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A Sandpile to Model the Brain

Kristina Rydbeck

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Matematisk statistik
Matematiska institutionen
Stockholms universitet
106 91 Stockholm

Mathematical Statistics
Stockholm University
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Abstract

In neuroscience there has emerged an interest in the hypothesis that the brain functions at a critical point, where it is balanced between a phase of too much disorder and a phase of too much order. The model that is the focus of this thesis - the Abelian sandpile model - is a model that was developed due to an interest in constructing a model exhibiting self-organized critical behaviour; where the critical behaviour arises over time in an open, dissipative system as opposed to through the tuning of a parameter in an equilibrium system as is the case in classical models of criticality. In this thesis we present the Abelian sandpile model before we explain the concept of criticality and why it is of interest to some neuroscientists. We also present our own two variations of the model where we alter the dissipative structure of the system. We subsequently perform some simulations on the models, comparing the behaviour of the classical model to the behaviour of the altered ones. We find that our alterations to the model changes the behaviour of the system to a degree that we do not expect it to exhibit critical behaviour.

*Postal address: Mathematical Statistics, Stockholm University, SE-106 91, Sweden. E-mail: kristin_rydbeck@hotmail.com. Supervisor: Maria Deijfen and Pieter Trapman.

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Preface

This work constitutes a thesis of 15 ECTS and leads to a Bachelors's degree in Mathematical Statistics at the Department of Mathematics at Stockholm University.

I would like to thank my thesis supervisors, Maria Deijfen och Pieter Trapman, for guiding the work on this paper. Thank you both for excellent advice along the way, and for giving me this interesting model to work with.

Introduction

In the 1987 article *Self-Organized Criticality: An Explanation of 1/f Noise* the theoretical physicists Per Bak, Chao Tang and Kurt Wiesenfeld introduced the concept of *self-organized criticality* and argued that it is a mechanism that underlies a variety of physical phenomena (Bak et al. 1987). By means of a simple mathematical model (Jensen, 1998) they tried to illustrate numerically their rather philosphical arguments concerning the behaviour of systems proposed to exhibit self-organized criticality. They also proposed that the building of a simple sandpile may be an example of a real physical system exhibiting this type of behaviour. Whereas the model itself does in fact display the properties that were expected by the authors, the same has been shown not to be true of real physical sandpiles (Jensen, 1998). However, their sandpile illustration explains why the model came to be called *the BTW sandpile model* as well as the language commonly used when describing it. The model which is the subject of this paper, *the Abelian sandpile model*, is a generalization of this model which has a simplifying mathematical structure - the structure of an Abelian group - that allows for analytical calculations of many of its properties (Dhar, 1999).

The concept of self-organized criticality has sparked a lot of interest in diverse scientific fields (Watkins et al. 2015), among these neuroscience; the field concerned with the study of the nervous system. What interests some neuroscientists is the hypothesis that the brain functions at *criticality*; at a point where the brain remains in a balanced state over time, separating a disordered phase from an ordered phase. In the popular science article *A Fundamental Theory to Model the Mind* the neuroscientist Dietmar Plenz is quoted as saying: *At one extreme, there is too much order, such as during an epileptic seizure; the interactions among elements are too strong and rigid, so the system cannot adapt to changing conditions. At the other, there is too much disorder; the neurons aren't communicating as much, or aren't as broadly interconnected throughout the brain, so information can't spread as efficiently and, once again, the system is unable to adapt* (Ouellette). In this paper we will take a special interest in this feature, criticality, which is of interest to some neuroscientists. We begin by describing the Abelian sandpile model and subsequently present our own two variations of the model. Our alterations to the model are inspired by the tendency of neurons not to fire again for some time when they have just done so. After having presented these we will direct our attention to the classical model, focusing on the concept of criticality before performing some simulations on the classical model as well as on our altered versions to see how they behave.

The Abelian Sandpile Model

We begin with an informal description of the model in the 2-dimensional case. Consider some finite subset V of the 2-dimensional integer lattice with each vertex (each lattice point) containing a number of sand grains. We call the four lattice points that are directly connected to a vertex through the grid lines, the nearest neighbours of the vertex. At discrete times we add a sand grain to one of the vertices, chosen at random. Each vertex can hold at most three grains to be stable and if the adding of a grain causes a vertex to receive a larger number of sand grains than this it topples, sending one grain to each of its four neighbours. This starts an avalanche which may spread, that is, cause other sites to topple. At the boundary grains fall of the edge, leaving the system. The avalanche will end when there is no longer any site exceeding the threshold value three. At this point the sandpile is said to be in a stable state, and we can again continue the process by adding a grain to a site. See Figure 1 below for an illustration of an avalanche.

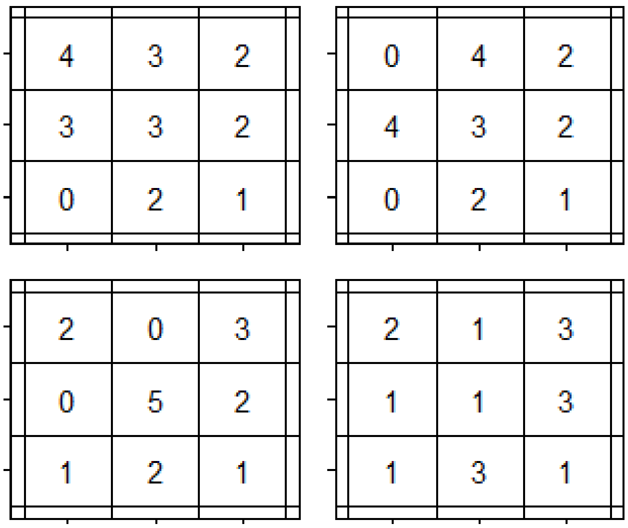

Figure 1. An avalanche on a 3×3 grid caused by a grain having been added to the top left site. Four grains in total leave the system during the avalanche.

Note that the time that it takes, from the addition of one grain to the addition of the next one, is always the same. This may strike us as counterintuitive since adding a grain may not lead to any toppling at all or it may lead to a sequence of topplings involving the whole system, and naturally these events differ in how long they take. The relaxation process is, from one perspective, viewed as instantaneus in the model. That we in some sense ignore the fast time scale of the model can be motivated by the application of the model to phenomena where some events of interest start and finish in a substantially shorter amount of time than other events of interest. (Jensen, 1998) We can visualize this by considering an earthquake where a stress in the earth's crust is built up during years as a result of the motion of the tectonic plates. The slowly built up stress can be released in an earthquake in the much shorter time period of minutes or seconds. Example from (Jensen, 1998).

The formal definition

In describing the model formally we use (Meester et al. 2003) where not otherwise stated.

Let V be a finite subset of the integer lattice $\mathbb{Z}^{\mathbb{d}}$ and Δ^V be an integer valued matrix with its elements denoted by Δ_{vw}. These elements are such that: as site v topples Δ_{vw} grains are removed from site w. Δ^V is a toppling matrix if it satisfies the following four conditions:

1. $\forall v, w \in V, v \neq w, \Delta_{vw} = \Delta_{wv} \leq 0$
2. $\forall v \in V, \Delta_{vv} \geq 1$
3. $\forall v \in V, \sum_{w \in V} \Delta_{vw} \geq 0$
4. $\sum_{v,w \in V} \Delta_{vw} > 0$

where the third condition ensures that no grains are created in the toppling process and the fourth guarantees that there are sites that loose more grains than they give to their neighbours when toppling. These sites are called *dissipative sites* and if we would like these sites to be the boundary sites of the lattice we choose the matrix Δ^V with its elements as described below:

$$\Delta_{vw} = \begin{cases} 2d & \text{if } v = w, \\ -1 & \text{if } v \text{ and } w \text{ are nearest neighbours} \\ 0 & \text{otherwise} \end{cases}$$

where we choose $d = 2$ since we will consider the 2-dimensional case only. Note that if we do not have a dissipative site the system will be closed and if we keep adding grains to such a system it will at some point fill up and we will keep on toppling sites without the system reaching a stable configuration. When condition four above is met however, the toppling process will end in finite time. For a proof of this, see (Holroyd, 2013).

A height configuration η - what we have previously termed "a sandpile" - is the assignment of a non-negative integer, $\eta(v) \in \mathbb{Z}^*$, to each site of V. A configuration $\eta \in \mathbb{Z}^*_{\mathbb{V}}$ is called stable if for all $v \in V$, $\eta(v) \leq 3$ and unstable if there exists some $v \in V$ such that $\eta(v) \geq \Delta_{vv}$. We can now define the toppling rules as

$$T_v(\eta)(w) = \begin{cases} \eta(w) - \Delta_{vw} & \text{if } \eta(v) \geq \Delta_{vv} \\ \eta(w) & \text{otherwise} \end{cases}$$

that is, a vertex topples if and only if it has gone above its maximal capacity and does this by losing Δ_{vv} grains and giving $-\Delta_{vw}$ grains to the sites for which $v \neq w$. The toppling rules commute on unstable configurations; that is, given that two sites are unstable it does not matter which site we topple first, the resulting configuration will be the same. The *toppling transformation* $\mathcal{T}_\Delta(\eta)$ is defined as the application of the toppling rule on each site belonging to some enumeration $\{v_1, \ldots, v_N\}$ of V until a stable configuration is reached:

$$\mathcal{T}_\Delta(\eta) = \prod_{i=1}^{N} T_{v_i}(\eta)$$

For $\eta \in \mathbb{Z}^*_{\mathbb{V}}$ and $v \in V$, let η^v denote the configuration obtained from η through adding one grain to site v. In this paper each of the N sites of V are chosen with uniform probability $p = \frac{1}{N}$. The *addition operator* a_v takes a stable configuration η to another stable configuration through the application of the toppling transformation to η^v

$$\mathcal{T}_\Delta(\eta) = \prod_{i=1}^{N} T_{v_i}(\eta^v)$$

Our sandpile is abelian in that the result of applying the toppling transformation to some arbitrary configuration is independent of the order in which we topple the sites in the set $\{v_1, \ldots, v_N\}$; the same sites will always be toppled the same number of times. See proof in (Meester et al. 2003). As a result of this, which configuration we get after applying the addition operator to two sites is not dependent on which of these two sites we apply the addition operator to first.

Altering the dissipative structure of the system

We have mentioned before, the importance of having grains leave the system, and it is now our intention to present our own variation of the model described above, where we alter the manner in which grains dissipate from the system. Instead of having specific sites through which grains leave the system we make it possible for all sites to "drop" grains under some specific circumstances. Our toppling rules are inspired by an aspect of the functioning of neurons - the cells responsible for information processing in the brain - and before we present our alterations it might be appropriate to consider some basic facts about the workings of neurons.

A sufficiently strong electric excitation of a neuron will result in it firing, meaning that the excitation will result in an *action potential* propagating along its cell membrane and to the junction between the output channel of the neuron and the input channel of another neuron. This junction is called a *synapse* and through a great number of these, the neurons communicate with each other. When an action potential arrives at a synapse the signal will increase or decrease the probability of the receiving neuron firing itself; in the first case the synapse is *exitatory* and in the second it is *inhibitory*. Actually, there is yet another possibility, which is that the action potential has no effect at all. Furthermore, whether or not a receiving neuron will fire or not will in the end depend on the cumulative effect of the signals that reach it. (Coolen, 1998) We now imagine that a vertex in our model corresponds to a neuron and that its neighbours are the neurons that it can communicate with directly through synapses. Through the addition of grains we signal to the neurons that they should fire, so that our neurons do not always fire upon receiving a signal but do so only when the signal results in them reaching a value higher than the threshold value three. The toppling of a site we interpret as a neuron signaling to its neighbours to fire in turn, and the greater the resulting sequence of firing, the greater the amount of activity taking place in our neuronal network.

After an action potential has propagated along a neuron there is a refractory period where the membrane can only be forced to generate an action potential through extremly strong excitation (Coolen, 1998). Our two alternative ways of having grains dissipate are inspired by this feature and both involve sites loosing their ability to topple once they have already done so during the ongoing avalanche. The rules can be described as follows:

When a site that has already toppled during the ongoing avalanche again goes above its maximal capacity three it

1. drops all its grains instead of toppling, reaching a value of zero,
2. drops its excess grains and takes the minimum value it can have, while still being stable, that is, three.

In the subsequent avalanche the sites are again able to topple. Since there is at least one site that has toppled in each avalanche, there is at least one dissipative site for each avalanche.

Given these rules there is now the question of what will happen at the boundary, since the boundary sites have up until now been our designated dissipative sites. We take the lattice and tie its boundaries together forming a torus structure, so that each corner site now gets two new neighbours and each edge site gets one new neighbour.

For our first rule of dissipation we have the following example of an avalanche:

3	4	3
0	3	0
0	3	2

4	0	4
0	4	0
0	4	2

1	0	1
2	1	2
2	1	4

1	0	2
2	1	3
3	2	0

Figure 2. An avalanche resulting in 4 grains leaving the system. Here, the boundary sites now have the other boundary sites as neighbours.

And similarly, for the second rule of dissipation we have the example below:

0	3	0
3	4	2
0	2	0

0	4	0
4	0	3
0	3	0

2	0	1
0	2	4
1	4	0

2	1	2
1	3	0
2	0	2

Figure 3. An avalanche starts by the addition of a grain at the site positioned in the center of the figure and results in one grain leaving the system.

Although our altered systems must always reach stable configurations since there is always at least one dissipative site - the site where the addition operator was applied - when an avalanche has started, we cannot guarantee that our altered systems are abelian. We therefore must clarify the order in which we topple sites: At the initial point in time, on the fast time scale, the site where the addition operator has resulted in a site going above its capacity is toppled and if this result in other sites going above their maximal capacity these are toppled simultaneously in the next time instant, and so on until a stable configuration is reached.

If we consider the first rule of dissipation, where a site drops all its grains upon going above its maximal capacity a second time, a threat to the abelianness of the model would be if we arrived at the subconfiguration below

4	3	4

where the site having the value 3 has toppled once so will drop all grains if it again goes above its maximal capacity and where both sites containing 4 have not yet toppled so they still have the capacity to do so. If we arrive at this subconfiguration we can get two different results depending on how we topple. If we topple all sites above their capacity at the same time we get the subconfiguration

0	0	0

and if we instead first topple one of the sites only and then the other we get the subconfiguration

0	1	0

and since the order in which we topple these two sites results in different configurations, the toppling transformation cannot be abelian. However, if we consider the subconfiguration leading to this situation we realize that arriving at it is impossible given our definition. To arrive at it the middle site needs to have toppled once, and since the avalanche did not end at this point it must have made one of its neighbours topple so that it has at least one neighbour that does not have the ability to topple again. From this neighbour it must have received one grain, but to arrive at a height of three, two other neighbours must have toppled into it, so this site must have three neighbours that are now unable to topple again. We now have one site left in its neighbourhood that could take the value four, and we conclude that the configuration that worried us is not possible to arrive at. In addition, we realize that in this variation of the model, since a site can topple at most once before it becomes dissipative it cannot become overcritical more than two times. This is true in the case of the second rule of dissipation as well.

Criticality

We now return to the original model, with sand leaving the system via the edge sites, and begin by noting that the configurations η belonging to the set of stable configurations is a discrete time stochastic process $\{\eta_t, t \geq 0\}$. Furthermore, since we can reasonably assume that a configuration η_t only depends on past configurations through their effect on the previous one η_{t-1}, we may consider it a markov chain as well (Ross, 2014) (Meester et al. 2003). A configuration η is recurrent for the chain if after starting in η it will return to this configuration infinitely often, and if it will not it is transient. If we have $p_v > 0$ for all $v \in V$, there is a unique class of recurrent states belonging to the markov chain and in the stationary state, all the recurrent configurations occur with equal probability. (Meester et al. 2003) (Dhar, 1999)

Of interest is also how the system behaves on the fast time scale, in between the jumps from one stable configuration to the next: specifically, how some measures of avalanche size behave, on average, when the system has reached a stationary state. Such measures are: the total number of topplings s, the number of distinct sites toppled s_d, the duration of the avalanche d and the diameter of the region affected by the avalanche R.(Dhar, 1999) Here, d is the number of time steps needed for the avalanche to end if we topple all the sites that are above their capacity at a certain time point simultaneously, and we denote it by d to clarify that we are not speaking of the slow time scale. We do not expect these magnitudes to be indepedent of each other, for example, when an avalanche affects a large region we may reasonably expect that it also consists of a large number of topplings. Yet another property of the system that is related to the above is the probability that upon adding a particle at site v this starts a sequence of topplings that affects another site, w at distance R from it. The two-point correlation function C_{vw} is defined as the expected number of topplings at site w upon adding a particle at site v in the steady state (Dhar, 1999). In viewing our system as a neuronal network we may see that correlations between sites are important since they represent the ability of neurons to communicate with each other (Beggs et al. 2012). We would like a network where a neuron has the ability to communicate with a neuron far away from it. However, as stated in the introduction, we want there to be some balance, that is, we do not want the spiking of a neuron to always result in the whole system firing up. In the introduction we also connected this balance to so called criticality.

To explain what is meant by criticality we consider *site percolation*, an area studied in statistical physics. Consider again the integer square lattice $\mathbf{Z}^2$, but this time the infinite, and as before let the nearest neighbours of a vertex be the vertices one step away from it in the vertical and horizontal directions. The vertices are "occupied" with probability p and "vacant" with probability 1-p, and these probabilities are independent of the states of the neighbouring sites. A group of occupied sites that are connected to each other by an unbroken chain of nearest neighbour sites is called a *cluster* and percolation theory deals with the properties and numbers of these clusters. Of particular interest is the questions of whether or not an infinite cluster exists for different values of p and if one does we say that *percolation occurs*. (Stauffers et al. 1994) The percolation threshold, p_c is (Slade, 2008) that value of p which separates two different phases: the supercritical phase where for all $p > p_c$ an infinite cluster exists from the subcritical phase where for all $p < p_c$ no such infinite cluster exists. We recall the quote of Plenz in the introduction and relate the supercritical phase, where an infinite cluster always exists, to the strong and too rigid interactions in the brain during an epileptic seizure and we relate the small clusters of the subcritical phase to a neuronal network that is not so good at processing information due to poor communiction. How does a system behave at the point that separates these phases?

In general for critical phenomena, several properties of such a system go through very sudden changes as some critical parameter is varied through a critical value and these properties are predicted by physicists to be dependent of each other, and have in some cases been shown to be. These properties of interest follow, or are believed to follow, so called *power laws* close to the critical value, whereas away from it they do not (Slade, 2008). Magnitudes that are best described by power laws (Newman, 2004) are characterized by that while the bulk of the distribution is centered around values of small sizes, the distribution is highly right-skewed so that values much larger than the typical value are included. For some parameter values the second moment does not exist and for some parameters the first moment does not exist, so for power laws it rarely the case that we can describe the magnitudes of interest by using the standard deviation and/or the expected value. (Newman, 2004)

The interest taken in the Abelian sandpile model is due to the fact that the measures of avalanche size mentioned above as well as related measures of correlations are believed to be described by power laws as the size of the system goes to infinity (Redig, 2006). The difference in this model (Dhar, 1999) as opposed to traditional models of critical phenomena is that we do not have an equilibrium system where a parameter needs to be tuned for the critical behaviour to arise, but we instead have a dynamical system that is open and dissipative and driven to a steady state by the addition and dissipation of grains. By some this is seen as a more robust underlying mechanism of criticality and thus more likely to be a mechanism through which critical behaviour arises in nature but this is a view that has received criticism. (Dhar, 1999)

Before we present some results from simulations in the next chapter we mention that although we in this paper look at a model with sandgrains added at random locations, the addition of sand grains only to the center of a 2-dimensional structure will yield som interesting patterns as seen in Figure 4 below.

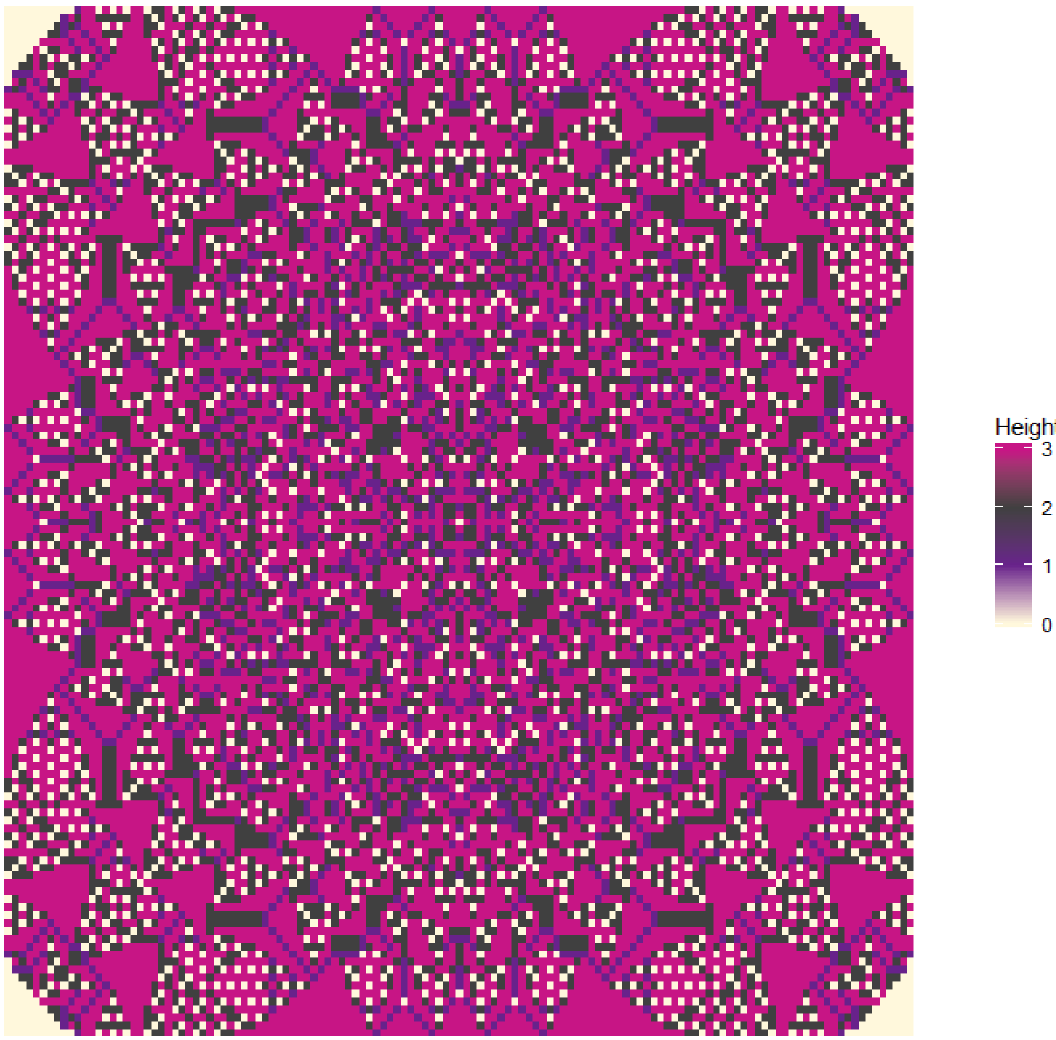

Figure 4. A deterministic sandpile on the 131×131 *square grid, with 60000 sandgrains dropped in the middle.*

Simulations

In our simulations we start from a configuration where all height variables are zero and we add grains with equal probability $p = \frac{1}{L^2}$ to each of the $L \times L$ sites. We collect data on the following measures:

- s, the total number of topplings
- s_d, distinct sites toppled
- d, the duration of the avalanche, that is, the number of time steps needed for the avalanche to end
- $\frac{\sum_i \eta(v_i)}{L^2}$, the average height of the sandpile
- the number of grains that leave the system at each point in time

For the models where sites only can topple once we collect data on the total number of topplings only.

Dissipation through grains falling of the edge

We first simulate a system where the dissipation takes place through grains falling of the edges of the lattice structure. We look at a 51×51 square grid and we add grains with equal probability $p = \frac{1}{51^2}$ to each site.

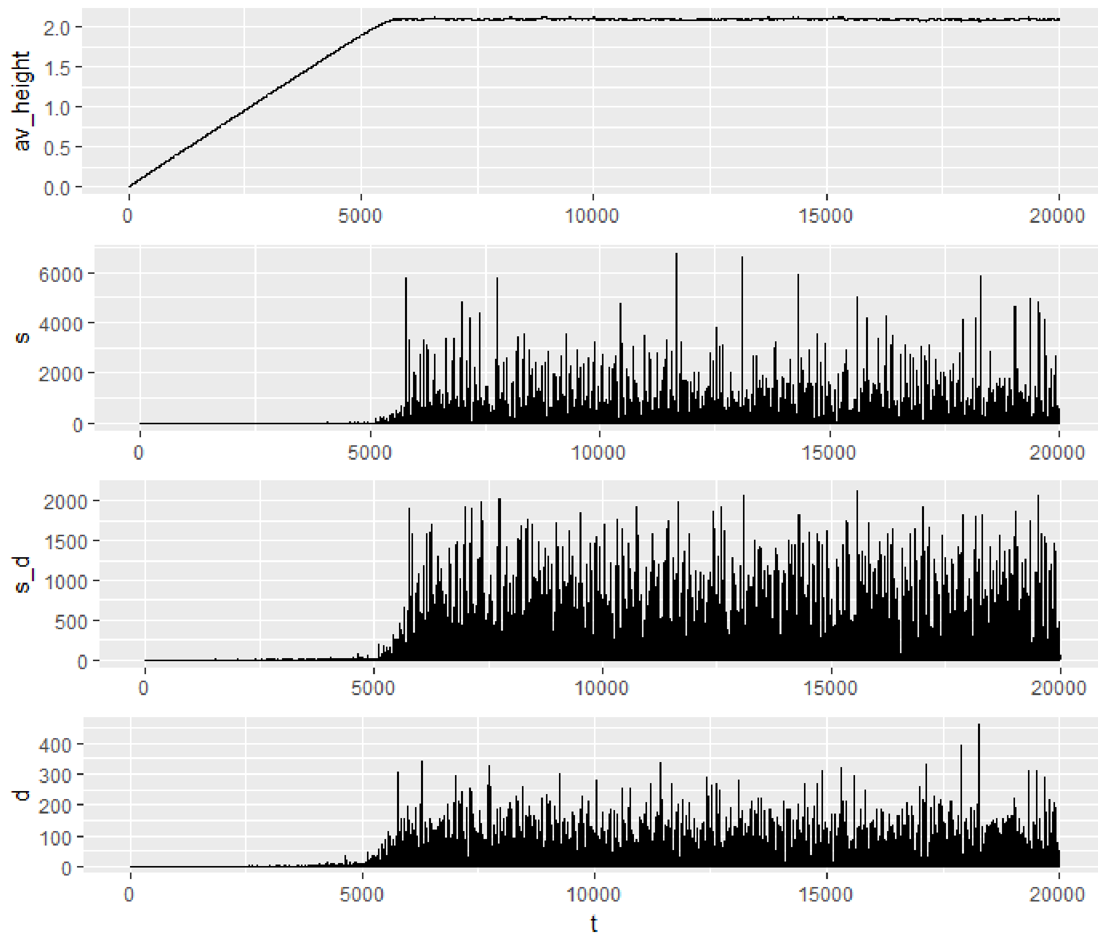

Figure 5. A simulation on 51×51 *square grid with 20000 grains dropped at random locations. The system seems to go into a steady state after approximately 5625 grains have been dropped.*

An avalanche can result in 2601 distinct sites toppling at most for a grid of our size. From viewing Figure 5 we see that most avalanches seem to involve less than 750 sites toppling whereas avalanches involving all the sites toppling do not take place during this simulation.

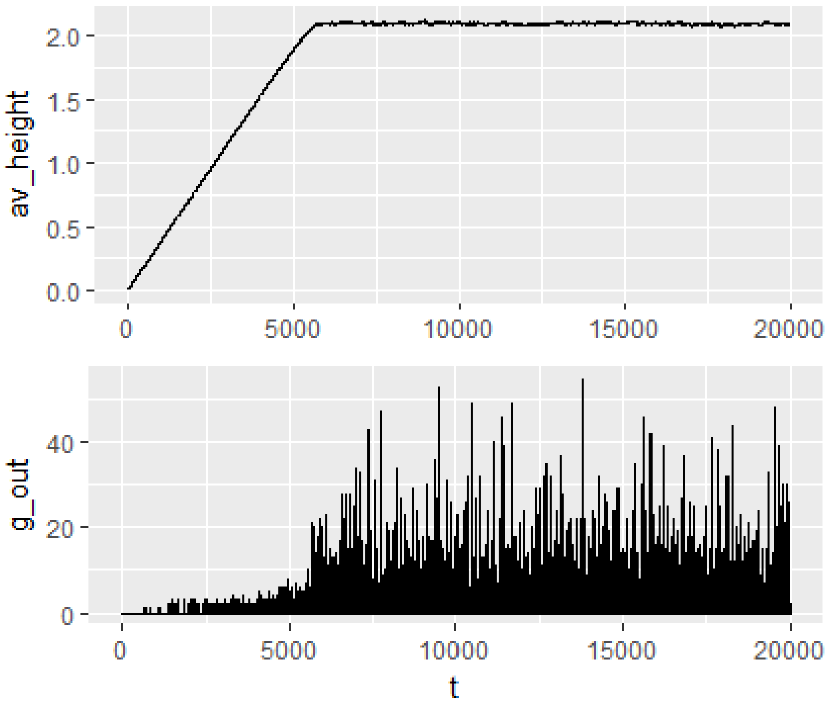

Figure 6. Structure where grains dissipate by falling of the edge. A simulation on 51×51 *square grid with 20000 grains dropped at randomly chosen locations. Average height, number of topplings and the duration of the avalanche are plotted over time.*

By looking at Figure 6 and the outflux of sand that the avalanches yield, we get a rough idea of the relative sizes of the dissipative events of the system. The measures of avalanche size in Figure 5 show the sizes of avalanches that reach the boundaries as well as those that do not reach the boundaries. A large avalanche may not reach the boundary but only involve a rearrangement of heights. Small avalanches may cause dissipation for example if they take place close to the boundary. In the figure we can see that most dissipative avalanches seem to result in less than 15 or 20 grains leaving the system whereas rare, dissipative events sometimes involve around 50 grains leaving.

First rule of dissipation

We look at the simulations of a system where a site that goes above its maximal capacity a second time drops all its grains. We again look at a 51×51 square grid and we add grains with equal probability $p = \frac{1}{51^2}$ to each site.

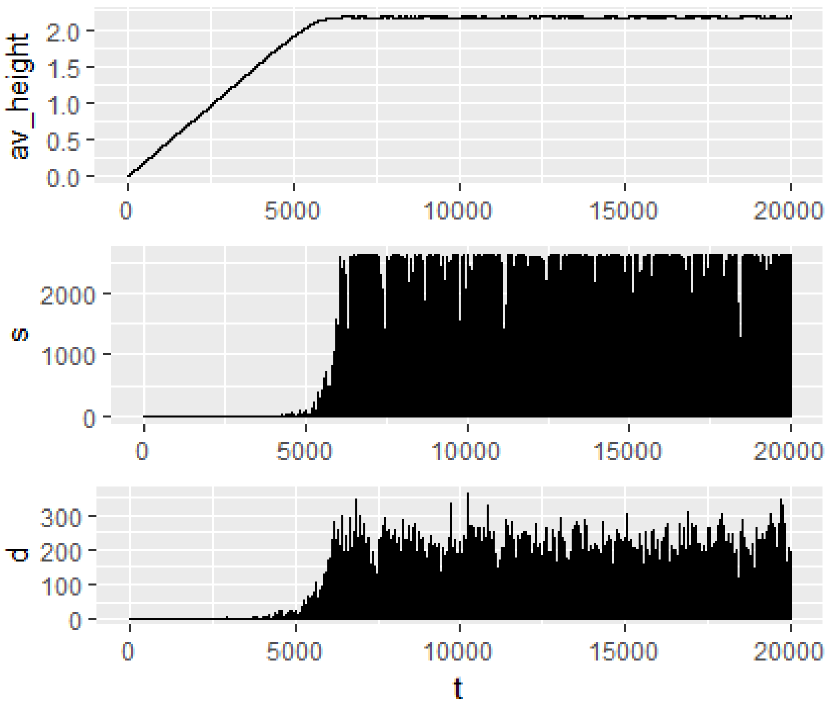

Figure 7. A simulation on 51×51 *square grid with 20000 grains dropped at random. Average height, number of topplings and the duration of the avalanche are plotted over time.*

Here we can see that the average height is quite close in the two models but with the average of the classical model slightly lower. Compare Figure 7 to Figure 5 and look at distinct sites toppled, s_d. Whereas in Figure 5 no avalanche took place that involved all the 2601 sites, for this model avalanches that involve all the sites or close to all the sites frequently.

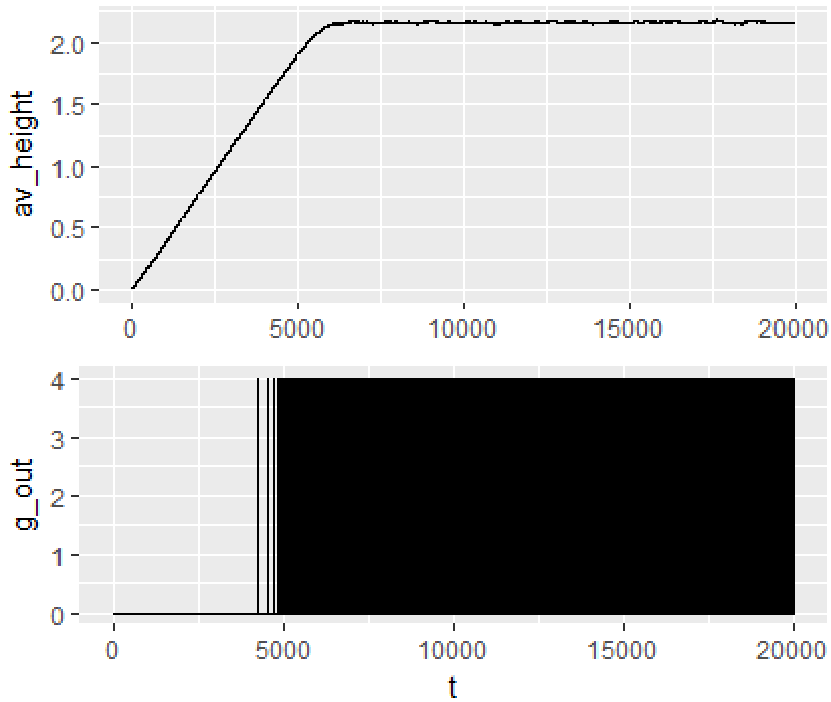

Figure 8. Above: a plot of average height. Below: the number of grains that leave the system at the different times. A simulation on 51×51 *square grid with 20000 grains dropped at randomly chosen sites.*

In Figure 8 we see that when there is a dissipative avalanche there are always four grains that leave the system. Compare this to the variation in the number of leaving grains when an avalanche takes place in the simulation for the classical model.

Second rule of dissipation

When considering the model where a site that goes above its maximal capacity a second time drops all its grains we look at a grid of 31×31 sites.

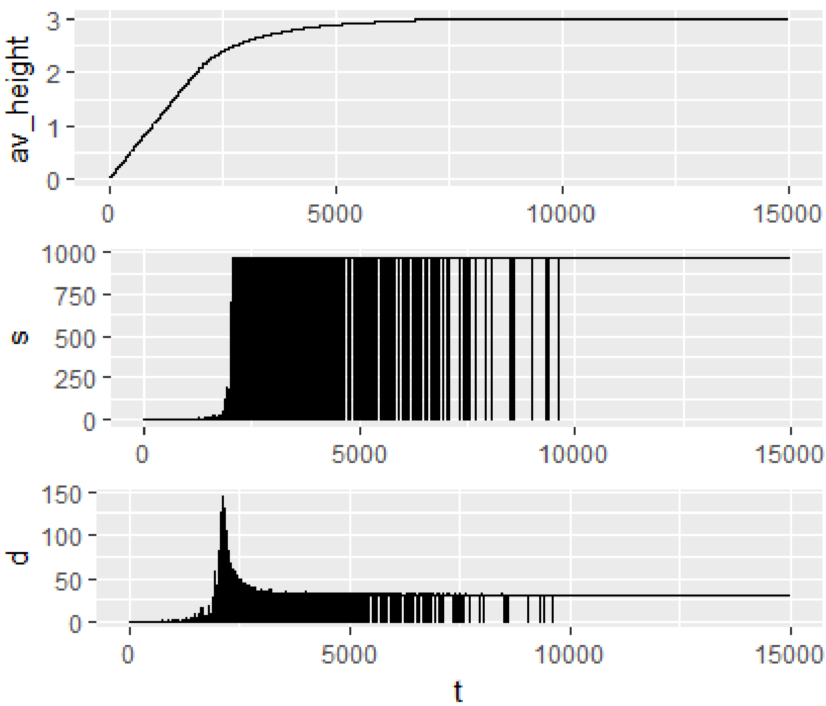

Figure 9. A simulation on a 31×31 *square grid with 15000 grains dropped at randomly chosen locations. Average height, number of topplings and the duration of the avalanche are plotted over time.*

We consider Figure 9 and note that a bit before 10000 grains have been dropped the system reaches a configuration where all the height variables are equal to three. From this point on in our simulation, as a grain is added, there is always an avalanche that causes a grain to leave and all the 961 sites to topple. Note that it takes quite a long time for the system to reach this steady state, compared to the other models, and our system here is still quite small.

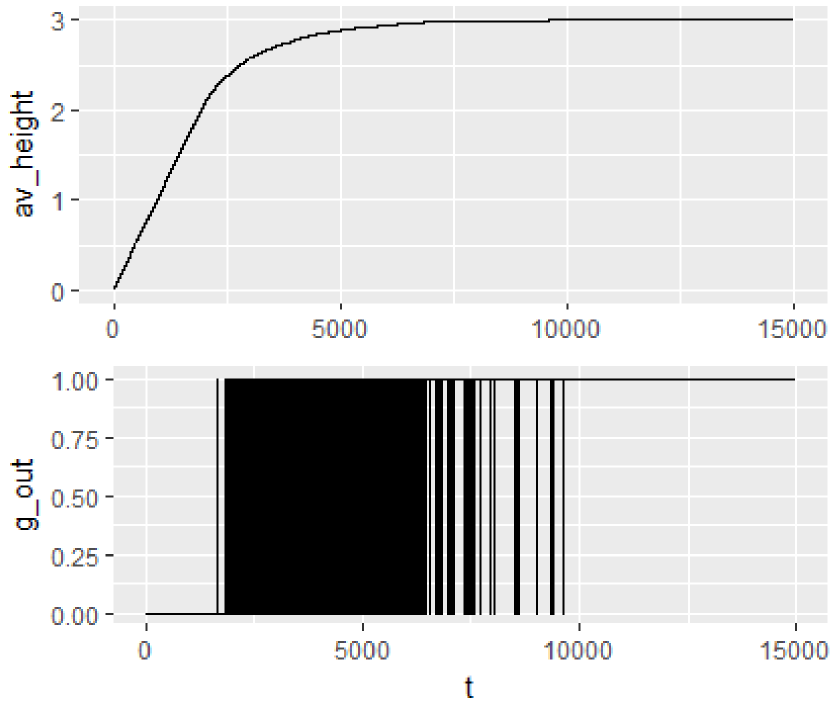

Figure 10. Above is a plot of average height over time. Below we see the number of grains that leave the system at each time step. A simulation on 31×31 *square grid with 15000 grains dropped at randomly chosen locations.*

If we look at a 3×3 grid where we have added a grain to the maximal configuration, that is, the configuration where all height variables have their highest values while still stable, we get

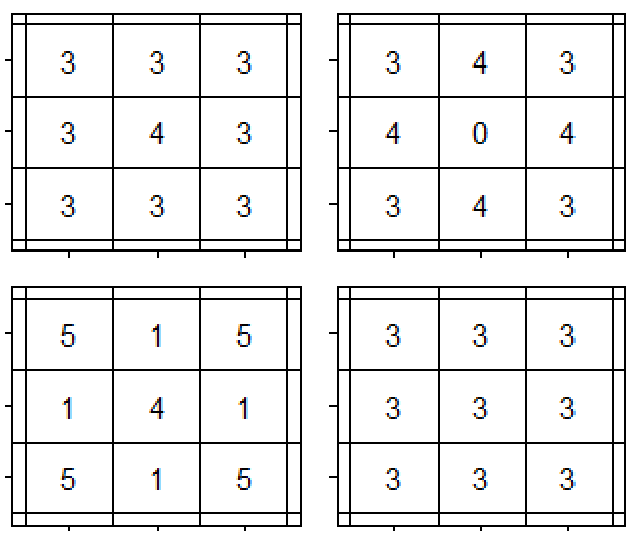

Figure 11. A grain added to the maximal configuration on the 3×3 *grid.*

that is, all sites topple in the avalanche, exactly one grain leaves the system, and this goes on as we keep adding grains. Since we have no thresholds left, that is, no values below three that function as thresholds, after the system has filled up, the random application of the addition operator results in no stochastic behaviour of the system, that becomes degenerate.

Analysis

As mentioned above, for the simulations that have been made here, it is the case, that when the avalanches in the altered models become dissipative they always loose either four grains - in the case of the system with the first dissipation rule - or one grain - in the case of the system with the second dissipation rule. Considering this, we would like to make the claim that it is only possible for exactly one site to become overcritical twice during an avalanche. Earlier in this paper we claimed that a site can go above its maximal capacity twice at most and in the reasoning that led to this conclusion we assumed that the site in question was the site where the avalanche had started as a result of the application of the addition operator. If we instead try to construct a situation where the site that becomes overcritical twice is not the site that has had its value increased through external perturbation, we realize that it is only the site at which an avalanche starts that has the capacity to go above the maximal height twice. Assume that a site has the maximal height three and that one of its neighbours topple into it, making it topple in turn. For this site to go above its maximal value once more it must receive four grains, but since only three of its neighbours have the capacity to topple and they can do so at most one time each, the site cannot reach a value of three during the ongoing avalanche. We thus have only one dissipative site per avalanche although our intention was to construct a model where all the sites have the capacity to become dissipative sites. In making all the sites unable to topple twice we have removed their ability to become dissipative since they cannot be reached twice by the avalanche cluster. A question that arises is whether the propagation of the avalanche cluster can block our one dissipative site and if this in addition could possibly cause a never ending avalanche; this has not happened in our simulations however.

The interest that led to the writing of this paper was in a potential model that could be used to model the way in which neurons function. If we consider the unpredictable behaviour of neurons described earlier in the paper this may reasonably convince us, that the steady state of the model that converges to the maximal configuration, is not appropriate for modeling networks of neurons. Furthermore, our particular interest in the Abelian sandpile model stemmed from its expected critical behaviour as the size of the system goes to infinity. If we compare our simulations of the different models we see that the measures of avalanche size for the classical model vary a lot more than the same measures do in the altered other models. For the other models the typical variation is between very large events and no events at all. Considering this, the neuroscientists interested in the hypothesis that the brain functions well at a critical point would not be happy with our model of neurons. We conclude that the dissipative structure plays an important part in determining the behaviour of the model.

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