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Abstract

Standard Markovian optimal stopping problems are consistent in the sense that the first entrance time into the stopping set is optimal for each initial state of the process. Clearly, the usual concept of optimality cannot in a straightforward way be applied to non-standard stopping problems without this time-consistent structure. This paper is devoted to the solution of time-inconsistent stopping problems with the reward depending on the initial state using a game-theoretic approach in which each state of the process corresponds to a player in the game. More precisely, we give a precise equilibrium definition — of the type subgame perfect Nash equilibrium based on pure Markov strategies. Such equilibria do not always exist. We, however, develop an iterative approach to finding such equilibrium stopping times for a general class of problems and apply this approach to one-sided stopping problems on the real line. We furthermore prove a verification theorem based on a set of variational inequalities which also allows us to find equilibria. As an application of the developed theory we study a selling strategy problem under exponential utility and endogenous habit formation.

Keywords: Markov process, Optimal stopping, Subgame perfect Nash equilibrium, Time-inconsistency, Variational inequalities.

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1 Introduction

Consider a Markovian process X with state space $E \subseteq \mathbb{R}^d$ and the problem of choosing a stopping time τ in order to maximize the expected discounted reward

$$\mathbb{E}_x(e^{-r\tau}F(X_\tau, x)), \quad \text{for each current state } x \in E.$$

Note that the dependence of the reward $F(X_\tau, x)$ on the current state x implies that this is not a standard optimal stopping problem. Specifically, the problem is inconsistent in the sense that we cannot generally expect the existence of an optimal stopping time that is independent of the current state x. In other words, if an optimal stopping time is optimal for the current state x, then it will generally not be optimal at a later time, call it t, after adjusting the reward, since the then current state X_t will typically be different from x. Optimal stopping and more general optimal control problems with this property are called *time-inconsistent*.

It is clear that the time-inconsistency implies that the usual notion of optimality cannot be applied straightforwardly — it must first be clarified how the time-inconsistent stopping problem should be interpreted. One way of dealing with this issue is of course to treat the problem as a parametrized, by the current state x, optimal stopping problem and ignore the issue that the corresponding optimal stopping time will not generally be optimal at later times. In the literature this is known as a *pre-commitment* strategy. In the present paper we instead interpret the time-inconsistent stopping problem using a game-theoretic approach where we let each state x correspond to an agent, who all play a sequential game against each other regarding when to stop the process X — and then we look for equilibrium strategies i.e. *equilibrium stopping times*. The type of equilibria we consider are subgame perfect Nash equilibria based on pure Markov strategies (known as pure Markov perfect equilibria), see Remark 2.5 for an explanation of these terms. The present paper contains, to our knowledge, the first general treatment of the game-theoretic approach to time-inconsistent stopping problems.

The structure of the paper is as follows. In Section 1.1 we describe previous literature related to time-inconsistent problems. In Section 2 we formulate the general time-inconsistent stopping problem introduced above in more detail and define the notions of pure Markov strategies and subgame perfect Nash equilibria in this setting. We also present an example which proves that an equilibrium (of the type we consider) does not always exist. In Section 3 we develop an iterative approach to finding equilibrium stopping times in a general setting under certain assumptions. As an application of this iterative approach we, in Section 4, study a class of one-sided problems on the real line. In Section 5 we present a verification theorem for time-inconsistent optimal stopping based on a set of variational inequalities that we call the *time-inconsistent variational inequalities*.

Illustrative examples are studied in Sections 3, 4, and 5. In Example 5.7 we apply the verification theorem to find equilibrium selling strategies for an investor with exponential utility and endogenous habit formation.

1.1   Previous literature

Time-inconsistency in financial economics typically arises for either of the following reasons:

(i) Endogenous habit formation,

(ii) Non-exponential discounting,

(iii) Mean-variance utility.

Stopping problems with (i) and (ii) can be formulated and studied in the framework of the present paper whereas (iii) goes beyond this framework. Stopping problems with (ii)—(iii) are described below. A stopping problem with (i) is studied in Example 5.7. See also [4] for a short description of (i)—(iii).

There is a substantial financial economics literature that studies specific time-inconsistent problems; in either continous or discrete time, for either stochastic or deterministic models, and using either game-theoretic or pre-commitment approaches. Historically important papers include [16, 30, 32, 33]. We remark that most of these papers consider problems of control type. Papers in financial economics studying time-inconsistent stopping problems include [1, 10, 11, 17, 18].

The first paper to use a game-theoretic approach — based on subgame perfect Nash equilibria — to time-inconsistency, there termed *consistent planning*, was [33], where a deterministic problem under non-exponential discounting in discrete time is studied. Further financial economics research in this direction can be found in [2, 16, 22, 30, 32].

Early papers of a more mathematical kind to consider the game-theoretic approach — based on subgame perfect Nash equilibria — in time-inconsistent problems in continuous time are [12, 14], who study optimal consumption and investment under non-exponential (hyperbolic) discounting. Inspired by the approach of e.g. [12, 14], the first papers to develop a general mathematical theory for finding subgame perfect Nash equilibria for time-inconsistent stochastic control problems in Markovian models are [3, 4]. The main feature of that theory is a generalization of the standard HJB equation called the extended HJB system and the main result is a verification theorem saying that if a solution to the extended HJB system exists then it corresponds to an equilibrium. In [24] it is shown that a *regular* equilibrium is necessarily a solution to an extended HJB system. Other

papers studying specific time-inconsistent control problems from a more mathematical perspective include [5, 9, 13, 19, 23].

Papers of a more mathematical kind to study time-inconsistent stopping include [20] who study a stopping problem with non-exponential discounting and [26, 29] (see the discussion below). We refer to [4, 20, 26, 29] for short surveys of the literature on time-inconsistent problems.

Endogenous habit formation problems (see Example 5.7) are time-inconsistent because the reward depends on the current state. Stopping problems of this kind can therefore be studied in the framework of the present paper. A version of the non-exponential discounting stopping problem corresponds to maximizing

$$\mathbb{E}_{t,x}(\delta(\tau - t)\tilde{F}(X_\tau)) \tag{1.1}$$

with respect to stopping times τ, where the discounting function $\delta : [0, \infty) \to [0, 1]$ is a decreasing (non-exponential) function satisfying $\delta(0) = 1$. Problem (1.1) can in our framework be obtained by letting one of the dimensions of X correspond to time, i.e., by considering the time-space process.

Mean-variance problems are, however, time-inconsistent for the fundamentally different reason that the expression to be maximized is a non-linear function of the expected value of a reward. Hence, mean-variance problems cannot be studied in the present framework. A version of the mean-variance stopping problem is to find a stopping time τ that maximizes

$$\mathbb{E}_x(X_\tau) - cVar_x(X_\tau), \quad \text{where } c > 0 \text{ is a fixed constant.} \tag{1.2}$$

In [29], this mean-variance stopping problem is studied for an underlying geometric Brownian motion (i.e. a mean-variance selling problem in a Black-Scholes market). The problem is interpreted and solved in two different ways, by the introduction of two different definitions of optimality. *Static optimality*, corresponds to finding, for a fixed $x > 0$, a stopping time that maximizes (1.2). The static optimality definition corresponds to a pre-commitment approach. *Dynamic optimality*, corresponds to finding a stopping time τ^* such that there is no other stopping time σ with $\mathbb{P}_x(\mathbb{E}_{X_{\tau^*}}(X_\sigma) - cVar_{X_{\tau^*}}(X_\sigma) > X_{\tau^*}) > 0$ for some $x > 0$. This is a novel interpretation of time-inconsistent problems, that does not rely on game-theoretic arguments, see, however, Remark 3.3 below. We refer to [29] for a discussion of the difference between the game-theoretic approach and the dynamic optimality approach. Time-inconsistent stopping problems with more general non-linear functions of the expected reward are studied in [26] using an approach which is inspired by [29].

2 Problem formulation

On the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$ we consider a strong Markov process $X = (X_t)_{t\geq 0}$ taking values in $(E, \mathcal{B})$ where $E \subseteq \mathbb{R}^d$ and $\mathcal{B}$ is the corresponding Borel σ-algebra and $X_0 = x \in E$. We assume that the filtration satisfies the usual conditions and X to have cádlág sample paths and to be quasi left continuous and that $x \mapsto \mathbb{P}_x(F)$ is measurable for each $F \in \mathcal{F}$. The associated expectations are denoted by $\mathbb{E}_x$. Without loss of generality we assume that $(\Omega, \mathcal{F})$ equals the canonical space so that the shift operator θ given by $\theta_t(\omega)(s) = \omega(t+s)$ for $\omega = (\omega(t))_{t\geq 0} \in \Omega$ and $t, s \geq 0$ is well-defined. The class of stopping times with respect to $(\mathcal{F}_t)_{t\geq 0}$ is denoted by $\mathcal{M}$.

Consider a function $F : E \times E \to \mathbb{R}$ and the problem of finding a stopping time τ such that it maximizes, over the class of stopping times $\mathcal{M}$,

$$J_\tau(x) := \mathbb{E}_x\left(e^{-r\tau} F(X_\tau, x)\mathbf{1}_{\{\tau < \infty\}}\right), \quad \text{for each } x \in E,$$

where $r \geq 0$ is a constant and — to guarantee that all expectations are well-defined — the function $F(\cdot, y)$ is measurable and bounded from below for each fixed $y \in E$. For the ease of exposition we will in the rest of the paper not explicitly write out indicator functions of the type $\mathbf{1}_{\{\tau < \infty\}}$ in expected values, but instead implicitly assume that they are there.

The difference in our formulation to usual Markovian optimal stopping problems is that the reward $F(X_\tau, x)$ explicitly depends on the initial state $X_0 = x$. In the standard formulation, the reward $F(X_\tau, x)$ is independent of x. In that classical case, it is well-known that — under minimal assumptions — an optimal stopping time is Markovian in the sense that it is a first entrance time into the stopping set, see, e.g., [31], I.2.2. In particular, this solution is consistent meaning that one rule is optimal for each initial state, i.e. such problems are consistent with Bellman's principle of optimality.

This kind of consistency can of course not be expected in our formulation. We therefore have to be careful how to reinterpret the concept of optimality. Clearly, we could choose different stopping times for different starting points x. This, however, does not represent the following interpretation of our problem:

We interpret the time-inconsistent stopping problem above as a stopping problem for a person whose preferences, identified with the reward function $F(\cdot, x)$, change as the state x changes. Based on this we think of the person as comprising versions of herself, one version for each state x. These versions of the person can then be thought of as agents who play a sequential game against each other, where the game regards when to stop the process X. Note that the number of players in this game is generally uncountable. Each agent, i.e. each x-version of

the person, then has the possibility, at x, to either stop, or not stop. A reasonable definition of an equilibrium strategy, in this case an *equilibrium stopping time* $\hat{\tau}$, should therefore be such that the following holds:

Under the assumption that each other version of the person uses $\hat{\tau}$ then,

(i) no x-version of the agent wants to stop in her state before $\hat{\tau}$, and

(ii) no x-version of the agent wants to continue for an "infinitesimal" time if $\hat{\tau}$ calls for stopping.

We thus define an equilibrium stopping time $\hat{\tau}$ using conditions which guarantee that no agent wants to deviate from $\hat{\tau}$. Furthermore, we demand that the decision whether to stop or not should depend directly only on the preferences of each agent x and not, for example, on the outcome of some randomization procedure, or on events from the past, cf. Remark 2.5. These conditions are, in reverse order, formalized in the following definitions.

Definition 2.1. A stopping time $\tau \in \mathcal{M}$ is said to be a pure Markov strategy stopping time if it is the entrance time of the state process into a set in the state space, more specifically, if $\tau = \inf\{t \geq 0 : X_t \in S\}$ for some measurable $S \subseteq E$. Denote the set of such stopping times by $\mathcal{N}$.

Definition 2.2. A stopping time $\hat{\tau} \in \mathcal{N}$ is said to be a (pure Markov strategy) equilibrium stopping time if, for all $x \in E$,

$$J_{\hat{\tau}}(x) - F(x,x) \geq 0, \quad \text{and} \tag{2.1}$$

$$\liminf_{h \searrow 0} \frac{J_{\hat{\tau}}(x) - J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x)}{\mathbb{E}_x(\tau_h)} \geq 0, \tag{2.2}$$

where $\tau_h = \inf\{t \geq 0 : |X_t - X_0| \geq h\}$.

Definition 2.3. If $\hat{\tau}$ is a (pure Markov strategy) equilibrium stopping time then the function $J_{\hat{\tau}}(x)$, $x \in E$, is said to be the (pure Markov strategy) equilibrium value function corresponding to $\hat{\tau}$. The function

$$f_{\hat{\tau}}(x,y) := \mathbb{E}_x(e^{-r\hat{\tau}} F(X_{\hat{\tau}}, y)), \quad (x,y) \in E \times E$$

is said to be the auxiliary function corresponding to $\hat{\tau}$.

It follows that the equilibrium value function satisfies

$$J_{\hat{\tau}}(x) = f_{\hat{\tau}}(x,x) = \mathbb{E}_x(e^{-r\hat{\tau}} F(X_{\hat{\tau}}, x)), \quad x \in E.$$

This paper is devoted to the question of how to find equilibrium stopping times (of the type in Definition 2.2).

The interpretation of (2.1) is that each x-agent should prefer the equilibrium strategy over stopping directly. The interpretation of (2.2) is that each x-agent should prefer the equilibrium strategy over not stopping on the short (stochastic) time interval $[0, \tau_h)$, over which we interpret the x-agent as being in charge, given that the equilibrium strategy is played from τ_h and onwards.

Remark 2.4. Definition 2.2 is inspired by time-inconsistent stopping problems in financial economics, see e.g. [11, 18], and the equilibrium definition for time-inconsistent stochastic control problems, see e.g. [3, 4].

Remark 2.5. Let us informally describe some of the game theoretic jargon used above, for a reference see e.g. [25]. A Markov strategy depends on past events that are payoff-relevant. Markov strategies can be pure or mixed. A pure strategy is one that determines the actions of the agents without randomization. In our setting, the actions of the agents are to stop or not to stop, hence first entrance times correspond to pure strategies. A mixed strategy is one that randomly selects pure strategies. In our situation, this could be realized by extending the underlying filtration in a suitable way and consider general stopping times with respect to this filtration. See Example 2.6 below for an illustration. A subgame perfect Nash equilibrium is a strategy that forms a Nash equilibrium at any time t, and a Markov perfect equilibrium is a subgame perfect Nash equilibrium in which all players use Markov strategies. Thus, Definition 2.2 corresponds to a subgame perfect Nash equilibrium, and more specifically a pure Markov perfect equilibrium.

Example 2.6. As mentioned above, in standard Markovian optimal stopping problems, we only have to consider first entrance times and the filtration generated by X. Moreover, any additional information included in some larger filtration cannot improve the optimal value function as long as the process is Markovian also with respect to the larger filtration. Similarly, in Markovian Dynkin-type stopping games it is also the case that equilibria can be found (under technical assumptions) as first-entrance times, see [15]. This is however not the case for equilibrium stopping problems in general as we will see in the following example.

Consider a discrete time process X that lives on the state space $E = \{\partial_1, a, b, \partial_2\} \subseteq \mathbb{R}$ where ∂_1 and ∂_2 are absorbing states and

$$\mathbb{P}_a(X_1 = \partial_1) = \mathbb{P}_a(X_1 = b) = \mathbb{P}_b(X_1 = a) = \mathbb{P}_b(X_1 = \partial_2) = \frac{1}{2}.$$

Let $r = 0$ and define $X_\infty = \lim_{t\to\infty} X_t$.
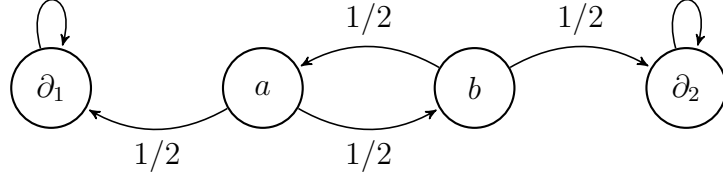
(X can of course be embedded into a continuous time Markov chain, so that we do not leave the setting of this paper). Let

$$F(x, a) = \begin{cases} 0, & x = \partial_1 \\ 1, & x = a \\ 3, & x = b \\ 0, & x = \partial_2, \end{cases}$$

$$F(x, b) = \begin{cases} 4, & x = \partial_1 \\ 0, & x = a \\ 1, & x = b \\ 0, & x = \partial_2 \end{cases}$$

and $F(\cdot, \partial_i) = 0$ for $i = 1, 2$. We will now show that no entrance time of the state process X into a subset $S \subseteq E$ can be an equilibrium stopping time, i.e. no pure Markov strategy equilibrium stopping time exists. We do this by investigating all such stopping sets S. Since ∂_1 and ∂_2 are absorbing we can without loss of generality assume that $\partial_1, \partial_2 \in S$. It remains to consider the following four sets:

(i) $S = \{\partial_1, \partial_2, a, b\}$: This stopping set corresponds to the rule that both agent a and agent b should always stop when they get the chance. But this rule cannot correspond to an equilibrium stopping time, since agent a would obtain 1 when stopping but she obtains $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 3 = \frac{3}{2} > 1$ (in expectation) if she deviates from the rule by never stopping.

(ii) $S = \{\partial_1, \partial_2, a\}$: This stopping set corresponds to the rule that a should stop and b should continue. But this cannot correspond to an equilibrium stopping time, since agent b obtains $\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 0 = 0$ when continuing and $1 > 0$ when stopping.

(iii) $S = \{\partial_1, \partial_2, b\}$: This stopping set corresponds to the rule that b should stop and a should continue. Let $V_{0,0}(b)$ denote the value that agent b obtains when not following this rule. Then $V_{0,0}(b) = \frac{1}{2} \cdot 0 + \frac{1}{2}(\frac{1}{2} \cdot 4 + \frac{1}{2} V_{0,0}(b)) \Rightarrow V_{0,0}(b) = \frac{4}{3}$. Note that agent b obtains $1 < V_{0,0}(b)$ when stopping. This means that the set $\{\partial_1, \partial_2, b\}$ cannot be the stopping set of an equilibrium stopping time.

(iv) $S = \{\partial_1, \partial_2\}$: This stopping set corresponds to the rule that both a and b should continue. Since agent a obtains zero in the absorbing states she prefers to stop since this gives her 1.

The above implies that there is no equilibrium stopping time in the set of pure Markov strategy stopping times. However, a mixed strategy equilibrium stopping time (in the sense defined below for this example) does exist, as we shall now see. Consider the stopping time $\tau_{p,q}$ defined as follows: for any $t \in \mathbb{N}_0$ given $\{\tau_{p,q} \geq t\}$, if $X_t \in \{\partial_1, \partial_2\}$ then $\tau_{p,q} = t$, if $X_t = a$ then $\tau_{p,q} = t$ with probability p, and if $X_t = b$ then $\tau_{p,q} = t$ with probability q (assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ is large enough for $\tau_{p,q}$ be be a stopping time with respect to $(\mathcal{F}_t)_{t\geq 0}$). Heuristically, the stopping time $\tau_{p,q}$ corresponds to the agents a and b flipping biased coins in order to decide whether to stop or not. Let $V_{p,q}(x), x \in \{a, b\}$, denote the (expected) value that agent x obtains when $\tau_{p,q}$ is used.

The following ad hoc definition, which is inspired by [34], will be used only in the present example:

A stopping time of the type $\tau_{p,q}$ (defined above) is said to be a mixed strategy stopping time. A mixed strategy stopping time $\tau_{p',q'}$ is said to be a mixed strategy equilibrium stopping time if $V_{p,q'}(a) \leq V_{p',q'}(a)$ for all $p \in [0,1]$ and $V_{p',q}(b) \leq V_{p',q'}(b)$ for all $q \in [0,1]$.

Heuristically, a mixed strategy equilibrium stopping time $\tau_{p',q'}$ is a strategy from which neither agent a (nor b) wants to deviate from by choosing another mixed strategy $\tau_{p,q'}$ ($\tau_{p',q}$), i.e. they do not want to deviate by choosing another biased coin (including degenerate biased coins, i.e. with $p, q \in \{0, 1\}$).

We obtain

$$V_{p,q}(a) = p \cdot 1 + (1-p) \left[\frac{1}{2} \cdot 0 + \frac{1}{2} \left(q \cdot 3 + (1-q) \left(\frac{1}{2} V_{p,q}(a) + \frac{1}{2} \cdot 0 \right) \right) \right] \Rightarrow$$

$$V_{p,q}(a) = \frac{p + \frac{3}{2}(1-p)q}{1 - \frac{1}{4}(1-p)(1-q)}, \quad \text{and}$$

$$V_{p,q}(b) = q \cdot 1 + (1-q) \left[\frac{1}{2} \cdot 0 + \frac{1}{2} \left(p \cdot 0 + (1-p) \left(\frac{1}{2} V_{p,q}(b) + \frac{1}{2} \cdot 4 \right) \right) \right] \Rightarrow$$

$$V_{p,q}(b) = \frac{q + (1-p)(1-q)}{1 - \frac{1}{4}(1-p)(1-q)}.$$

Choose $p' = \frac{1}{5}$ and $q' = \frac{3}{5}$, i.e. consider the mixed strategy stopping time $\tau_{\frac{1}{5}, \frac{3}{5}}$. The corresponding expected values are $V_{\frac{1}{5}, \frac{3}{5}}(a) = V_{\frac{1}{5}, \frac{3}{5}}(b) = 1$. All we need to do in order to verify that $\tau_{\frac{1}{5}, \frac{3}{5}}$ is a mixed strategy equilibrium stopping time, is to check that neither agent a nor agent b wants to deviate from it, i.e. we need to verify that $V_{p, \frac{3}{5}}(a) \leq V_{\frac{1}{5}, \frac{3}{5}}(a) = 1$ for all $p \in [0, 1]$ and that $V_{\frac{1}{5}, q}(b) \leq V_{\frac{1}{5}, \frac{3}{5}}(b) = 1$

9

for all $q \in [0, 1]$. This is easily done as in fact $V_{p, \frac{3}{5}}(a) = V_{\frac{1}{5}, q}(b) = 1$ for all $p, q \in [0, 1]$. It follows that the stopping time $\tau_{\frac{1}{5}, \frac{3}{5}}$ is indeed a mixed strategy equilibrium stopping time.

3   A forward iteration approach

The previous example illustrates that there is no hope to come up with a general method to find equilibrium stopping times (of the pure Markov strategy type, see Definition 2.2). In particular cases, this can however be done. We now propose an approach for constructing a candidate for an equilibrium stopping time by solving a — possibly terminating — sequence of ordinary optimal stopping problems. More precisely, we construct a set $\hat{S}$ and prove that — under certain assumptions — the first entrance time $\tau_{\hat{S}}$ into $\hat{S}$ is an equilibrium stopping time. To this end, write

$$S_0 := \emptyset, \quad v_0(x, y) := \sup_{\tau} \mathbb{E}_x(e^{-r\tau} F(X_\tau, y)).$$

and define recursively for all $n \geq 1$

$$S_n := \{x \in E : v_{n-1}(x, x) = F(x, x)\},$$
$$v_n(x, y) := \sup_{\tau \leq \tau_{S_n}} \mathbb{E}_x(e^{-r\tau} F(X_\tau, y)).$$

Note that $v_n(\cdot, y)$ is the value function of an ordinary optimal stopping problem for the process X absorbed in S_n. It holds that $S_1, S_2, \ldots$ is an increasing sequence of sets and we assume that

$$S_1, S_2, \ldots \text{ are closed sets.} \tag{A1}$$

We denote the closure of the union $\bigcup_{n=0}^{\infty} S_n$ in E by $\hat{S}$. Moreover, v_n is decreasing in n and therefore converges to a limit v_∞. By the construction of the problem, it is furthermore natural to assume that

$$v_\infty(x, x) = \sup_{\tau \leq \tau_{\hat{S}}} \mathbb{E}_x(e^{-r\tau} F(X_\tau, x)) \text{ for all } x \in E. \tag{A2}$$

Our candidate for the equilibrium stopping time is now the first entrance time $\tau_{\hat{S}}$ into $\hat{S}$. The heuristic motivation is as follows: In case it is rational for the agent in state $y = x$ to stop immediately in the starting state $X_0 = x$ in problem $v_n(x, x)$, n minimal, say, there is no reason for her not to stop immediately in x under the global time $\tau_{\hat{S}}$ as $\tau_{\hat{S}} \leq \tau_{S_n}$. Hence, the agent should accept $\tau_{\hat{S}}$ when

10

$x \in \hat{S}$.

On the other hand, in the case $x \notin \hat{S}$, there exists a stopping time $\tau \leq \tau_{\hat{S}}$ that gives strictly more expected reward than to stop immediately. In case the structure of the problem is such that

$$\text{(2.2) is satisfied with } \hat{\tau} = \tau_{\hat{S}} \text{ for all } x \in \hat{S} \backslash \bigcup_{n \in \mathbb{N}} S_n, \tag{A3}$$

and

$$F(x, x) \leq \mathbb{E}_x\big(e^{-r\tau_{\hat{S}}} F(X_{\tau_{\hat{S}}}, x)\big) \text{ for all } x \notin \hat{S}, \tag{A4}$$

we see that it is also in this case optimal for the agent to accept $\tau_{\hat{S}}$. Indeed:

Theorem 3.1. *Under the assumptions* (A1) − (A4)*, the stopping time* $\tau_{\hat{S}}$ *defined above is an equilibrium stopping time.*

Proof. Write $\hat{\tau} = \tau_{\hat{S}}$ for short. Let us first consider $x \notin \hat{S}$. As $\hat{S}$ is closed, we find $h_0 > 0$ such that the open ball $B(x, h_0)$ around x with radius h_0 is a subset of $\hat{S}^c$. Therefore,

$$J_{\hat{\tau}}(x) = J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x)$$

for all $h \leq h_0$, so that (2.2) is fulfilled automatically. Furthermore, (A4) warrants (2.1).

For $x \in \hat{S} \backslash \bigcup_{n \in \mathbb{N}} S_n$, (2.1) holds trivially as $\hat{\tau}$ calls for immediate stopping, and (A3) yields (2.2).

It remains to check that the equilibrium conditions (2.1) and (2.2) are fulfilled for $x \in \bigcup_{n \in \mathbb{N}} S_n$. In this case (2.1) holds trivially. For the second property, find $n \in \mathbb{N}$ such that $x \in S_n \setminus S_{n-1}$. As S_{n-1} is closed, there exists $\epsilon_0 > 0$ such that the ball $B(x, \epsilon_0)$ around x with radius ϵ_0 is a subset of S_{n-1}^c. Then, for each $h < \epsilon_0$ it holds that $\hat{\tau} \circ \theta_{\tau_h} + \tau_h \leq \tau_{S_{n-1}}$ and therefore

$$F(x, x) = v_n(x, x) = \sup_{\tau \leq \tau_{S_{n-1}}} \mathbb{E}_x\big(e^{-r\tau} F(X_\tau, x)\big)$$

$$\geq \mathbb{E}_x\big(e^{-r\hat{\tau} \circ \theta_{\tau_h} + \tau_h} F(X_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}, x)\big) = J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x),$$

where we used that $x \in S_n$ implies $v_n(x, x) = F(x, x)$ and $v_{n-1}(x, x) = F(x, x)$. This yields

$$\liminf_{h \searrow 0} \frac{J_{\hat{\tau}}(x) - J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x)}{\mathbb{E}_x(\tau_h)} = \liminf_{h \searrow 0} \frac{F(x, x) - J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x)}{\mathbb{E}_x(\tau_h)} \geq 0.$$

$\square$

Remark 3.2. We now discuss the assumptions above.

- On (A1): Optimal stopping sets are well-known to be closed under weak assumptions, see [31], I.2.2. In particular, (A1) is obviously fulfilled if

$$x \mapsto F(x,x), \ x \mapsto v_n(x,x) \text{ are continuous.}$$

- On (A2): This assumption warrants that $-$ in the sense described above $-$ the optimal stopping sets of the problems related to v_n converge to the optimal stopping set of the limiting problem. In particular, if the procedure terminates, i.e., there exists $n_0 \in \mathbb{N}$ such that $S_{n_0} = S_{n_0+1}$, then assumption (A2) is automatically fulfilled.

- On (A3): This assumption is trivially fulfilled when the procedure terminates. In general, it can be understood as a version of a smooth fit property for the limiting problem.

- On (A4): In contrast to the previous conditions, (A4) is more than a technical regularity assumption. As mentioned above, it is by construction clear that for $x \notin \hat{S}$, there exists a stopping time $\tau \leq \tau_{\hat{S}}$ with strictly larger expected reward than to stop immediately. But it is not clear in general that $\tau_{\hat{S}}$ also has this property. As discussed at the end of this section, Example 3.5 is a counterexample.

Remark 3.3. In some cases of interest, for example in Section 4, the procedure terminates already after one step, i.e.

$$\hat{S} = \{x \in \ E : F(x,x) = \sup_{\tau} \mathbb{E}_x(e^{-r\tau}F(X_\tau, x))\}.$$

Under (A1) it holds that $X_{\hat{\tau}} \in \hat{S}$, where we write $\hat{\tau} = \tau_{\hat{S}}$ as above. Hence, in the case of termination after one step, we obtain, using the strong Markov property, for all $x \in \ E$ and all stopping times σ,

$$\mathbb{P}_x \left(\mathbb{E}_{X_{\hat{\tau}}} \left(e^{-r\sigma} F(X_\sigma, X_0) \right) > F(X_{\hat{\tau}}, X_{\hat{\tau}}) \right) = 0.$$

This may be interpreted as an adaptation of the notion of dynamic optimality (see Section 1.1) to our setup. Hence, in this case the equilibrium (a local property) is in this sense also dynamically optimal (a global property). We remark that the equilibrium in Example 3.4 below is not dynamically optimal.

We close this section by discussing two examples. A general class of examples with a one-sided equilibrium stopping time found by this approach is discussed separately in Section 4.

Example 3.4. We now consider an underlying one-dimensional Wiener process X and fix a discount rate $r > 0$. To illustrate the theory with an explicit example, we look at the reward function

$$F(x, y) := \begin{cases} x^+ & , \ y \geq 0, \\ (-x)^+ & , \ y < 0. \end{cases}$$

A (somewhat artificial) financial interpretation is that the holder of a perpetual American option with strike 0 in a Bachelier market is uncertain whether she has bought a put or a call option. She is inherently optimistic and changes her belief depending on the state the process is in. If the current state is non-negative, i.e. $y \geq 0$, she believes that the derivative is a call, and a put otherwise. Using standard approaches to the solution of optimal stopping problems, such as a free boundary approach or the harmonic function technique of [6], it is straightforward to find that for fixed $y \geq 0$

$$v_0(x, y) = \sup_\tau \mathbb{E}_x(e^{-r\tau} F(X_\tau, y)) = \begin{cases} x & , \ x \geq x_1, \\ a_1 e^{cx} & , \ x < x_1, \end{cases}$$

where $c = \sqrt{2r}$, $x_1 = 1/c$ and $a_1 = 1/(ec)$. Due to symmetry, we have for $y < 0$ that $v_0(x, y) = v_0(-x, -y)$. Therefore,

$$S_1 = (-\infty, -x_1] \cup [x_1, \infty).$$

Now, we can go on iteratively to find v_n and S_n again using standard arguments for optimal stopping problems for diffusions. Writing

$$f_n(x) = a_n e^{-cx} + b_n e^{cx},$$

we try to find x_{n+1}, a_n, b_n such that

$$f_n(-x_{n-1}) = 0,$$
$$f_n(x_n) = x_n,$$
$$f_n'(x_n) = 1.$$

This system is indeed solvable and the solution is given by

$$a_n = \frac{1}{2} e^{cx_n} \left(x_n - \frac{1}{c} \right),$$
$$b_n = \frac{1}{2} e^{-cx_n} \left(x_n + \frac{1}{c} \right)$$

and $x = x_n$ is the unique solution in $(0, x_{n-1})$ of

$$e^{2cx} = e^{-2cx_{n-1}} \frac{\frac{1}{c} + x}{\frac{1}{c} - x}.$$

Then,

$$S_n = (-\infty, -x_n] \cup [x_n, \infty)$$

and for $y \geq 0$

$$v_n(x, y) = \begin{cases} x & , \ x \geq x_{n+1}, \\ f_n(x) & , \ -x_{n+1} \leq x < x_{n+1}, \\ 0 & , \ x < -x_{n+1} \end{cases}$$

and, as above, for $y < 0$ it holds that $v_n(x, y) = v_n(-x, -y)$. It is easily seen that x_n converges monotonically to the unique solution $x = x^*$ in $(0, 1/c)$ of

$$e^{4cx} = \frac{\frac{1}{c} + x}{\frac{1}{c} - x},$$

so that

$$\hat{S} = (-\infty, -x^*] \cup [x^*, \infty)$$

and for $y \geq 0$

$$v_\infty(x, y) = \begin{cases} x & , \ x \geq x^*, \\ f_\infty(x) & , \ -x^* \leq x < x^*, \\ 0 & , \ x < -x^* \end{cases}$$

with

$$f_\infty(x) = \frac{1}{2} e^{cx^*} \left(x^* - \frac{1}{c} \right) e^{-cx} + \frac{1}{2} e^{-cx^*} \left(x^* + \frac{1}{c} \right) e^{cx}.$$

Again, $v_\infty(x, y) = v_\infty(-x, -y)$ for $y < 0$.

It is straightforwardly verified that (A1) and (A2) are fulfilled. (A3) holds by the smoothness of the function v_∞ in x^*. (A4) could be verified using the theory developed in the following section. Here, it is however immediately checked elementary due to the convexity of f_∞ for $x \geq 0$ and $f'_\infty(x^*) = 1$. Hence, Theorem 3.1 yields that

$$\hat{\tau} = \inf\{t \geq 0 : |X_t| \geq x^*\}$$

is an equilibrium stopping time.

Example 3.5. We now come back to Example 2.6. We already know that there is no pure Markov strategy equilibrium stopping time, so that the approach described in this section cannot be successful. Indeed, $S_1 = \{\partial_1, \partial_2\}$ and the procedure terminates after this step. As argued in Example 2.6 (iv), this is no equilibrium stopping time. More precisely, condition (A4) fails to hold true.
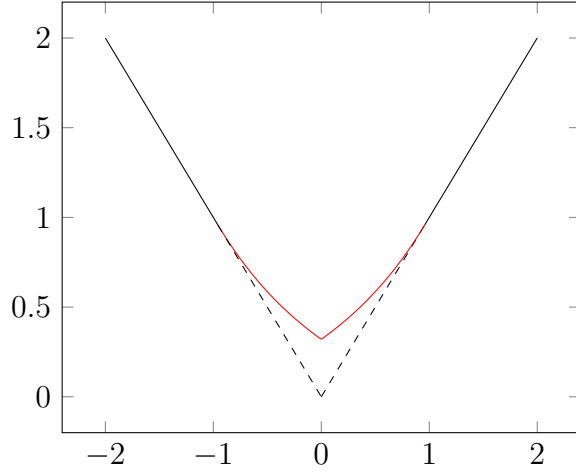
14

Figure 1: The functions $x \mapsto v_\infty(x, x)$ and $x \mapsto F(x, x)$ for $c = 1$. Here, $x^* \approx 0.9575$.

4 A class of one-sided solvable problems with potential jumps

As a more advanced application of the method in the previous example, we consider a Markov process on the real line. To construct an equilibrium stopping time for a wide class of examples, we consider the general setting of [8] for the auxiliary optimal stopping problems with value function

$$v_y(x) = \sup_\tau \mathbb{E}_x(e^{-r\tau} F(X_\tau, y)), \ y \in \mathbb{R}. \tag{4.1}$$

That is, we assume that each function $F(\cdot, y)$ has a representation of the form

$$F(x, y) = \mathbb{E}_x \Big(Q_y(M_T) \Big),$$

where $M_t := \sup_{0 \le s \le t} X_s, \ t \ge 0$, denotes the running maximum process of X and T is an exponentially with parameter r distributed random variable independent of X. In this section we assume that $r > 0$. At first sight, it is not clear at all why such a representation should exist. However, as detailed in Section 2.2 of [8], it always exists under suitable integrability and smoothness assumptions. More explicitly, it is given by

$$Q_y(z) := \frac{1}{r} \int_{-\infty}^z (r - A_X) F(u, y) \mathbb{P}_x(X_T \in du | M_T = z),$$

$$\mathbb{P}_x(X_T \in du | M_T = z) := \mathbb{P}_x(X_T \in du, \ M_T \in dz) / \mathbb{P}_x(M_T \in dz),$$

15

where A_X denotes the (extended) infinitesimal generator of X.

Remark 4.1. If A_X is applied to a function $E \times E \to \mathbb{R}$ then A_X should, here and in the following, be understood to only act on the first variable.

The conditional density used above can be found (semi-)explicitly for general Lévy processes and diffusions, so that also Q_y is given in analytical terms in these cases. The following result, which follows directly from Theorem 2.5 in [8], then leads to the solution of the auxiliary optimal stopping problems in case they are of a one-sided form:

Lemma 4.2. *Assume that for each y there exists a point x_y^* such that*

(B1) $Q_y(x) \leq 0$ *for $x \leq x_y^*$,*

(B2) $Q_y(x)$ *is positive and non-decreasing for $x > x_y^*$.*

Then, the value function of the auxiliary optimal stopping problem (4.1) *is given by*

$$v_y(x) = \mathbb{E}_x \left(Q_y(M_T) 1_{\{M_T \geq x_y^*\}} \right)$$

and

$$\tau_y^* := \inf\{t \geq 0 : X_t \geq x_y^*\}$$

is an optimal stopping time.

Now, using the approach described in Section 3, we obtain the following verification theorem for problems where the underlying auxiliary optimal stopping problems are one-sided.

Theorem 4.3. *In the one-dimensional setting of this section assume that for each y there exists a point x_y^* such that (B1) and (B2) hold true. Furthermore, assume that there exists a point x^* such that $x_y^* \leq y$ for $y \geq x^*$ and $x_y^* \geq x^*$ for $y \leq x^*$. Then*

$$\hat{\tau} := \inf\{t \geq 0 : X_t \geq x^*\}$$

is an equilibrium stopping time.

Remark 4.4. The conditions of Theorem 4.3 imply that if the function $y \mapsto x_y^*$ is continuous then x^* is the unique fixed point of that function.

Proof. Note that Lemma 4.2 yields that the forward iteration sequence of Section 3 is given by

$$S_1 = [x^*, \infty)$$

16

and the procedure then terminates, i.e. $S_1 = S_2 = S_3 = ... = \hat{S}$ and S_1 is closed (this can easily be seen directly and it also follows from the following argument). To apply Theorem 3.1, it remains to check (A4), i.e.

$$F(x, x) \leq \mathbb{E}_x(e^{-r\hat{\tau}} F(X_{\hat{\tau}}, x)) \text{ for all } x < x^*.$$

This, however, holds as

$$F(x, x) = \mathbb{E}_x\left(Q_x(M_T)\right) \leq \mathbb{E}_x\left(Q_x(M_T) 1_{\{M_T \geq x^*\}}\right),$$

where we used that $Q_x(M_T)$ is non-positive on $\{M_T < x^*\} \subseteq \{M_T < x_x^*\}$ by (B1). We conclude by noting that Lemma 2 in [7] yields

$$\mathbb{E}_x\left(Q_x(M_T) 1_{\{M_T \geq x^*\}}\right) = \mathbb{E}_x(e^{-r\hat{\tau}} F(X_{\hat{\tau}}, x)).$$

$\square$

Remark 4.5. By applying the previous results to $-X$, we immediately obtain an analogous result for the case that the auxiliary optimal stopping sets are of left-sided type $(-\infty, x_y^*]$.

Example 4.6. To illustrate the general approach above, we consider a perpetual American call problem with state-dependent strike $K(y)$ in a general Lévy market. One interpretation is an investor who has forgotten the concerted strike of the option. Depending on the state of the price process, she changes her opinion on the concerted strike. More concretely, her reward function for the log-price process X has the structure

$$F(x, y) = (e^x - K(y))^+,$$

where we assume that the function $K : \mathbb{R} \to (0, \infty)$ is continuous and non-increasing; the interpretation of this is that the investor believes the strike to be lower when the asset price is higher. Let X be a general Lévy process. To avoid trivial cases, we assume X not to be a subordinator and to fulfill $E_0(e^{X_1}) < e^r$. For technical reasons, we first ignore the $(\cdot)^+$, i.e. we change the reward function to

$$\tilde{F}(x, y) = e^x - K(y),$$

which makes some arguments and notations in the following shorter. Using the approach from [8], or just by guessing, we see that the function Q_y is given by

$$Q_y(x) = ae^x - K(y),$$

where $a = 1/\mathbb{E}_0 e^{M_T} < 1$, see also [27]. The value of a can be found more explicitly for many classes of processes. For example, for Lévy processes without

17

positive jumps, M_T is exponentially distributed. In the case of a Wiener process X, we obtain $a = \frac{\sqrt{2r}-1}{\sqrt{2r}}$. The optimal stopping boundary for the auxiliary problem is therefore, by Lemma 4.2, given by $x_y^* = \log(K(y)/a)$. Now use the properties of $K(\cdot)$ to verify that that the conditions of Theorem 4.3 are satisfied and that there exits a (unique) fixed point

$$x^* = \log(K(x^*)/a).$$

It follows from Theorem 4.3 that the equilibrium stopping time for the reward function $\tilde{F}(x,y)$ is given by

$$\hat{\tau} = \inf\{t \geq 0 : X_t \geq x^*\}. \tag{4.2}$$

We may therefore conclude that the corresponding equilibrium value function $\tilde{J}_{\hat{\tau}}(x) := \mathbb{E}_x\left(e^{-r\hat{\tau}}\tilde{F}(X_{\hat{\tau}}, x)\right)$ and the reward function $\tilde{F}(x,y)$ satisfy the equilibrium properties (2.1) and (2.2).

In order to show that (4.2) is an equilibrium stopping time also for the original reward function $F(x,y) = (e^x - K(y))^+$, let us verify that also $J_{\hat{\tau}}(x) := \mathbb{E}_x\left(e^{-r\hat{\tau}}F(X_{\hat{\tau}}, x)\right)$ and $F(x,y)$ satisfy (2.1) and (2.2): First, note that if x is such that $e^{x^*}-K(x) \geq 0$, then $\tilde{J}_{\hat{\tau}}(x) = \mathbb{E}_x\left(e^{-r\hat{\tau}}(e^{X_{\hat{\tau}}} - K(x))\right) = \mathbb{E}_x\left(e^{-r\hat{\tau}}(e^{X_{\hat{\tau}}} - K(x))^+\right) = J_{\hat{\tau}}(x)$, and similarly $\tilde{J}_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}(x) = J_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}(x)$. Moreover, if x is such that $e^{x^*} - K(x) < 0$, then we are in the continuation region and hence $\hat{\tau} = \hat{\tau} \circ \theta_{\tau_h} + \tau_h$ for sufficiently small h. It follows that $J_{\hat{\tau}}(x) = J_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}(x)$ for sufficiently small h. We conclude that $J_{\hat{\tau}}(x)$ satisfies (2.2). Second, note that if $x \geq x^*$ then $J_{\hat{\tau}}(x) = F(x,x)$ and (2.1) follows trivially. Let us deal with the case $x < x^*$. If x is such that $e^x - K(x) < 0$ then $F(x,x) = 0$, and since $J_{\hat{\tau}}(x) \geq 0$, it follows that (2.1) satisfied. If x is such that $e^x - K(x) \geq 0$ then $e^{x^*} - K(x) > 0$ which implies that $J_{\hat{\tau}}(x) = \tilde{J}_{\hat{\tau}}(x)$ and $\tilde{F}(x,x) = F(x,x)$. We conclude that $J_{\hat{\tau}}(x)$ and $F(x,x)$ satisfy (2.1).

We have thus shown that the equilibrium stopping time for the original reward $F(x,y) = (e^x - K(y))^+$ is also given by (4.2) and it follows that the corresponding equilibrium value function can be written as

$$J_{\hat{\tau}}(x) = \begin{cases} e^x - K(x), & x \geq x^*, \\ \mathbb{E}_x\left(e^{-r\hat{\tau}}\left(e^{X_{\hat{\tau}}} - K(x)\right)^+\right), & x < x^*, \end{cases}$$

where more explicitly, for x with $\log K(x) < x^*$,

$$\mathbb{E}_x\left(e^{-r\hat{\tau}}\left(e^{X_{\hat{\tau}}} - K(x)\right)^+\right) = \mathbb{E}_x\left(e^{-r\hat{\tau}}\left(e^{X_{\hat{\tau}}} - K(x)\right)\right)$$
$$= a\,\mathbb{E}_x\left(e^{M_T}1_{\{M_T \geq x^*\}}\right) - K(x)\mathbb{P}_x(M_T \geq x^*),$$

Remark 4.7. A put-version of Example 4.6 can be interpreted — economically more meaningful — as an equilibrium selling problem under endogenous habit formation and exponential utility in a Bachelier market. That problem is, however, analyzed and discussed in the more realistic Black-Scholes market in Example 5.7.

5 The time-inconsistent variational inequalities

In the rest of the paper we assume that the state process X is the strong solution to the d-dimensional SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x \in E, \tag{5.1}$$

where W is an r-dimensional Wiener process, the state space $E \subseteq \mathbb{R}^d$ is an open set, and the deterministic functions μ and σ are continuous. Standard conditions for the existence of a strong solution to (5.1) can be found in e.g. [21]. Note that we do not exclude the possibility that $E = \mathbb{R}^d$. The generator A_X is now given by the differential operator

$$A_X = \sum_i^d \mu_i(x)\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j}^d a_{i,j}(x)\frac{\partial^2}{\partial x_i x_j}, \quad a(x) := \sigma(x)\sigma^T(x).$$

5.1 A heuristic derivation of the time-inconsistent variational inequalities

In this subsection we heuristically derive the time-inconsistent variational inequalities. We remark that this section is only of motivational value and that there are no claims of rigor in the derivation. In this subsection we consider $r = 0$ for the ease of exposition. Suppose an equilibrium stopping time $\hat{\tau}$ exists, see Definition 2.2. Recall that $\tau_h = \inf\{t \geq 0 : |X_t - X_0| \geq h\}$ and let $f_{\hat{\tau}}(X_{\tau_h}, x)$ denote the auxiliary function that uses the equilibrium stopping time given the starting value X_{τ_h}. Given sufficient regularity, we use the strong Markov property to see that

$$\begin{aligned}
J_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}(x) &:= \mathbb{E}_x(F(X_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}, x)) \\
&= \mathbb{E}_x(\mathbb{E}_{X_{\tau_h}}(F(X_{\hat{\tau}}, x))) \\
&= \mathbb{E}_x(f_{\hat{\tau}}(X_{\tau_h}, x))
\end{aligned} \tag{5.2}$$

and Itô's formula to obtain

$$\mathbb{E}_x(f_{\hat{\tau}}(X_{\tau_h}, x)) = f_{\hat{\tau}}(x, x) + \mathbb{E}_x\left(\int_0^{\tau_h} A_X f_{\hat{\tau}}(X_t, x)dt\right), \tag{5.3}$$

where we recall that the differential operator A_X operates only on the first variable. We now use the dominated convergence theorem, Lebesgue's differentiation theorem, (5.2), (5.3) and $J_{\hat{\tau}}(x) = f_{\hat{\tau}}(x, x)$ (cf. Definition 2.3) to obtain, under sufficient regularity,

$$\liminf_{h \searrow 0} \frac{J_{\hat{\tau}}(x) - J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x)}{\mathbb{E}_x(\tau_h)} = \liminf_{h \searrow 0} \frac{f_{\hat{\tau}}(x, x) - \mathbb{E}_x(f_{\hat{\tau}}(X_{\tau_h}, x))}{\mathbb{E}_x(\tau_h)}$$
$$= \liminf_{h \searrow 0} \frac{\mathbb{E}_x \left(\int_0^{\tau_h} -A_X f_{\hat{\tau}}(X_t, x) dt \right)}{\mathbb{E}_x(\tau_h)}$$
$$= -A_X f_{\hat{\tau}}(x, x).$$

This, of course, reflects the well-known characterization of the infinitesimal generator due to Dynkin. The definition of an equilibrium stopping time in Definition 2.2 therefore translates to $A_X f_{\hat{\tau}}(x, x) \leq 0$ and $J_{\hat{\tau}}(x) - F(x, x) \geq 0$. Now note that $J_{\hat{\tau}}(x) = f_{\hat{\tau}}(x, x)$ implies that the equilibrium value function $J_{\hat{\tau}}(x)$ is completely determined by the auxiliary function $f_{\hat{\tau}}(x, y)$. We therefore summarize the above in terms of the auxiliary function:

$$f_{\hat{\tau}}(x, x) \geq F(x, x), \quad x \in E \tag{5.4}$$
$$A_X f_{\hat{\tau}}(x, x) \leq 0, \quad x \in E.$$

For any x, stopping yields the value $F(x, x)$. Using (5.4) we see that it is therefore optimal, for the x-agent, to stop if and only if $f_{\hat{\tau}}(x, x) = F(x, x)$. Suppose that the set

$$C = \{x \in E : f_{\hat{\tau}}(x, x) > F(x, x)\}$$

is open, where C is said to be the continuation region. It follows that the corresponding equilibrium stopping time is the first exit time from C, or analogously the first entrance time into the stopping region $E \backslash C$, i.e

$$\tau_{E \backslash C} = \inf\{t \geq 0 : X_t \in E \backslash C\} = \inf\{t \geq 0 : X_t \notin C\}, \tag{5.5}$$

which implies that $f_{\hat{\tau}}(x, y) = \mathbb{E}_x(F(X_{\tau_{E \backslash C}}, y))$ for all x and y. By (5.5) it is also clear that if $x \in E \backslash C$, then $\tau_{E \backslash C} = 0$ which implies that $f_{\hat{\tau}}(x, y) = \mathbb{E}_x(F(X_{\tau_{E \backslash C}}, y)) = F(x, y)$. It therefore holds, for any y, that

$$f_{\hat{\tau}}(x, y) = F(x, y), \quad x \in E \backslash C.$$

Moreover, since $f_{\hat{\tau}}(x, y) = \mathbb{E}_x(F(X_{\tau_{E \backslash C}}, y))$ it follows that $f_{\hat{\tau}}(X_t, y)$ is a martingale on C, for any fixed y, given sufficient regularity. Hence, for any fixed y,

$$A_X f_{\hat{\tau}}(x, y) = 0, \quad x \in C.$$

20

Let us summarize our findings. If an equilibrium stopping time exists then, under the assumption of sufficient regularity, it is given by $\tau_{E\setminus C}$ defined in (5.5) and the auxiliary function $f_{\hat{\tau}}(x, y) = \mathbb{E}_x(F(\tau_{E\setminus C}, y))$ satisfies

$$A_X f_{\hat{\tau}}(x, x) \le 0, \quad x \in E,$$

and for any fixed $y \in E$

$$A_X f_{\hat{\tau}}(x, y) = 0, \quad x \in C,$$
$$f_{\hat{\tau}}(x, y) - F(x, y) = 0, \quad x \in E\setminus C,$$

where

$$C = \{x \in E : f_{\hat{\tau}}(x, x) > F(x, x)\}.$$

We call the expressions above the *time-inconsistent variational inequalities*.

5.2   A verification theorem

Let us define the time-inconsistent variational inequalities in more detail.

Definition 5.1. A function $f : E \times E \to \mathbb{R}$ is said to satisfy the *time-inconsistent variational inequalities* if[*]

$$A_X f(x, x) - r f(x, x) \le 0, \quad x \in E\setminus\partial C, \tag{5.6}$$

and, for each fixed $y \in E$,

$$A_X f(x, y) - r f(x, y) = 0, \quad x \in C, \tag{5.7}$$
$$f(x, y) - F(x, y) = 0, \quad x \in E\setminus C, \tag{5.8}$$

where

$$C := \{x \in E : f(x, x) > F(x, x)\}.$$

Moreover, the function $f(\cdot, y) : E \to \mathbb{R}$ must, for each fixed $y \in E$, satisfy:

(i) $f(\cdot, y) \in \mathcal{C}(\overline{C}) \cap \mathcal{C}^2(C)$, where $\overline{C}$ denotes the closure of C in E,

(ii) $f(\cdot, y) \in \mathcal{C}^1(B(y, \epsilon)) \cap \mathcal{C}^2(B(y, \epsilon)\setminus\partial C)$ for some $\epsilon > 0$, where the second order derivative is locally bounded (near ∂C),

(iii) $f(\cdot, y)$ is bounded on $\overline{C}$.

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[*]Recall that the differential operator A_X operates only on the first variable, in e.g. $f(x, x)$.

Lastly, we also demand that:

(iv) C is open and $\partial C \neq \emptyset$ is a Lipschitz surface[†].

(v) $\limsup_{z \notin \partial C \to y} (A_X f(z,y) - r f(z,y)) \leq 0$, for $y \in \partial C$.

Theorem 5.2. *Suppose that a function $f : E \times E \to \mathbb{R}$ solves the time-inconsistent variational inequalities. Suppose that the state process X that solves the SDE* (5.1) *spends almost no time on the boundary ∂C, i.e.*

$$\int_0^\infty I_{\partial C}(X_t) dt = 0 \quad a.s., \tag{5.9}$$

and that

$$\hat{\tau} := \inf\{t \geq 0 : X_t \notin C\} < \infty \quad a.s., \tag{5.10}$$

for each starting value $X_0 = x \in E$. Then,

- *$J : E \to \mathbb{R}$, with $J(x) := f(x,x)$, is an equilibrium value function,*

- *$f : E \times E \to \mathbb{R}$ is the corresponding auxiliary function, and*

- *the stopping time $\hat{\tau}$ in* (5.10) *is the corresponding equilibrium stopping time.*

Proof. Recall that the state space $E \subseteq \mathbb{R}^d$ is here assumed to be an open set and note that E can here, without loss of generality, be taken to be connected, since X has continuous sample paths. Let $\{C_k\}_{k=1}^\infty$ be an increasing sequence of open, bounded and connected sets with $\overline{C}_k \subseteq C$ and $\cup_{k=1}^\infty C_k = C$. Consider arbitrary $y \in E$ and $x \in C$, which implies that $x \in C_k$ for any $k \geq k'$, for some k'. Let $\tau_k = \inf\{t \geq 0 : X_t \notin C_k\} \wedge k$. Use (i), Itô's formula and (5.7) to obtain

$$f(x,y) = \mathbb{E}_x \left(e^{-r\tau_k} f(X_{\tau_k}, y) - \int_0^{\tau_k} e^{-rt}(A_X f(X_t, y) - r f(X_t, y)) dt \right)$$
$$= \mathbb{E}_x \left(e^{-r\tau_k} f(X_{\tau_k}, y) \right)$$

where the Itô integral has vanished by the continuity of $\sigma(x)$, the continuity of the trajectories of X, the continuity of $\frac{\partial^2 f(x,y)}{\partial x^2}$ on C, and the boundedness of X_s on the bounded stochastic interval $[0, \tau_k]$. Note that (5.8), (i) and (iv) imply, for fixed $y \in E$, that $f(\cdot, y)$ is continuous on $\overline{C}$ with $f(x,y) = F(x,y)$ on ∂C, where $\partial C \neq \emptyset$ by assumption. Since, $f(\cdot, y)$ is bounded on $\overline{C}$ (cf. (iii)) we may

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[†]For a definition see [28, ch. 10].

thus use the bounded convergence theorem, and that $\tau_k \to \hat{\tau}$ a.s. as $k \to \infty$ (cf. (5.10)), to obtain

$$f(x,y) = \lim_{k\to\infty} \mathbb{E}_x\left(e^{-r\tau_k}f(X_{\tau_k},y)\right) = \mathbb{E}_x\left(e^{-r\hat{\tau}}F(X_{\hat{\tau}},y)\right). \qquad (5.11)$$

Using (5.8) and (5.10) we see that this implies that

$$f(x,y) = \mathbb{E}_x(e^{-r\hat{\tau}}F(X_{\hat{\tau}},y)) \quad \text{on } E \times E. \qquad (5.12)$$

Definition 2.2, Definition 2.3 and (5.12) yield the following: if we can prove that

$$f(x,x) - F(x,x) \geq 0, \quad \text{for each } x \in E, \text{ and} \qquad (5.13)$$

$$\liminf_{h\searrow 0} \frac{f(x,x) - J_{\hat{\tau}\circ\theta_{\tau_h}+\tau_h}(x)}{\mathbb{E}_x(\tau_h)} \geq 0, \quad \text{for each } x \in E, \qquad (5.14)$$

then it follows that $\hat{\tau}$ in (5.10) is an equilibrium stopping time, that $J(x) := f(x,x)$ is the corresponding equilibrium value function and that $f(x,y)$ is the corresponding auxiliary function. Thus, all we have left to do is to show that (5.13) and (5.14) are satisfied.

Since $f(x,y)$ solves the time-inconsistent variational inequalities we know that $f(x,x) > F(x,x)$ on C and $f(x,x) = F(x,x)$ on $E\backslash C$. It follows that (5.13) holds.

For each fixed $y \in E$, observe that $B(y,\epsilon)$ and $B(y,\epsilon)\cap C$ are open, $B(y,\epsilon)\backslash\partial(B(y,\epsilon)\cap C) = B(y,\epsilon)\backslash\partial C$ and that $\partial(B(y,\epsilon)\cap C)$ is a Lipschitz surface (cf. (iv)). It therefore follows from (ii) and Theorem D.1 in [28, App. D] that there exists, for each fixed $y \in E$ and some $\epsilon > 0$, a sequence of functions $\{f_i(\cdot,y)\}_{i=1}^{\infty}$ such that

(a) $f_i(\cdot,y) \in \mathcal{C}(\overline{B}(y,\epsilon)) \cap \mathcal{C}^2(B(y,\epsilon))$ for each i,

(b) $f_i(\cdot,y)$ converges to $f(\cdot,y)$ uniformly on compact subsets of $\overline{B}(y,\epsilon)$ as $i \to \infty$,

(c) $A_X f_i(\cdot,y)$ converges to $A_X f(\cdot,y)$ uniformly on compact subsets of $B(y,\epsilon)\backslash\partial C$ as $i \to \infty$,

(d) $\{A_X f_i(\cdot,y)\}_{i=1}^{\infty}$ is locally bounded on $\overline{B}(y,\epsilon)$.

For any fixed $x \in E$, $h \in (0,\epsilon)$, i and $k > 0$ it thus follows from Itô's formula that

$$f_i(x,x) = \mathbb{E}_x\left(e^{-r\tau_h\wedge k}f_i(X_{\tau_h\wedge k},x) - \int_0^{\tau_h\wedge k} e^{-rt}(A_X f_i(X_t,x) - rf_i(X_t,x))dt\right).$$

23

where the Itô integral has vanished for reasons analogous to the above. Because of the continuity in (a) and X_t being bounded on $[0, \tau_h]$ we can use the bounded convergence theorem to obtain

$$
f_i(x, x) = \lim_{k \to \infty} f_i(x, x)
$$

$$
= \mathbb{E}_x \left(\lim_{k \to \infty} \left(e^{-r\tau_h \wedge k} f_i(X_{\tau_h \wedge k}, x) - \int_0^{\tau_h \wedge k} e^{-rt} (A_X f_i(X_t, x) - r f_i(X_t, x)) dt \right) \right)
$$

$$
= \mathbb{E}_x \left(e^{-r\tau_h} f_i(X_{\tau_h}, x) - \int_0^{\tau_h} e^{-rt} (A_X f_i(X_t, x) - r f_i(X_t, x)) dt \right).
$$

Set the undefined $\frac{\partial^2 f(x,y)}{\partial x^2}, x \in \partial C$, to zero. Now use the convergence and boundedness properties in (b), (c), and (d), and the regularity in (5.9), and the bounded convergence theorem to obtain

$$
f(x, x) = \lim_{i \to \infty} f_i(x, x)
$$

$$
= \mathbb{E}_x \left(\lim_{i \to \infty} \left(e^{-r\tau_h} f_i(X_{\tau_h}, x) - \int_0^{\tau_h} e^{-rt} (A_X f_i(X_t, x) - r f_i(X_t, x)) dt \right) \right)
$$

$$
= \mathbb{E}_x \left(e^{-r\tau_h} f(X_{\tau_h}, x) - \int_0^{\tau_h} e^{-rt} (A_X f(X_t, x) - r f(X_t, x)) dt \right). \quad (5.15)
$$

Now use (5.12) and the strong Markov property to see that

$$
\mathbb{E}_x \left(e^{-r\tau_h} f(X_{\tau_h}, x) \right) = \mathbb{E}_x \left(e^{-r\tau_h} \mathbb{E}_{X_{\tau_h}} \left(e^{-r\hat{\tau}} F(X_{\hat{\tau}}, x) \right) \right)
$$

$$
= \mathbb{E}_x \left(e^{-r(\hat{\tau} \circ \theta_{\tau_h} + \tau_h)} F(X_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}, x) \right)
$$

$$
= J_{\hat{\tau} \circ \theta_{\tau_h} + \tau_h}(x), \quad (5.16)
$$

where we also relied on (iii) and (5.8). Using (5.15) and (5.16) we rewrite the left hand side of (5.14) as

$$
\liminf_{h \searrow 0} \frac{-\mathbb{E}_x \left(\int_0^{\tau_h} e^{-rt} (A_X f(X_t, x) - r f(X_t, x)) dt \right)}{\mathbb{E}_x(\tau_h)}
$$

$$
= -\limsup_{h \searrow 0} \frac{\mathbb{E}_x \left(\int_0^{\tau_h} e^{-rt} (A_X f(X_t, x) - r f(X_t, x)) dt \right)}{\mathbb{E}_x(\tau_h)}. \quad (5.17)
$$

Hence, all we have left to do in order to show that (5.14) is true, i.e. to conclude the proof, is to show that (5.17) is non-negative for all $x \in E$. Let us do this.

First consider an arbitrary $x \in E \backslash \partial C$. Recall that $\mu(x), \sigma(x)$ and the trajectories of X are continuous. Note that $h \in (0, \epsilon)$ implies that the process $(X_t)_{0 \le t \le \tau_h}$ with $X_0 = x$ stays in $B(x, \epsilon)$. The regularity properties in (ii) therefore imply

24

that the integrand in (5.17), i.e. $e^{-rt}(A_X f(X_t, x) - rf(X_t, x))$, is a.e. continuous in t a.s. Recall that X_s is bounded on $[0, \tau_h]$ when $h \in (0, \epsilon)$. Note also that that if we pick a sufficiently small $h = h(\omega)$ then the integrand in (5.17) is continuous in t a.s, since we can for sufficiently small $h = h(\omega)$ avoid the issue that $\frac{\partial^2 f(x,y)}{\partial x^2}$ is arbitrarily set to 0 at ∂C. It follows that we may use the bounded convergence theorem and Lebesgue's differentiation theorem to obtain that (5.17) is equal to

$$- (A_X f(x, x) + rf(x, x)) \geq 0, \text{ for } x \in E \backslash \partial C$$

where the inequality follows from (5.6). Now consider an arbitrary $x \in \partial C$. Replace the integrand in (5.17) with a right-continuous version (in t a.s.) and use (v) and (5.9) in the following way

$$- \limsup_{h \searrow 0} \frac{\mathbb{E}_x \left(\int_0^{\tau_h} e^{-rt}(A_X f(X_t, x) - rf(X_t, x))dt \right)}{\mathbb{E}_x(\tau_h)}$$

$$= - \limsup_{h \searrow 0} \frac{\mathbb{E}_x \left(\int_0^{\tau_h} \lim_{k \searrow 0} \sup_{0 < l \leq k} e^{-r(t+l)}(A_X f(X_{t+l}, x) - rf(X_{t+l}, x))dt \right)}{\mathbb{E}_x(\tau_h)}$$

$$\geq - \limsup_{z \notin \partial C \to x} (A_X f(z, x) - rf(z, x)) \geq 0, \text{ for } x \in \partial C.$$

We have thus shown that (5.17) is non-negative for all $x \in E$. $\qquad \square$

Remark 5.3. In the case $r > 0$ then the condition in (5.10) is not necessary in order for the verification theorem to be true, since in this case (iii) is sufficient to obtain (5.11) (using also our convention regarding expected values and infinite stopping times, as described in the beginning of Section 2).

Remark 5.4. The continuous differentiability requirement (ii) and requirement (iv) imply that we can approximate the function $f(\cdot, y)$ by the sequence of $\mathcal{C}^2$ functions $f_i(\cdot, y)$, on which we can apply the standard Itô formula. After this we let $i \to \infty$ and effectively find that Dynkin's formula (5.15) holds. We remark that the continuous differentiability requirement (ii) could in some settings be relaxed if we instead of using the current approach were to use a more general version of Itô's formula based on the concept of local time, see e.g. [21, 31].

Remark 5.5. Let us underline that we can now apply the standard procedure to use the verification theorem in order to find equilibrium value functions and equilibrium stopping times in particular cases. More precisely:

(i) Make an ansatz, i.e. make an educated guess of how the solution $f(x, y)$ to the time-inconsistent variational inequalities should look like. The guess $f(x, y)$ should typically have traits in common with $F(x, y)$ and involve unspecified parameters, see e.g. the a, b and x^* in Example 5.6 below.

(ii) Use the verification theorem to verify that $f(x, y)$ can solve the time-inconsistent variational inequalities (and the regularity conditions of the verification theorem) for some specific values of the parameter(s). Note that the point of step (ii) is two-fold 1) to make sure that the guess $f(x, y)$ has any chance of solving the time-inconsistent variational inequalities, and 2) to determine the unspecified parameters of $f(x, y)$.

(iii) If the previous steps were successful then you may use the verification theorem to conclude that the guess $f(x, y)$, with the specified parameter(s), is indeed the auxiliary function, that $J(x) := f(x, x)$ is the corresponding equilibrium value function and that $\hat{\tau} := \inf\{t \geq 0 : X_t \notin C\}$ is the equilibrium stopping time, where $C := \{x \in E : f(x, x) > F(x, x)\}$.

Example 5.6. Let us re-analyze the optimistic holder of the perpetual American option from Example 3.4 using the verification theorem. The advantage here is that we do not have to solve a sequence of free boundary problems, but can make a direct ansatz for the value function. Since the state process is a Wiener process it follows that $E = \mathbb{R}$. As $x \mapsto e^{-cx}$, $x \mapsto e^{cx}$, $c = \sqrt{2r}$, are the fundamental solutions to $A_X f = rf$ and due to symmetry, a natural guess for a solution to the time-inconsistent variational inequalities is

$$f(x, y) = \begin{cases} x & , \ x \geq x^*, \\ ae^{-cx} + be^{cx} & , \ -x^* < x < x^*, y \geq 0 \\ 0 & , \ x \leq -x^*, \end{cases}$$

and $f(x, y) = f(-x, -y)$, $y < 0$, with the continuation region $C = (-x^*, x^*)$, for some parameters a, b and x^* to be determined. For $f(\cdot, y)$ to be continuous, we need

$$ae^{-cx^*} + be^{cx^*} = x^*,$$
$$ae^{cx^*} + be^{-cx^*} = 0.$$

For sufficient smoothness of $f(\cdot, x^*)$ we furthermore need

$$-cae^{-cx^*} + cbe^{cx^*} = 1.$$

Elementary arguments yield that this system of equations indeed has a solution given by

$$f(x, y) = \begin{cases} x & , \ x \geq x^*, \\ \frac{1}{2}e^{cx^*}(x^* - \frac{1}{c})e^{-cx} + \frac{1}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{cx} & , \ -x^* < x < x^*, y \geq 0 \\ 0 & , \ x \leq -x^*, \end{cases}$$

where $x^* \in (0, 1/c)$ satisfies

$$-e^{4cx^*}(x^* - \frac{1}{c}) = x^* + \frac{1}{c}. \tag{5.18}$$

Using elementary methods one can now verify that:

- $x \mapsto f(x, y)$ is convex and $\frac{\partial f(x^*, y)}{\partial x} = 1$ for $x, y \geq 0$ which implies that $C = (-x^*, x^*) = \{x \in \mathbb{R} : f(x, x) - F(x, x) > 0\}$, which also implies that (5.9) and the condition in (5.10) are fulfilled (we remark that the last condition is not necessary since $r > 0$, cf. Remark 5.3),

- conditions (5.8), (i), (iii) and (iv) hold.

Let us explicitly verify condition (ii). Naively taking derivatives gives us

$$\frac{\partial f(x, y)}{\partial x} = \begin{cases} 1 & , x \geq x^*, \\ -\frac{c}{2}e^{cx^*}(x^* - \frac{1}{c})e^{-cx} + \frac{c}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{cx} & , -x^* < x < x^*, y \geq 0 \\ 0 & , x \leq -x^*, \end{cases}$$

$$\frac{\partial f(x, y)}{\partial x} = \begin{cases} 0 & , x \geq x^*, \\ \frac{c}{2}e^{cx^*}(x^* - \frac{1}{c})e^{cx} - \frac{c}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{-cx} & , -x^* < x < x^*, y < 0 \\ -1 & , x \leq -x^*. \end{cases}$$

It is easy to check that these derivatives are well defined except at points (x, y) satisfying $(x, y) = (x^*, y)$ with $y < 0$ or $(x, y) = (-x^*, y)$ with $y \geq 0$. Hence, for fixed y, $f(\cdot, y) \in \mathcal{C}^1(B(y, \epsilon))$, for a sufficiently small $\epsilon > 0$. Naively taking derivatives again gives us

$$\frac{\partial^2 f(x, y)}{\partial x^2} = \begin{cases} 0 & , x > x^*, \\ \frac{c^2}{2}e^{cx^*}(x^* - \frac{1}{c})e^{-cx} + \frac{c^2}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{cx} & , -x^* < x < x^*, y \geq 0 \\ 0 & , x < -x^*, \end{cases}$$

$$\frac{\partial^2 f(x, y)}{\partial x^2} = \begin{cases} 0 & , x > x^*, \\ \frac{c^2}{2}e^{cx^*}(x^* - \frac{1}{c})e^{cx} + \frac{c^2}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{-cx} & , -x^* < x < x^*, y < 0 \\ 0 & , x < -x^*. \end{cases}$$

We thus see that $f(\cdot, y) \in \mathcal{C}^2(B(y, \epsilon) \backslash \partial C)$, for any $y \in E$, and that this derivative is locally bounded. We have thus verified (ii). Now use that $c = \sqrt{2r}$ and the above to obtain
$A_X f(x, y) - rf(x, y) = \frac{1}{2}\frac{\partial^2 f(x,y)}{\partial x^2} - rf(x, y)$

$$= \begin{cases} -rx < 0 & , x > x^*, \\ 0 & , -x^* < x < x^*, y \geq 0 \\ 0 & , x < -x^*, \end{cases}$$

27

$$A_X f(x, y) - rf(x, y) = \frac{1}{2}\frac{\partial^2 f(x,y)}{\partial x^2} - rf(x,y)$$

$$= \begin{cases} 0 & , \ x > x^*, \\ 0 & , \ -x^* < x < x^*, y < 0 \\ rx < 0 & , \ x < -x^*. \end{cases}$$

This means that (5.6), (5.7) and (v) are also satisfied. We have thus verified that the function $f(x, y)$ with x^* determined in (5.18) is a solution to the time-inconsistent variational inequalities. The verification theorem therefore implies that $\hat{\tau} = \inf\{t \geq 0 : X_t \notin (-x^*, x^*)\}$ is an equilibrium stopping time and that the corresponding equilibrium value function is

$$J(x) = \begin{cases} x & , \ x \geq x^*, \\ \frac{1}{2}e^{cx^*}(x^* - \frac{1}{c})e^{-cx} + \frac{1}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{cx} & , \ 0 \leq x < x^*, \\ \frac{1}{2}e^{cx^*}(x^* - \frac{1}{c})e^{cx} + \frac{1}{2}e^{-cx^*}(x^* + \frac{1}{c})e^{-cx} & , \ -x^* < x < 0, \\ -x & , \ x \leq -x^*. \end{cases}$$

Example 5.7. Equilibrium selling strategies under endogenous habit formation and exponential utility. We will now study a model for selling strategies under exponential utility and endogenous habit formation, using the verification theorem. Section 1.1 contains information about previous literature on related problems.

Consider an investor who wishes to optimally dispose of an asset in a Black-Scholes market. Specifically, the price of the asset, measured in e.g USD or MUSD, is given by the process X satisfying

$$dX_t = \sigma X_t dW_t.$$

We model the utility of the investor as exponential, but we also let her utility be inversely related to the present price of the asset, which makes the problem time-inconsistent. Specifically, we assume that the agent wishes to maximize

$$\mathbb{E}_x\left(e^{-r\tau}\left(1 - e^{-a(X_\tau + g(x) - k)}\right)\right),$$

where $a, r, k > 0$ are constants and $g : [0, \infty) \to \mathbb{R}$ is a non-increasing bounded function such that $x \mapsto x + g(x)$ is non-decreasing and $g(0) = 0$.

We will study this endogenous habit formation selling problem without making any functional assumptions for $g(\cdot)$. In Figure 2, we present the solution to the problem for a particular specification of $g(\cdot)$.

Remark 5.8. The reward function of the present model corresponds to the function $F(x, y) := 1 - e^{-a(x+g(y)-k)}$, which is clearly time-inconsistent and bounded on $[0, \infty)^2$. If $g(\cdot) = 0$ and $k = 0$ then we recover a standard exponential utility function.

Remark 5.9. We interpret this model as the investor having formed a habit regarding what she thinks the asset should be worth, and the larger the current value of the asset is the less happy she will be for a given selling price in the future. The parameter a is a measure of the risk aversion of the investor: a larger a means more risk aversion. The parameter r is a measure of the impatience of the investor: a larger r means more impatience. The nonstandard feature of this model is the function $g(\cdot)$ which we interpret to be a measure of the habit formation of the investor. The assumption that $g(\cdot)$ is non-increasing is interpreted as follows: the smaller the current price x is, the happier the investor is given the same future selling price. The assumption that $x \mapsto x + g(x)$ is increasing means that the investor cannot become less happy for a larger selling price given immediately selling. The parameter k allows the possibility for negative utility.

A reasonable starting point is to try with a one-sided solution $C = (0, x^*)$. We therefore guess that the auxiliary function is

$$
f(x, y) = \begin{cases} 1 - e^{-a(x+g(y)-k)} & , \ x \geq x^*, \\ \mathbb{E}_x \left(e^{-r\tau_{[x^*,\infty)}} \left(1 - e^{-a\left(X_{\tau_{[x^*,\infty)}} + g(y) - k \right)} \right) \right) & , \ 0 < x < x^*, \end{cases}
$$

for some x^* to be determined. Using standard theory, see e.g [28, ch. 9,10], we note that the function $f(x, y)$ can be simplified using

$$
\mathbb{E}_x \left(e^{-r\tau_{[x^*,\infty)}} \left(1 - e^{-a\left(X_{\tau_{[x^*,\infty)}} + g(y) - k \right)} \right) \right) = \left(\frac{x}{x^*} \right)^{\gamma} \left(1 - e^{-a(x^*+g(y)-k)} \right)
$$

where $\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2r}{\sigma^2}}$. Naively taking derivatives therefore gives us

$$
\frac{\partial f(x, y)}{\partial x} = \begin{cases} ae^{-a(x+g(y)-k)} & , \ x \geq x^*, \\ \gamma \frac{x^{\gamma-1}}{x^{*\gamma}} \left(1 - e^{-a(x^*+g(y)-k)} \right) & , \ 0 < x < x^*. \end{cases}
$$

In order for (ii) to be fulfilled x^* must satisfy

$$
x^* ae^{-a(x^*+g(x^*)-k)} = \gamma \left(1 - e^{-a(x^*+g(x^*)-k)} \right) \tag{5.19}
$$

which means that x^* must be the zero of the function

$$
H(x) = \gamma - e^{-a(x+g(x)-k)}(\gamma + ax), \tag{5.20}
$$

which must be verified to exist uniquely in $(0, \infty)$ for the particular choice of $g(\cdot)$. Note that a unique x^* exists if there is no habit formation i.e. with $g(\cdot) = 0$; to see this note that if $g(\cdot) = 0$ then $H(0) = \gamma(1 - e^{ak}) < 0$ and $H'(x) = ae^{-a(x-k)}(ax + \gamma - 1) > 0$ on $[0, \infty)$, since $\gamma > 1$.

Taking derivatives again gives us

$$\frac{\partial^2 f(x,y)}{\partial x^2} = \begin{cases} -a^2 e^{-a(x+g(y)-k)} & , \ x > x^*, \\ \gamma(\gamma-1)\frac{x^{\gamma-2}}{x^{*\gamma}}\left(1 - e^{-a(x^*+g(y)-k)}\right) & , \ 0 < x < x^*. \end{cases}$$

Using $A_X f(x,y) = \frac{1}{2}x^2\sigma^2\frac{\partial^2 f(x,y)}{\partial x^2}$ and $\gamma(\gamma-1) = \frac{2r}{\sigma^2}$, we obtain
$A_X f(x,y) - rf(x,y) =$

$$= \begin{cases} -\frac{x^2}{2}\sigma^2 a^2 e^{-a(x+g(y)-k)} - r\left(1 - e^{-a(x+g(y)-k)}\right) & , \ x > x^*, \\ \frac{x^2}{2}\sigma^2 \frac{2r}{\sigma^2}\frac{x^{\gamma-2}}{x^{*\gamma}}\left(1 - e^{-a(x^*+g(y)-k)}\right) - r\left(\frac{x}{x^*}\right)^\gamma\left(1 - e^{-a(x^*+g(y)-k)}\right) & , \ 0 < x < x^*, \end{cases}$$

$$= \begin{cases} -\frac{x^2}{2}\sigma^2 a^2 e^{-a(x-g(y)-k)} - r\left(1 - e^{-a(x+g(y)-k)}\right) & , \ x > x^*, \\ 0 & , \ 0 < x < x^*, \end{cases}$$

which implies that $f(x,y)$ satisfies (5.7). It follows from (5.19) that $x^* + g(x^*) - k > 0$, which since $x + g(x)$ is non-decreasing implies that

$$A_X f(x,x) - rf(x,x) = -\frac{x^2}{2}\sigma^2 a^2 e^{-a(x+g(x)-k)} - r\left(1 - e^{-a(x+g(x)-k)}\right) < 0, \ x > x^*.$$

Hence, (5.6) is satisfied. Condition (v) is verified in the same way. Conditions (i), (ii), (iii) and (iv) are directly verified. Now, if $g(\cdot)$ is such that

$$C := (0, x^*) = \{x \in \mathbb{R} : f(x,x) - F(x,x) > 0\} \tag{5.21}$$

holds then conditions (5.8) and (5.9) follow, and all the conditions of the verification theorem are hence fulfilled (the condition in (5.10) is not necessary in this case, cf. Remark 5.3).

To show that (5.21) holds for the case $g(\cdot) = 0$ it is sufficient to show that

$$\left(\frac{x}{x^*}\right)^\gamma\left(1 - e^{-a(x^*-k)}\right) > 1 - e^{-a(x-k)}, \ 0 < x < x^*.$$

This is trivially true if the right side is non-positive since the left side is positive by (5.19), and we may thus treat the right side and the left side as positive. It is therefore sufficient to show that

$$\kappa(x) := \frac{x^{*\gamma}}{1 - e^{-a(x^*-k)}}x^{-\gamma}\left(1 - e^{-a(x-k)}\right) < 1, \ 0 < x < x^*.$$

We obtain that $\kappa'(x) = \frac{x^{*\gamma}}{1-e^{-a(x^*-k)}}x^{-\gamma-1}(-H(x))$, and since x^* is assumed to be the unique zero of $H(\cdot)$ and $H(0) < 0$ it follows that $\kappa'(x) > 0$, where we also used (5.19) to see that the first fraction in $\kappa(x)$ is positive. Since $\kappa(x^*) = 1$ it
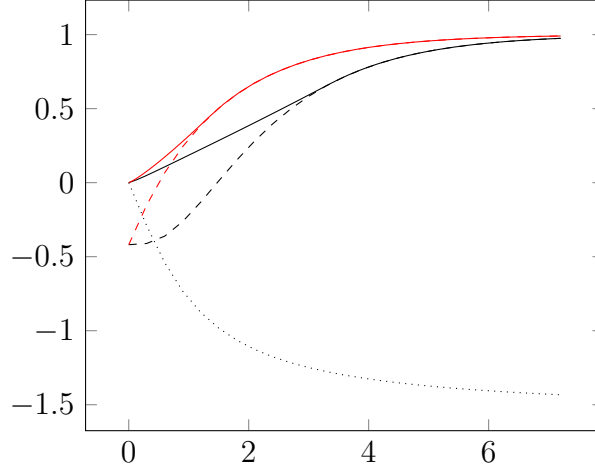
Figure 2: $x \mapsto J(x)$ (solid, black) and $x \mapsto F(x,x)$ (dashed, black), with $g(x) = \text{arccot}(x) - \frac{\pi}{2}$, here $x^* \approx 3.3524$. $x \mapsto J(x)$ (solid, red) and $x \mapsto F(x,x)$ (dashed, red), with $g(x) = 0$, here $x^* \approx 1.3412$. $x \mapsto \text{arccot}(x) - \frac{\pi}{2}$ (dotted, black). $a = 0.7, r = 0.1, k = 0.5$ and $\sigma = 1$.

follows that $\kappa(x^*) < 1$ for $0 < x < x^*$ and we are done. In order to show that (5.21) holds when $g(\cdot) \neq 0$, we must show that

$$\left(\frac{x}{x^*}\right)^\gamma \left(1 - e^{-a(x^*-k)}e^{-ag(x)}\right) > 1 - e^{-a(x-k)}e^{-ag(x)}, \ \ 0 < x < x^*.$$

This is trivially true if the right side is non-positive; to see this use that the left side is positive by (5.19) and since $g(\cdot)$ is non-increasing. We may thus treat both the left and the right sides as positive; and since $1 - e^{-a(x-k)} > 1 - e^{-a(x-k)}e^{-ag(x)}$ we may also treat $1 - e^{-a(x-k)}$ as positive. It is thus enough to show that $\left(\frac{x^*}{x}\right)^\gamma \frac{1-e^{-a(x-k)}e^{-ag(x)}}{1-e^{-a(x^*-k)}e^{-ag(x)}} < 1, \ \ 0 < x < x^*$. But $\frac{1-e^{-a(x-k)}e^{-ag(x)}}{1-e^{-a(x^*-k)}e^{-ag(x)}} < \frac{1-e^{-a(x-k)}}{1-e^{-a(x^*-k)}}$ and the result follows from the case when $g(\cdot) = 0$.

We conclude that if $g(\cdot)$ is such that $H(\cdot)$ in (5.20) has a unique zero x^*, then the equilibrium stopping time is $\hat{\tau} = \inf\{t \geq 0 : X_t \geq x^*\}$ and the equilibrium value function is

$$J(x) = \begin{cases} 1 - e^{-a(x+g(x)-k)} & , \ x \geq x^*, \\ \left(\frac{x}{x^*}\right)^\gamma \left(1 - e^{-a(x^*+g(x)-k)}\right) & , \ 0 < x < x^*. \end{cases}$$

31

References

[1] N. Barberis. A model of casino gambling. *Management Science*, 58(1):35–51, 2012.

[2] R. J. Barro. Ramsey meets Laibson in the neoclassical growth model. *The Quarterly Journal of Economics*, 114(4):1125–1152, 1999.

[3] T. Björk, M. Khapko, and A. Murgoci. On time-inconsistent stochastic control in continuous time. *Finance and Stochastics*, 21(2):331–360, 2017.

[4] T. Björk and A. Murgoci. A theory of Markovian time-inconsistent stochastic control in discrete time. *Finance and Stochastics*, 18(3):545–592, 2014.

[5] T. Björk, A. Murgoci, and X. Y. Zhou. Mean-variance portfolio optimization with state-dependent risk aversion. *Mathematical Finance*, 24(1):1467–9965, 2014.

[6] S. Christensen and A. Irle. A harmonic function technique for the optimal stopping of diffusions. *Stochastics*, 83(4-6):347–363, 2011.

[7] S. Christensen and P. Salminen. Impulse control and expected suprema. *Advances in Applied Probability*, 49(1):238–257, 2017.

[8] S. Christensen, P. Salminen., and B. Ta. Optimal stopping of strong Markov processes. *Stochastic Process. Appl.*, 123:1138–1159, 2013.

[9] C. Czichowsky. Time-consistent mean-variance portfolio selection in discrete and continuous time. *Finance and Stochastics*, 17(2):227–271, 2013.

[10] S. Ebert and P. Strack. Until the bitter end: on prospect theory in a dynamic context. *The American Economic Review*, 105(4):1618–1633, 2015.

[11] S. Ebert, W. Wei, and X. Y. Zhou. Discounting, diversity, and investment. *SSRN preprint 2840240*, 2017.

[12] I. Ekeland and A. Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. arXiv preprint math/0604264, 2006.

[13] I. Ekeland, O. Mbodji, and T. A. Pirvu. Time-consistent portfolio management. *SIAM Journal on Financial Mathematics*, 3(1):1–32, 2012.

[14] I. Ekeland and T. Pirvu. Investment and consumption without commitment. *Mathematics and Financial Economics*, 2(1):57–86, 2008.

[15] E. Ekström and G. Peskir. Optimal stopping games for Markov processes. *SIAM Journal on Control and Optimization*, 47(2):684–702, 2008.

[16] S. M. Goldman. Consistent plans. *The Review of Economic Studies*, 47(3):533–537, 1980.

[17] S. R. Grenadier and N. Wang. Investment under uncertainty and time-inconsistent preferences. *Journal of Financial Economics*, 84(1):2–39, 2007.

[18] C. Harris and D. Laibson. Instantaneous gratification. *The Quarterly Journal of Economics*, 128(1):205–248, 2012.

[19] Y. Hu, H. Jin, and X. Y. Zhou. Time-inconsistent stochastic linear–quadratic control. *SIAM journal on Control and Optimization*, 50(3):1548–1572, 2012.

[20] Y.-J. Huang and A. Nguyen Huu. Time-consistent stopping under decreasing impatience. *arXiv preprint 1502.03998*, 2017.

[21] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[22] P. Krusell and A. A. Smith Jr. Consumption–savings decisions with quasi–geometric discounting. *Econometrica*, 71(1):365–375, 2003.

[23] X. Li, X. Y. Zhou, and A. E. Lim. Dynamic mean-variance portfolio selection with no-shorting constraints. *SIAM Journal on Control and Optimization*, 40(5):1540–1555, 2002.

[24] K. Lindensjö. Time-inconsistent stochastic control: solving the extended HJB system is a necessary condition for regular equilibria. arXiv preprint 1611.02902, 2017.

[25] E. Maskin and J. Tirole. Markov perfect equilibrium: I. observable actions. *Journal of Economic Theory*, 100(2):191–219, 2001.

[26] C. W. Miller. Nonlinear PDE approach to time-inconsistent optimal stopping. *SIAM Journal on Control and Optimization*, 55(1):557–573, 2017.

[27] E. Mordecki. Optimal stopping and perpetual options for Lévy processes. *Finance Stoch.*, 6(4):473–493, 2002.

[28] B. Øksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[29] J. L. Pedersen and G. Peskir. Optimal mean–variance selling strategies. *Mathematics and Financial Economics*, 10(2):203–220, 2016.

[30] B. Peleg and M. E. Yaari. On the existence of a consistent course of action when tastes are changing. *The Review of Economic Studies*, 40(3):391–401, 1973.

[31] G. Peskir and A. Shiryaev. *Optimal stopping and free-boundary problems.* Springer, 2006.

[32] R. A. Pollak. Consistent planning. *The Review of Economic Studies*, 35(2):201–208, 1968.

[33] R. Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955.

[34] N. Touzi and N. Vieille. Continuous-time Dynkin games with mixed strategies. *SIAM Journal on Control and Optimization*, 41(4):1073–1088, 2002.