u}}(s,x))$ is bounded and right-continuous (Section 2 and Assumption 5.2), and the derivatives satisfy Assumption 5.3. Moreover, recall that $X^{\hat{\mathbf{u}}}$ has continuous trajectories and that $X_s^{\hat{\mathbf{u}}}$ is bounded on $[t, \tau_n]$.

$\omega \in \Omega$, view the numerator of (14) as a deterministic Lebesgue integral of a one-dimensional right-continuous bounded deterministic function (in $s$).

Recall that $\tau_n \to t$ in the same way as $t + h_n \to t$ as $n \to \infty$ a.s. It follows from Lebesgue's differentiation theorem that sending $n \to \infty$ in (14) gives us

$$\lim_{n\to\infty} \frac{\int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(s, X^{\hat{\mathbf{u}}}_s)ds}{h_n} = \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(t, x) \quad a.s. \tag{15}$$

The random variables in (14) are bounded by a constant (uniformly in $n$[4]) and they correspond to an a.s. convergent sequence (cf. (15)) and we may therefore use dominated (or bounded) convergence when sending $n \to \infty$ in (13), which with (15) gives us

$$\begin{aligned} 0 &= \lim_{n\to\infty} E_{t,x}\left[\frac{\int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(s, X^{\hat{\mathbf{u}}}_s)ds}{h_n}\right] = E_{t,x}\left[\lim_{n\to\infty} \frac{\int_t^{\tau_n} \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(s, X^{\hat{\mathbf{u}}}_s)ds}{h_n}\right] \\ &= E_{t,x}\left[\mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(t, x)\right] = \mathcal{A}^{\hat{\mathbf{u}}} f^y_{\hat{\mathbf{u}}}(t, x) \end{aligned}$$

which, since $(t, x, y)$ was arbitrary, means that $f_{\hat{\mathbf{u}}}(t, x, y)$ solves the Kolmogorov part of the extended HJB system.

*Part 3.* Using (6) we obtain $\mathcal{A}^{\hat{\mathbf{u}}} V(t, x) = \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}(t, x, x)$, which with the conclusion of Part 2 gives us

$$\mathcal{A}^{\hat{\mathbf{u}}} V(t, x) - \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}(t, x, x) + \mathcal{A}^{\hat{\mathbf{u}}} f^x_{\hat{\mathbf{u}}}(t, x) = 0, \quad (t, x) \in [0, T) \times \mathbb{R}^n. \tag{16}$$

*Part 4*: The conclusions of Part 1, Part 2 and Part 3 together with Assumption 5.3 imply that we may view our regular equilibrium $(\hat{\mathbf{u}}, V(t, x), f_{\hat{\mathbf{u}}}(t, x, y)$ as a *candidate solution* of the extended HJB system in the sense that the only thing we now have left to do is to show that our equilibrium control $\hat{\mathbf{u}}$ is maximal in (16), which we will now do.

Consider an arbitrary point $(t, x, u) \in [0, T) \times \mathbb{R}^n \times U$ and the sequence of stopping times given by

$$\tilde{\tau}_n = \inf\{s > t : (s, X^u_s) \notin [t, t + h_n) \times B[x]\} \wedge T. \tag{17}$$

[4]The boundedness for a specific $n$ is obvious (since $X^{\hat{\mathbf{u}}}_s$ is bounded by a constant on $[t, \tau_n]$, cf. (11), the derivatives of $f^y_{\hat{\mathbf{u}}}(t, x)$ satisfy Assumption 5.3, and $\mu$ and $\sigma$ are bounded (Section 2)). The uniform boundedness can explained as follows: for (almost all) fixed $\omega$ (and our fixed $(t, x, y)$ and $\hat{\mathbf{u}}$), the integrals in (14) correspond to a convergent sequence (in $\mathbb{R}$, with index $n$) where each element is bounded by a constant $M_n \in \mathbb{R}$ which does not depend on $\omega$ (cf. (11)), where the sequence $\{M_n\}$ can be taken to converge (cf. the right side of (15)). It is easy to see that there for such a sequence exists a constant $M$ dominating each element of the sequence (where $M$ then also does not depend on $\omega$), which is what we claimed.

Let $u_{\tilde{\tau}_n}$ denote a control that is equal to the arbitrary constant $u$ on $[t, \tilde{\tau}_n]$ and equal to the equilibrium control $\hat{\mathbf{u}}$ outside of this interval. Using (17), we note that $u_{\tilde{\tau}_n}$ is equal to $u$ on $[t, t+h_n] \times B[x]$ and equal to $\hat{\mathbf{u}}$ on $\{[t, t+h_n] \times B[x]\}^c$, and hence that $u_{\tilde{\tau}_n}$ is, for each $n$, a (deterministic) function of the type $\mathbf{u}_{t+h}$ in Definition 3.2, specifically $u_{\tilde{\tau}_n} = u_{t+h_n}$.

Note that

$$E_{\tilde{\tau}_n, X^{u_{\tilde{\tau}_n}}}[F(X_T^{\hat{\mathbf{u}}}, x)] = E_{\tilde{\tau}_n, X^{u_{\tilde{\tau}_n}}}[F(X_T^{u_{\tilde{\tau}_n}}, x)] \quad a.s. \tag{18}$$

since the starting values of the two expected values are both $(\tilde{\tau}_n, X^{u_{\tilde{\tau}_n}})$, and since the controls (i.e. $u_{\tilde{\tau}_n}$ and $\hat{\mathbf{u}}$) are equal after the starting time, i.e. on $[\tilde{\tau}_n, T]$ except at exactly the starting time $\tilde{\tau}_n$. The only difference between the controls is thus on a set with Lebesgue measure zero, and the expected values in (18) therefore coincide.

Now use Itô's formula, in the same way as in Part 2, to see that

$$E_{t,x}\left[f_{\hat{\mathbf{u}}}^x(\tilde{\tau}_n, X_{\tilde{\tau}_n}^{u_{\tilde{\tau}_n}})\right] = f_{\hat{\mathbf{u}}}^x(t,x) + E_{t,x}\left[\int_t^{\tilde{\tau}_n} \mathcal{A}^{u_{\tilde{\tau}_n}} f_{\hat{\mathbf{u}}}^x(s, X_s^{u_{\tilde{\tau}_n}})ds\right]. \tag{19}$$

The integral limits in (19) are $t$ and $\tilde{\tau}_n$ and the control $u_{\tilde{\tau}_n}$ is equal to $u$ on $[t, \tilde{\tau}_n]$. It follows that $\mathcal{A}^{u_{\tilde{\tau}_n}} = \mathcal{A}^u$ and $X_s^{u_{\tilde{\tau}_n}} = X_s^u$ a.s. on $[t, \tilde{\tau}_n]$. We may therefore replace the superscript in the differential operator and in the controlled process in the integral in (19) with $u$. Using this replacement, the identities $f_{\hat{\mathbf{u}}}^y(t,x) = E_{t,x}[F(X_T^{\hat{\mathbf{u}}}, y)]$ and $f_{\hat{\mathbf{u}}}^x(t,x) = V(t,x)$, and (18), we see that (19) can be rewritten as

$$\begin{aligned}
&V(t,x) + E_{t,x}\left[\int_t^{\tilde{\tau}_n} \mathcal{A}^u f_{\hat{\mathbf{u}}}^x(s, X_s^u)ds\right] = E_{t,x}\left[f_{\hat{\mathbf{u}}}^x(\tilde{\tau}_n, X_{\tilde{\tau}_n}^{u_{\tilde{\tau}_n}})\right] \\
&= E_{t,x}\left[E_{\tilde{\tau}_n, X^{u_{\tilde{\tau}_n}}}[F(X_T^{\hat{\mathbf{u}}}, x)]\right] = E_{t,x}\left[E_{\tilde{\tau}_n, X^{u_{\tilde{\tau}_n}}}[F(X_T^{u_{\tilde{\tau}_n}}, x)]\right] \\
&= E_{t,x}\left[F(X_T^{u_{\tilde{\tau}_n}}, x)\right] = J(t, x, u_{\tilde{\tau}_n}).
\end{aligned} \tag{20}$$

The identity $V(t,x) = J(t,x,\hat{\mathbf{u}})$ and (20) imply that, for each $n$,

$$J(t,x,\hat{\mathbf{u}}) - J(t,x,u_{\tilde{\tau}_n}) = -E_{t,x}\left[\int_t^{\tilde{\tau}_n} \mathcal{A}^u f_{\hat{\mathbf{u}}}^x(s, X_s^u)ds\right]. \tag{21}$$

Let us now verify the inequality and equalities of (22) below. The inequality in (22) follows from the assumption that $\hat{\mathbf{u}}$ is an equilibrium control (Definition 3.2). The first equality follows the fact that $u_{\tilde{\tau}_n} = u_{t+h_n}$ (see the arguments directly after (17)), which implies that we may replace $u_{t+h}$ with $u_{\tilde{\tau}_n}$, $h$ with $h_n$ and $h \searrow 0$ with $n \to \infty$ (since this implies that $h_n \searrow 0$). The three following equalities follow from (21), dominated convergence and Lebesgue's differentiation theorem (which we may use for reasons analogous to those in Part 2). The last equality follows the fact $V(t,x) = f_{\hat{\mathbf{u}}}(t,x,x)$ by definition.

$$
\begin{aligned}
0 \qquad & \leq \lim_{h \searrow 0} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,u_{t+h})}{h} \\
& = \lim_{n\to\infty} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,u_{\tilde{\tau}_n})}{h_n} \\
& = \lim_{n\to\infty} \frac{-E_{t,x}\left[\int_t^{\tilde{\tau}_n} \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(s,X^u_s)ds\right]}{h_n} \\
& = E_{t,x}\left[\lim_{n\to\infty} \frac{-\int_t^{\tilde{\tau}_n} \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(s,X^u_s)ds}{h_n}\right] \\
& = -\mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x) \\
& = -(\mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x) + \mathcal{A}^u f^x_{\hat{\mathbf{u}}}(t,x)). \qquad (22)
\end{aligned}
$$

Recall that $(t,x,u)$ was arbitrary. Therefore, (22) implies that $\hat{\mathbf{u}}$ is maximal in (16), which is all we had left to prove. ∎

# 6 A more general case

In this section we let the time-inconsistency in the terminal payoff function $F$ depend not only on the current state of the controlled process $x$ but also on current time $t$. We also a add a time-inconsistent *running payoff function* $H$. Specifically, we consider the following auxiliary and value functions.

**Definition 6.1** (Auxiliary function II & Value function II )**.**

$$
f_{\mathbf{u}}(t,x,s,y) = E_{t,x}\left[\int_t^T H(r,X^{\mathbf{u}}_r,\mathbf{u}(r,X^{\mathbf{u}}_r),s,y)dr + F(s,X^{\mathbf{u}}_T,y)\right],
$$

$$
J(t,x,\mathbf{u}) = f_{\mathbf{u}}(t,x,t,x) = E_{t,x}\left[\int_t^T H(r,X^{\mathbf{u}}_r,\mathbf{u}(r,X^{\mathbf{u}}_r),t,x)dr + F(t,X^{\mathbf{u}}_T,x)\right],
$$

*where* $F:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ *is continuous with* $|F(t,x,y)| \leq C_0(1+|x|^2)$ *for some constant* $C_0$ *which is independent of t (but which may depend on y) and* $H:[0,T]\times\mathbb{R}^n\times U\times[0,T]\times\mathbb{R}^n\to\mathbb{R}$ *is continuous and bounded.*

It is easy to see that an analogous version of Lemma 2.4 holds in this more general case. Specifically, in this version of Lemma 2.4 we replace $f_{\mathbf{u}}(t,x,y)$ with $f_{\mathbf{u}}(t,x,s,y)$ and note that $C_2$ is independent also of $s$.

The definition of equilibrium is analogous to Definition 3.2. The only difference is that the equilibrium triplet is $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,s,y))$ with

$$V(t,x) = J(t,x,\hat{\mathbf{u}}) = f_{\hat{\mathbf{u}}}(t,x,t,x).$$

The general case extended HJB system is the following natural extension (where $f_{\hat{\mathbf{u}}}(t,x,s,y) = f_{\hat{\mathbf{u}}}^{s,y}(t,x)$ and $\mathcal{A}^{\hat{\mathbf{u}}}$ still operates only on variables in parenthesis).

**Definition 6.2** (The extended HJB system II). *For $(t,x,s,y) \in [0,T) \times \mathbb{R}^n \times [0,T) \times \mathbb{R}^n$,*

$$\begin{aligned} \mathcal{A}^{\hat{\mathbf{u}}} f_{\hat{\mathbf{u}}}^{s,y}(t,x) + H(t,x,\hat{\mathbf{u}}(t,x),s,y) &= 0 \\ f_{\hat{\mathbf{u}}}^{s,y}(T,x) &= F(s,x,y) \\ \sup_{u \in U} \left\{ \mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,t,x) + \mathcal{A}^u f_{\hat{\mathbf{u}}}^{t,x}(t,x) + H(t,x,u,t,x) \right\} &= 0 \\ V(T,x) &= F(T,x,x) \end{aligned}$$

*where $\hat{\mathbf{u}}$ satisfies*

$$\hat{\mathbf{u}}(t,x) \in \arg sup_{u \in U} \left\{ \mathcal{A}^u V(t,x) - \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,t,x) + \mathcal{A}^u f_{\hat{\mathbf{u}}}^{t,x}(t,x) + H(t,x,u,t,x) \right\}$$

*and where any solution triplet $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,s,y))$ must satisfy*

- $\hat{\mathbf{u}} \in \mathbf{U}$,
- $f_{\hat{\mathbf{u}}}^{s,y}(t,x) \in C^{1,2}([0,T) \times \mathbb{R}^n) \cap C([0,T] \times \mathbb{R}^n)$ *for any fixed* $(s,y) \in [0,T] \times \mathbb{R}^n$,
- $V(t,x)$ *and* $f_{\hat{\mathbf{u}}}(t,x,t,x)$ *have existing derivatives to the extent that* $\mathcal{A}^u V(t,x)$ *and* $\mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,t,x)$ *exist, for any* $(t,x) \in [0,T) \times \mathbb{R}^n$.

The definition of a regular equilibrium in the general case is almost the same as in the less general case, the difference is that Assumption 5.3 here regards the differentiability of $f_{\hat{\mathbf{u}}}^{s,y}(t,x)$ for fixed $(s,y)$.

We are now ready to present the main result in the general case. The proof is very similar to the proof of Theorem 5.5 and is therefore omitted.

**Theorem 6.3** (Main result II). *A regular equilibrium $(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,s,y))$ necessarily solves the extended HJB system (Definition 6.2). In particular, if a regular equilibrium exists then the extended HJB system has a solution.*

# 7 Example: a time-inconsistent quadratic regulator

The linear-quadratic regulator is a classic problem in control theory. A typical stochastic linear-quadratic regulator problem in dimension $n = 1$ can be described as follows. The drift of the controlled process is linear and the volatility $\sigma > 0$ is a constant,

$$dX_s^{\mathbf{u}} = (c_1 X_s^{\mathbf{u}} + c_2 \mathbf{u}(s, X_s^{\mathbf{u}}))ds + \sigma dW_s, \quad X_t^{\mathbf{u}} = x. \tag{23}$$

The running and terminal payoffs are quadratic. Specifically, the value function is

$$E_{t,x}\left[\int_t^T (c_3 \mathbf{u}(s, X_s^{\mathbf{u}})^2 + c_4 (X_s^{\mathbf{u}})^2) ds + c_5 (X_T^{\mathbf{u}})^2\right]. \tag{24}$$

All $c_i$ are constants and $c_3, c_4, c_5 > 0$. The optimization problem is to *minimize* the value function (24).

**A time-inconsistent quadratic regulator**
In this section we consider a time-inconsistent version of the regulator problem which we call *a time-inconsistent quadratic regulator*. Let $n = 1$ and $U = [-a, a]$ for some constant $a > 0$. Let the terminal payoff be $F(x, y) = (x - y)^2$ and let there be no running payoff, $H(t, x, u, s, y) = 0$. Let $\sigma > 0$ be constant and $\mu(t, x, u) = -u^2$. It is easy to verify that this model satisfies the general conditions of Section 2. The resulting value function that the agents wish to *maximize* is

$$J(t, x, \mathbf{u}) = E_{t,x}\left[(X_T^{\mathbf{u}} - x)^2\right],$$

the auxiliary function is

$$f_{\mathbf{u}}(t, x, y) = E_{t,x}\left[(X_T^{\mathbf{u}} - y)^2\right],$$

and the dynamics are

$$dX_s^{\mathbf{u}} = -\mathbf{u}(s, X_s^{\mathbf{u}})^2 ds + \sigma dW_s, \quad X_t^{\mathbf{u}} = x. \tag{25}$$

Now, *guess* that $\hat{\mathbf{u}} = 0$ is an equilibrium control. This implies that $\mu(t, x, \hat{\mathbf{u}}(t, x)) = -\hat{\mathbf{u}}(t, x)^2 = 0$ for each $(t, x)$ and that

$$\begin{aligned}
&f_{\hat{\mathbf{u}}}(t, x, y) = E_{t,x}\left[(X_T^{\hat{\mathbf{u}}} - y)^2\right] = E_{t,x}\left[(x + \sigma(W_T - W_t) - y)^2\right] \\
&= E_{t,x}\left[(x + \sigma(W_T - W_t))^2 - 2y(x + \sigma(W_T - W_t)) + y^2\right] \\
&= E_{t,x}\left[x^2 + 2x\sigma(W_T - W_t) + \sigma^2(W_T - W_t)^2\right] - 2xy + y^2 \\
&= x^2 + \sigma^2(T - t) - 2xy + y^2 = (x - y)^2 + \sigma^2(T - t).
\end{aligned}$$

Moreover,

$$\frac{\partial f_{\hat{\mathbf{u}}}(t,x,y)}{\partial t} = -\sigma^2, \quad \frac{\partial f_{\hat{\mathbf{u}}}(t,x,y)}{\partial x} = 2x - 2y, \quad \frac{\partial^2 f_{\hat{\mathbf{u}}}(t,x,y)}{\partial x^2} = 2. \tag{26}$$

**Remark 7.1.** *We need the following general observation. Let $(t,x)$ be an arbitrary point and let $\mathbf{u}$ be an arbitrary control. Consider a control $\mathbf{u}_{\theta_n}$ defined as being equal to $\mathbf{u}$ on the stochastic interval $[t,\theta_n]$ and equal to $\hat{\mathbf{u}}$ outside of this interval, where*

$$\theta_n = \inf\{s > t : (s, X_s^{\mathbf{u}}) \notin [t, t+h_n) \times B[x]\} \wedge T.$$

*In the same way as in the proof of Theorem 5.5 (where $h_n$ and $B[x]$ are defined) one can see that $\mathbf{u}_{\theta_n}$ is a control of the type defined in Definition 3.2, specifically $\mathbf{u}_{\theta_n} = \mathbf{u}_{t+h_n}$. Using the same arguments as in Part 4 of the proof of Theorem 5.5 we can, for any sufficiently differentiable function $f_{\hat{\mathbf{u}}}(t,x,y)$, show that*

$$J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{\theta_n}) = -E_{t,x}\left[\int_t^{\theta_n} \mathcal{A}^{\mathbf{u}(s,X_s^{\mathbf{u}})} f_{\hat{\mathbf{u}}}^x(s,X_s^{\mathbf{u}})ds\right] \tag{27}$$

*(where (27) is analogues to (21)).*

Let us return to our example and show that our guess, $\hat{\mathbf{u}} = 0$, is correct by showing that our $J(t,x,\hat{\mathbf{u}})$ satisfies the equilibrium condition (5). Consider an arbitrary point $(t,x)$ and an arbitrary control $\mathbf{u}$. Using that $n = 1$, $\sigma$ is a constant, $\mu(t,x,u) = -u^2$ and (26) we see that

$$\begin{aligned} &\mathcal{A}^{\mathbf{u}(s,X_s^{\mathbf{u}})} f_{\hat{\mathbf{u}}}^x(s,X_s^{\mathbf{u}}) \\ &= \frac{\partial f_{\hat{\mathbf{u}}}^x(s,X_s^{\mathbf{u}})}{\partial t} + \mu(s,X_s^{\mathbf{u}},\mathbf{u}(s,X_s^{\mathbf{u}}))\frac{\partial f_{\hat{\mathbf{u}}}^x(s,X_s^{\mathbf{u}})}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f_{\hat{\mathbf{u}}}^x(s,X_s^{\mathbf{u}})}{\partial x^2} \\ &= -\sigma^2 - \mathbf{u}(s,X_s^{\mathbf{u}})^2(2X_s^{\mathbf{u}} - 2x) + \frac{\sigma^2}{2}2. \end{aligned} \tag{28}$$

Now use (27), (28) and $U = [-a,a]$ to obtain

$$\begin{aligned} &J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{\theta_n}) \\ &= E_{t,x}\left[\int_t^{\theta_n} \mathbf{u}(s,X_s^{\mathbf{u}})^2(2X_s^{\mathbf{u}} - 2x)ds\right] \\ &\geq -2a^2 E_{t,x}\left[\int_t^{\theta_n} |X_s^{\mathbf{u}} - x|ds\right]. \end{aligned} \tag{29}$$

Arguments analogous to those in the proof of Theorem 5.5 imply that we may now use Lebesgue's differentiation theorem and dominated convergence as follows. $X^{\mathbf{u}}$ has continuous trajectories and $X_t^{\mathbf{u}} = x$, using Lebesgue's differentiation theorem we therefore obtain: $\frac{1}{h_n}\int_t^{\theta_n} |X_s^{\mathbf{u}} - x| ds$ converges to 0 as $n \to \infty$ for a.e. $\omega \in \Omega$. Using this and dominated convergence (note that $X_s^{\mathbf{u}}$ is bounded on $[t, \theta_n]$) we obtain from (29) that

$$\liminf_{n\to\infty} \frac{J(t,x,\hat{\mathbf{u}}) - J(t,x,\mathbf{u}_{\theta_n})}{h_n} \geq 0.$$

Since $(t,x)$ and the alternative control $\mathbf{u}$ (and $B[x]$) were arbitrary it follows that $\hat{\mathbf{u}}$ is an equilibrium control (see Definition 3.2, and recall that $\mathbf{u}_{\theta_n} = \mathbf{u}_{t+h_n}$ and $\lim_{n\to\infty} h_n = 0$). Our guess was therefore correct and

$$(\hat{\mathbf{u}}, V(t,x), f_{\hat{\mathbf{u}}}(t,x,y)) = (0, \sigma^2(T-t), (x-y)^2 + \sigma^2(T-t)) \qquad (30)$$

is an equilibrium for the time-inconsistent quadratic regulator.

**The corresponding extended HJB system**

It is easy to see that our equilibrium (30) is regular (Definition 5.1): the equilibrium control $\hat{\mathbf{u}} = 0$ is right-continuous and the auxiliary function $f_{\hat{\mathbf{u}}}(t,x,y) = (x-y)^2 + \sigma^2(T-t)$ and the equilibrium value function $V(t,x) = \sigma^2(T-t)$ are clearly sufficiently differentiable. Since our equilibrium is regular we may use Theorem 5.5 to draw the conclusion that it necessarily solves the corresponding extended HJB system. Let us verify, however, that this is the case.

Our equilibrium (30) satisfies the boundary conditions of the extended HJB system (Definition 4.1), since $V(T,x) = \sigma^2(T-T) = 0 = (x-x)^2 = F(x,x)$ and $f_{\hat{\mathbf{u}}}(T,x,y) = (x-y)^2 + \sigma^2(T-T) = (x-y)^2 = F(x,y)$. With $\mu$ and $\sigma$ defined as above, and using that the equilibrium value function and the auxiliary function satisfy $\mathcal{A}^u V(t,x) = \mathcal{A}^u f_{\hat{\mathbf{u}}}(t,x,x)$ (which is always true, cf. (6)), the rest of extended HJB system simplifies to

$$\begin{aligned} \frac{\partial f_{\hat{\mathbf{u}}}^y(t,x)}{\partial t} - \hat{\mathbf{u}}(t,x)^2 \frac{\partial f_{\hat{\mathbf{u}}}^y(t,x)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f_{\hat{\mathbf{u}}}^y(t,x)}{\partial x^2} &= 0 \qquad (31) \\ \sup_{u\in U}\left\{ \frac{\partial f_{\hat{\mathbf{u}}}^x(t,x)}{\partial t} - u^2 \frac{\partial f_{\hat{\mathbf{u}}}^x(t,x)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f_{\hat{\mathbf{u}}}^x(t,x)}{\partial x^2} \right\} &= 0 \qquad (32) \end{aligned}$$

$$\hat{\mathbf{u}}(t,x) \in \arg sup_{u\in U} \left\{ \frac{\partial f_{\hat{\mathbf{u}}}^x(t,x)}{\partial t} - u^2 \frac{\partial f_{\hat{\mathbf{u}}}^x(t,x)}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 f_{\hat{\mathbf{u}}}^x(t,x)}{\partial x^2} \right\} \qquad (33)$$

**Remark 7.2.** *We remark that (31), (32) and (33) correspond to an example of the simplified version of the extended HJB system in Corollary 5.6.*

The derivatives of the auxiliary function in the equilibrium $f_{\hat{\mathbf{u}}}(t,x,y) = (x-y)^2 + \sigma(T-t)$ are given in (26). Using these we write (31) as

$$-\sigma^2 - \hat{\mathbf{u}}(t,x)^2(2x-2y) + \frac{1}{2}\sigma^2 2 = 0$$

where the left side is equal to zero since our equilibrium control is $\hat{\mathbf{u}} = 0$. This verifies that our equilibrium solves the Kolmogorov part (31). Similarly, (32) can be written as

$$\sup_{u \in U}\left\{-\sigma^2 - u^2(2x-2x) + \frac{1}{2}\sigma^2 2\right\} = 0$$

where $u = 0$ clearly attains the supremum 0 in the left side. This means that (32) and (33) are satisfied by $f_{\hat{\mathbf{u}}}(t,x,y) = (x-y)^2 + \sigma(T-t)$ and $\hat{\mathbf{u}} = 0$. We have thus verified that our equilibrium solves the corresponding extended HJB system.

**A time-consistent version of the time-inconsistent regulator**

The time-inconsistent quadratic regulator problem studied above is genuinely time-inconsistent in the sense that the equilibrium control $\hat{\mathbf{u}} = 0$ is not an optimal control in following time-consistent version of this problem: Consider the time-inconsistent quadratic regulator problem, but now let the agent at the initial point $(0, x_0)$ be able to dictate which control $\mathbf{u}$ should be used until $T$. This is clearly a standard stochastic control problem with optimal value function

$$\sup_{\mathbf{u} \in \mathbf{U}} E_{t,x}[(X_T^{\mathbf{u}} - x_0)^2]. \tag{34}$$

Therefore, if $\mathbf{u} = 0$ were an optimal control then the optimal value function would be (cf. the calculations above (26))

$$G(t,x) = E_{t,x}[(X_T^{\mathbf{u}} - x_0)^2] = (x-x_0)^2 + \sigma^2(T-t). \tag{35}$$

Therefore, if $\mathbf{u} = 0$ were an optimal control then we would be able to use the standard (time-consistent) stochastic control version of Theorem 5.5 (see e.g. [19, ch. 11]) to deduce that the function $G$ in (35) would solve the standard HJB corresponding to (34) (and (25)),

$$\sup_{u \in U}\left\{\frac{\partial G(t,x)}{\partial t} - u^2\frac{\partial G(t,x)}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 G(t,x)}{\partial x^2}\right\} = 0, \quad G(T,x) = (x-x_0)^2. \tag{36}$$

It is easy to verify that the function $G$ in (35) *does not* solve (36) which therefore implies that $\mathbf{u} = 0$ cannot be an optimal control for this time-consistent version of our problem.

**Remark 7.3.** *Another time-inconsistent version of the regulator problem is studied in [2].*

# 8 Conclusions and remarks

We study time-inconsistent stochastic control. Our model is that of a general (Markovian) Itô diffusion and a general time-inconsistent terminal payoff function, see Section 2. We also study a more general case including a time-inconsistent running payoff function, see Section 6. The standard notion of optimality is not suitable for time-inconsistent optimization and, in Section 3, we therefore define the time-inconsistent stochastic control *equilibrium* (found also in [2, 10, 11]). In Section 4, we define a system of PDEs called *the extended HJB system* (proposed in [2]). Our main result is Theorem 5.5 (and the more general Theorem 6.3), which says that a regular equilibrium (*if it exists*) is necessarily a solution to the corresponding extended HJB system. The proof is based on probabilistic methods.

Let us put our main result more in context. A verification theorem is formulated and proved in [2]. In the setting of the present paper it can be formulated as follows: *if* a (sufficiently regular) solution to the extended HJB system exists, then this solution is in fact an equilibrium. Unfortunately, there are no general results saying anything about the *existence* of solutions to extended HJB systems, see Remark 8.1 below. However, our main result implies that if a regular equilibrium exists, then the corresponding extended HJB system has a solution. Thus, our main result adds knowledge to the open problem of the existence of solutions to extended HJB systems, by relating it to the question of the existence of regular equilibria.

The remarks below reveal a potential for future research in time-inconsistent stochastic control.

**Remark 8.1** (Existence of equilibrium controls, Assumption 5.2). *The problem of finding general model conditions (i.e. for $\sigma, \mu, U$, $F$ and $H$) which guarantee the existence of an equilibrium control, and a solution to the extended HJB system, is still open. For related remarks see [2, section 4.2] and [4, section 6].*

**Remark 8.2** (Differentiability, Assumption 5.3). *For any particular model one should not a priori expect the differentiability in Assumption 5.3 to be satisfied. It involves a lot of work to find useful model conditions such that similar assumptions are satisfied for standard (time-consistent) stochastic control (for such conditions see e.g. [12, 13, 18, 23]). Indeed, Fleming and Rishel (2012) [12, p. 154] write the following about a similar differentiability assumption:* "Substantial difficulties are encountered if one seeks to put all of the above formal discussion on a precise basis. It has been implicitly assumed that $W^0(t,\cdot)$ *[the optimal value function]* is in the domain of $\mathcal{A}^{\mathbf{u}}(t)$ *[the differential operator]*, which is not always the case."

*We also remark that we would clearly have to impose more regularity for the variable $y$ in the terminal payoff function $F$ in (4) in order for Assumption 5.3 to be reasonable.*

**Remark 8.3** (On the possibility of several equilibria)**.** *Generally, in game theory there is no reason to expect anything else than that there may exist several equilibria, or no equilibrium. Note, however, that the investigations in the present paper regard any fixed equilibrium control $\hat{\mathbf{u}}$, and if, in a particular case, there should be more than one equilibrium control, then our results are true for any such fixed equilibrium control. Moreover, we do not expect that different equilibrium controls necessarily correspond to different equilibrium value functions, although they may do so in particular cases.*

**Remark 8.4** (Viscosity solutions)**.** *For the standard time-consistent stochastic control theory, it is well-known that rather strong conditions for the model are needed in order for the value function to be sufficiently differentiable to be able to solve the HJB in the classical sense. This is true also when considering a.e. differentiability. Clearly, there is no reason to expect anything else for time-inconsistent stochastic control. For the standard time-consistent stochastic control theory, this problem was famously solved by the introduction of a type of weak solutions to PDEs known as* viscosity solutions*, see [6]. Viscosity solutions for extended HJB systems have so far not been studied.*

**Remark 8.5** (More examples)**.** *Several particular time-inconsistent stochastic control problems studied in a mathematical way can be found in e.g. [2, 4, 5, 7, 10, 20].*

# Acknowledgments

The author is grateful to Tomas Björk and Jan-Erik Björk for helpful discussions.

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