Mathematical Statistics

Stockholm University

Statistical Inference for the Beta Coefficient

Taras Bodnar

Arjun K. Gupta

Valdemar Vitlinskyy

Taras Zabolotskyy

Research Report 2019:03

ISSN 1650-0377

Postal address:

Mathematical Statistics
Dept. of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden

Internet:

http://www.math.su.se

Mathematical Statistics
Stockholm University
Research Report **2019:03**,
http://www.math.su.se

Statistical Inference for the Beta Coefficient

April 2019

Taras Bodnar[a,*], Arjun K. Gupta[b], Valdemar Vitlinskyy[c] and Taras Zabolotskyy[d]

[a] *Department of Mathematics, Stockholm University, Roslagsvägen 101, SE-10691 Stockholm, Sweden*

[b] *Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA*

[c] *Department of Economic and Mathematical Modelling, Kyiv National Economic University, Peremoga Avenue 54/1, 03680 Kyiv, Ukraine*

[d] *Department of Programming, Ivan Franko Lviv National University, Universytetska str. 1, 79000 Lviv, Ukraine*

Abstract

The beta coefficient plays a crucial role in finance as a risk measure of a portfolio in comparison to the benchmark portfolio. In the paper, we investigate statistical properties of the sample estimator for the beta coefficient. Assuming that both the holding portfolio and the benchmark portfolio consist of the same assets whose returns are multivariate normally distributed, we provide the finite-sample and the asymptotic distributions of the sample estimator for the beta coefficient. These findings are used to derive a statistical test for the beta coefficient and to construct a confidence interval for the beta coefficient. Moreover, we show that the sample estimator is an unbiased estimator for the beta coefficient. The theoretical results are implemented in an empirical study.

Keywords: Beta coefficient, sampling distribution, test theory, Wishart distribution

* Corresponding author. E-mail address: taras.bodnar@math.su.se

1 Introduction

The method of portfolio selection presented by Markowitz (1952) became very popular through practitioners and scientists of financial market due to its simplicity and good tractability of the results. The approach can be presented in two equivalent ways: (i) to maximize the portfolio expected return under the condition that its variance is equal to a predefined level or (ii) to minimize the portfolio variance under the condition that its expected return is equal to a predefined level. The solutions of these two optimization problems constitute a set of mean-variance optimal portfolios which is a parabola in the mean-variance space and is known as the efficient frontier (see, e.g., Merton (1972), Bodnar and Schmid (2009), Bauder et al. (2019)). Several optimal portfolios obtained as solutions of other optimization problems also belong to the efficient frontier, like the global minimum variance portfolio (Frahm and Memmel (2010), Bodnar et al. (2018)), the minimum value-at-risk (VaR) and the minimum conditional value-at-risk (CVaR) optimal portfolios (see, e.g., Alexander and Baptista (2002), Alexander and Baptista (2004), Bodnar et al. (2012)), the maximum Sharpe ratio portfolio (Sharpe (1994), Schmid and Zabolotskyy (2008)).

Another important direction of research in portfolio theory is devoted to the explanation of the dynamics in the stochastic behaviour of the asset returns with the capital asset pricing model (CAMP) playing a crucial role. The CAPM model was suggested in the seminal papers of Sharpe (1964), Lintner (1965), Mossin (1966) where the asset returns are assumed to follow a one-factor model with the return of the market portfolio being the factor. Berk (1997) proved that one of the necessary conditions for the CAPM is the assumption that the asset returns follow an elliptical distribution, while Zhou (1993) extended findings of Gibbons et al. (1989) by applying their test of the validity of the CAPM to elliptically distributed returns. A further test on the validity of this model was suggested by Hodgson et al. (2002).

The beta coefficient determines the relation between the asset return or the return of the holding portfolio to the return of the market portfolio and plays the central role in the theory of the CAPM. In order to calculate the beta coefficient for an individual asset, a linear regression of the asset return against the market portfolio return should be fitted. In contrast, the portfolio beta can be computed in one of the following two methods (see, e.g., Damodaran (2012, p.120)): The beta of the portfolio is determined by either taking the weighted average of the beta coefficients calculated for each asset included into the portfolio or by regressing the portfolio return against the return of the market portfolio. Alexander (2001, p.231) pointed out that these two methods are equivalent by using ordinary least square (OLS) estimator for the parameters of the linear regression model.

It appears that the portfolio beta is fully determined by the covariance matrix of the asset returns which is an unobservable quantity in practice. As a result, it is estimated by using the historical data of the asset returns. Due to the randomness in the behaviour of the asset returns, the estimated covariance matrix appears to be random and, consequently, the estimated portfolio beta is random as well. We contribute in this paper by deriving the exact finite-sample distribution of the portfolio beta. We show that the estimated beta coefficient follows a t-distribution. This finding allows us to quantify the estimation error presented in the beta coefficient in practice as well as to derive a statistical test for the portfolio beta.

The rest of the paper is organized as follows. In the next section, the theoretical findings of the paper are presented. Here, we derive both the finite-sample and the asymptotic distributions of the estimated portfolio beta as well as provide its interval estimation. These finding are applied to real data based on the returns of 30 stocks included into the DAX index in Section 3. Section 4 discusses the robustness of the obtained results to the violation of the normality assumption used in the derivation of the theoretical findings.

2 Estimated beta coefficient and its distributional properties

Let the weights of investor portfolio be given by $\mathbf{w} = (w_1, w_2, ..., w_k)'$ and let $\mathbf{w}_b = (w_{1b}, w_{2b}, ..., w_{kb})'$ be the weights of the benchmark (market) portfolio. The vector of asset returns at time point t we denote by $\mathbf{X}_t = (X_{1t}, X_{2t}, ..., X_{kt})'$. Following the CAPM, the beta coefficient is given by

$$\beta_t = \frac{\mathrm{Cov}(R_{\mathbf{w}t}, R_{\mathbf{w}_b t})}{\mathrm{Var}(R_{\mathbf{w}_b t})}, \tag{1}$$

where

$$R_{\mathbf{w}_b t} = \sum_{i=1}^{k} w_{ib} X_{it} = \mathbf{w}_b' \mathbf{X}_t$$

stands for the return of the benchmark portfolio at time point t and

$$R_{\mathbf{w}t} = \sum_{i=1}^{k} w_i X_{it} = \mathbf{w}' \mathbf{X}_t$$

denotes the return of the investor portfolio with the weights $\mathbf{w}$ at time point t.

We assume that the vector of asset returns $\mathbf{X}_t$ follows weakly stationary process and denote by $\boldsymbol{\mu} = \mathrm{E}(\mathbf{X}_t)$ and $\mathrm{Var}(\mathbf{X}_t) = \boldsymbol{\Sigma}$ its mean vector and covariance matrix, respectively. Then (1) can be rewritten by

$$\beta_t = \frac{\mathrm{Cov}(\mathbf{w}_b' \mathbf{X}_t, \mathbf{w}' \mathbf{X}_t)}{\mathrm{Var}(\mathbf{w}_b' \mathbf{X}_t)} = \frac{\mathbf{w}_b' \boldsymbol{\Sigma} \mathbf{w}}{\mathbf{w}_b' \boldsymbol{\Sigma} \mathbf{w}_b} = \beta, \tag{2}$$

which appears to be independent of t.

Although (2) provides a simple way how to compute the beta coefficient of the portfolio with weights $\mathbf{w}$ in practice, unfortunately this formula cannot be directly applied in practice since the parameters of the asset returns distribution involved in (2), namely $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, are unknown quantities and have to be estimated by using the historical data of the asset returns. Given the sample $\mathbf{X}_1, ..., \mathbf{X}_n$, their estimators become

$$\hat{\boldsymbol{\mu}} = \frac{1}{n}\sum_{j=1}^{n}\mathbf{X}_j \quad \text{and} \quad \hat{\boldsymbol{\Sigma}} = \frac{1}{n-1}\sum_{j=1}^{n}(\mathbf{X}_j - \hat{\boldsymbol{\mu}})(\mathbf{X}_j - \hat{\boldsymbol{\mu}})', \tag{3}$$

which are the sample counterparts of the corresponding population values.

Substituting $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\Sigma}}$ from (3) in (2), the sample estimator for the β-coefficient is obtained and it is expressed as

$$\hat{\beta} = \frac{\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}}{\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b}. \tag{4}$$

2.1 Sample distribution of the estimated beta coefficient

Since $\mathbf{X}_1, ..., \mathbf{X}_n$ are random, we also obtain that the estimator of the beta coefficient is a random quantity whose computed value based on historical realizations of the asset returns can considerably deviate from the population value as given in (2). In order to quantify the possible differences between β and $\hat{\beta}$, we derive the exact finite-sample distribution of β in Theorem 1 under the assumption that the asset returns are multivariate normally distributed.

Theorem 1. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent and identically distributed random vectors with $\mathbf{X}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $n > k$. Then it holds that*

$$\frac{\sqrt{n-1}\sqrt{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}}{\sqrt{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - (\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}_b)^2/\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}}(\hat{\beta} - \beta) \sim t_{n-1}\,.$$

Proof of Theorem 1: Let $\mathbf{M} = (\mathbf{w}, \mathbf{w}_b)'$ denote $2 \times k$ matrix. Since $(n-1)\hat{\boldsymbol{\Sigma}} \sim \mathcal{W}_k(n-1, \boldsymbol{\Sigma})$, then the application of Theorem 3.2.5 in Muirhead (1982) implies that

$$(n-1)\mathbf{M}\hat{\boldsymbol{\Sigma}}\mathbf{M}' = \begin{pmatrix} \mathbf{w}'\hat{\boldsymbol{\Sigma}}\mathbf{w} & \mathbf{w}'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b \\ \mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w} & \mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b \end{pmatrix} \sim \mathcal{W}_2(n-1, \mathbf{M}\boldsymbol{\Sigma}\mathbf{M}')\,. \tag{5}$$

From (5) and Theorem 3.2.10 of Muirhead (1982) we get

$$(n-1)\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b \sim \mathcal{W}_1(n-1, \mathbf{w}_b\boldsymbol{\Sigma}\mathbf{w}_b') \tag{6}$$

and

$$(n-1)\mathbf{w}'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b|(n-1)\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b \sim \mathcal{N}\left(\frac{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}_b}{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}(n-1)\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b; \left(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \frac{(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}_b)^2}{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}\right)(n-1)\mathbf{w}_b'\hat{\boldsymbol{\Sigma}}\mathbf{w}_b\right). \tag{7}$$

In using

$$\beta = \frac{\mathbf{w}'\mathbf{\Sigma}\mathbf{w}_b}{\mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b} \quad \text{and} \quad \hat{\beta} = \frac{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b}{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b},$$

we get

$$\hat{\beta}|(n-1)\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b \sim \mathcal{N}\Big(\beta, \frac{1}{(n-1)\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}A\Big).$$

with

$$A = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} - \frac{(\mathbf{w}'\mathbf{\Sigma}\mathbf{w}_b)^2}{\mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b}.$$

Let

$$\sigma_b^2 = \mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b.$$

Then,

$$\begin{aligned} f_{\hat{\beta}}(x) &= \int_0^\infty f_{\hat{\beta}|(n-1)\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}(x|y) f_{(n-1)\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}(y)dy \\ &= \int_0^\infty \frac{1}{\sqrt{2\pi A}}\sqrt{y}e^{-\frac{(x-\beta)^2}{2A}y}\frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)\sigma_b^{n-1}}e^{-\frac{1}{2}\sigma_b^{-2}y}y^{(n-3)/2}dy \\ &= \frac{1}{2^{n/2}\sqrt{\pi}\sqrt{A}\Gamma((n-1)/2)\sigma_b^{n-1}}\int_0^\infty y^{n/2-1}e^{-\frac{1}{2}\left(\frac{(x-\beta)^2}{A}+\frac{1}{\sigma_b^2}\right)y}dy. \end{aligned}$$

The application of the equality

$$\Gamma(n/2) = \Big(\frac{1}{2}\Big(\frac{(x-\beta)^2}{A}+\frac{1}{\sigma_b^2}\Big)\Big)^{n/2}\int_0^\infty y^{n/2-1}e^{-\frac{1}{2}\left(\frac{(x-\beta)^2}{A}+\frac{1}{\sigma_b^2}\right)y}dy$$

implies

$$\begin{aligned} f_{\hat{\beta}}(x) &= \frac{\Gamma(n/2)}{2^{n/2}\sqrt{\pi}\sqrt{A}\Gamma((n-1)/2)\sigma_b^{n-1}}\Big(\frac{1}{2}\Big(\frac{(x-\beta)^2}{A}+\frac{1}{\sigma_b^2}\Big)\Big)^{-n/2} \\ &= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\frac{1}{\sqrt{\pi}\sqrt{A}}\frac{1}{2^{n/2}\sigma_b^{n-1}}2^{n/2}\sigma_b^n\Big(\frac{(x-\beta)^2\sigma_b^2}{A}+1\Big)^{-n/2} \\ &= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\frac{1}{\sqrt{\pi}}\frac{\sigma_b}{\sqrt{A}}\Big(\frac{(x-\beta)^2\sigma_b^2}{A}+1\Big)^{-n/2} \\ &= \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\frac{1}{\sqrt{\pi(n-1)}}\frac{\sigma_b\sqrt{n-1}}{\sqrt{A}}\Big(\frac{1}{n-1}(x-\beta)^2\frac{\sigma_b^2(n-1)}{A}+1\Big)^{-n/2}. \end{aligned}$$

which is the density of the univariate t-distribution with $(n-1)$ degrees of freedom, location parameter β, and scale parameter $\frac{A}{\sigma_b^2(n-1)}$. □

The results of Theorem 1 possess several important application. First of all, it provides the exact finite-sample distribution of the estimated beta coefficient. In particular, it shows that the $\hat{\beta}$ has a shifted and scaled univariate t-distribution which is one of the most widely used distributions in practice. The application of the properties leads to the observation that

$$\mathrm{E}\left(\hat{\beta}\right) = \beta, \tag{8}$$

that is the sample estimator $\hat{\beta}$ is an unbiased estimator of β. Second, it allows to quantify the uncertainty of the estimated beta. For $n > \max\{3, k\}$ it holds that

$$\mathrm{Var}\left(\hat{\beta}\right) = \frac{n-1}{n-3} \frac{\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} - \frac{(\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}_b)^2}{\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b}}{(n-1)\mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b} = \frac{1}{n-3}\left(\frac{\sigma^2}{\sigma_b^2} - \beta^2\right), \tag{9}$$

where

$$\sigma^2 = \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \quad \text{and} \quad \sigma_b^2 = \mathbf{w}_b'\boldsymbol{\Sigma}\mathbf{w}_b.$$

Moreover, (8) and (9) imply that $\hat{\beta}$ is a consistent estimator for β.

Finally, since the univariate t-distribution tends to the standard normal distribution when its degrees of freedom tend to infinity, we also get as a direct consequence of Theorem 1 that

$$\sqrt{n}\frac{\hat{\beta} - \beta}{\sqrt{\sigma^2/\sigma_b^2 - \beta^2}} \xrightarrow{d} \mathcal{N}(0, 1), \tag{10}$$

where the symbol $\xrightarrow{d}$ denotes the convergence in distribution.

2.2 Interval estimation and test theory

The results of Theorem 1 provides the finite-sample distribution of $\hat{\beta}$ which depends on the unknown quantities β, σ^2 and σ_b^2. Although this finding together with asymptotic result (10) is very useful to assess both the finite-sample and the asymptotic properties of the suggested estimator $\hat{\beta}$ of β, they do not provide the complete information needed to construct a confidence interval, an interval estimation of β, or to develop a test theory. A further research in this topic is needed and will be discussed in the current section.

From the view point of asymptotic statistics, a consistent estimator of $\sigma^2/\sigma_b^2 - \beta^2$ would provide us a possibility to obtain an asymptotic confidence interval for β which can be used when the sample size is relatively large. The situation is more complicated when an exact finite-sample confidence interval should be derived. As a starting point of the derivation we use the duality between the test theory and the interval estimation by considering a test on the beta coefficient (see, e.g., Aitchison (1964)). Namely, we test the hypotheses

$$H_0: \ \beta = \beta_0 \quad \text{against} \quad H_1: \ \beta \neq \beta_0 \tag{11}$$

for some target value β_0.

Let

$$\hat{\sigma}^2 = \mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} \quad \text{and} \quad \hat{\sigma}_b^2 = \mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b.$$

The results of Theorem 1 motivates the application of the following test statistic

$$T = \sqrt{n-2}\frac{\hat{\beta}-\beta_0}{\sqrt{\hat{\sigma}^2/\hat{\sigma}_b^2 - \hat{\beta}^2}} = \sqrt{n-2}\frac{\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}(\hat{\beta}-\beta_0) \tag{12}$$

The finite-sample null distribution of T is derived in Theorem 2.

Theorem 2. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent and identically distributed random vectors with $\mathbf{X}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ and $n > k$. Then it holds that*

$$T \sim t_{n-2}\,.$$

under the null hypothesis in (11).

Proof of Theorem 2: We define

$$\sigma_b^2 = \mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b \quad \text{and} \quad A = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} - \frac{(\mathbf{w}'\mathbf{\Sigma}\mathbf{w}_b)^2}{\mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b}.$$

and their corresponding sample estimators by

$$\hat{\sigma}_b^2 = \mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b \quad \text{and} \quad \hat{A} = \mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - \frac{(\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2}{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}.$$

The application of Theorem 3.2.10 in Muirhead (1982) leads to

$$(n-1)\hat{A} \sim \mathcal{W}_1(n-2, A) \implies \frac{(n-1)\hat{A}}{A} \sim \chi^2_{n-2},$$

which is independent of $\hat{\beta}$ and $\hat{\sigma}_b^2$.

From the proof of Theorem 1 we get

$$\hat{\beta}|(n-1)\hat{\sigma}_b^2 \sim \mathcal{N}\Big(\beta_0, \frac{1}{(n-1)\hat{\sigma}_b^2}A\Big).$$

and, hence,

$$(\hat{\beta}-\beta_0)\hat{\sigma}_b|(n-1)\hat{\sigma}_b^2 \sim \mathcal{N}\Big(0, \frac{1}{(n-1)}A\Big).$$

Consequently, it holds that

$$(\hat{\beta}-\beta_0)\hat{\sigma}_b \sim \mathcal{N}\Big(0, \frac{1}{(n-1)}A\Big).$$

Since $\hat{A}$ is independent of $\hat{\beta}$ and $\hat{\sigma}_b^2$, it is also independent of $(\hat{\beta}-\beta)\hat{\sigma}_b$ and, as a result, we obtain

$$\frac{(\hat{\beta}-\beta_0)\hat{\sigma}_b\sqrt{n-1}/\sqrt{A}}{\sqrt{(n-1)\hat{A}}/\sqrt{A(n-2)}} = \sqrt{n-2}\frac{\hat{\sigma}_b(\hat{\beta}-\beta_0)}{\sqrt{\hat{A}}} \sim t_{n-2}$$

which completes the proof of the theorem. □

The application of Theorem 2 leads to the expression of the two-sided $(1-\gamma)$ confidence interval for the beta coefficient β of the portfolio with weights $\mathbf{w}$ expressed as

$$CI_{1-\gamma}(\beta) = \left[\hat{\beta} - t_{1-\gamma/2,n-2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}, \hat{\beta} + t_{1-\gamma/2,n-2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}\right],$$

where $t_{1-\gamma/2,n-2}$ stands for the $(1-\gamma/2)$ quantile of the t-distribution with $n-2$ degrees of freedom. Similarly, we obtain the one-sided confidence intervals. The lower one-sided $(1-\gamma)$ confidence interval is given by

$$CI_{1-\gamma}(\beta) = \left(-\infty, \hat{\beta} + t_{1-\gamma,n-2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}\right],$$

while the upper one-sided $(1-\gamma)$ confidence interval is the following one

$$CI_{1-\gamma}(\beta) = \left[\hat{\beta} - t_{1-\gamma,n-2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}, +\infty\right).$$

Moreover, since the univariate t-distribution converges to the standard normal distribution as the degrees of freedom tend to infinity, we can also use the asymptotic confidence interval for β. For example, the two-sided asymptotic $(1-\gamma)$ confidence interval is given by

$$CI_{1-\gamma}(\beta) = \left[\hat{\beta} - z_{1-\gamma/2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}, \hat{\beta} + z_{1-\gamma/2}\frac{\sqrt{\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w} - (\mathbf{w}'\hat{\mathbf{\Sigma}}\mathbf{w}_b)^2/\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}{\sqrt{n-2}\sqrt{\mathbf{w}_b'\hat{\mathbf{\Sigma}}\mathbf{w}_b}}\right],$$

where $z_{1-\gamma/2}$ denotes the $(1-\gamma/2)$ quantile of the standard normal distribution.

Finally, we present the distribution of T under the alternative hypothesis in (11) in Theorem 3.

Theorem 3. *Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be independent and identically distributed random vectors with $\mathbf{X}_1 \sim \mathcal{N}_k(\boldsymbol{\mu}, \mathbf{\Sigma})$ and $n > k$. Then the density of T under the alternative hypothesis is given by*

$$f_T(x) = \int_0^\infty f_{t_{n-2}(\delta\sqrt{y})}(x) f_{\chi^2_{n-1}}(y) dy$$

with

$$\delta = \frac{(\beta - \beta_0)}{\sqrt{\sigma^2/\sigma_b^2 - \beta^2}}$$

where $f_{\chi^2_{n-1}}(.)$ denotes the density function of the χ^2-distribution with $(n-1)$ degrees of freedom and $f_{t_{n-2}(\delta\sqrt{(y)})}(.)$ stands for the density of the non-central t-distribution with $(n-2)$ degrees of freedom and non-centrality parameter $\delta\sqrt{y}$.

Proof of Theorem 3: Following the proof of Theorem 2 we get

$$(n-1)\hat{A} \sim \mathcal{W}_1(n-2, A) \Longrightarrow \frac{(n-1)\hat{A}}{A} \sim \chi^2_{n-2},$$

and

$$\sqrt{n-1}\frac{(\hat{\beta}-\beta_0)\hat{\sigma}_b}{\sqrt{A}}|(n-1)\hat{\sigma}_b^2 \sim \mathcal{N}\Big(\frac{(\beta-\beta_0)\sigma_b}{\sqrt{A}}\xi, 1\Big).$$

where

$$\xi^2 = \frac{(n-1)\hat{\sigma}_b^2}{\sigma_b^2} \sim \chi^2_{n-1}$$

and $\hat{A}$ is independent of $\hat{\beta}$ and ξ^2.

Since ξ^2 depends only on $\hat{\sigma}_b^2$, we get that the conditional distribution of T given $\hat{\sigma}_b^2$ coincides with the conditional distribution of T given ξ^2 and it is given by

$$T|\xi^2 \sim t_{n-2}(\delta\xi) \quad \text{with} \quad \delta = \frac{(\beta-\beta_0)\sigma_b}{\sqrt{A}}.$$

Hence, the unconditional density of T is expressed as

$$f_T(x) = \int_0^\infty f_{t_{n-2}(\delta\sqrt{y})}(x) f_{\chi^2_{n-1}}(y)dy$$

where $f_{\chi^2_{n-1}}(.)$ denotes the density function of the χ^2-distribution with $(n-1)$ degrees of freedom and $f_{t_{n-2}(\delta\sqrt{(y)})}(.)$ stands for the density of the non-central t-distribution with $(n-2)$ degrees of freedom and non-centrality parameter $\delta\sqrt{y}$. $\square$

The results of Theorem 3 simplify considerably the study of the power of the suggested test for the beta coefficient. In particular, these findings lead to the expression of the power function expressed as

$$\begin{aligned}
G(\delta) &= P(|T| > t_{1-\gamma/2,n-2}) = 1 - \int_{-t_{1-\gamma/2,n-2}}^{t_{1-\gamma/2,n-2}} \Big(\int_0^\infty f_{t_{n-2}(\delta\sqrt{y})}(x) f_{\chi^2_{n-1}}(y)dy\Big)dx \\
&= 1 - \int_0^\infty \Big(F_{t_{n-2}(\delta\sqrt{y})}\left(t_{1-\gamma/2,n-2}\right) - F_{t_{n-2}(\delta\sqrt{y})}\left(-t_{1-\gamma/2,n-2}\right)\Big) f_{\chi^2_{n-1}}(y)dy \\
&= 1 - \int_0^\infty \Big(2F_{t_{n-2}(\delta\sqrt{y})}\left(t_{1-\gamma/2,n-2}\right) - 1\Big) f_{\chi^2_{n-1}}(y)dy \\
&= 2\left(1 - \int_0^\infty F_{t_{n-2}(\delta\sqrt{y})}\left(t_{1-\gamma/2,n-2}\right) f_{\chi^2_{n-1}}(y)dy\right),
\end{aligned}$$

where the third line follows from the symmetry of the t-distribution (see, e.g., Gupta et al. (2013)) and the symbol $F_{t_{n-2}(\delta\sqrt{y})}(.)$ denotes the cumulative distribution function of the non-central t-distribution with $(n-2)$ degrees of freedom and non-centrality parameter $\delta\sqrt{y}$. Moreover, it appears that the power function depends on $\mathbf{\Sigma}$ only through δ. As a result, it allows to assess the power function as a function of δ only for a given value of n.

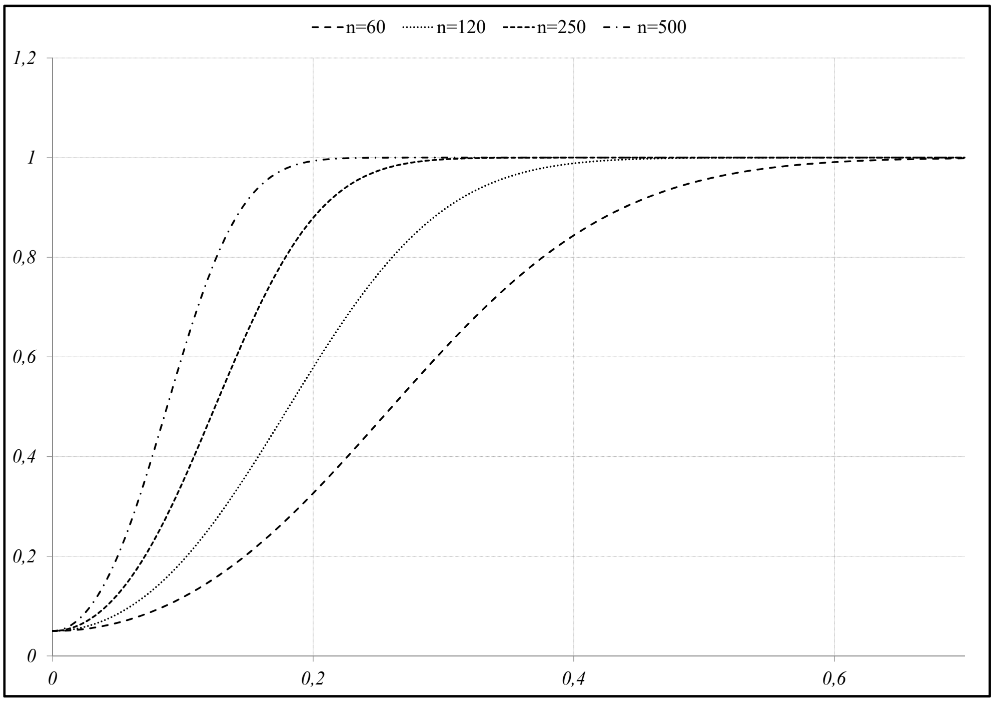

Figure 1: Power function for $n \in \{60,\ 120,\ 250,\ 500\}$ and $\gamma = 0.05$.

In Figure 1 we plot the power of the suggested test as a function of δ for $n \in \{60,\ 120,\ 250,\ 500\}$. We observe that even for a small sample size, like $n = 60$, the test is able to reject the null hypothesis already for small deviations from the target value β_0. For instance, the power of the test is larger than 60% when δ is around 0.3 only and it is approximately equal to 1 if $\delta = 0.6$. Furthermore, the power of the test increases as the sample size becomes larger. For instance, the power is around 1 already for δ close to 0.3 when $n = 250$ and for δ around 0.2 when $n = 500$.

3 Empirical illustration

In this section we implement the theoretical findings of Section 2 to real data based on the daily returns on the assets included into the DAX index for the period from 01.01.2018 to 31.12.2018 (251 observations). As a benchmark portfolio $\mathbf{w}_b$, the DAX index is chosen whose weights are provided on the official website of the German capital market, namely on www.boerse.de. As a target portfolio for which we aim to infer the beta coefficient we take the equally-weighted portfolio of six dimensions, namely $k \in \{5, 10, 15, 20, 25, 30\}$, which consists of the corresponding first k assets included into the DAX index in the alphabetical order. Using the running window with length equal to $n = 100$, we construct confidence intervals for six considered equally-weighted portfolios at significance level $1 - \gamma \in \{90\,\%, 95\,\%, 99\,\%\}$ and plot them in

Figure 2.

Figure 2: Sample estimators and confidence intervals for the beta coefficient of the equally-weighted portfolio constructed for the first $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (middle left), $k = 20$ (middle right), $k = 25$ (bottom left), $k = 30$ (bottom right) assets included into the DAX index in the alphabetical order.

In Figure 2 we observe that the beta coefficient of the equally-weighted portfolios does not deviate considerably from one for almost all of the considered dimensions. The smallest values of the beta coefficient are present in the case of the $k = 5$ dimensional equally-weighted portfolio, especially at the end of 2018 where the estimated beta coefficient drops to 0.85. In contrast, the beta coefficient is almost always larger than one for the $k = 10$ dimensional portfolio. When the portfolio dimension increases, then the beta coefficient becomes less volatile in time. For instance, almost all values of $\hat{\beta}$ belong to the interval $[0.96, 1]$ when $k = 30$. Finally, the beta coefficients of the equally-weighted portfolios drop at the second part of October 2018 which is in-line with the opinion that October, 2018 was the worst month in more than six years (since May 2012) for global capital markets since the global financial crisis in 2008. Some events, like disappointing earnings from big tech companies and a budget row between Italy and the European Union, seem to have a large influence on the overall performance of the German capital market.

Finally, confidence intervals constructed for the beta coefficients of the considered equally-weighted portfolios cover one in almost all of the considered cases with the exception present during several weeks in October, 2018 – December, 2018 for the $k = 5$, for the $k = 25$, and for the $k = 30$ dimensional equally-weighted portfolio. At the end of 2018 the estimated beta coefficients increase and only the confidence intervals constructed for the 5-dimensional portfolio do not include the value of one.

4 Robustness to the violation of the normality assumption

The theoretical findings of Section 3 were obtained under the assumption that the asset returns are normally distributed. In this section we investigate how crucial is the assumption of normality on the performance of the beta estimator and on the distributional properties of the test statistic T introduced in Section 3.2. The results are obtained via Monte Carlo study by drawing random sample of size n from the k-dimensional multivariate t-distribution with $d \in \{5, 10\}$ degrees of freedom. Several values of k and n are considered, namely, $k \in \{5, 10, 15, 20, 25, 30\}$ and $n \in \{60, 120, 250, 500, 1000, 2000\}$. For each possible choice of (k, n) we draw a sample from the t-distribution with 5 and 10 degrees of freedom, with the location vector and covariance matrix equal to the sample mean vector and to the sample covariance matrix computed from the daily returns on the assets included into the DAX index for the period from 01.01.2018 to 31.12.2018 (251 observation). From the simulated data the standardized realization of the estimated beta coefficient is computed as well as the value of the test statistic T provided in Section 3.2. Finally, the whole procedure is repeated $B = 50000$ times for each possible value of d, k and n. The resulting samples of the standardized estimators of the beta coefficient and of

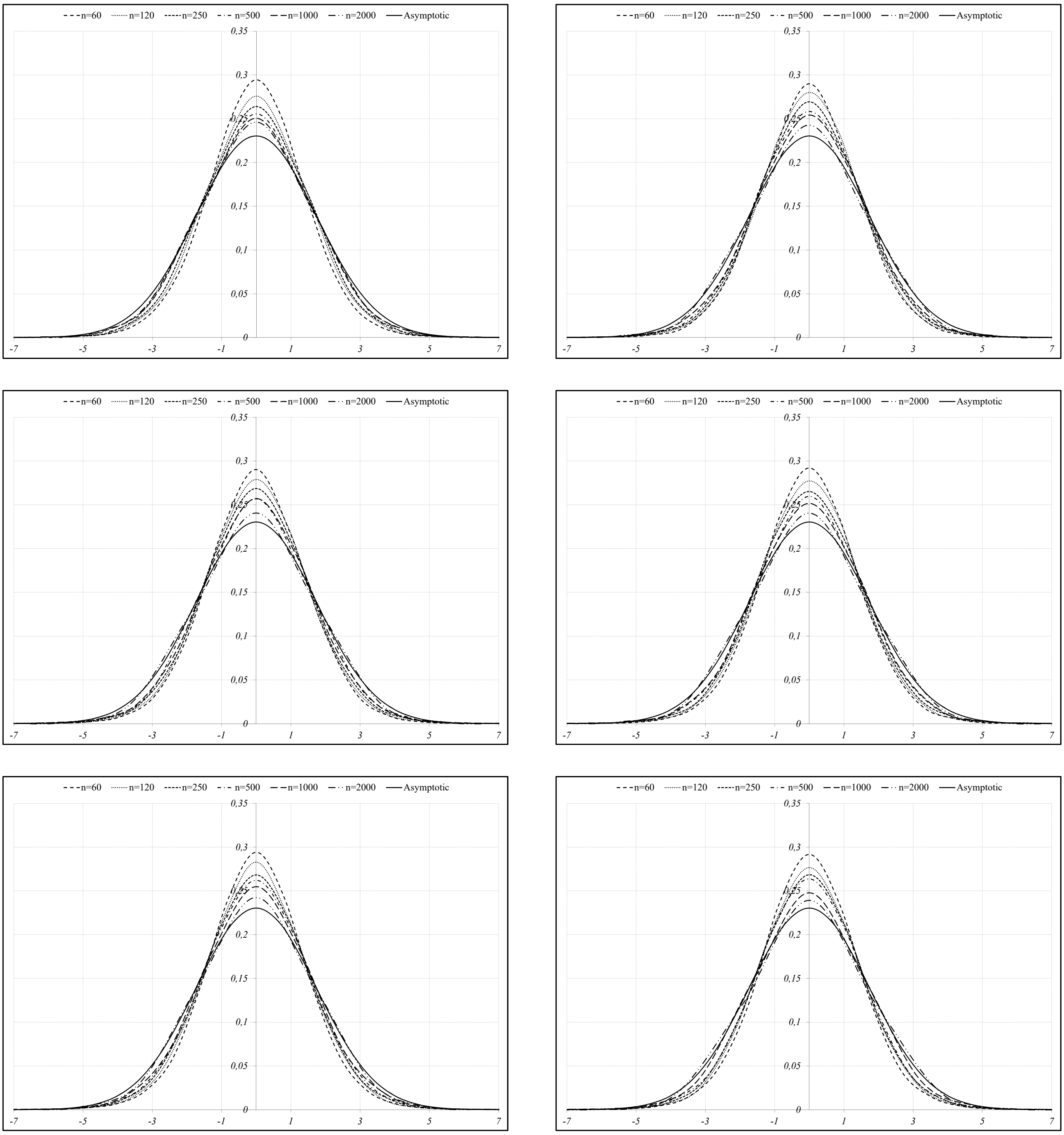

Figure 3: Kernel density estimators for $n \in \{60, 120, 250, 500, 1000, 2000\}$ and the asymptotic density of the standardized estimator for the beta coefficient in the case of the $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (middle left), $k = 20$ (middle right), $k = 25$ (bottom left), $k = 30$ (bottom right) dimensional equally-weighted portfolio. The asset returns are drawn from the multivariate t-distribution with 5 degrees of freedom.

the values of the test statistic T are used to estimate their sampling distribution by employing the kernel density estimator with Epanechnikov kernel.

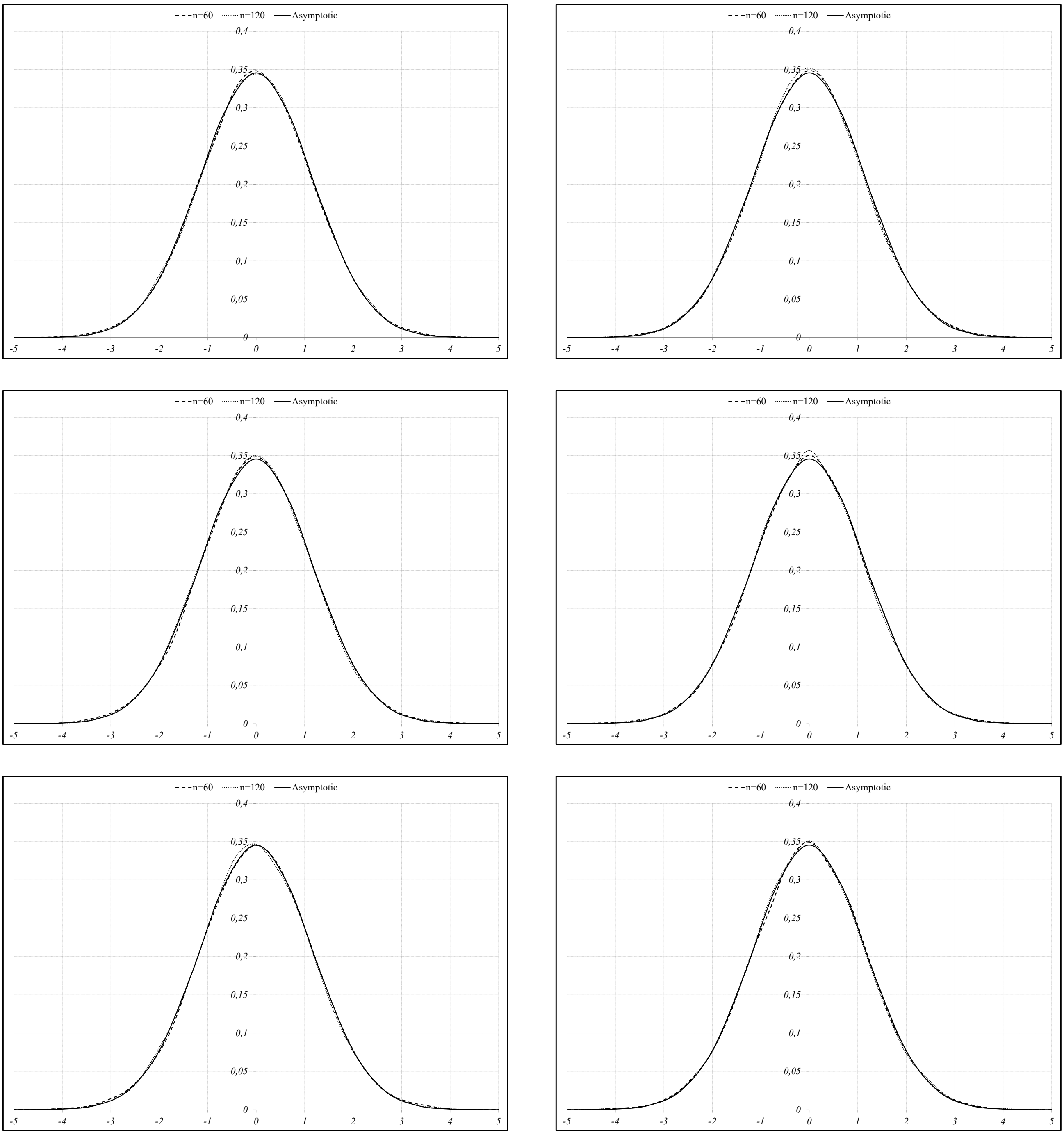

Figure 4: Kernel density estimators for $n \in \{60, 120\}$ and the asymptotic density of the standardized estimator for the beta coefficient in the case of the $k = 5$ (top left), $k = 10$ (top right), $k = 15$ (middle left), $k = 20$ (middle right), $k = 25$ (bottom left), $k = 30$ (bottom right) dimensional equally-weighted portfolio. The asset returns are drawn from the multivariate t-distribution with 10 degrees of freedom.

In Figures 3 and 4 the results are shown for the t-distribution with 5 and 10 degrees of freedom. In both figures we present the kernel density estimators and the asymptotic density of the standardized estimator for the beta coefficient given by

$$\sqrt{n-1}\frac{\hat{\beta}-\beta}{\sqrt{\sigma^2/\sigma_b^2-\beta^2}} \xrightarrow{d} \mathcal{N}(0,1),$$

with

$$\sigma^2 = \mathbf{w}'\mathbf{\Sigma}\mathbf{w} \quad \text{and} \quad \sigma_b^2 = \mathbf{w}_b'\mathbf{\Sigma}\mathbf{w}_b.$$

and note that similar results are also present for the values of the test statistic T. The plots in the case of T are not included in the paper and are available from the corresponding author on request. Independently of the sample size, the kernel densities of the standardized estimator for the beta coefficient are roughly symmetric around zero. Small departures from the asymptotic distribution, which is the standard normal distribution, are present in the case of kernel densities computed when the sample size is relatively small and asset returns are generated from the t-distribution with 5 degrees of freedom. If data are generated from the t-distribution with 10 degrees of freedom, then the convergence of the kernel densities are already observed for a small sample size equal to 120 and for that reason the kernel densities computed for large sample size are not present in Figure 4.

References

Aitchison, J. (1964). Confidence-region tests. *Journal of the Royal Statistical Society: Series B (Methodological)*, 26(3):462–476.

Alexander, C. (2001). *Market models: A guide to financial data analysis*. John Wiley & Sons.

Alexander, G. J. and Baptista, A. M. (2002). Economic implications of using a mean-var model for portfolio selection: A comparison with mean-variance analysis. *Journal of Economic Dynamics and Control*, 26(7-8):1159–1193.

Alexander, G. J. and Baptista, A. M. (2004). A comparison of var and cvar constraints on portfolio selection with the mean-variance model. *Management science*, 50(9):1261–1273.

Bauder, D., Bodnar, R., Bodnar, T., and Schmid, W. (2019). Bayesian estimation of the efficient frontier. *Scandinavian Journal of Statistics*, to appear.

Berk, J. B. (1997). Necessary conditions for the capm. *Journal of Economic Theory*, 73(1):245–257.

Bodnar, T., Parolya, N., and Schmid, W. (2018). Estimation of the global minimum variance portfolio in high dimensions. *European Journal of Operational Research*, 266(1):371–390.

Bodnar, T. and Schmid, W. (2009). Econometrical analysis of the sample efficient frontier. *The European journal of finance*, 15(3):317–335.

Bodnar, T., Schmid, W., and Zabolotskyy, T. (2012). Minimum var and minimum cvar optimal portfolios: estimators, confidence regions, and tests. *Statistics & Risk Modeling with Applications in Finance and Insurance*, 29(4):281–314.

Damodaran, A. (2012). *Investment valuation: Tools and techniques for determining the value of any asset*. John Wiley & Sons.

Frahm, G. and Memmel, C. (2010). Dominating estimators for minimum-variance portfolios. *Journal of Econometrics*, 159(2):289–302.

Gibbons, M. R., Ross, S. A., and Shanken, J. (1989). A test of the efficiency of a given portfolio. *Econometrica*, 57:1121–1152.

Gupta, A. K., Varga, T., and Bodnar, T. (2013). *Elliptically Contoured Models in Statistics and Portfolio Theory*. New York, NY: Springer.

Hodgson, D. J., Linton, O., and Vorkink, K. (2002). Testing the capital asset pricing model efficiently under elliptical symmetry: A semiparametric approach. *Journal of Applied Econometrics*, 17(6):617–639.

Lintner, J. (1965). The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets. *The review of economics and statistics*, 47(1):13–37.

Markowitz, H. (1952). Portfolio selection. *The Journal of Finance*, 7:77–91.

Merton, R. C. (1972). An analytic derivation of the efficient portfolio frontier. *Journal of financial and quantitative analysis*, 7(4):1851–1872.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica: Journal of the econometric society*, pages 768–783.

Muirhead, R. J. (1982). *Aspects of Multivariate Statistical Theory*. Wiley, New York.

Schmid, W. and Zabolotskyy, T. (2008). On the existence of unbiased estimators for the portfolio weights obtained by maximizing the sharpe ratio. *AStA Advances in Statistical Analysis*, 92(1):29–34.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *The journal of finance*, 19(3):425–442.

Sharpe, W. F. (1994). The sharpe ratio. *Journal of portfolio management*, 21(1):49–58.

Zhou, G. (1993). Asset-pricing tests under alternative distributions. *The Journal of Finance*, 48(5):1927–1942.