Stockholms universitet

Modelling and forecasting the financial volatility of H&M, Hennes&Mauritz stock prices

Magnifique Harushimana

Kandidatuppsats i matematisk statistik
Bachelor Thesis in Mathematical Statistics

Kandidatuppsats 2016:18
Matematisk statistik
Juni 2016

www.math.su.se

Matematisk statistik
Matematiska institutionen
Stockholms universitet
106 91 Stockholm

Mathematical Statistics
Stockholm University
Bachelor Thesis **2016:18**
http://www.math.su.se

Modelling and forecasting the financial volatility of H&M, Hennes&Mauritz stock prices

Magnifique Harushimana*

June 2016

Abstract

This paper treats the dynamic modelling and forecast performance for financial volatility of Hennes&Mauritz assets returns. The presence of volatility clustering within the returns series required the use of the Autoregressive Conditional Heteroscedasticity (ARCH) model to fit the financial data.

The ARCH(1)-GARCH(1,1) models have been applied to the financial volatility of Hennes&Mauritz assets returns. The Akaike Information Criteria indicates that GARCH(1,1) has a suitable number of lags. The Minimum Mean Square Error estimate (MMSE) shows that conditional heteroscedastic variance approaches the unconditional variance.

Under the assumption of both normal and student's t-distributions, the fitted GARCH(1,1) with an assumed student's t-distribution appears to be the best model in volatility forecasts.

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: hamagni05@yahoo.fr. Supervisor: Mathias Lindholm, Karl Rökaeus and Joanna Tyrcha.

Contents

Chapter 1

Theory Background

1.1 Introduction

Financial markets cover periods of crisis followed by periods of calm.
During a crisis period the stock prices tend to fluctuate very much while in calm period the stock prices remain stable. This phenomenon is known as volatility clustering. Volatility describes the relative rate at which the stock prices move up and down. This is measured by estimating the conditional standard deviation of the daily change in price.

If the price of a stock moves up and down rapidly over short time-periods, it has high volatility. If the price practically never changes, it has low volatility. When volatility is high, the level of expected returns is greater as is the risk of loss.

It is in the light of economic development that several authors such as Robert F. Engle, Tim Bollerslev and others have studied the relationship between the risk of losses and the expectation of profitability. In their discussion, they have proposed different autoregressive conditional heteroscedastic variance models, useful to model volatility.

In this thesis volatility describes the conditional standard deviation of the asset returns. In order to understand how volatility of the asset return changes over time, we will study some methods and econometric models appropriate for modelling the volatility of asset returns. The used volatility model will be able to forecast volatility as required by financial theory. Forecasting volatility is needed in risk management, asset allocation and precision for value of future volatility.

We will follow the volatility modelling hypothesis drafted 30 years ago by Robert F. Engle. In 1982, Robert F.Engle introduced the Autoregressive Conditional Heteroscedasticity (ARCH) model. The ARCH model is used to model financial time series with time-varying volatility, such as stock prices.

In 1986, Tim Bollerslev extended the ARCH process to the Generalized Autoregressive Conditional Heteroscedasticity (GARCH). Both ARCH and GARCH models are designed to model volatility clustering.

By applying the ARCH-GARCH models on our data, we will test the goodness of ARCH-GARCH models in forecast performance.

1.2 Data

The data used in this thesis is the stock prices of Hennes&Maurirz traded every day on NASDAQ. There are 2767 trading days observed over 10 years (2004-2014), and expressed as daily time series. The chosen period covers periods of high and low volatility.

In 1947, the Hennes&Mauritz group opened their first women's clothing store in Västerås, Sweden. The store started by selling women's wear and grew with a reputation for quality, sustainability and high profitability, see [19]. Today the Hennes&Mauritz group offers fashion for every one and is growing with new stores the world over and on line. There are six different brands: H&M, COS, Monki, Weekday, Cheap Monday and &Other Stories in 3500 stores all around the World. Periods of success and solid growth have resulted in a strong financial position. Since 1974 is Hennes&Maurirz listed on the Stockholm Stock Exchange.

In order to investigate the model of dynamic volatility, we will use Hennes&Maurirz's dairly closing price. The original data is available and has been downloaded from the NASDAQ's website.

1.3 Aim

This paper will focus on the dynamic modeling of financial volatility of H&M asset returns. We start by analyzing the distribution of return series residuals. We will check if there is variability within the return series and whether the required characteristics are satisfied.

If the sampled data exhibits the non-constant variance known as the ARCH-effect, they will be suitable to fit the Autoregressive Conditional Heteroscedasticity (ARCH) and the Generalized-ARCH models. By using the Akaike Criterion, we will determine the suitable number of lags needed to estimate the parameters of the ARCH and GARCH models. The estimated model will be used to predict the absolute magnitude of return series.

1.4 Characteristics of financial times series

Generally, financial time series are based on the theory and practice of asset valuation over time. In this section we will describe different characteristics of financial time series. These will be extracted from the data of the daily closing price of Hennes&Mauritz's stock. In a simple way, we will organize this data set and transform the daily closing price in daily asset returns according to the linear financial time series theory.

1.4.1 Asset Returns and Volatility

We start by an overview of the statistical properties of the asset returns at a given frequency. According to Tsay [16], the simple returns $R_t, t = 1, 2...T$ at time t is given by

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \tag{1.1}$$

where P_t is the stock price at time t with T number of observations and $P_t - P_{t-1}$ the price change over the period $t-1$ to t. In the case of small relative price changes the equation above will be defined as the log price change expressed by

$$r_t = \ln(1 + R_t) \tag{1.2}$$

The measure of the relative changes during the time series period is called volatility and defined as an instantaneous deviation of stock returns. As mentioned in Gulisashvil [3], during epochs of low volatility the stock price does not change much, while large movements of the stock price may be expected during periods with high volatility. In Höglund [8], volatility is estimated as the square root of the sample variance.

$$\sigma = \sqrt{\frac{1}{T}\sum_{t=1}^{T}(r_t - \mu)^2} \tag{1.3}$$

where μ stands for the mean of returns. In statistics, the standard deviation is used to quantify the magnitude of variation or dispersion of a set of data values. When the standard deviation is close to zero, the data points tend to be very close to the mean value of the set, while a high standard deviation illustrates that the data points are spread out over a wider range of values.

Chapter 2

Data and methodology

This section covers the basic theory of some methods and econometric tools used for modelling the volatility of asset returns. Volatility measures the variation of a financial instrument over time. According to the theory of financial mathematics, volatility is an important factor in option trading, portfolio and risk management. It is predictable and does not change with time. It is assumed to be approximately normally distributed if the time period is sufficiently large, r_t are assumed to be a purely random process with independent increments, see [16]. We have to investigate if the used data meet these requirements.

2.1 Test of independence

We assume that r_t represents the random process with independent increment, ε_t the standard errors, independent, identically distributed random variables and t the number of observations. We then define the following formula:

$$r_t = \mu + \varepsilon_t \tag{2.1}$$

where μ is the mean of r_t and ε_t has the expected value zero and variance σ^2.

2.1.1 Autocorrelation test

The autocorrelation of a random process expresses the correlation between the values at different times. Autocorrelation occurs when the errors are correlated. The serial correlation or lagged correlation describes the dependence between the observations of a series of numbers arranged in a time period of length l called lag.

Consider a return series r_t, the lag autocorrelation of r_t is the correlation coefficient between r_t and r_{t-l}. The autocorrelation function for r_t is denoted by ρ_l and is estimated as follow with respect to [16].

$$\hat{\rho}_l = \frac{\sum_{t=l+1}^{T}(r_t - \mu)(r_{t-l} - \mu)}{\sum_{t=1}^{T}(r_t - \mu)^2} \tag{2.2}$$

where $0 \leq l < T$ and μ stands for the expected value of r_t. For a sufficiently large sample, the estimated sample autocorrelation function is assumed to be asymptotically normally distributed with mean zero and variance $\frac{1}{T}$ for any fixed positive integer l.

The main important tools for assessing the autocorrelation of a time series are the autocorrelation function (ACF) and the Partial autocorrelation function (PACF). The ACF is important in linear time series analysis and is constructed to capture the linear dynamic of the data. The PACF of stationary time series is a function of its ACF and is a commonly used tool for identifying the order of an autoregressive model, see [16].

The Ljung and Box test is also used to detect whether the autocorrelations in the data are different from zero, see [16]. It suggests the null hypothesis that a serie of residuals has no autocorrelation for a fixed number of lags, m, so that $\rho_l = 0$, $l = 1, 2, ..., m$ is equal to zero, against the alternative hypothesis that the correlation coefficient $\rho_l \neq$, $l = 1, 2, ..., m$ is different from zero. The test statistic is defined according to the following formula,

$$Q(m) = T(T+2)\sum_{l=1}^{m}\frac{\hat{\rho_l}^2}{T-l} \tag{2.3}$$

where T is the sample size, m is the number of autocorrelation lags, and $\hat{\rho}_l$ is the sample autocorrelation at lag l. According to [16], $Q(m)$ is asymptotically chi-square distributed with m degrees of freedom.
As shown by Tsay, the valid way to choose m number of lags is to use the natural log of the total number of the observations.
In our study, $m \approx \ln(T)$ and corresponds to

$$\ln(2767) \approx 7.9255 = 8$$

The same test is useful to detect if the sampled data have equal variance called homoscedasticity.

2.1.2 Heteroscedasticity test

Consider the classical linear regression model defined as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon} \tag{2.4}$$

where $\mathbf{Y} = (Y_1, Y_2, ..., Y_n)^T$, $\mathbf{X}$ is $n \times m$ design matrix, $\boldsymbol{\beta} = (\beta_1, \beta_2, ..., \beta_m)^T$ stands for the slope of the line, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)^T$ represents the residuals.
The variance of ε_n is assumed to be constant and equal to σ^2. This assumption is called homoscedasticity and applies the Ordinary Least Squares to estimate the parameters β. Under homscedasticity the residuals are
uncorrelated with an expected value of

$$E[\varepsilon\varepsilon^T|\mathbf{X}] = \sigma^2\mathbf{I}_n \tag{2.5}$$

where $\mathbf{I}$ is the identical matrix. If the variance of the residuals is non
constant or is heteroscedastic, then the Weighted Least Squares formula will be desirable to estimate our parameters. The heteroscedastic variance of the conditional

values of $\mathbf{X}$ is expressed in a matrix as follow:

$$E[\varepsilon\varepsilon^T|\mathbf{X}] = \begin{pmatrix} \sigma_1^2 & 0 & ... & 0 \\ 0 & \sigma_2^2 & ... & 0 \\ . & . & ... & . \\ . & . & ... & . \\ 0 & . & ... & \sigma_n^2 \end{pmatrix}$$

In order to check the heteroscedasticity, one can apply the test statistic introduced by Ljung and Box and described by (2.3) and the Lagrange multiplier test of Engle (1982).

The test of Engle (1982) recognized as the test of the ARCH-effect, is equivalent to the F statistic defined by [16] for testing homoscedasticity in the linear regression. Under the assumption of $E[\varepsilon\varepsilon^T|\mathbf{X}] = \sigma^2\mathbf{I}_n = \zeta_t^2$, the
Lagrange multiplier test of Engle (1982), assesses the null hypothesis
$H_0 : \alpha_i^2 = 0,\ i = 1, 2, ..., m$
in the linear regression
$\zeta_t^2 = \alpha_0 + \alpha_1\zeta_{t-1}^2 + ... + \alpha_m\zeta_{t-m}^2 + \varepsilon_t$,
$t = m + 1, ..., T$, where ε_t denotes the error term, m is a specified positive integer, and T is the sample size.
The test statistic is calculated as

$$F = \frac{(SS_0 - SSR_1)/m}{SSR_1/(T - 2m - 1)} \tag{2.6}$$

which is asymptotically distributed as a χ-squared distribution with m degrees of freedom under the null hypothesis.

2.2 Test of normality

2.2.1 Skewness and Kurtosis

Let $\{x_1, ..., x_N\}$ be a random sample of X with N observations. The sample skewness is calculated as:

$$\hat{S}(X) = \frac{1}{(N-1)\hat{\sigma}_x^3}\sum_{n=1}^{N}(x_n - \hat{\mu}_x)^3 \tag{2.7}$$

This measures the symmetry of X in respect to its mean. The distribution of X is said to be *positively skewed, negatively skewed or zero* depending on the sign of estimated value (positive, negative or zero), i.e. if the distribution is positively skewed then the probability density function has a long tail to the right, and if the distribution is negatively skewed the probability density function has a long tail to the left. Such distributions are called *leptokurtic*. A symmetric distribution has a skewness equal to 0 and is said to be unskewed.

In order to provide information of the tail behaviour of X, we will introduce a fourth central moment called Kurtosis. This is calculated as

$$\hat{K}(X) = \frac{1}{(N-1)\hat{\sigma}_x^4}\sum_{n=1}^{N}(x_n - \hat{\mu}_x)^4 \tag{2.8}$$

The Kurtosis is a statical measure used to describe the distribution of observed data around the mean. It measures the degree to which a distribution is more or less peaked than a normal distribution. Positive kurtosis indicates a relatively peaked distribution. Negative kurtosis indicates a relatively flat distribution. A normal distribution has a kurtosis of 3.

2.2.2 Jarque-Bera Test

This test is known as the test of normality. Consider testing the null hypothesis:
H_0: $\hat{S}(x) = 0$ and $\hat{K}(x) - 3 = 0$;
against the alternative hypothesis:
$H_1 : \hat{S}(x) \neq 0$ and $\hat{K}(x) - 3 \neq 0$
The Jarque-Bera test formula is:

$$JB = \frac{\hat{S}^2(x)}{6/N} + \frac{(\hat{K}(x) - 3)^2}{24/N} \tag{2.9}$$

where N is the sample size, $\hat{S}(x)$ is the sample skewness, and $\hat{K}(x)$ is the sample kurtosis. For large sample sizes, the test statistic is asymptotically distributed as a chi-squared random variables with 2 degrees of freedom. The null hypothesis of normality is rejected if the p value of Jarque Bera is less than the significance level.

2.3 Time series structure

In this section we will discuss the methods employed for analysing time series data in order to extract meaningful statistics and other characteristics of the data.

2.3.1 Stationarity

Stationarity is the basic characteristic to consider in time series analysis. Referring to [16], a stochastic process $\{X(t), t \geq 0\}$ is said to be a strictly stationary process if for all $n, s, t, ..., t_n$ the random vectors $X(t_1), ..., X(t_n)$ and $X(t_1 + s), ..., X(t_n + s)$ have the same joint distribution. This means that the joint probability distribution remains constant when shifted in time. Consequently, parameters such as the mean and the variance, if present, also do not change over time and do not follow any trends. This property has been difficult to verify empirically.

In case of time series $\{r_t\}$, a weak stationarity is considered. This assumption holds if both the mean of $\{r_t\}$ and the covariance between r_t and r_{t-l} are time invariant, and where l is an arbitrary integer. Statistically, the series $\{r_t\}$ is weakly stationary if the first and the second moments are constant over time. This means that $E[r_t] = \mu$ and $Cov(r_t, r_{t-l}) = \kappa_l$ which only depends on l. The covariance κ_l is called the lag-l autocovariance of r_t and the correlation coefficient between r_t and r_{t-l} is called the lag-l autocorrelation and is denoted by ρl.

2.3.2 White Noise

A stochastic process $\{X_t\}$ is called white noise if the sequence of independent and identically distributed random variables with the finite mean μ, variance σ^2 and covariance $Cov(X_t, X_{t+s}) = 0$ for $s > 0$. If such $\{X_t\}$ is normally distributed with mean zero and variance σ^2 it is recognized as *Gaussian White Noise*
In addition of those characteristics, there are many others properties of time series to take into account in this study, such as trends and seasonality.

2.4 Dynamic modelling of financial volatility

2.4.1 ARCH-GARCH

The ARCH-GARCH models and their various extensions are mostly employed in financial applications and may treat the financial data as return series in which the variance of the errors terms is non-constant, while the expected value of assets returns tends to fluctuate much for periods of turbulence and then revert to the same value as at the periods before crisis, see [12].

Referring to [14], returns values are assumed to be unpredictable with fat tails and volatility clustering. In order to meet this issues, Robert Engle (1982) introduced the first model known as Autoregressive Conditional Heteroscedasticity (ARCH), developed later by Bollerslev (1986) as a Generalized Autoregressive Conditional Heteroscedastic model.

The ARCH-GARCH process was designed to model the conditional variance in volatility. The process is valid when there are autoregressivity in squared returns and the next period's volatility is conditioned by information of the previous period.

Let I_{t-1} denotes the information set of all information through time $t-1$.
In order to put the volatility models in proper perspective, consider the conditional mean $\mu_t = E[r_t|I_{t-1}]$, variance $h_t^2 = Var(r_t|I_{t-1})$ and assume that r_t follows a simple time series model such as

$$r_t = \mu_t + \zeta_t = \mu_t + h_t \varepsilon_t \tag{2.10}$$

Returns are described by a mean equation and a volatility equation.
According to [16], the expected value μ_t could be equal to zero or follow an ARMA(p, q) process. The best way to check the model of returns r_t , is to test the assumption that the expected value is zero. This will be done by using a t-test, see [2].

Under the assumption of $\mu = 0$, the returns process would be
$r_t = \zeta_t = h_t \varepsilon_t$
where ζ_t denotes the shock of an asset return which is assumed to be serially uncorrelated but dependent.

Referring to financial applications as described in [16], we will use the squared series ζ_t^2 to check the Conditional Heteroscedasticity recognized as the ARCH effect.

ARCH(q)

The autoregressive conditional heteroscedasticity process defines the conditional variance as a linear regression model of lagged squared errors.

$$h_t^2 = \alpha_0 \zeta_{t-1}^2 + ... + \alpha_q \zeta_{t-q}^2 \tag{2.11}$$

where $\alpha_0 > 0$ and $\alpha_i \geq 0$ for $i = 1, ..., q$ to ensure positive variance.

Generalized ARCH(p,q)

Bollerslev extended the ARCH model to a Generalized-ARCH and described it as the sum of lagged squared errors and the lagged conditional variance.

$$h_t^2 = \alpha_0 + \alpha_1 \zeta_{t-1}^2 + ... + \alpha_q \zeta_{t-q} + \beta_1 h_{t-1}^2 + ... + \beta_p h_{t-p}^2 \tag{2.12}$$

where $\alpha_0 > 0$ $\alpha_i, \beta_j \geq 0$ for $i = 0, 1, ..., q$, $j = 1, 2, ...p$
and $(\alpha_i + \beta_j) < 1$ which implies that the unconditional variance of ζ_t is finite, whereas the conditional variance h_t^2 evolves over time. ε_t is the sequence of iid random variables with mean 0 and variance 1 and is often assumed to follow a standard normal or standardized Student-t distribution.

The GARCH model faces several limitations relative to the parameters estimation. The parameters in GARCH models are required to be positive in order to ensure that h_t^2 remains positive for all t *with probability 1*, see [12]. The GARCH model is symmetric and assume that positive and negative shocks have the same effect on volatility.

2.4.2 Distribution of error terms

For conditional variance models, the innovation's process is written

$$\zeta_t = h_t \varepsilon_t$$

1. If the error terms ε_t are assumed to be identically distributed and independent with mean zero and unit variance, then the ε_t are normally distributed with the density function

$$f(\zeta) = \frac{1}{\sqrt{2\pi h^2}} e^{-\frac{(\zeta-\mu)^2}{2h^2}}$$

where μ is the mean value and h^2 is the variance. Referring to probability theory when $\mu = 0$ and $h^2 = 1$ the distribution is called Standard Normal distribution.

2. If the error terms ε_t are assumed to be indentically distributed and independent with mean zero and unit variance, then ε_t are Student t-distributed with the density function

$$f(\zeta) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma}(1+\frac{\zeta^2}{\nu})^{-(\dfrac{\nu+1}{2})}$$

where ν stands for the number of degrees of freedom and Γ is the gamma function. As the number of degrees of freedom grows, the Student t-distribution approaches the normal distribution with mean 0 and variance 1 and is called standardized Student-t distribution.

2.4.3 Akaike Information Criterion

In this study, we will use the Akaike information Criterion for two reasons. Firstly, the AIC is useful in order to determine the suitable number of lags in Autoregressive Heteroscdedastic Conditional Variance. Secondly, we will use it to identify the goodness of models. The Akaike information criterion (AIC) for a parametric model is defined as:

$$AIC(k) = -2\log(L) + 2k$$

$\log(L)$ is the log likelihood function for the model, L are the Maximum Likelihood estimates for the parameters, and k is the number of parameters in the model. The lower the AIC, the better the model class, see [7].

2.5 Model specification

In financial time series theory, see [16] return series are assumed to be stationary and show no or little autocorrelation. Many financial data use the standard deviation as the main descriptive tool. This is used to quantify the magnitude of variation or dispersion of a set of data values.

2.5.1 ARCH (1)

ARCH models assume that the variance of the current error term is related to the size of the previous periods' error terms, giving rise to volatility clustering. This phenomenon is widely observable in financial markets, where periods of low volatility are followed by periods of high volatility and vice versa. The ARCH model describes the forecast variance in terms of current observables.

Assume that

$$\zeta_t = h_t\varepsilon_t$$

According to Engle (1992), the ARCH(1) model follows

$$h_t^2 = \alpha_0 + \alpha_1\zeta_{t-1}^2 \tag{2.13}$$

where $\alpha_0 > 0$, $\alpha_1 \geq 0$ to describe a positive variance.
ζ_t is the residual of the mean equation. The squared series ζ_t^2 is then useful to detect the conditional heteroscedasticity which is also called the ARCH-effect.

2.5.2 GARCH(1,1)

The GARCH(1,1) regression model developed by Bollerslev (1986) is given by

$$h_t^2 = \alpha_0 + \alpha_1 \zeta_{t-1}^2 + \beta_1 h_{t-1}^2 \tag{2.14}$$

where $\alpha_0 > 0$, $\alpha_1 > 0$, $\beta_1 > 0$ and $\alpha_1 + \beta_1 < 1$, so that the next period forecast of variance is a mix of the last period forecast and the last period's squared return. The unconditional variance of ζ_t^2 is equal to

$$\begin{aligned} Var(\zeta_t) &= E[\zeta_t]^2 - (E[\zeta_t])^2 \\ &= E[\zeta_t^2] \\ &= E[h_t^2 \varepsilon_t^2] \\ &= E[h_t^2] \\ &= \alpha_0 + \alpha_1 E[\zeta_{t-1}^2] + \beta_1 h_{t-1}^2 \end{aligned}$$

and since ζ_t is a stationary process

$$Var(\zeta_t) = \frac{\alpha_0}{1 - \alpha_1 - \beta_1}$$

the unconditional variance of returns becomes,

$$E[\sigma_t^2] = E[\zeta_t^2] = \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \tag{2.15}$$

The GARCH(1,1) modell provides a simple parametric function useful to describe the volatility evolution. It is often useful to forecast the next period's variance of returns. Assume that the forecast origin is t with l steps to expiration, see [A.2].

For l-step-ahead forecast, we have

$$h_{t+l}^2 \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \tag{2.16}$$

as $l \to \infty$,
From the above equation one can see that $h_{t+l}^2 \to h_t^2$ as $l \to \infty$, then the variance forecast approaches the unconditional variance of ζ_t, see [5].
Looking at the l-step ahead variance forecast, we observe that $(\alpha_1 + \beta_1)$ determines how quickly the variance forecast converges to the unconditional variance.

2.5.3 Parameters estimation

The GARCH(1,1) regression model includes some unknown parameters $(\alpha_0, \alpha_1, \beta_1)$, which will be estimated. We are interested in the best estimate to use in forecasting the variance. The statistical theory supplies various methods applicable in parameters estimation such as OLS (*Ordinary Least Squares*), *Moment Method* and MLE (*The Maximum Likelihood Estimation*).

In this study, we will use *The Maximum Likelihood Estimation* method. According to [16], several likelihood functions are commonly applied in ARCH- GARCH estimation, depending on the distributions assumption of ε_t.

Under the normality assumption of ε_t, the likelihood function of an ARCH(m) model is

$$L = f(\zeta_{m+1}, ..., \zeta_T|\boldsymbol{\alpha}, \zeta_1, ..., \zeta_m) = \prod_{t=m+1}^{T} \frac{1}{\sqrt{2\pi h_t^2}} exp\left(\frac{-\zeta_t^2}{2h_t^2}\right) \tag{2.17}$$

where $\boldsymbol{\alpha} = (\alpha_0, \alpha_1, ..., \alpha_m)'$ and $f(\zeta_{m+1}, ..., \zeta_T|\boldsymbol{\alpha})$ is the joint probability density function of $\zeta_1, ..., \zeta_m$, see [16]

The log likelihood function is

$$l = \ln(L) = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=m+1}^{T} \ln(h_t^2) - \frac{1}{2}\sum_{t=m+1}^{T} \frac{\zeta_t^2}{h_t^2} \tag{2.18}$$

we plug in GARCH(1,1) and we get

$$I_{GARCH(1,1)} = -\frac{1}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{T} \ln(\alpha_0 + \alpha_1\zeta_{t-1}^2 + \beta_1 h_{t-1}^2) - \frac{1}{2}\sum_{t=1}^{T} \frac{\zeta_t^2}{(\alpha_0 + \alpha_1\zeta_{t-1}^2 + \beta_1 h_{t-1}^2)} \tag{2.19}$$

Under the assumption of Student t-distribution of ε_t , the likelihood function of ARCH(m) is

$$L = f(\zeta_{m+1}, ..., \zeta_T|\boldsymbol{\alpha}, \zeta_1, ..., \zeta_m) = \prod_{t=m+1}^{T} \frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu)/2)\sqrt{(\nu-2)\pi}} \frac{1}{h_t} \left[1 + \frac{\zeta_t^2}{(\nu-2)h_t^2}\right]^{-(\nu+1)/2} \tag{2.20}$$

and the loglikelihood function is

$$l = (T-m)\ln\left(\frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu)/2)\sqrt{(\nu-2)\pi}}\right) - \frac{1}{2}\sum_{t=m+1}^{T} \ln({h_t}^2) - \frac{(\nu+1)}{2}\sum_{t=m+1}^{T} \ln\left(1 + \frac{\zeta_t^2}{(\nu-2)h_t^2}\right) \tag{2.21}$$

The loglikelihood function for GARCH(1,1) is

$$I_{GARCH(1,1)} = T * \ln\left(\frac{\Gamma[(\nu+1)/2]}{\Gamma(\nu)/2)\sqrt{(\nu-2)\pi}}\right) - \frac{1}{2}\sum_{t=1}^{T}\ln(\alpha_0 + \alpha_1\zeta_{t-1}^2 + \beta_1 h_{t-1}^2)$$
$$- \frac{(\nu+1)}{2}\sum_{t=1}^{T}\ln(1 + \frac{\zeta_t^2}{(\alpha_0 + \alpha_1\zeta_{t-1}^2 + \beta_1 h_{t-1}^2)}$$

2.5.4 Minimum mean square error (MMSE) Forecasts

A common objective of conditional variance modeling is generating forecasts for the conditional variance process over a future time horizon. We assume the conditional variance process $h_1^2, h_2^2, ..., h_T^2$, a forecast horizon l, and generate predictions for $h_{t+1}^2, h_{t+2}^2 ..., h_{T+l}^2$.

Let $\hat{h}_{t+1}^2$ denote a forecast for the variance at time $t+1$, conditional on the history of the process up to time t, I_t see[23]. The minimum mean square error (MMSE) forecast is the forecast $\hat{h}_{t+1}^2$ that minimizes the conditional expected square loss, $E(h_{t+1}^2 - \hat{h}_{t+1}^2 | I_t)$

Minimizing this loss function yields the MMSE forecast,
$\hat{h}_{t+1}^2 = E(h_{t+1}^2 | I_t) = E(\varepsilon_t^2 | I_t)$

Chapter 3

Data analysis

3.1 Result

3.1.1 H&M Historical Stock Prices

The aim of this study is to fit the ARCH(1) and GARCH(1,1) models on the data set from Hennes&Mauritz historical stock prices over 10 years. There is a large number of observations so that our calculations will be made by running the numerical functions from the Matlab Software. We start by plotting the daily closing price.

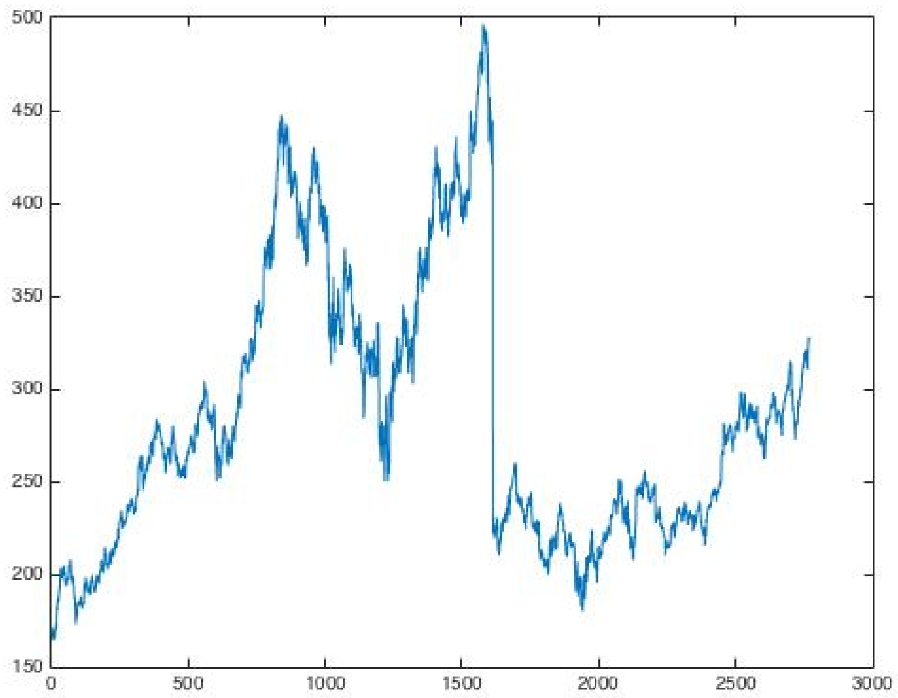

Figure 3.1: H&M Daily Closing Price

Figure 3.1 illustrates the stock price for each trading day. We observe that the stock price varies between 164 and 496.50. The observed data show that there are periods with higher fluctuations, followed by periods with lower movements. Over the period 1396-1613 days, we see that the stock price fluctuates very much and doubles in value. After 1613 days the stock price moves down rapidly and falls to half. This event occurred June 1, 2010 when H&M implemented the share split in which each share was split into two shares of the same class.

As mentioned in [6], we consider the returns instead of the stock price. The main reason is that prices are non-stationary, whereas returns are stationary. As defined by [9] asset returns have attractive statistical properties with respect to financial time series.

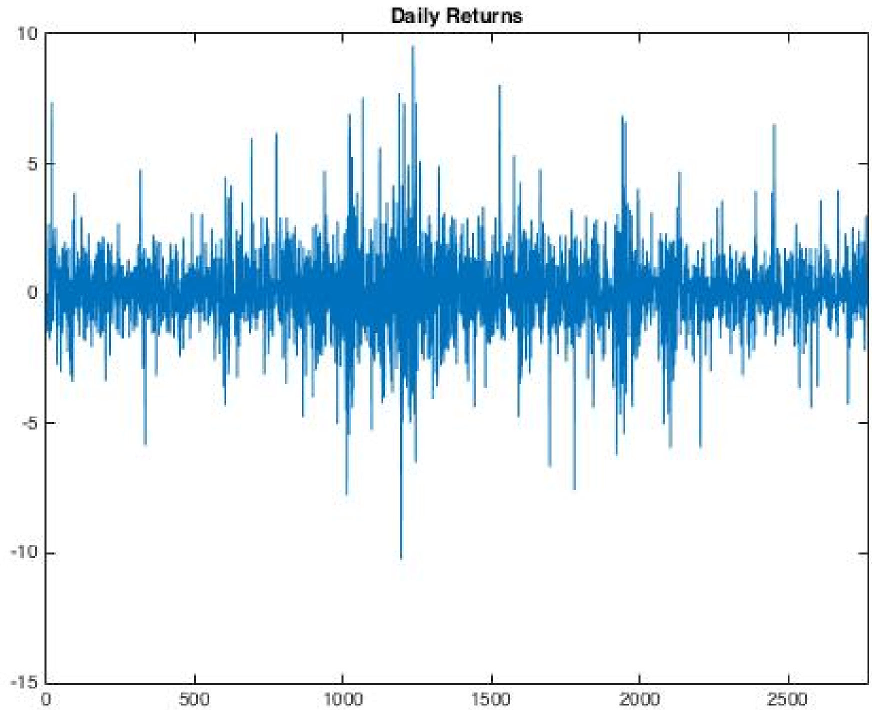

Figure 3.2: Returns value

Figure 3.2 This plot reveals returns values with respect to the confidence interval of [-10,10]. We observe periods of low movement followed by periods of significant fluctuation. There is variability within the return series. The share split has no impact on our data. We see that large changes in the returns tend to cluster together and similarly small changes cluster together. This property indicates the presence of a non constant variance called heteroscedastic variance. We continue by examining the distribution of r_t.

3.1.2 Descriptive Data

Table 3.1: Descriptive Data

Returns	Estimate
Mean	0.0483
Variance	2.4136
Std	1.5536
Kurtosis	7.2237
Skewness	0.0607

Table 3.1 shows statistics values of r_t useful to estimate the parameters in the financial volatility model. The daily mean value of r_t equal to 0.0483 is significant with 95% significance level.

The kurtosis is strictly positive and thus indicates if the data is flat or top with respect to the normal distribution. The kurtosis of H&M daily returns is 7.2237 exceeding twice the theoretical value for the normal distribution of 3. Data volumes with high kurtosis tend to have a distinct peak near the mean, to decrease fairly rapidly, and to have heavy tails .
A large kurtosis indicates that the time series data have a sharper peak and flatter tails character.

The skewness measures the symmetry with respect to the mean value of the data. A skewness with a value equal to 0.0607 describes a positively skewed distribution. This means that the sample distribution is not symmetric around zero, thus the distribution expresses the non-normality in the return series.

3.2 Test of Normality

3.2.1 Jarques Bera test

The Jarque-Bera test asseses the null hypothesis of normally distributed return series against the alternative hypothesis of non normality within return series, see formula (2.9)

Table 3.2: Jarque Bera Test

h	p-value	jbstat	critval
1	1.0000e-03	2.0577e+03	5.9717

Table 3.2 indicates the result of a Monte Carlo simulation. With the Jarques Bera test we obtain a Jbstat much larger than the critical value with a p-value close to zero. The h value equal to 1 indicates that the jbtest rejects the null hypothesis of

normality at the 5% significance level.

3.2.2 Histogram of returns and Normal QQ-plot

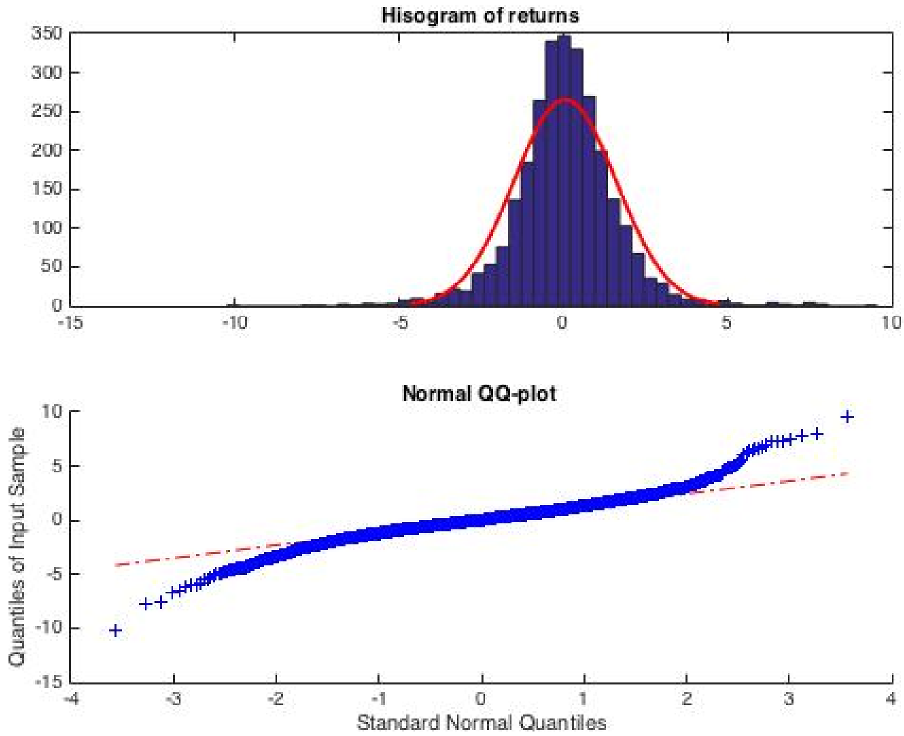

Figure 3.3: Histogram of daily returns and Normal QQ-plot

Figure 3.3 This plot describes a data set with high kurtosis. The histogram shows a distinct peak around zero, which subsides fairly quickly and has a heavy tail to the right. There is a sharp peak which is different from the density function of a normal distribution.

We now check the distribution of r_t by using the graphical technique called Quantile-Quantile Plot. If our data is normally distributed, then the plot will show that all observations lie on the straight line.

In the figure, the Normal QQ-plot presents some extreme values which deviate from the straight line. This suggests that the sampled data violate the assumption of normally distributed serial residuals and that the assumption of independence and non-symmetric distribution is appropriate to our data.

3.2.3 Test of dependency

Autocorrelation as a measure of dependence between the value today and the value some days ago occurs when errors are correlated. In this section, we apply the Autocorrelation Function of assets returns on our data.

Model of returns

Let $r_t = \mu_t + \zeta_t$ be the mean equation for the return process r_t
We assume that r_t could be represented by a mean model expressed in μ_t and a volatility model. We assume that the returns are normally distributed and test the null hypothesis that $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$.

We get a test statistic equal to 1.6354 with 2765 degrees of freedom, a confidence interval $(-0.0096, 0.106)$ and a p-value equal to 0.102. The returned value of $h = 0$ indicates that the t-test does not reject the null hypothesis at the 5% significance level.
The result indicates that $r_t = \zeta_t$, thus the squared series ζ_t^2 would be used to check the conditional heteroscedasticity.

Autocorrelation Function of Returns

Figure 3.4 shows both the estimated autocorrelation of the returns and the autocorrelation of the squared returns. We observe that the return series indicates that few of the values outrun the confidence interval at the 5% significance level, while the squared returns show a significant dependency in the second moment of returns. All the estimated values are positive and significant.

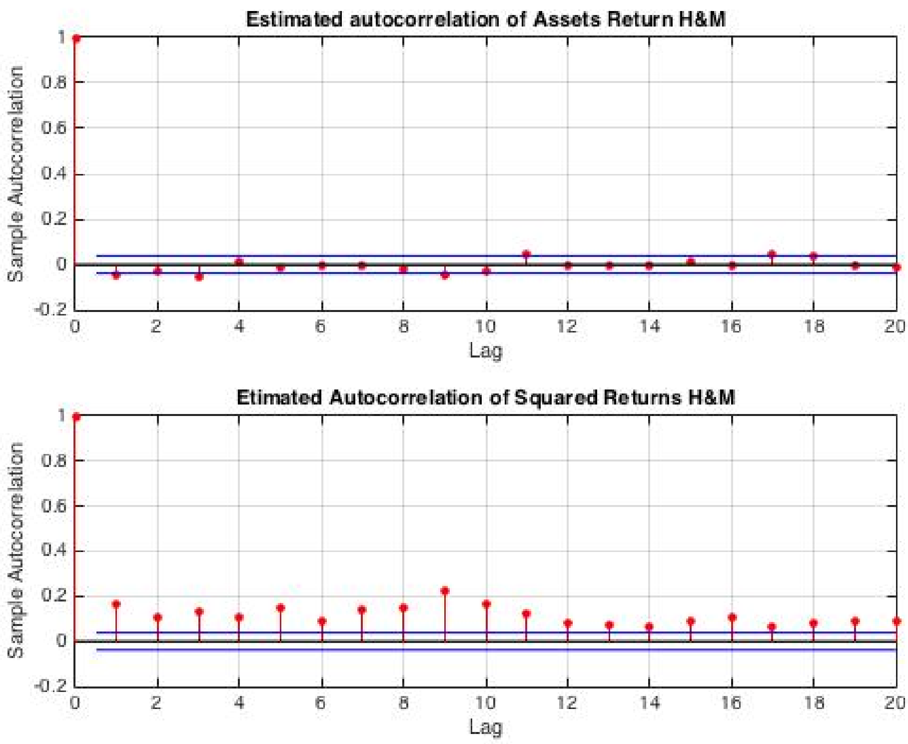

Figure 3.4: *Sample autocorrelation function of return series/Sample autocorrelation of squared returns*

Ljung-Box Q-test

The Ljung-Box Q-test tests the null hypothesis that there is no autocorrelation within return series, against the alternative hypothesis that there is autocorrelation within return series. The Q-statistc of Q(8) is 16.6322 with a p-value of 0.03325 less than 5%.

We then run the same test with the squared return series. We obtain a Q-value of 381.9427, with a P-value equal to zero and a corresponding critical value of 15.5073. According to financial theory, both of these test statistics are asymptotically chi-squared distributed with $m = 8$ degrees of freedom. The result shows that both Q-values are larger than the critical values. We get $h = 1$ with a p-value close to zero. This affirms that both the squared returns and the returns are significantly autocorrelated at the 5% significance level.

3.2.4 Heteroscedasticity

As defined by [16], the squared series ζ_t^2 is used to check for conditional heteroscedasticity, which is also known as the ARCH effect. TSAY suggests two tests for the ARCH effect, the Ljung-Box Q-test and the Lagrange multiplier test . The result of Ljung-Box Q-test applied on the squared returns indicates that there are ARCH-effect within return series.

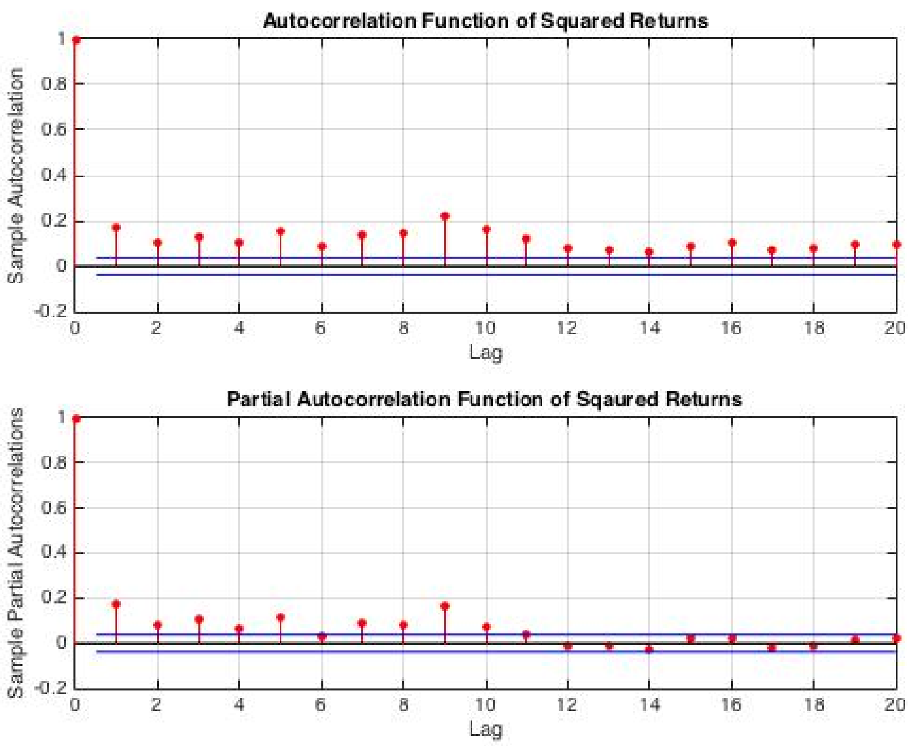

Figure 3.5: ARCH-Effect

Figure 3.5 illustrates the estimated ACF and PACF of the squared residuals. They illustrate significant autocorrelation in the series and thus, the presence of volatility clustering within the residuals series.

We also check the ARCH-effect with the Lagrange multiplier test called Engle's ARCH test. The Lagrange multiplier tests the linear model for residuals defined in (2.6)
$\zeta_t^2 = \alpha_0 + \alpha_1\zeta_{t-1}^2 + ... + \alpha_l\zeta_{t-l}^2$, then $t = l + 1 + ...T$
l is a positive integer and T is the sample size.

We will test the null hypothesis homoschedasticity against the alternative hypothesis conditional heteroscedasticity up to lag 8 .

The series has significant serial correlation so that it can be directly used to test the ARCH-effect. As seen earlier, the $Q(m)$ statistics of the return series give $Q(8) = 16.7798$ with a p-value =0.0325 and $h = 1$ indicates a strong serial correlation in the data.
The Lagrange multiplier test shows a strong ARCH-effect with the F statistic equal to 213.7592, much larger than the critical value of the χ^2 distribution with eight degrees of freedom, 15.5073 and a p-value equal to zero. This result suggests that the null hypothesis of homoscedasticity is rejected in favour of the alternative hypothesis conditional heteroscedasticity. The result confirms that the volatility of the returns is heteroscedastic.

We then determine the suitable number of lags for the model. This is done by fitting the model over $m = 1, 2..., 8$ lags and then compare the fitted models by using the AIC method.

Table 3.3 indicates that lag 1 has the smallest AIC. It is then reasonable to conduct the ARCH test using lag one.

Table 3.3: Suitable Number of Lags

Smallest AIC								
Lag	**1**	2	3	4	5	6	7	8
AIC	**9.9250**	9.9270	9.9290	9.9310	9.9330	9.9350	9.9370	9.9390

We run the same test by using lag 1. The F statistic for the test is 78.9770, much larger than the critical value of the χ^2 distribution with one degree of freedom, 3.8415. This result suggests that the null hypothesis of homoscedasticity can be rejected. This means that the volatility of return is heteroscedastic.

We conclude that there is significant volatility clustering in the residual series and that the ARCH(1) and GARCH(1,1) processes can be applied to model returns volatility.

3.2.5 Modelling financial volatility

The observed test statistics suggest the presence of volatility clustering. This is an important property required by financial theory relative to the conditional variance models. Engle's test shows that lag 1 is suitable for the conditional variance model. The GARCH(1,1) model includes α_0, α_1 which are also the components in the ARCH(1) linear model. The first number in parentheses refers to how many autoregressive lags, or ARCH terms appear in the equation, while the second number refers to how many moving average lags are specified or the number of GARCH, see [12]. Both ARCH(1) and GARCH(1,1) are able to predict the financial volatility. Thus we decide to fit the GARCH(1,1) on our data. By using the Matlab-Software, we maximize the log likelihood function of the GARCH(1,1) model.

GARCH(1,1)

Table 3.4 shows the estimated GARCH(1,1) values under the assumption of Gaussian and Student-t distributions. The Student t-distribution expresses the smallest AIC .

Table 3.4: GARCH(1,1)

Parameter	Gaussian	Student-t
α_0	0.0533252	0.0305644
α_1	0.0446668	0.0470664
β_1	0.931613	0.940584
DoF		4.81434
AIC $10^3 *$	9.8463	9.5582
$(\alpha_1 + \beta_1)$	9.9763	0.9877
h_t^2	2.2481	2.4749

The result illustrates that both Q-statistics are less than the critical value and h= 0 suggests that we can not reject the null hypothesis of no autocorrelation within the stantardized residuals.

Model checking

For GARCH(1,1), the standardized residuals are calculated as $\hat{\zeta}_t = \dfrac{\zeta_t}{h_t}$, where $\hat{\zeta}_t$ is a sequence of iid random variables. We can check the adequacy of the fitted GARCH(1,1) model by analyzing the series $\hat{\zeta}_t$, see[16]. We use Ljung-Box statistics of $\hat{\zeta}_t$ to check the adequacy of the mean equation and, $\hat{\zeta}_t$, to check the validity of volatility equation. The Q-statistics of $\hat{\zeta}_t$ test the null hypothesis of no autocorrelation in the standardized residuals series.

Table 3.5 shows the result of the Ljung-Box test applied on standardized residuals with normal distribution and on standardized residuals with normal Student's t-distribution. One can see that both Q-statistics are less than the critical value

Table 3.5: GARCH(1,1)

$\hat{\zeta}_t$	Gaussian	Student-t
h	0	0
p	0.8295	0.8199
Qstat	0.0474	0.0518
Critical value	3.8415	3.8415

and h= 0 indicates that we can not reject the null hypothesis of no autocorrelation within the standardized residuals.

Figure 3.11-3.12 illustrate ACF of GARCH(1,1) model with normal distributed standardized residuals, respectively Student's t-distributed standardized residuals. The figures indicate that there is no autocorrelation in the standardized residuals series. See [A.1].

We now use the $\hat{\zeta}_t^2$ to test the null hypothesis that there are homoscedasticity within the squared standardized residuals series with assumed normal distribution and the squared residuals with Student's t-distribution.

Table 3.6: GARCH(1,1)

$\hat{\zeta}_t^2$	Gaussian	Student-t
h	0	0
p	0.8438	0.9878
Qstat	0.0388	2.3266e-04
Critical value	3.8415	3.8415

Table 3.6 illustrates the result of the Ljung-Box test applied on squared standardized residuals with normal distribution and standardized residuals with normal Student's t-distribution. Both Q-statistics are less than the critical value and h= 0 indicates that we can not reject the null hypothesis of homoscedasticity within the squared stantardized residuals.

Figures 3.13-3.14 show ACF of GARCH(1,1) model with normally distributed squared standardized residuals, resprctively Student's t-distributed squared standardized residuals. The figure affirm that there is homoscedasticity in the squared standardized residuals series. See [A.1].

Figure 3.6 shows the standardized residuals including the conditional variance with an assumed normal distribution. We observe more large values than expected in a standard normal distribution. The financial time series is not white noise. This suggests that a non symmetric distribution might be more appropriate for the residuals distribution.

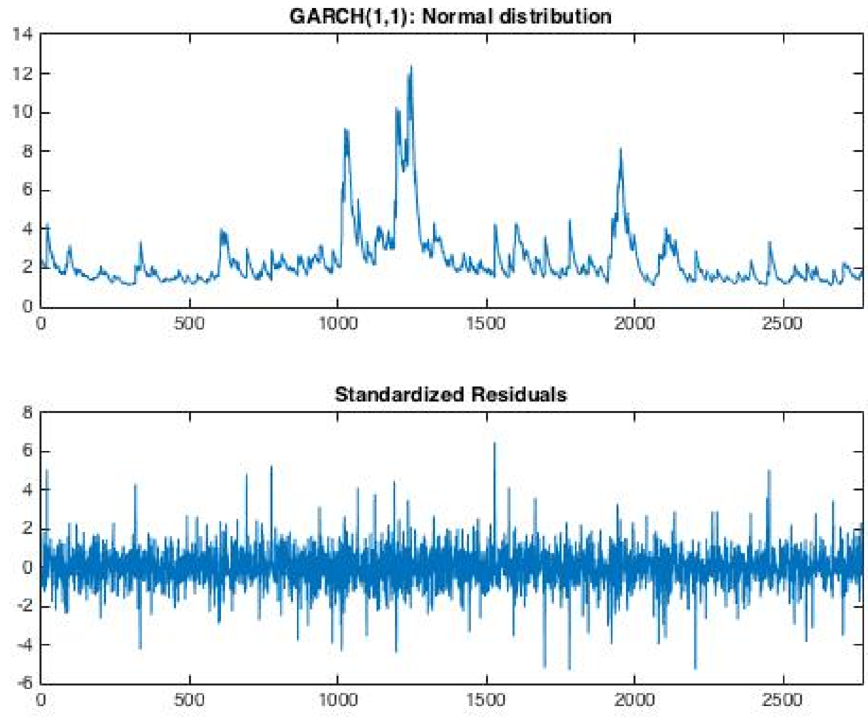

Figure 3.6: Standardized residuals with GARCH(1,1), Normal distribution

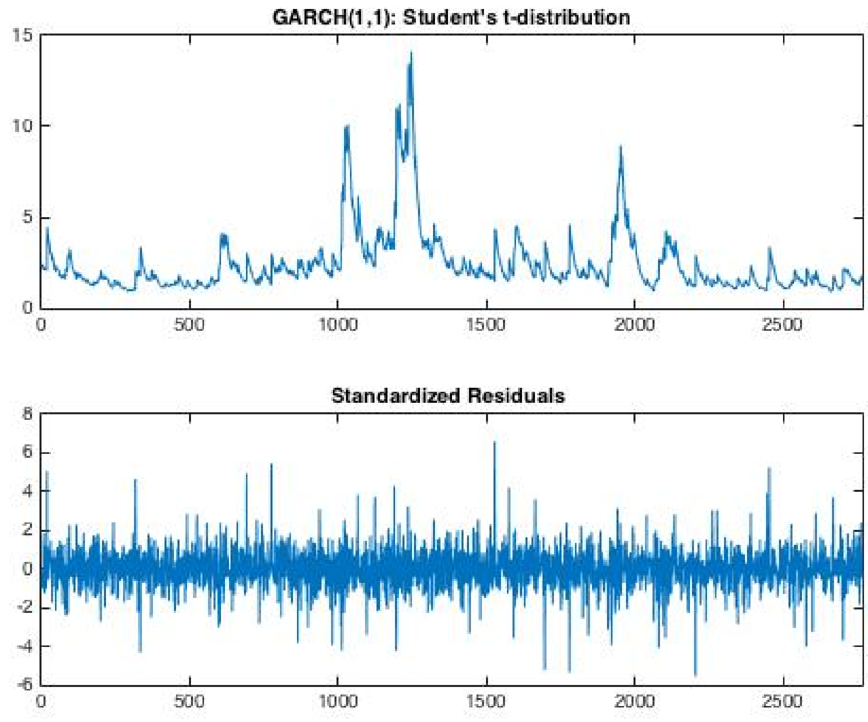

Figure 3.7: Standardized residuals with GARCH(1,1), Student's t- distribution

Figure 3.7 illustrates the standardized residuals including the conditional variance with an assumed Student's t- distribution. We also observe more large values than expected in a standard student's t-distribution.

In order to identify the validity of the distribution assumption for GARCH(1,1) models, we analyze the QQ-plots of standardized residuals with assumed normal distribution and standardized residuals with assumed Student's t-distribution.

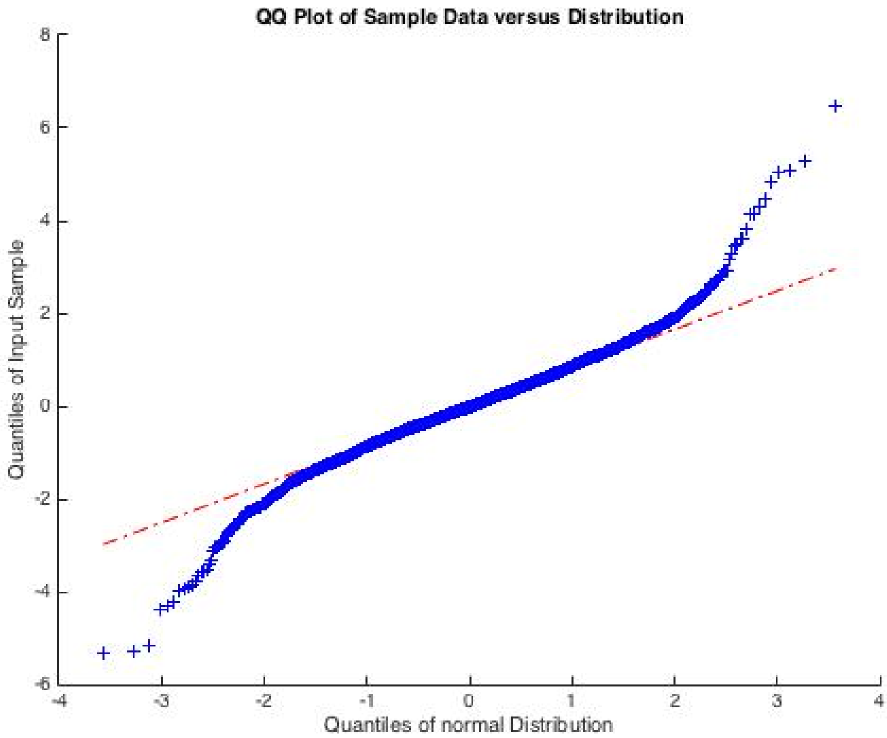

Figure 3.8: QQ-plot of standardized residuals with GARCH(1,1), Normal distribution

Figure 3.8 We see that the standardized residuals series does not follow the dashed straight line marked in the figure. There are many observations which deviate from this line. The standardized residuals form an s-shape different from normally distributed standardized residuals series.

Figure 3.9 indicates that the standardized residuals series follows the dashed straight line marked in the figure.The standardized residuals with assumed Student's t-distribution seem to be more linear. There are few points which diverge from the dashed straight line. By combining the results from table 3.4 and this plot, GARCH(1,1) with assumed Student's t-distribution is adequate to model volatility of return series.

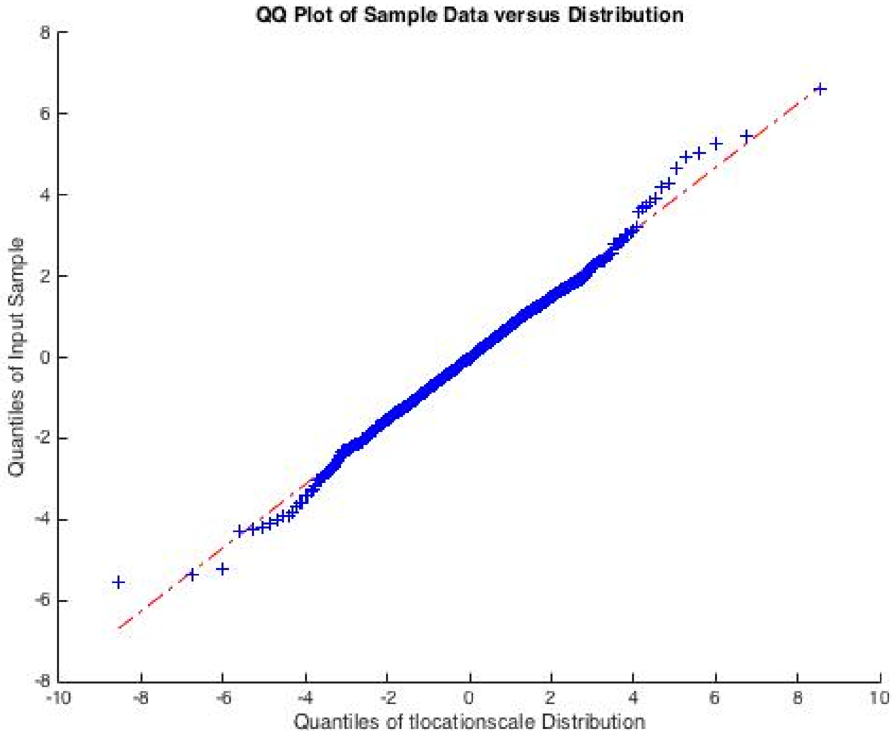

Figure 3.9: QQ-plot with standardized residuals with GARCH(1,1), Student's t-distribution

3.2.6 Forecast Performance

Time series forecasting implies the use of a model to predict future values based on previously observed values. The aim of this study has been to provide some knowledge about the goodness of the fitted model.
GARCH(1,1) and assume that the forecast origin is $t = 1$ day-step-ahead. As defined by (3.4), the 300-step ahead volatility at the forecast origin t corresponds to

$$\hat{h}^2_{t+300} = h^2(\hat{\alpha_1} + \hat{\beta_1})^{300}(h^2_t - h^2)$$

$$\hat{h}^2_{t+300} \to h^2 = \frac{\hat{\alpha_0}}{1 - \hat{\alpha_0} - \hat{\beta_1}} = 2.4749$$

as $l \to \infty$. This is the theoretical or unconditional variance.
By using the Minimum Mean Square Error(MMSE) forecast over the 300 days-period horizon ,we get the following plot.

Figure 3.15 indicates that conditional variance converges to the unconditional variance as $l \to \infty$. See [A.1]

3.3 Conclusion

This paper discusses modelling and forecasting the financial volatility of H&M stock prices, by using the ARCH(1)-GARCH(1,1) models. The sampled data includes daily return series of Hennes&Mauritz stock prices over a period of 10 years between 2004-2014.

In 2010 the return series indicate extreme values differing from the expected value of the observed returns. This event was caused by the share split in which each share was split into two shares of the same class.

We found that the share split had no great impact on the returns series analysis so that we could use our data to investigate the distribution of the returns residuals series. We have reviewed the properties of the returns series by applying different statistical methods.

The estimated kurtosis and skewness values exceed more the theoretical value of the normal distribution. The QQ plot, histogram and Jarque Bera Test suggest that there is non-normality within the returns values.
The Autocorrelation function of the returns series illustrates a significant dependency in the second moment of returns. In accordance to financial theory, we choose the natural logarithm of the observed days as reference degree of freedom. The Q(8) test suggests that the squared returns and the returns are significantly autocorrelated at the 5% significance level.

The Autocorrelation function and the Partial Autocorrelation of the squared returns indicate some volatility clustering. This feature requires the use of an autoregressive conditional heteroscedastic process to model the volatility of returns.

By comparing the AIC values for a different number of lags, we see that GARCH(1,1) has a suitable number of lags. The first number in parentheses refers to how many autoregressive lags, or ARCH terms appear in the equation, while the second number refers to how many moving average lags are specified or the number of GARCH.[12]

As is well known the GARCH(1,1) model imposes some limitations which can lead us to underestimate the sampled data. Sometimes models with more than one lag are desirable to estimate good variance forecasts. The GARCH model is symmetric and assumes that positive and negative shocks have the same impact on volatility.

Financial time series include periods characterized by good and bad news, i.e. financial crises, wars or natural disasters. A period of bad news would negatively affect the stock price. The change in stock price tends to be negatively correlated with the change in volatility, the so called Leverage effect, see[6]. In order to capture this feature, it would be better to use the extended Exponential GARCH model of Nelson(1991) or the GJR-GARCH model of Glosten, Jagannathan and Runken(1993).

In our study, we have used the GARCH(1,1) model and found that the estimated $\alpha_1 + \beta_1 < 1$ which means that distant horizon forecast is the same for all time periods. The MMSE method shows that the conditional heteroscedastic variance converges towards the unconditional variance as the lag number goes towards infinity. This means that the model has the property to forecast the development of future volatility. According to financial theory, a good volatility model must be able to forecast the future volatility. The plot of standardized residuals including conditional variance indicates that returns series are not a white noise.

By comparing the GARCH(1,1) model with an assumed Gauss distribution and a Student's t-distribution, we see that GARCH (1,1) with an assumed Student's t-distribution has the smallest AIC value, and that the QQ-plot of standardized residuals series seems to be more linear.
Finally we find that the GARCH(1,1) model with assumed Student's t-distribution is adequate to model volatility of return series.

3.4 Appendix

3.4.1 A.1

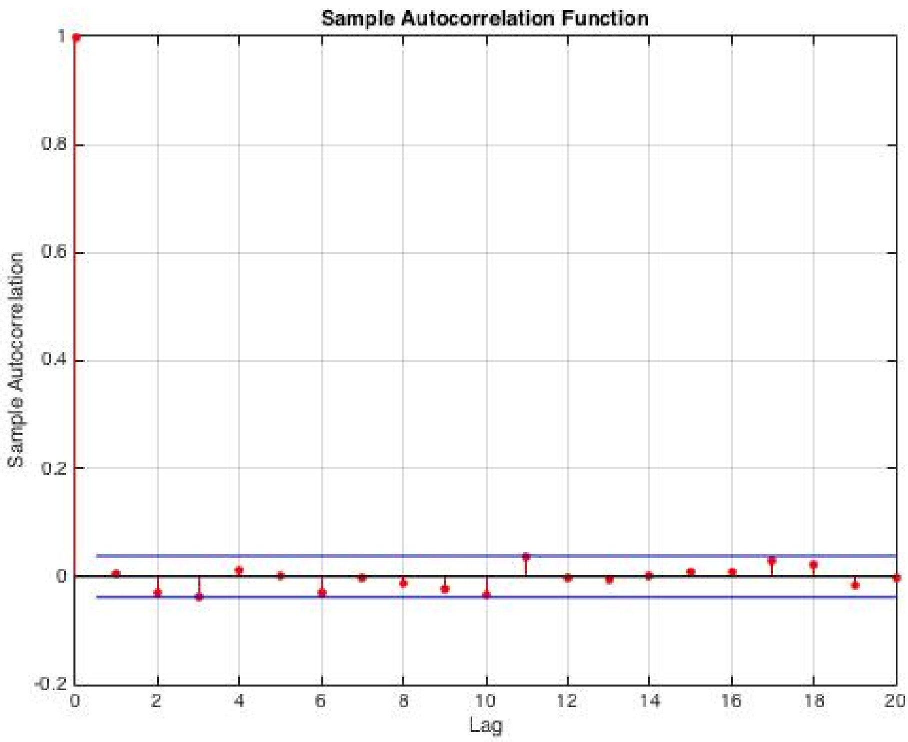

Figure 3.10: Autocorrelation Function of $\hat{\zeta}_t$ normal distribution

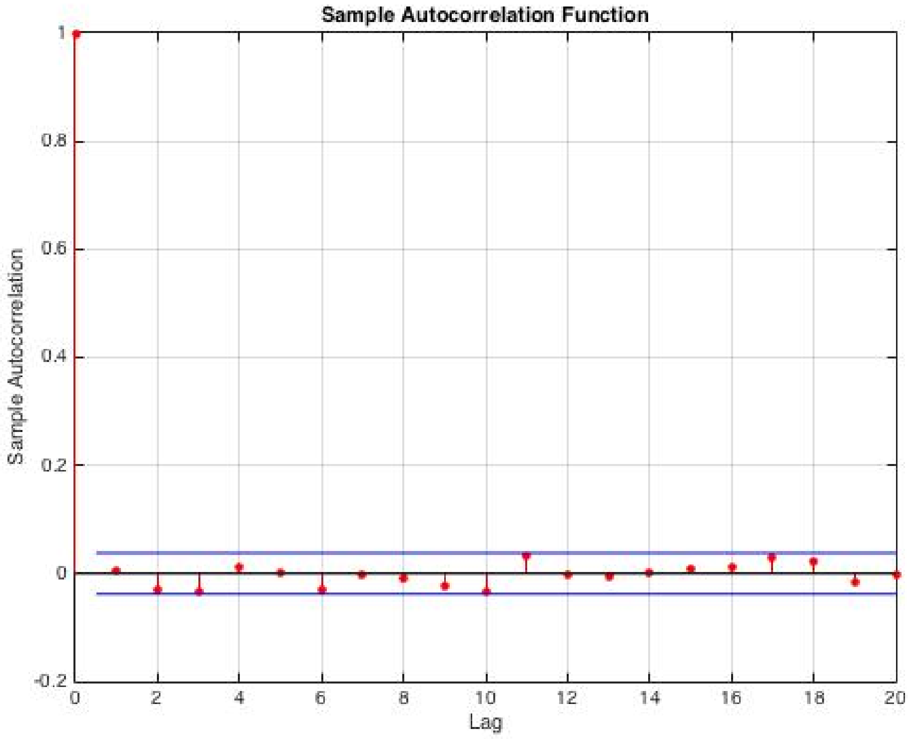

Figure 3.11: Autocorrelation Function of $\hat{\zeta}_t$, Student's t-distribution

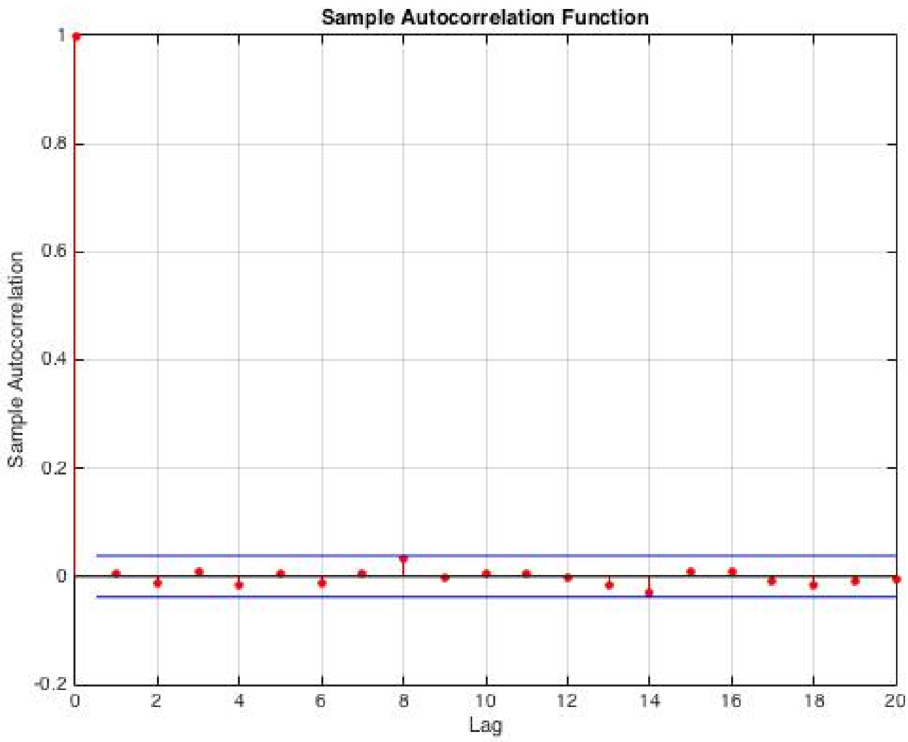

Figure 3.12: Autocorrelation function of $\hat{\zeta}_t^2$, normal distribution

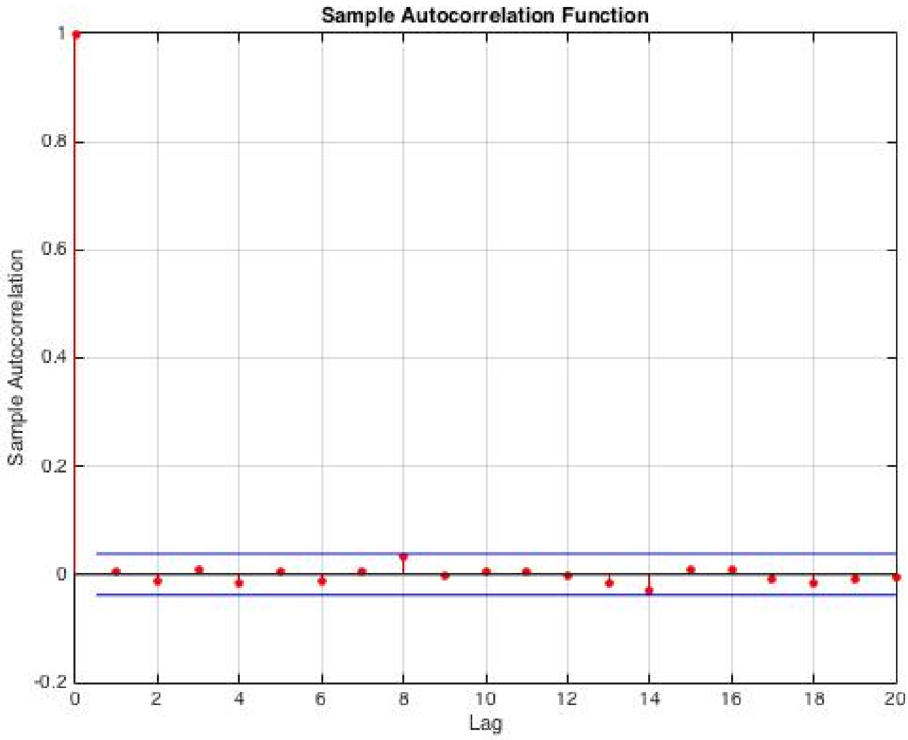

Figure 3.13: Autocorrelation function of $\hat{\zeta}_t^2$, Student's t-distribution

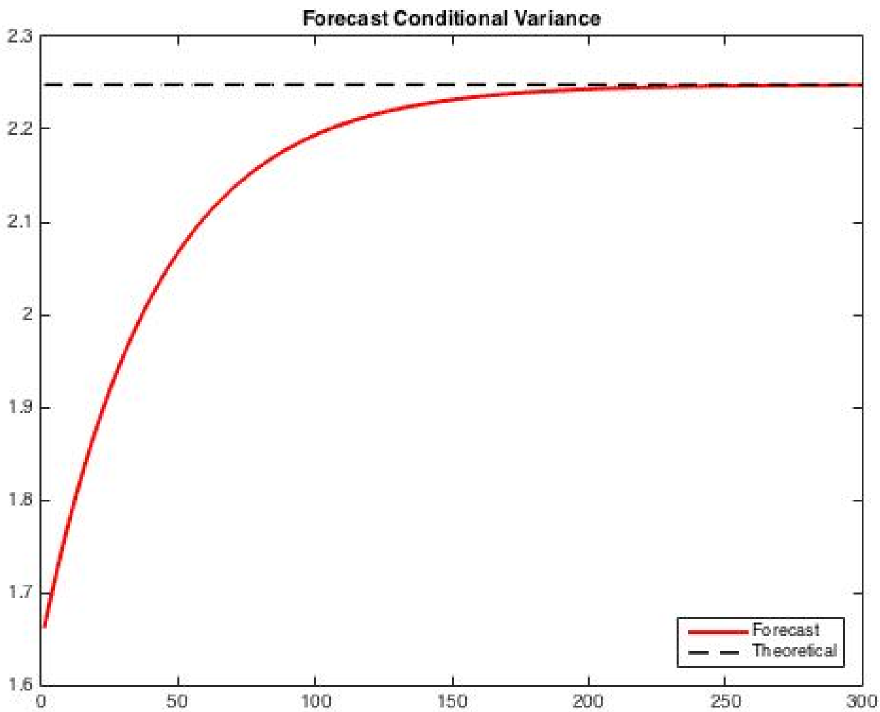

Figure 3.14: Forecast Conditional variance over 300 days-period

3.4.2 A.2

1. Forecasting GARCH(1,1)
   Assume that the forecast origin is t with l steps to expiration.

   For 1-step-ahead forecast, we have

$$\begin{aligned} h_t^2 &= \alpha_0 + \alpha_1 \zeta_{t-1}^2 + \beta_1 h_{t-1}^2 \\ h_{t+1}^2 &= \alpha_0 + \alpha_1 E[\zeta_t^2 | I_{t-1}] + \beta_t h_t^2 \\ &= \alpha_0 + \alpha_1 h_t^2 + \beta_1 h_t^2 \\ &= \alpha_0 + (\alpha_1 + \beta_1) h_t^2 \\ &= h_{t+1}^2 + (\alpha_1 + \beta_1)(h_{t+1}^2 - h_t^2) \end{aligned}$$

$$\begin{aligned} h_{t+2} &= \alpha_0 + \alpha_1 E[\zeta_{t+2}^2 | I_{t-1}] + \beta_1 E[h_{t+1}^2 | I_{t-1}] \\ &= \alpha_0 + (\alpha_1 + \beta_1) h_{t+1}^2 \\ &= h_{t+2}^2 + (\alpha_1 + \beta_1)^2 (h_{t+2}^2 - h_{t+1}^2) \end{aligned}$$

$$\vdots$$

   For l-step-ahead forecast, we have

$$\begin{aligned} h_{t+l}^2 &= \alpha_0 + (\alpha_1 + \beta_1) h_{t+l-1}^2 \\ &= h_{t+l}^2 + (\alpha_1 + \beta_1)^l (h_{t+l}^2 - h_{t+l-1}^2) \end{aligned}$$

   where $l > 1$
   and

$$h_{t+l}^2 \to \frac{\alpha_0}{1 - \alpha_1 - \beta_1} \tag{3.1}$$

   as $l \to \infty$,

Bibliography

[1] Adrian R Pagan and G William Schwert(1990), *Journal of Econometrics 45:* Alternative Models for Conditional Stock Volatility, pp.267-290.

[2] Anderson P.,Tyrcha J. (2012): *Notes of Econometrics.* Department of Mathematics, Stockholm University

[3] Archil Gulisashvili (2012). *Analytically Tractable Stochastic Stock Price Model.*

[4] Daniel B.Nelson(1991) *Econometrica: Conditional Heteroscedasticity in Asset Returns.* A New Approach, Volume 59 No.2 pp.247-370

[5] Edwardo Rossi (2004) *Lecture notes on GARCH model* University of Pavia

[6] Eric Jondeau, Ser Huang Poon, Michael Rockinger: *Financial Modelling Under Non-Gaussian Distributions*

[7] Hirotugu Akaike(1974); *Transactions On Automatic Control: A new look at The Statistical Model Identification,* Volume AC-19 No.6, pp 716-723

[8] Höglund T. (2008): *Mathematical Asset Management,* John Wiley and Sons

[9] John Y.Campbell, Andrew W.Lo, and A.Craig Mackinlay: *The Econometrics of Financial*

[10] Petra Posedel(2005): *Properties and Estimation of GARCH (1,1) Model,* Metodoloski zvezki, vol 2,No.2

[11] P. Reinhard Hansen and A. Lunde (2001): A comparison of volatility models: Does anything beat a GARCH(1,1), *Working Paper Series No. 84*

[12] Robert Engle Econometrics, GARCH 101: The Use of ARCH/GARCH Models in Applied Econometrics *Journal of Economic Perspectives Volume 15, Number 4, Fall 2001Pages 157168*

[13] Robert F Engle and Andrew J Patton(2001): *What good is Volatility Model?* Quantitative Finance Volume 1 pp.237-245

[14] Robert F Engle III(2003) : *Risk and Volatility;* Econometric Models and Financial Practice

[15] Rob Reider(2009): *Volatility Forecasting I: GARCH Models* Rob Reider(2009): *Volatility Forecasting I: GARCH Models*

[16] Ruey S. Tsay *Analysis of* Financial Time Series,Third Edition

[17] Sheldon M. Ross; *Introduction to* Probability Models
9^{th} *edition*

[18] Tim Bollerslev (1986); Generalized Autoregressive Conditional Heteroscedasticity, pp.307-327 *Journal of Econometric*

[19] http://www.hm.com

[20] http://www.math.uah.edu/stat/expect/Covariance.html

[21] www.onlinecourses.science.psu.edu

[22] http://www.okstate.edu/sas/v7/sashtml/books/ets/chap8/sect22.html

[23] http://se.mathworks.com/help/econ/mmse-forecasting-for-arima-models.html