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Estimating dynamic volatility of returns for Deutsche Bank

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Abstract

In financial market, the volatility of returns varies over time. The purpose of this study is to estimate the dynamic volatility of returns for Deutsche Bank by using an autoregressive conditional heteroscedasticity model. Both ARCH and GARCH models have been used to model observed time series and determine future volatility based on previous values and volatility. We have applied the assumptions of both normal distribution and Student t-distribution of error terms to models. ARCH(m) and GARCH(1,1) have been compared by abilities of forecasting volatility based on AIC and Ljung-Box tests. The result has shown that GARCH(1,1) with normal distribution is the model that we need use for estimating volatility of returns.

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1 Introduction

With the development of economy, the financial market has gradually become an important part. Volatility in financial markets increases along with the economic globalization and investment liberalization. Financial theory is based on the relationship between risk and earnings. Fluctuations in asset prices to a certain extent reflect the risk characteristics of assets. Understanding how price fluctuations change over time is one of the main problems investors are facing in the decision-making process. Market investors can take advantage of volatility forecasting in risk management, derivatives pricing, hedging and portfolio choice.

This thesis is inspired by Robert F. Engle's contribution to the economic science. The autoregressive conditional heteroscedastic (ARCH) model was introduced by Engle (1982), see [8]. The ARCH model is a kind of dynamic non-linear time series model and it is used in modeling time-varying volatility. Besides the ARCH model, the generalized autoregressive conditional heteroskedasticity(GARCH) model, which was developed by Bollerslev(1986) based on ARCH model,see [8], will also be presented in this thesis. Both of models are able to predict the same thing, it is interesting to compare these models in their behavior in forecasting financial volatility and use the one with the best predictive ability in the future.

A presentation of the theory is given in section 4 with a short overview of tests for determining the properties of data. In section 5 we will introduce how ARCH and GARCH models are constructed and how the models can be compared. The results of data analysis, modeling of volatility and comparison of the two models will be presented in section 6.

2 Aim

The aim of this thesis is to estimate dynamic volatility of returns for Deutsche Bank. The estimated volatility will be determined by comparing performance of abilities to forecast financial volatility among ARCH(m) and GARCH(1,1) models.

3 Data

Deutsche bank is Germany's biggest bank and one of the most major financial institutions in the world. It is listed on the stock exchanges around world. In this thesis we have chosen Deutsche bank's stock price on National Association of Securities Dealers Automated Quotations (NASDAQ), New York, to examine the models of dynamic volatility. Since the stocks are

traded every trading day on global financial market, they are presented as daily time series data. In this thesis the original data from NASDAQ is represented by daily stock prices from 2012-01-30 to 2015-01-30, i.e. 756 trading days. The reason we chose the three-years-period is because, it covers both calm and turbulent conditions on the market. These different conditions are sufficient for model estimation and hence to get a better predication of volatility. We use daily close price in our analysis. The original data of daily close prices is downloaded from NASDAQ's website, see [10].

4 Theory

In this section we will describe the theory which is used in different parts in our analysis. It will also be used for reasoning and modeling. The original data, which is downloaded from NASDAQ's website, is daily close stock price for Deutsche Bank. The data is presented as time series. It shows how much the value of Deutsche Bank's price has changed every day. The continuous daily returns, also called daily log-returns, are used for mathematical modeling.

4.1 Return and Variance

Return is a basic concept in finance. It presents the gain or loss of an asset in a specific period, it is usually quoted as a percentage, but we use decimal form in this thesis. We start with the definition of return. Let P_t denote the price of a stock at time t , $t = 1, 2, \dots, n$

One-period simple(net)return:

$$\begin{aligned} R_t^* &= \frac{P_t - P_{t-1}}{P_{t-1}} \\ &= \frac{P_t}{P_{t-1}} - 1 \end{aligned} \tag{1}$$

Continuous or log-return:

$$R_t = \ln\left(\frac{P_t}{P_{t-1}}\right) \tag{2}$$

The difference $P_t - P_{t-1}$ is the revenue or profit during the holding period. The denominator P_{t-1} is the stock price at one day before time t . The simple return is always greater or equal than the continuous return. Since $\ln(x) \approx x - 1$ for $0.9 \leq x \leq 1.1$, it holds that $R_t^* = \frac{P_t - P_{t-1}}{P_{t-1}} \approx \ln\left(\frac{P_t}{P_{t-1}}\right)$, see [2].

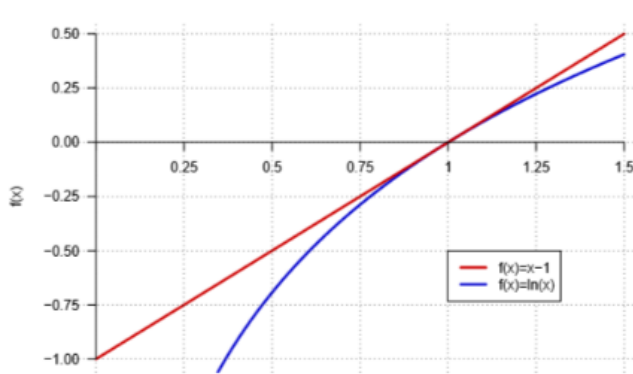


Figure 1: Comparison of simple return and continuous return

In finance, volatility is a measure for variation of price of an asset over time. A high volatility means that the stock value could change dramatically over a time period. The typical statistic used to measure volatility is standard deviation. It is defined as the square root of the average squared deviation of the data from its mean. The mathematical formula for standard deviation calculation is

$$\hat{\sigma} = \sqrt{\frac{1}{n} \sum_{i=1}^n (R_t - \hat{\mu})^2}, \quad (3)$$

where μ is the mean value of returns, n is number of observations.

4.2 Normality test

It is typically assumed that data is normally distributed. Based on skewness and kurtosis, we can assess the fit of normal distribution to our data. Further, two normality tests, Shapiro-Wilk Normality Test and Jarque-Bera Test, will be also used in order to check normality of data.

4.2.1 Skewness and kurtosis

Kurtosis is a descriptive statistic of the degree of steepness for the distribution of all values in overall. This statistic needs to be compared with the normal distribution. If kurtosis equals to three, it indicates that the data has the same degree of steepness as a normal distribution, see [2]. Kurtosis, greater than three, indicates that the data has a relatively steeper distribution than normal distribution, as a sharp peak. On the contrary, the distribution of data is relatively flat compared with the normal distribution, if the kurtosis is less than three. The greater the absolute value of kurtosis value is, the greater difference of the degree of steepness between data's

distribution and normal distribution. The kurtosis value can be calculated according the following formula

$$\hat{\gamma}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})^4}{ns^4}, \quad (4)$$

where s is the estimated standard deviation.

Skewness is also a descriptive statistic of data distribution. It describes the asymmetry of the distribution. Normal distribution has skewness equal to zero, see[2]. Skewness, which is greater than zero, indicates that the distribution has a fatter or longer tail on the right side than the left side. It means that there are more extreme value of data in the right side from the mean. On the contrary, the tail on the left side of the probability density function is fatter or longer than the right side if the skewness is less than zero. The following formula calculates skewness value

$$\hat{\gamma}_2 = \frac{\sum_{i=1}^n (x_i - \bar{x})^3}{ns^3}, \quad (5)$$

where s is the estimated standard deviation.

4.2.2 Shapiro-Wilk Normality Test

In 1965, Samuel Sanford Shapiro and Martin Wilk published a test of normality, see [9]. The Shapiro-Wilk Normality test uses the null hypothesis principle to check whether a sample comes from a normally distributed population or not. The test has an assumption of normal distribution of data, see [2]. Thus

Assumption:

$$F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

$\phi(x)$ is the cumulative distribution function of standard normal distribution.

Test statistic:

$$W = \frac{\sum_{i=1}^n (x_i - \bar{x}) \phi^{-1}((i - 1/2)/n)^2}{\sum_{i=1}^n (x_i - \bar{x})^2 \sum_{i=1}^n \phi^{-1}((i - 1/2)/n)^2} \quad (6)$$

4.2.3 Jarque-Bera Test

The Jarque-Bera test is a goodness-of-fit test of whether the skewness and kurtosis of data matching a Normal distribution. The statistic JB is defined as

$$JB = \frac{n}{6}(S^2 + \frac{1}{4}(K - 3)^2) \quad (7)$$

where n is the number of observations, S is the sample's skewness and K is the sample's kurtosis, see [2].

4.3 t-test

The t-test is used in order to check whether the expected value of a sample data from normal distribution is equal to zero. Assume that the sample $X = (x_1, x_2, \dots, x_n)$ is from a normal distribution with expected value μ and variance σ^2 . We have the null hypothesis $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. The test statistic is

$$T = \frac{\bar{X} - \mu_0}{s(X)/\sqrt{n}} \sim t(n - 1) \quad (8)$$

where $s(X)$ is standard deviation of sample. The test statistic is Student t-distributed with $(n - 1)$ degrees of freedom. If $|T| > t_{\alpha/2}(n - 1)$, then we reject the null hypothesis. $t_{\alpha/2}(n - 1)$ is the $\alpha/2$ t-distributed quantile, see [7]

4.4 Autocorrelation test

The autocorrelation describes the dependency between two observations. Thus, correlation of random variables at different times. The time periods between two observations is called lag, for example, lag 1 is between X_t and X_{t-1} , lag 2 is between X_t and X_{t-2} , see [3].

We start by defining the process of returns as

$$r_t = \mu + a_t \quad (9)$$

where μ is the expected value of the process r_t and a_t is an independent random variable with expected value zero and variance σ^2 .

Consider a return series r_t , the lag- ℓ autocorrelation of r_t is the correlation coefficient between r_t and $r_{t-\ell}$, and it is denoted by ρ_ℓ . The autocorrelation function for r_t is estimated as follow, see [4].

$$\hat{\rho}_\ell = \frac{\sum_{t=\ell+1}^T (r_t - \bar{r})(r_{t-\ell} - \bar{r})}{\sum_{t=1}^T (r_t - \bar{r})^2}, \quad (10)$$

where $0 \leq \ell \leq T - 1$, T is size of the sample and \bar{r} is estimated mean value

of r_t .

The estimator $\hat{\rho}$ is asymptotically normally distributed with mean equals to zero and variance equals to $\frac{1}{T}$ for any fixed positive integer under the condition that r_t is an independent identical sequence and $E(r_t^2) < \infty$.

The Ljung-Box test is used to test whether a series of observations over time are random and independent based on a number of lags, see [4]. The Ljung-Box test has a null hypothesis that the series of residual has no autocorrelation for a fixed number of lag, which means $\rho(k)$, $k = 1, 2, \dots$ are equal to zero, against the alternative hypothesis that the correlation coefficient $\rho(k)$, $k = 1, 2, \dots$ are different from zero. The test statistic has the following formula for calculation of autocorrelation coefficient, see [4].

$$Q(m) = T(T+2) \sum_{\ell=1}^m \frac{\hat{\rho}_\ell^2}{T-\ell}, \quad (11)$$

where T is the sample size, m is the number of autocorrelation lags, and ρ_ℓ is the sample autocorrelation at lag ℓ . The test statistic Q is asymptotically χ^2 distributed with m degrees of freedom.

Partial Autocorrelation Function (PACF) is a function of its ACF and used to determine the order of model. We introduce the PACF in the following way. The return is

$$\begin{aligned} r_t &= \phi_{0,1} + \phi_{1,1}r_{t-1} + e_{1t} \\ r_t &= \phi_{0,2} + \phi_{1,2}r_{t-1} + \phi_{2,2}r_{t-2} + e_{2t} \\ &\vdots \\ &\vdots \\ &\vdots \end{aligned}$$

where the first term $\phi_{0,1}$, $\phi_{0,2}$ are constant, $\phi_{i,j}$, $i \neq j$ is the coefficient of r_{t-i} and e_{jt} is error term. The estimated $\hat{\phi}_{i,j}$, $i = j$ is called the lag- j sample PACF of r_t . For example, in the first equation, $\hat{\phi}_{1,1}$ is the lag-1 sample PACF of r_t . In the second equation, $\hat{\phi}_{2,2}$ is the lag-2 sample PACF of r_t , see [4].

4.5 Distribution of error term

In modeling of volatility, the distribution of error term plays an important roll. The most common assumption of financial volatility is normal distribution. But if the error term is fat tailed, the normal distribution assumption

is not suitable. In this thesis, besides the normal distribution, we will also introduce Student t-distribution.

4.5.1 Normal distribution

The density function of a random variabel X_t is given by

$$f_X(x_t) = \frac{1}{\sigma\sqrt{2\pi}} \exp^{-\frac{(x_t-\mu)^2}{2\sigma^2}}$$

where μ is the expected value and σ^2 is the variance. The standard normal distribution has mean $\mu = 0$ and variance $\sigma^2 = 1$.

4.5.2 Student-t distribution

Student-t distribution's probability density function is given by

$$f_X(x_t) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} (1 + \frac{x_t^2}{\nu})^{-\frac{\nu+1}{2}}$$

where ν is the number of degree of freedom and Γ is the gamma function. When $\nu \rightarrow \infty$ the Student -t distribution approach to the normal distribution.

5 Methodology

In this section we will present how ARCH and GARCH models are constructed, how parameters are estimated and how the models are compared. Several packages in program *R* will be listed in the end of section.

5.1 ARCH(m)

In 80's, Robert.F Engel introduced the AutoRegressive Conditional Heteroscedasticity model(ARCH) and successfully applied it to the research of United Kingdom inflation index volatility, see [5]. It is also the first model of conditional heteroscedasticity.

With the referens to Tsay, see [4], "the basic idea of ARCH models is that (a) the a_t term of an asset return is serially uncorrelated, but dependent, and (b) the dependence of a_t can be described by a simple quadratic function of its lagged values". ARCH(m) model assumes that the residual term a_t is a product of standard deviation σ_t and errors ϵ_t , thus

$$a_t = \sigma_t \epsilon_t,$$

where ϵ_t is a sequence of independent and identically distributed random variables.

The squared standard deviation σ_t^2 is the one-step ahead forecast of the conditional variance. It depends on the previous squared residuals with a certain weight. Mathematically it can be written as

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \dots + \alpha_m a_{t-m}^2, \quad (12)$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$ for $i > 0$ are the weights of previous squared residuals, m is the number of the squared residuals included in the calculus of the conditional variance. ARCH model models the volatility as a function of previous squared residuals.

5.2 GARCH(m,s)

GARCH(Generalized Autoregressive Conditional Heteroscedasticity) model is proposed by Bollerslev based on Engel's ARCH model, see [8].

The GARCH(m,s)model is given by

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^m \alpha_i a_{t-i}^2 + \sum_{j=1}^s \beta_j \sigma_{t-j}^2 \quad (13)$$

Where again ϵ is a sequence of independent and identically distributed random variables, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ and $\sum_{i=1}^{max(m,s)} (\alpha_i + \beta_i) \leq 1$, m is the order of the ARCH terms a^2 and s is the order of the GARCH terms σ^2 , α_i are the weights of squared residuals and β_i are the weights of squared standard deviations, see [4]. GARCH model models the volatility as a function of both previous squared residuals and previous squared standard deviations.

Determining the order of a GARCH model is not easy, in most applications, only lower order GARCH models are used, i.e. GARCH(1,1), GARCH(2,1) and GARCH(1,2), see [4]. In our case, we only use GARCH(1,1) to estimate the volatility of returns. The GARCH(1,1) is given by

$$\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2 \quad (14)$$

It means the squared standard deviation is made up of consist of a constant α_0 , one-step previous squared residual with weight α_1 and one-step previous squared standard deviation with weight β_1 .

5.3 Parameter estimation

Maximum Likelihood method is a common approach to estimate parameter. In ARCH(m) estimation, for a set of T independent and identically distributed random variables data a_1, a_2, \dots, a_T , which comes from a

distribution with a known density function $f(x)$ and unknown parameter $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_m)$, see [4]

Under the assumption of normally distribution of ϵ_t , the likelihood function is

$$\begin{aligned} L &= f(a_{m+1}, \dots, a_T | \boldsymbol{\alpha}, a_1, \dots, a_m) \\ &= \prod_{t=m+1}^T \frac{1}{\sqrt{2\pi\sigma_t^2}} \exp\left(-\frac{a_t^2}{2\sigma_t^2}\right) \end{aligned} \quad (15)$$

and the loglikelihood function is

$$\begin{aligned} \ell &= \ln(L) \\ &= -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=m+1}^T \ln(\sigma_t^2) - \frac{1}{2} \sum_{t=m+1}^T \frac{a_t^2}{\sigma_t^2} \end{aligned} \quad (16)$$

Under the assumption of Student t-distribution of ϵ_t , the likelihood function is

$$\begin{aligned} L &= f(a_{m+1}, \dots, a_T | \boldsymbol{\alpha}, a_1, \dots, a_m) \\ &= \prod_{t=m+1}^T \frac{\Gamma((\nu+1)/2)}{\Gamma(\nu/2)\sqrt{(\nu-2)\pi}} \frac{1}{\sigma_t} \left(1 + \frac{a_t^2}{(\nu-2)\sigma_t^2}\right)^{-(\nu+1)/2} \end{aligned} \quad (17)$$

and the loglikelihood function is

$$\begin{aligned} \ell &= \ln(L) \\ &= (T-m) \ln\left(\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}}\right) - \frac{1}{2} \sum_{t=m+1}^T \ln(\sigma_t^2) \\ &\quad - \frac{\nu+1}{2} \sum_{t=m+1}^T \ln\left(1 + \frac{a_t^2}{(\nu-2)\sigma_t^2}\right) \end{aligned} \quad (18)$$

where $\sigma_t^2 = \alpha_0 + \alpha_1 a_{t-1}^2 + \dots + \alpha_m a_{t-m}^2$ can be evaluated recursively for an ARCH model.

In the same way, the log-likelihood function fo GARCH(1,1) model is

Under the normality assumption:

$$\begin{aligned} \ell &= -\frac{1}{2} \ln(2\pi) - \frac{1}{2} \sum_{t=1}^T \ln(\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \\ &\quad - \frac{1}{2} \sum_{t=1}^T \frac{a_t^2}{\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2} \end{aligned} \quad (19)$$

Under the Student t assumption:

$$\begin{aligned} \ell = & T * \ln\left(\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi(\nu-2)}}\right) - \frac{1}{2} \sum_{t=1}^T \ln(\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2) \\ & - \frac{\nu+1}{2} \sum_{t=1}^T \ln\left(1 + \frac{a_t^2}{(\nu-2)\alpha_0 + \alpha_1 a_{t-1}^2 + \beta_1 \sigma_{t-1}^2}\right) \end{aligned} \quad (20)$$

5.4 AIC

AIC, Akaike information criterion, is a measure of godness of statistical model fitting for a given set of data. The AIC is defined as

$$AIC = \frac{-2}{T} \ln(L) + \frac{2}{T} m \quad (21)$$

where $\ln(L)$ is log-likelihood function T is the sample size and m is the number of parameters, see [4]. The same data is used in models and the one with smallest AIC value is the best, see[6]. We will compare different models with AIC value to conclude which model has the best ability to forecast the financial volatility.

5.5 Back test

5.5.1 Confidence interval

We will construct 95% confidence intervals based on the estimated volatility from ARCH and GARCH models with normal distribution and Student t-distribution.

For a stochastic variabel Q , with $E[Q] = \mu$ and $Var(Q) = \theta^2$, we have

$$P(Q \leq q_\alpha) = \alpha.$$

where q_α is the quantile of α and α is a percentile. To standardized the distribution, we subtract the mean and divide by the standard deviation.

$$P\left(\frac{Q-\mu}{\theta} \leq \frac{q_\alpha-\mu}{\theta}\right) = P(\tilde{Q} \leq \tilde{q}_\alpha) = \alpha.$$

The stochastic variable \tilde{Q} has $E[\tilde{Q}] = 0$ and $Var(\tilde{Q}) = 1$. In order to determine the quantile from the distributions of the estimated volatility which are calculated with our models, we need multiply them in previous formula. Thus,

$$P(\tilde{Q}\hat{\sigma} \leq \tilde{q}_\alpha\hat{\sigma}) = P(W \leq w_\alpha) = \alpha.$$

The interval of quantile w_α and $-w_\alpha$ is the confidence interval we want. The stochastic variable W has $E[W] = 0$ and $Var(W) = \hat{\sigma}^2$

5.5.2 Binomial test

The binomial test is an exact test of the statistical significance of deviations. If we have a binomial distributed stochastic variable Y and let k denote the number of "success", s denote the expected proportion of successes if the null hypothesis is true and n is the total number of observations, then $Y \sim Bin(n, s)$. The probability of getting k successes in n is

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$

The test has the null hypothesis that the real probability of success is equivalent to s , and the alternative hypothesis that it is not equivalent to s .

5.5.3 Back-testing

Let T denote the number of observations of a data-set and τ denote a specific position in the interval $[1, T]$ such that $[k, k + \tau]$ belongs to $[1, T]$. In this thesis we use the observations in $[1, \tau]$ to estimate volatility next time point, i.e $\tau + 1$. In this way, we get a new data-set of estimated volatility from position $\tau + 1$ to position T . We construct the 95% confidence interval based on the estimated volatility. Let K_t denote the estimated confidence interval, R_t denote the real log-returns and define the indicator variable I indicate violations, then we have the process I_t is a process of i.i.d. Bernoulli variables with success (i.e violation) probability $1 - \alpha$, see [1].

$$I_t = \begin{cases} 1 & R_t \notin K_t \\ 0 & R_t \in K_t \end{cases}$$

Then we can calculate the per cent of observed violations $p = \frac{\sum_{t=\tau+1}^T I_t}{T-\tau}$. Finally, we can test the null hypothesis $H_0 : p = 1 - \alpha$ against alternative hypothesis $H_1 : p \neq 1 - \alpha$ by a binomial test.

5.6 Packages and Commands in R

In this thesis, we are using program *R* for data analysis and modeling. The benefit of using program *R* lies in the fact that it brings a lot of complete packages. Two main packages are *fBasics* and *fGarch* for our analysis. *fBasics* package collects functions to investigate basic properties of data. The main function of this package for our data analysis is to do hypothesis testing. With commandos *shapiroTest* and *jarqueberaTest*, One Sample Normality Tests, i.e. Shapiro-Wilks normality test respectively Jarque-Bera normality test could be done. *fGarch* is a specific package for ARCH/GARCH modeling. The procedure *garchFit* is used for parameter

estimation of ARCH/GARCH models by the maximum log-likelihood approach. With an addition of $cond.dist = std$, we can change the default value, which is the normal distribution, to Student t-distribution. This procedure will also calculate the AIC value of each model.

6 Results

6.1 Analysis of data

We start to plot our data in program *R*. Figure 2 below presents the development of Deutsche Bank stock daily close price on NASDAQ during the period of three years, i.e. 756 observations from 2012-01-30 to 2015-01-30. Between index 50 and index 125 i.e. around April 2012 and July 2012, the stock price is on the decline, reaches the lowest price in three years, then the stock price rebounds, is on the rise, and reaches the highest price at index 500 i.e. around February 2014. After that, price falls. The data appears to have quite high amplitude of the fluctuation over certain time period.

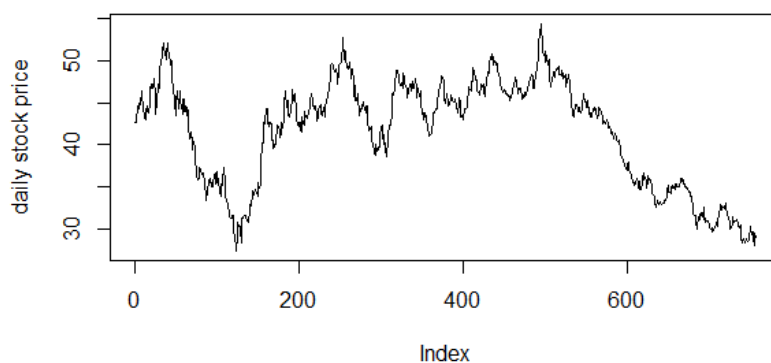


Figure 2: Daily close price for Deutsche Bank

Figure 3 is the daily log-returns of our data. Remember, the calculation of return used here, is defined by $R_{t*} = \frac{P_t - P_{t-1}}{P_{t-1}} \approx \ln\left(\frac{P_t}{P_{t-1}}\right)$. It is easy to see return changes with time. Most observations is on the interval $[-0.05, 0.05]$. Amplitude of return data has changes over time, it is relatively high in the beginning of period, then it tends to be calm, fluctuate within interval.

Since kurtosis and skewness are descriptive statistics of data distribution, it is useful for testing the sample data's distribution with normal distribution. As we present in *Skewness and Kurtosis* subsection, normal distribution's skewness and kurtosis equal to zero and three respectively. We list mean,

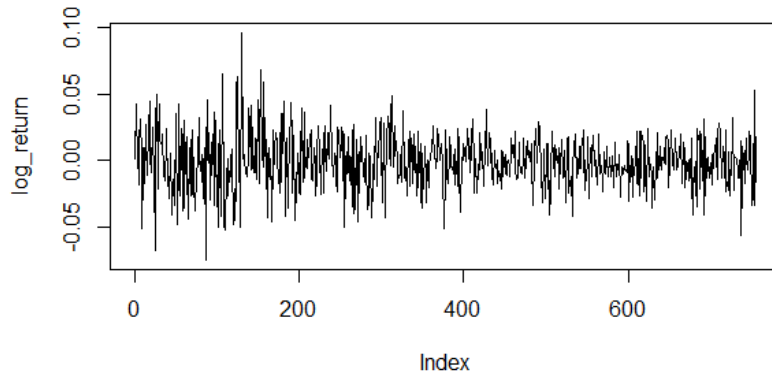


Figure 3: Daily log-return for Deutsche Bank

standard deviation, skewness, kurtosis, Shapiro-wilk test and Jarque-Bera test in figure 4.

Sample size : 755	
Mean	-5.1151E-4
Standard deviation	2.0277E-2
Skewness	2.0396E-3
Kurtosis	1.1299
Shapiro-Wilk test	p-value=5.361E-05
Jarque-Bera test	p-value=6.783E-03

Figure 4: Table of basic data analysis

The skewness of log-return series of Deutsche Bank stock price is 0.00204, which is very close to zero. It indicates that observations are relatively symmetrically distributed round the mean value however with more data placed on the left side from zero than right side. The kurtosis of log-return series is 1.1299, which is less than three. It indicates that data has a relatively steeper distribution than normal distribution, excess peakedness. Therefore, it could be assumed that the sample is symmetric distributed and has fat-tail characteristic. P -value of both Shapiro-wilk test and Jarque-Bera test are significant, which means we should reject null hypothesis. This is to say, the sample is not normally distributed.

Another way to observe data is Quantile-Quantile plot (QQ-plot). By plotting the log-returns, we see easily that the sample data is not normal distributed due to that some observations deviate from the straight line, see figure 5. Above all, we can conclude that the data is not normal distributed. For the financial time series data, other distributions could be considered. For example, Student t-distribution could be assumed for data. Figure 6 is a QQ-plot for Student t-distribution of log-return data. More observations tend to lie on the line comparing with normal QQ-plot. However, it is difficult to conclude that data is Student t-distributed due to several observations deviate from the straight line.

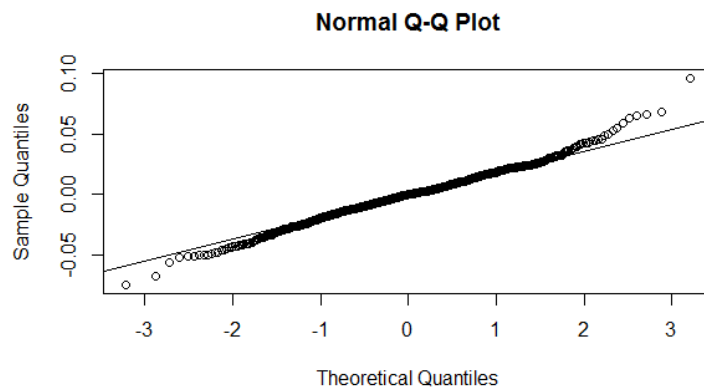


Figure 5: QQ-plot of log-return series for normal distribution

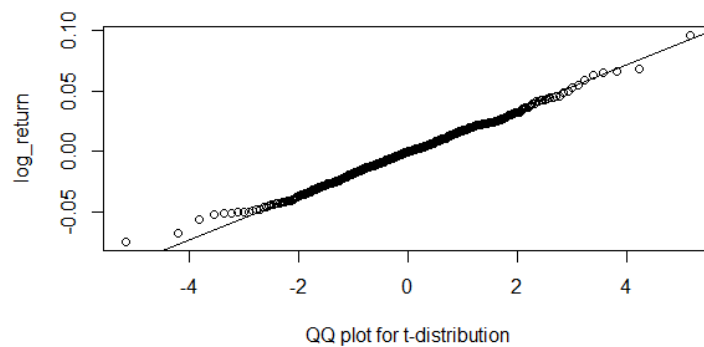


Figure 6: QQ-plot of log-return series for Student t-distribution

6.2 Model building

We refer to Tsay's method to build a volatility model, see [4]. Serial dependence in data will be tested and a mean equation will be specified to remove any linear dependence if it is necessary. Then we will test residuals of the mean equation for ARCH effects. A volatility model will be established in next step. Finally, we do model checking.

6.2.1 Specifying a mean equation

We start to specify a mean equation for the return process r_t ,

$$r_t = \mu_t + a_t.$$

Assume that r_t could be described by a mean model and a volatility model. The mean model describe term μ_t . We plot the autocorrelation function of log-return series in figure 7. The purpose of the autocorrelation function is to measure dependency between the value in the present and the value a few days in the past.

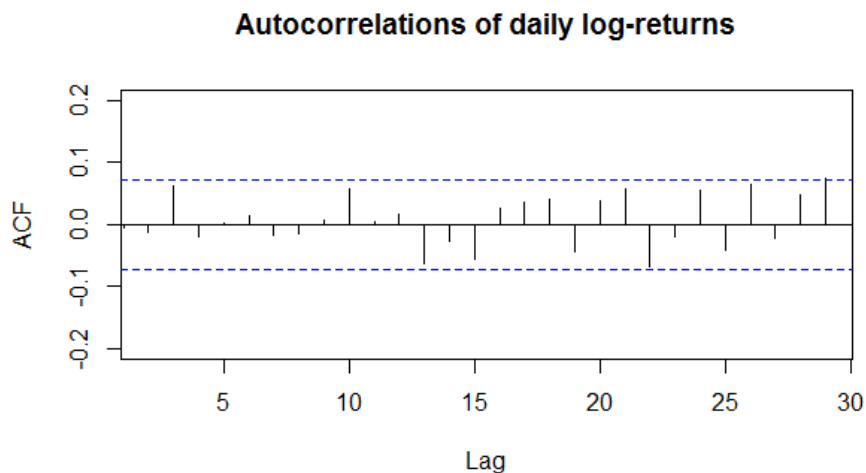


Figure 7: Autocorrelations of daily log-return series of Deutsche Bank

As we see, all sample ACFs are close to zero, the series is a white noise series. Clearly, the serie has no serial correlation. In addition, in order to detect the serial correlations, we can also test the sample data, log-return series, with Ljung-Box test. According to the earlier knowledge, the number of lags is given by $\ln(\text{the number of observations})$, i.e. $\ln(756)$. So lags up to 7 are chosen for calculation of Q-statistic of Ljung-box test. Q-statistic

is defined by formula (10). Ljung-Box test has a null hypothesis $H_0 : \rho = 0$ against alternative hypothesis $H_1 : \rho \neq 0$. Ljung-Box test gives us a p -value equal to 0.8004, which is non-significant with 95% significance level, we don't reject null hypothesis. Thus, the log-return series has no serial correlation. Another test needs to be done is to check if the series' mean is significantly different from zero. We have a null hypothesis $H_0 : \mu = 0$ against alternative hypothesis $H_1 : \mu \neq 0$ with 95% significance level. The t -test gives us a p -value equal to 0.4884, greater than 5%, which is non-significant. The estimated mean value is in 95% confidence interval, [-0.0019602137, 0.0009372022]. So we do not reject the null hypothesis. Thus, our mean equation equals to zero. The return process can be represented as $r_t = a_t$

6.2.2 Testing for ARCH effect

We have already determined that the return process can be written as $r_t = a_t$, now we need test it for ARCH effect. The squared series a_t^2 is used to check for conditional heteroscedasticity, which is also known as the ARCH effect. There are two main test for ARCH effect, Ljung-Box test and Lagrange multiplier test[4]. Here we focus on the Ljung-Box test. Lags up to 7 are chosen for calculation Q-statistic of Ljung-Box test. The Q-statistic is defined by formula (10). The $Q(7)$ value is 63.517. As we mentioned earlier, the test statistics is asymptotically chi-squared, χ^2 distributed with $k = 7$ degrees of freedom. The p -value is 2.985E-11, which is extremely close to zero. It means that ARCH-effect exists. The ACF figure of squared log-returns below (figure 8) also confirms that the series shows strong ARCH-effect.

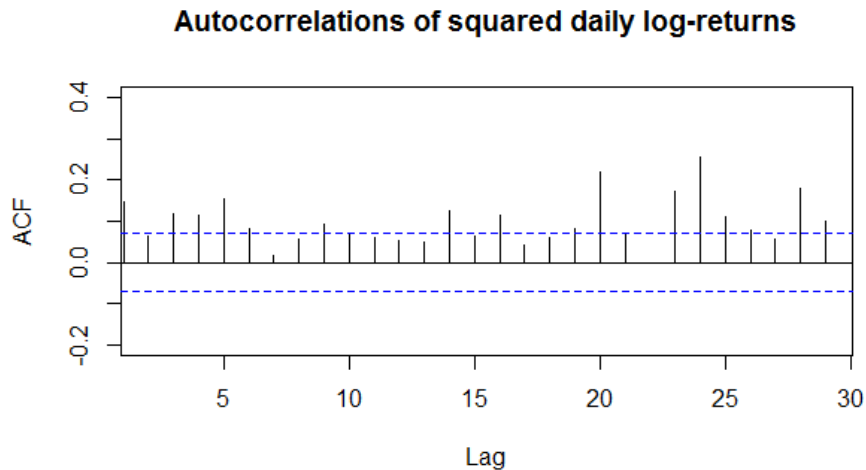


Figure 8: Autocorrelations of squared log-return series of Deutsche Bank

6.2.3 Specifying a volatility model

In this step we will examine the ARCH and GARCH models with two kinds of different distribution of the error term, i.e normal distribution and Student-t distribution, to estimate and forecast the financial volatility using the stock return for Deutsche Bank. We are especially interested in the coefficients, whether they are statistically significant or not. For comparison of models, we are interested in AIC values.

We start to determine the order in ARCH model. With reference to Tsay, see [4], the partial autocorrelation function (PACF) of a_t^2 can be used for order determination. PACF is plotted on figure 9.

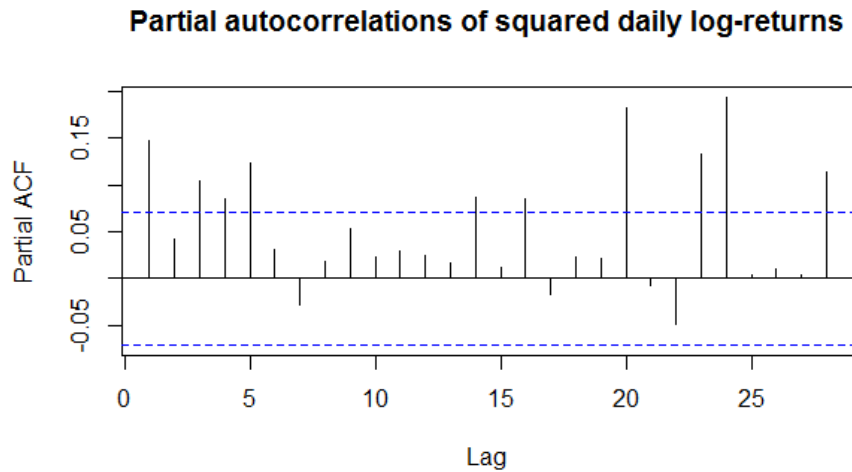


Figure 9: Partial autocorrelation of squared error term

As we see, it is difficult to determine the order from PACF plot. With the help of a producer in program *R*, we get a estimated order of 5. ARCH(5) with normal distribution and Student-t distribution respectively has the following estimation of coefficients and AIC, see figure 10. Note that values with "***" marking are statistically significant at 10% significance level.

	Normal	Student-t
α_0	2.132E-04**	2.048E-04**
α_1	1.244E-01**	1.328E-01**
α_2	6.716E-02	5.509E-02
α_3	1.277E-01**	1.479E-01**
α_4	9.193E-02**	1.263E-01**
α_5	7.829E-02**	7.525E-02
AIC	-5.01717	-5.024615

Figure 10: Parameter-estimating and AIC for ARCH(5)

With help of GARCH(1,1) model, we have following estimation of coefficients and AIC on the figure 11. Note that values with "***" marking are statistically significant at 10% significant level.

	Normal	Student-t
α_0	2.464E-06	2.773E-06
α_1	4.122E-02**	4.459E-02**
β_1	9.527E-01**	9.507E-01**
AIC	-5.068485	-5.060576

Figure 11: Parameter-estimating and AIC for GARCH(1,1)

Since the constant term means the long-term volatility, the non-significant α_0 is not logical. The constant term α_0 can be calculated by hands. We know that

$$\alpha_0 = (1 - \alpha_1 - \alpha_2)V_L$$

where α_1 , α_2 and the long-term volatility V_L are known. Hence, the calculated α_0 is 2.491665e-06 for GARCH(1,1) with normal distribution and 1.936591e-06 for GARCH(1,1) with Student-t distribution. In summary, the following four equations of volatility models are determined.

ARCH(5) with normal distribution:

$$\sigma_t^2 = 0.0002132 + 0.1244a_{t-1}^2 + 0.1277a_{t-3}^2 + 0.09193a_{t-4}^2 + 0.07829a_{t-5}^2$$

ARCH(5) with Student t-distribution:

$$\sigma_t^2 = 0.0002048 + 0.1328a_{t-1}^2 + 0.1479a_{t-3}^2 + 0.1263a_{t-4}^2$$

GARCH(1,1) with normal distribution:

$$\sigma_t^2 = 2.491665 * 10^{-6} + 0.04122a_{t-1}^2 + 0.9527\sigma_{t-1}^2$$

GARCH(1,1) with Student t-distribution:

$$\sigma_t^2 = 1.936591 * 10^{-6} + 0.04459a_{t-1}^2 + 0.9507\sigma_{t-1}^2$$

There are several way to compare the advantage of models. Here we determine to contrast the performance of models by comparing AIC values and results from model checking. We consider the smaller AIC model has, the better model is.

6.2.4 Model checking

To check the validity of volatility equations, we use the Q-statistics of \hat{a}_t and \hat{a}_t^2 , where \hat{a}_t is a sequence of independent identical random variables of the standardized residuals, see [4].

$$\hat{a}_t = \frac{a_t}{\sigma_t}$$

With the standardized residuals \hat{a}_t we test the null hypothesis that there are no autocorrelations in the standard residuals series, and test the null hypothesis that homoscedasticity exists in series with the squared standardized residuals \hat{a}_t^2 . Results are summarized in figure 12 and 13. Ljung-Box test gives us all p -value greater than 5%, which means we should not reject both null hypothesis.

	ARCH(5) Normal	ARCH(5) Student t	GARCH(1,1) Normal	GARCH(1,1) Student t
Q-statistics	2.5127	2.5236	7.9911	8.2729
p-value	0.9261	0.9253	0.3334	0.3091

Figure 12: Ljung-box test of \hat{a}_t

	ARCH(5) Normal	ARCH(5) Student t	GARCH(1,1) Normal	GARCH(1,1) Student t
Q-statistics	1.997	1.9992	2.1822	1.2392
p-value	0.96	0.995	0.9491	0.99

Figure 13: Ljung-box test of \hat{a}_t^2

ACF plots of standardized residuals \hat{a}_t on figures 17-20 in appendix for all four models show that no autocorrelations exist in standardized residual series. ACF plots of squared standardized residuals \hat{a}_t^2 on figures 21-24 in appendix show that GARCH models perform better than ARCH models in homoscedasticity test.

We can see GARCH models perform better than ARCH models in both AIC and homoscedasticity test. ACF plots of standardized residuals and squared standardized residuals are almost identical for both GARCH(1,1) model with normal distribution and GARCH(1,1) model with Student t-distribution, but GARCH(1,1) with normal distribution has a smaller AIC value. In order to check the validity of the distribution assumption for GARCH(1,1) models, we plot QQ-plots of standardized residuals with normally distribution and Student t distribution on figure 14 and figure 15 respectively.

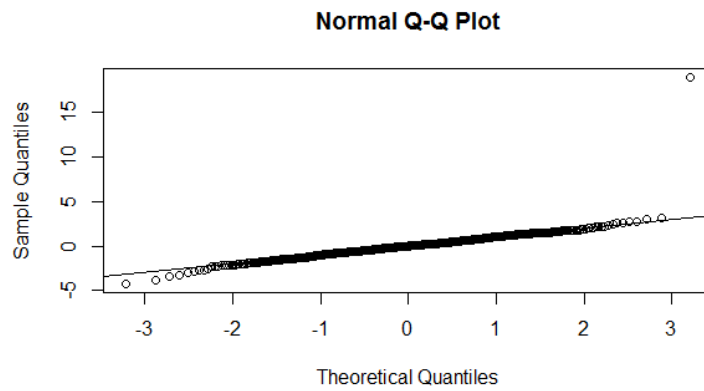


Figure 14: QQ-plot of standardized residuals with GARCH(1,1), normal distribution

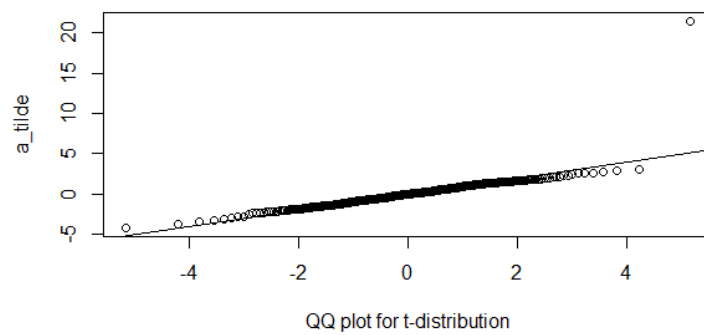


Figure 15: QQ-plot of standardized residuals with GARCH(1,1), Student t-distribution

We can see there is one extreme observation that lies far away from the straight line in both plots. It is difficult to conclude which distribution assumption is better.

6.3 Back-testing

For back-testing purposes the data is split into two sub-samples. We have the full sample of the Deutsche Bank log-returns of 755 observations in total. We use the sample of the first 500 observations to estimate volatility in the rest of positions, i.e. from 501 to 755, 255 observations. The sample of the estimate volatility will be compared with the sample of the real value, i.e. the last 255 observations in the full sample. We estimate the confidence intervals of the level 95% based on the estimated volatility from our models, which means that we assume that 5% will not be in the interval. All models have an amount of log-returns violate the estimated confidence interval. We calculate per cent of violations from each model and test the null hypothesis $H_0 : p = 0.05$, i.e per cent of violations equals to 0.05, against the alternative hypothesis $H_1 : p \neq 0.05$ with binomial test. The results of the number of violations, per cent of violations and p -value from binomial test of each model are summarized in figure 16.

Model	Distribution	Violations	Percent	p-value from Binomial test
ARCH(5)	Normal	7	0.027	0.1125
ARCH(5)	Student t	7	0.027	0.1125
GARCH(1,1)	Normal	13	0.051	0.8856
GARCH(1,1)	Student t	13	0.051	0.8856

Figure 16: Results of backtesting

We can see that we have the same values in both normal distribution and Student t-distribution of ARCH and GARCH models, so the distributions have no big impact. GARCH(1,1) models are closest to the wanted level and ARCH(5) models have almost a too low level of violations. P -values are non-significant, which means with 5% significance level we could not reject the null hypothesis.

7 Discussion and conclusion

The purpose of this thesis is to estimate dynamic volatility of returns for Deutsche Bank based on previous information by using ARCH(5) and GARCH(1,1) models. The sample is a set of series of Deutsche Bank's daily log-returns during the past three years. We have reviewed if the log-returns could be considered to be normal distributed or Student t-distributed. Since we can

not determine which distribution is most suitable, we have applied both distributions to these two models.

ARCH(5) models under normal- and Student t-distribution assumptions have five and four significant parameters respectively. GARCH models have significant parameters of both ARCH terms and GARCH terms but a non-significant parameter of the constant term. Since the constant term represents the long-term volatility which the expected value of variance will converge to, it is required to be in volatility equation.

Further, we have checked validity of volatility equations by using the standardized residuals. The Ljung-Box tests of autocorrelation and heteroscedasticity in the models have shown that neither autocorrelation nor heteroscedasticity existed in these models. But if we take a closer look to the ACF plots of squared standardized residuals of ARCH(5) models, we would discover several numbers of lags exceed the “blue marking” limits. When it comes to AIC, the GARCH(1,1) models have a lower AIC value than ARCH(5) models. It means that GARCH(1,1) models have generally better ability in forecasting financial volatility. However, the difference of AIC values between two GARCH(1,1) models are little. Furthermore, we checked the validity of the distribution assumption of GARCH(1,1) models with QQ-plots of standardized residuals. Both QQ-plots showed that the sample was not certain normal- or Student t-distributed.

We have split the full sample of Deutsche Bank log-returns into two subsamples. By using the sample of the first 500 observations we have estimated volatility of next 255 trading days. The sample of last 255 observations from the full sample has been used for comparison. We also have constructed confidence intervals based on the estimated volatility and done back-testing by calculating amount of log-returns that violate the estimated confidence interval and tested 95% confidence interval of per cent of violations by binomial test. The results have shown us that there is difference in distributions among ARCH(5) and GARCH(1,1). All models have almost 5% violations but GARCH(1,1) models are closer to the wanted level.

Further investigation of GARCH model with more lags could be considered in next time. An additional alternative could be to estimate volatility by using other advanced models for example EGARCH, IGARCH and GJR-GARCH. In our research GARCH(1,1) models are relatively better than ARCH(5) model based on AIC values, autocorrelation and heteroscedasticity plots of standardized residuals and back-testing. The differences in distributions are little, the assumption of distribution has no big impact. But GARCH(1,1) with assumed normal distribution has the smallest AIC value. Based on the analysis, we can conclude that GARCH(1,1) with assumed nor-

mal distribution is the best choice of model for estimating volatility among four models.

8 Appendix

The autocorrelation functions of standardized residuals for four models.

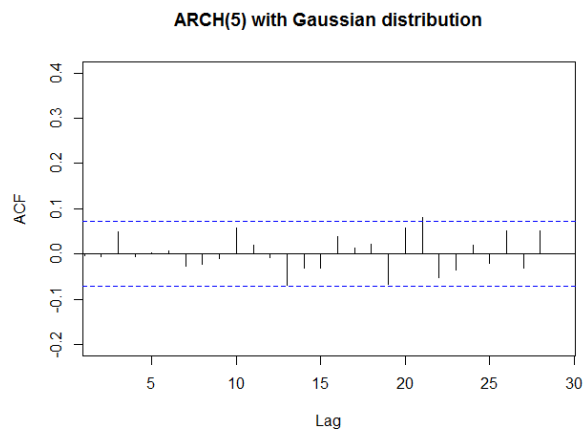


Figure 17: Autocorrelation function of \hat{a}_t , ARCH(5) Gaussian

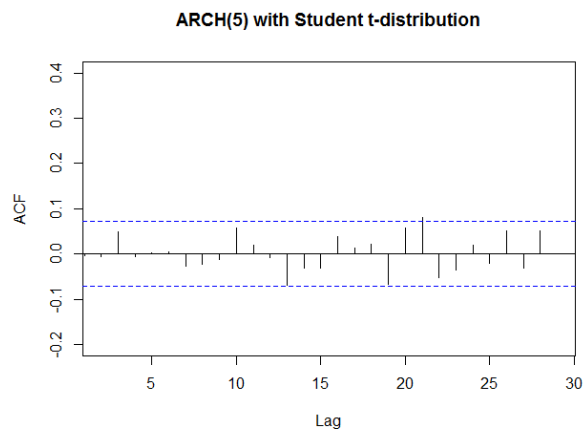


Figure 18: Autocorrelation function of \hat{a}_t , ARCH(5) Student t

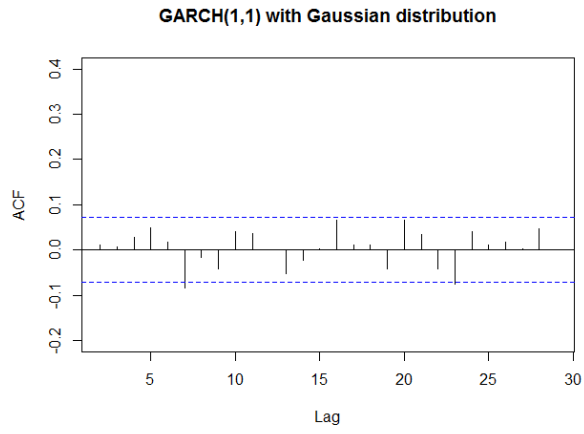


Figure 19: Autocorrelation function of \hat{a}_t , GARCH(1,1) Gaussian

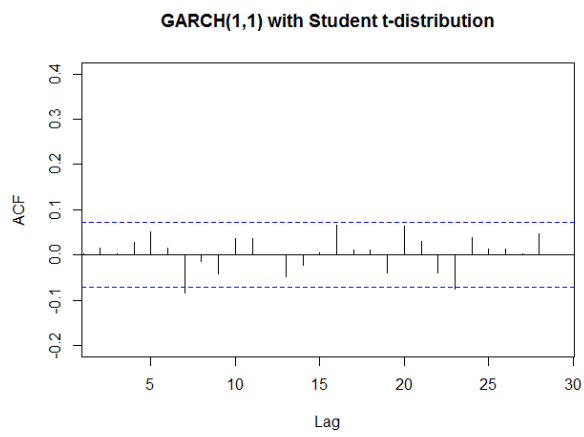


Figure 20: Autocorrelation function of \hat{a}_t , GARCH(1,1) Student t

The autocorrelation functions of squared standardized residuals for four models.

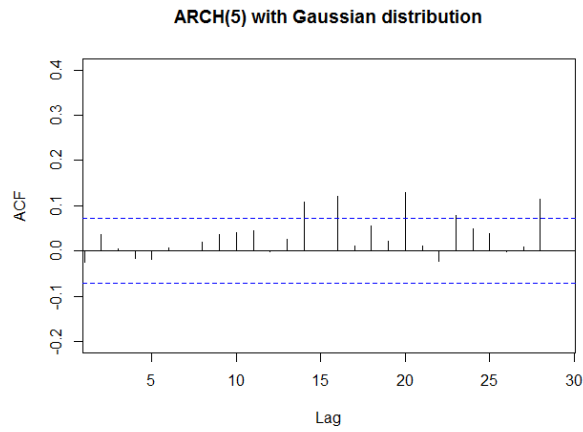


Figure 21: Autocorrelation function of \hat{a}_t^2 ARCH(5) Gaussian

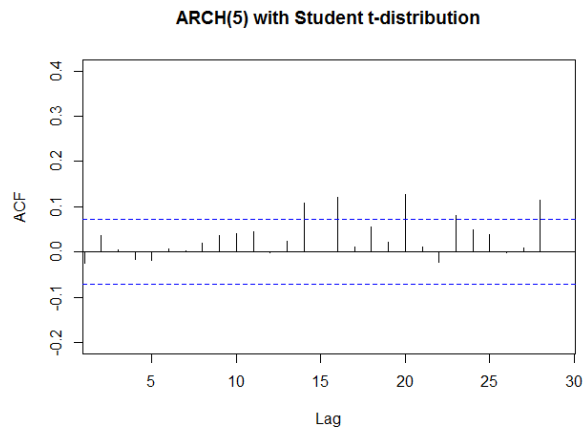


Figure 22: Autocorrelation function of \hat{a}_t^2 ARCH(5) Student t

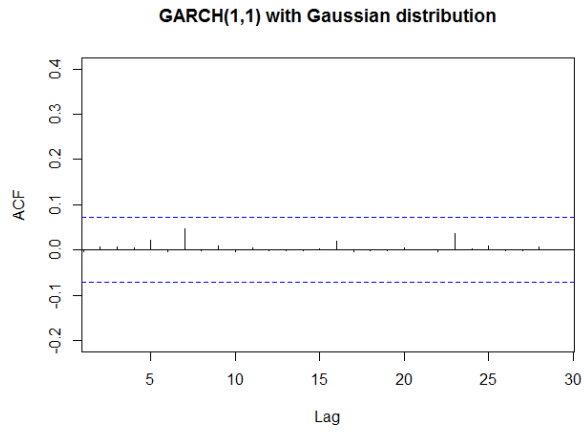


Figure 23: Autocorrelation function of \hat{a}_t^2 GARCH(1,1) Gaussian

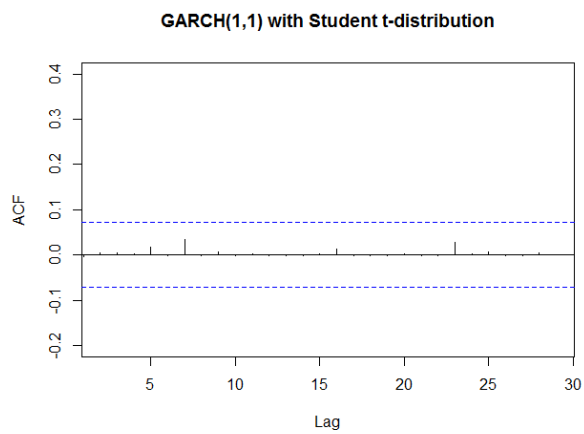


Figure 24: Autocorrelation function of \hat{a}_t^2 GARCH(1,1) Student t

References

- [1] McNeil.A.J, Frey.R, Embrechts.P, *Quantitative Risk Management: Concepts, Techniques, and Tools*
- [2] Ruppert. D, *Statistics and data analysis for financial engineering*
- [3] Tabachnick.B.G, Fidell.L.S, *Using Multivariate Statistics:International Edition*
- [4] Tsay.R.S, *Analysis of Financial Time Series:Third Edition*
- [5] Engle.R.F *Econometrica: Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation*
- [6] Aho.K, Derryberry.D.W, Peterson.T, *Ecology: Model selection for ecologists: the worldviews of AIC and BIC*
- [7] Alm.S, Britton.T, *Stokastik, sannolikhets teori och statistik teori med tillämpningar*
- [8] Bollerslev.T, *Journal of Econometrics 31, Generalized autoregressive conditional heteroskedasticity*
- [9] Shapiro.S.S, Wilk.M.B, *Biometrika 52 (3-4), An analysis of variance test for normality (complete samples)*
- [10] <http://www.nasdaq.com/symbol/db/historical>