Mathematical Statistics
Stockholm University
Research Report **2019:1**,
http://www.math.su.se/matstat

The two-type Richardson model in the half-plane

Daniel Ahlberg Maria Deijfen Christopher Hoffman

February 2019

Abstract

The two-type Richardson model describes the growth of two competing infection types on the two or higher dimensional integer lattice. For types that spread with the same intensity, it is known that there is a positive probability for infinite coexistence, while for types with different intensities, it is conjectured that infinite coexistence is not possible. In this paper we study the two-type Richardson model in the upper half-plane $\mathbb{Z} \times \mathbb{Z}_+$, and prove that coexistence of two types starting on the horizontal axis has positive probability if and only if the types have the same intensity.

1 Introduction

In 1998, Häggström and Pemantle [8] introduced a model for competing growth on $\mathbb{Z}^2$ known as the two-type Richardson model. Two competing entities, here referred to as type 1 and type 2 infection, initially occupy one site each of the $\mathbb{Z}^2$ nearest-neighbor lattice. As time evolves each uninfected site is occupied by type i at rate λ_i times the number of type i neighbors. An infected site remains in its state forever, implying that the model indeed defines a competition scheme between the types.

Regardless of the values of the intensities, both types clearly have a positive probability of winning by surrounding the other type at an early stage. Attention hence focuses on the event $\mathscr{C}$ that both types simultaneously grow to occupy infinitely many sites; this is referred to as *coexistence* of the two types. Deciding whether or not $\mathscr{C}$ has positive probability is non-trivial since it cannot be achieved on any finite part of the lattice. By time-scaling and symmetry we may restrict to the case $\lambda_1 = 1$ and $\lambda_2 = \lambda > 1$. The conjecture, due to Häggström and Pemantle [8], then is that $\mathscr{C}$ has positive probability if and only if $\lambda = 1$. The *if*-direction of the conjecture was proved in [8], and extended to higher dimensions independently by Garet and Marchand [6] and Hoffman [10], using different methods. As for the *only if*-direction, Häggström and Pemantle [9] showed in 2000 that coexistence is possible for at most countably many values of λ. Ruling out coexistence for *all* $\lambda > 1$ remains a seemingly challenging open problem.

In this paper we study the analogous problem in the upper half-plane $\mathbb{Z}\times\mathbb{Z}_+ = \{(x, y) : y \geqslant 0\}$ with $(0, 0)$ initially occupied by type 1 and $(1, 0)$ initially occupied by type 2, and show that coexistence has positive probability if and only if $\lambda = 1$. That coexistence is possible for $\lambda = 1$ follows from similar arguments as in the full plane, so the novelty lies in proving the *only if*-direction.

Theorem 1. *Consider the two-type Richardson model on $\mathbb{Z} \times \mathbb{Z}_+$ with $(0, 0)$ and $(1, 0)$ initially of type 1 and 2, respectively. Then we have that $\mathbb{P}(\mathscr{C}) > 0$ if and only if $\lambda = 1$.*

Some readers might suspect that the arguments used to prove this result could be adaptable to settle the Häggström-Pemantle conjecture in the full plane. This however is most likely not