Portfolio Value-at-Risk Using Regular Vine Copulas

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Abstract

Value at Risk (VaR) is a tool commonly used to measure market risk. However, it can only be precise as long as the return distribution is accurately modelled. For portfolios consisting of many dependent assets, standard models for the dependence structure are not precise enough. Dependence modelling based on regular vine copulas is a flexible and useful alternative for high-dimensional portfolios. We have applied a modelling approach based on regular vine copulas for a six dimensional portfolio and tested how the model performed with regards to predicting VaR over 16 years. A multivariate student t copula was used for comparison. Firstly, we backtested the model on simulated data in order to assess the performance under ideal circumstances. Secondly, we backtested the model on six-dimensional financial return data. We considered two approaches: for the first one we used a fixed dependence model and for the second backtest we re-estimated the dependence model in each step. We found that the method with fixed dependence structure performed adequately during normal market conditions. However, when applied to financial return data from the time period of the recent global financial crisis, it failed to adjust and under-estimated the risk. Last but not least, we discovered that the model with a moving window did not sufficiently well adjust to extreme market conditions.

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Introduction

In risk management there are several approaches to quantify risk. The most modern and consistent approaches are quantile based and therefore it is of interest to understand a portfolio return distribution. Complications may arise when dealing with higher dimensions.

There are several methods that allow modeling joint distributions and a large fraction of them are using linear correlation. However, linear correlation can be misleading and it is important to know a *dependence structure* between the assets of interest. In this case copulas can be a very useful concept, due to the fact that they reveal dependence structure on a quantile scale and therefore are a useful tool when describing dependence between the assets.

However, if the portfolio consists of several assets (more than two), often standard parametric copula families are not flexible enough, as all pairs have dependency structure defined by one multivariate copula, usually Gaussian, Student-t or Archimedean. Moreover, widely used multivariate Gaussian copula does not take into account tail dependencies or asymmetry. Likewise, Student-t copula also does not account for asymmetries, as well as has a single degrees of freedom parameter. Although Archimedean copulas can take into account asymmetry, they are criticized for only having one or two parameters for a possibly large number of variables. As a result, in high dimensions standard parametric copula families can contain too many limitations.

On the other hand, bivariate copulas have a wider variety of families to choose from. For tail independent pairs there are Gaussian and Plackett copulas, while Student t can be used for symmetric tail dependent pairs. Gumbel, Rotated Gumbel, BB7 and Symmetric Joe-Clayton can account for asymmetric tail dependence.

Bedford & Cooke (2002) have proposed the regular vine copula method, which connects graph theory together with pair copula constructions. It allows us benefit from the rich choice of bivariate copula families, while adding flexibility to the model. We will estimate joint distribution of a 6 dimensional portfolio using regular vine copula. Afterwards we can predict Value at Risk measure and use a back-testing procedure in order to evaluate whether our estimation was accurate.

2

Risk

Risk is related to randomness and uncertainty. For example, insurance companies, home owners or investors all (to a different degree) face uncertainty in the future value of their products. We can analyze risk in the context of the risk type. The three biggest categories of financial risks are market risk, credit risk and operational risk.

Market risk is especially well known in banking and it is a risk of changes in value of a financial position, which is caused by the changes in value of underlying assets. Changes in value of underlying assets may occur for several reasons, such as interest rate, currency, equity price or commodity price fluctuations, additionally risks occurring due to investments that cannot be traded fast enough in order to minimize/prevent losses (liquidity risk). Furthermore, **credit risk** is the risk of not receiving promised payments, such as repayment of loan or other obligations. Finally, **operational risk** is a risk occurring due to failure of internal processes (people and systems), or the external ones, such as fraud, security, legal, political situation etc. Operational risk differs from other categories of risk, because it is not revenue driven.

It is of interest to measure and manage risk and have safety mechanisms in place for possibly adverse future events. Risk management can be also used to optimize the return for the given risk level. We can model risk using random variables that have some distribution function and assign probabilities for certain outcomes (More about risk can be found in McNeil et al. (2005)). We will be analyzing the distribution of a portfolio that consists of several types of assets and assess the risk for some future time.

2.1 Risk Measures

We will continue talking about ways to quantify risk and analyze existing measures of risk.

Artzner et al. (1999) (p. 209-210) defined what is called a coherent risk measure, which clarifies the concept of risk and provides properties that risk measure might have. Let G denote a vector space of random variables representing portfolio values at a fixed future time. Let X and Y be random variables denoting a set of future net worth of an investment. *A coherent risk measure* is a function $\rho : G \to \mathbb{R}$ that satisfies the following properties:

- **Translation invariance**. For all $X \in G$ and all real numbers α, reference instrument total rate of return r, we have:

$$\rho(X + \alpha \cdot r) = \rho(X) - \alpha$$

Translation invariance axiom means that adding the amount α to the initial position and investing it in a reference instrument with return r decreases risk measure by α.

- **Subadditivity**. If $X, Y \in G$, then

$$\rho(X + Y) \leq \rho(X) + \rho(Y).$$

In other words, merger does not create risk. This property mathematically shows a benefit of diversification.

- **Positive homogeneity**. For all $\lambda \geq 0$ and all $X \in G$

$$\rho(\lambda X) = \lambda \rho(X)$$

The risk scales together with size of a position.

- **Monotonicity**. For all $X \in G$ and $Y \in G$, with $X \leq Y$

$$\rho(Y) \leq \rho(X)$$

That means if the future net worth X is smaller than Y, then X is more risky.

We will now introduce the main categories of methods used to evaluate risk, as well as briefly analyze them in terms of the properties of coherent risk measure and discuss the advantages and disadvantages of each method (McNeil et al. (2005), p. 34-36).

Notional-amount approach. Risk of portfolio is defined by adding up the notional values of the individual investments in the portfolio. This approach is very old and main advantage is its simplicity, however, it has many flaws. Risk measure calculated this approach would not reflect subadditivity property. Well diversified portfolio consisting of a collection of companies could be interpreted as risky as investing the whole amount just to one company.

Factor-sensitivity measures. It measures the change in portfolio value for a given change in one of the risk factors. This measure can be used to analyze the robustness of portfolio for some predefined events. On the other hand, this method cannot be used to evaluate the overall riskiness of a position.

Risk measures based on return distributions. This category of methods are most modern. The main idea is analyzing return distribution of the portfolio over some predetermined period of time and quantifying the risk in terms of return distribution quantiles. The main disadvantage is that distribution is estimated according to historical data. Laws and

regulations are constantly changing, as a result the historical data should be used carefully when predicting future. Moreover, there is a need for improved methods predicting large portfolio distribution, as current methods can be crude (for example, some methods require assumption of normality) and hence affect accuracy of quantile based risk measures. Altogether, quantile based risk measures can be objectively used from evaluating the uncertainty of a single asset to determining the overall position of financial institution.

Scenario-based risk measures. Here we consider a set of possible future scenarios. Such scenarios are changes in risk factors, for example, 5% rise in interest rate, or 20% drop in main stock market indexes. A risk of a portfolio is considered to be the maximum loss out of all scenarios. It can be useful if a portfolio is affected only by a small number of risk factors. However, it is difficult to determine the appropriate set of scenarios and to compare risk of portfolios that have different risk factors.

Throughout this thesis we analyze a method to estimate a joint distribution of a six dimensional portfolio. The success of this task can be determined by the accuracy of the quantile based risk measure.

2.1.1 Value at Risk

Value at Risk is a quantile based method. Given normal market conditions, it is used to estimate the amount of capital needed to account for possible losses for a given risk level.

It is common to analyze risk in terms of discounted loss $L = -\frac{X}{r}$ (where r is a reference instrument total rate of return) and use loss distribution. However, we will follow an approach more commonly practiced in academia by using the distribution of X, which in our case is a distribution of change in portfolio value in a given day. Therefore when talking about risk we are interested in the left tail of this distribution.

Value at Risk at level $\alpha \in (0,1)$ of a portfolio with value X at time 1 is defined in a following way (Hult et al. (2012), p. 165-166):

$$\text{VaR}_\alpha(X) = min\{m : \mathbb{P}(m \cdot r + X < 0) \le \alpha\}$$

VaR can also be expressed as $(1 - \alpha)$ quantile of discounted loss $L = -\frac{X}{r}$. Suppose F_L is loss distribution function, then we can express VaR as:

$$\text{VaR}_\alpha(X) = F_L^{-1}(1 - \alpha) = -F_X^{-1}(\alpha)$$
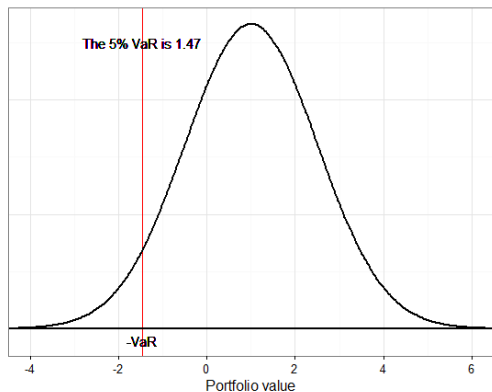
Figure 2.1: $\text{VaR}_{0.05}$ illustration

$VaR_{0.05}$ is illustrated in Figure 2.1 for some hypothetical portfolio that has $N(1, 1.5)$ distribution for some future time and red vertical line marks the 5% quantile of return distribution $F_X^{-1}(0.05) = -\text{VaR}_{0.05}$. We can observe that VaR does not take into account the tail of the distribution below α level and therefore in some cases can violate the sub-additivity property. That is why it is not a coherent risk measure by definition. However, it is widely used, because it is easier to work with than for example Expected Shortfall (defined below). There are several ways to calculate portfolio VaR. We will use the Monte Carlo simulation for estimating value at risk due to the nature of our model.

2.1.2   Expected Shortfall

Expected shortfall is linked to Value at Risk. It is deemed to be a superior risk measure to VaR, because it takes into account the left tail beyond α level. ES does not violate sub-additivity property and is a coherent risk measure.

Expected Shortfall (ES) is defined in a following way:

$$ES_\alpha(X) = \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_\gamma(x) d\gamma$$

If X has a continuous distribution function, then for discounted loss $L = -\frac{X}{r}$ we can define ES in a following way:

$$ES_\alpha(X) = \mathbb{E}[L | L \geq \mathrm{VaR}_\alpha(X)]$$

In other words, ES_α is an average of VaR_γ values where γ is in the interval $[0, \alpha]$. Figure 2.2 illustrates expected shortfall at level $\alpha = 0.05$ for a hypothetical portfolio that has $N(1, 1.5)$ return distribution at some future time. Both for VaR and ES it important to be able to estimate the tail of distribution precisely.
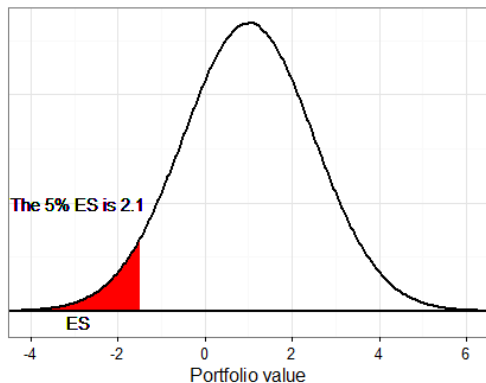
Figure 2.2: $ES_{0.05}$ Illustration

2.2   Portfolio Risk

In our case, we have a 6-dimensional portfolio, where we invested equally in each underlying investment (weights $w_1 = w_2 = \cdots = 1$) and we want to measure the risk for the next day. The change in value of our investment in any given day can be expressed as $X = \sum_{i=1}^6 w_i X_i = \sum_{i=1}^6 X_i$, where X_i is value change of the i-th underlying investment.

We can determine the marginal distributions of each underlying investment. However, it is more difficult to know the portfolio distribution. We are interested in finding portfolio distribution in order to measure the risk with the methods described above. This includes finding a joint model for random vector of investments $X = (X_1, \ldots, X_6)$.

Further we will talk about what it means to measure dependence, what is dependence structure and how it is important for joint distribution. Additionally, we will introduce r-vine copula method that can be used to find high dimensional portfolio distribution.

3

Copulas and Dependence

Copulas can help to perceive and visualize the nuances of dependence which is useful when describing the dependence of extreme events. In Figure 3.1 we can see that for the same linear correlation parameter the dependence structure can look very different. Since copulas reveal dependence on a quantile scale, it is especially useful in the context of quantile based risk measures. In this chapter we will introduce important mathematical concepts related to copula theory and explain how we can model joint portfolio distribution using bivariate copulas.
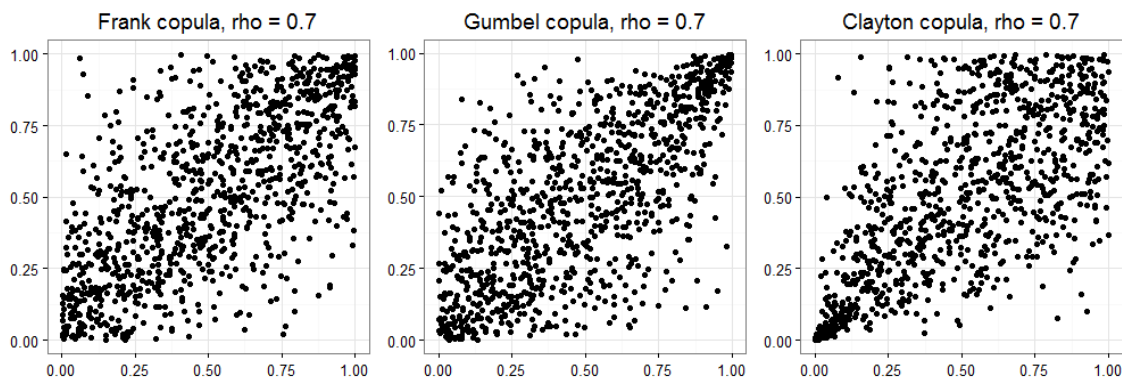
Figure 3.1: Dependence structure

Throughout this thesis we will present definitions and theorems that are the basis for regular vine copula theory. We will use some notation from Dißmann (2010) thesis, where the author described in detail the theoretical background concerning regular vines and applied the method to different data sets.

3.1 Copulas

Let $X = (x_1, \ldots, x_n)$ be a n-dimensional random variable. Let $f(x_1, \ldots, x_n)$ be its joint probability density function and $F(x_1, \ldots, x_n)$ be its joint cumulative distribution function. Further, let $F_1(x_1), \ldots, F_n(x_n)$ be the corresponding (strictly increasing, continuous) marginal cumulative distribution functions of $x_1, \ldots, x_n$ and $f_1(x_1), \ldots, f_n(x_n)$ be the corresponding probability distribution functions.

Definition 3.1.1. A n-dimensional Copula C is a joint cumulative distribution function on $[0, 1]^n$, $C : [0, 1]^n \to [0, 1]$ with univariate uniform distributed marginals.

A very important result is Sklar's theorem that states that joint distribution can be written using marginal distributions and copula. We will use this result when constructing pair copula constructions later in this thesis.

Theorem 3.1.1. (Sklar (1959)) Any joint cumulative distribution function $F(x_1, \ldots, x_n)$ with marginal cumulative distribution functions $F_1(x_1), \ldots, F_n(x_n)$ satisfies

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{3.1}$$

where C is the copula of $F(x_1, \ldots, x_n)$.

If joint cumulative distribution function is absolutely continuous and marginal cumulative distribution functions are strictly increasing and continuous then copula density can be expressed as:

$$c(F_1(x_1), \ldots, F_n(x_n)) = \frac{\partial^n C(F_1(x_1), \ldots, F_n(x_n))}{\partial F_1(x_1) \ldots \partial F_n(x_n)} \overset{3.1}{=} \frac{\partial^n F(x_1, \ldots, x_n)}{\partial F_1(x_1) \ldots \partial F_n(x_n)} = \frac{f(x_1, \ldots, x_n)}{\prod_{i=1}^n f_i(x_i)}$$

As a result, we can express $f(x_1, \ldots, x_n)$ with the following equation:

$$f(x_1, \ldots, x_n) = \prod_{i=1}^n f_i(x_i) c(F_1(x_1), \ldots, F_n(x_n)) \tag{3.2}$$

3.2 Dependence Measures

In this section we will describe measures of dependence Pearson's ρ, Spearman's ρ and Kendall's τ. It is important to understand difference between these dependence measures as we will later use them to express copula families. We will also introduce coefficients of tail dependence. More details on dependence measures in risk management can be found in Embrechts et al. (2002).

3.2.1 Pearson's ρ

Definition 3.2.1. Pearson's correlation coefficient $\rho \in [-1, 1]$ is defined as

$$\rho = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Pearson's ρ measures linear dependence between X and Y. Pearson's correlation can be interpreted as the slope of the regression line of X and Y. Pearson's ρ between copula variables U and V is:

$$\rho = 12 \int \int uv dC(u, v) - 3$$

Criticism of Pearson's ρ: the fact that it measures a linear dependence can be a drawback in cases where dependence is non-linear and zero correlation in general does not imply independence. Moreover, is it not invariant under non-linear strictly increasing transformations, meaning that value of Pearson's ρ depends on marginal distributions.

3.2.2 Spearman's ρ

Definition 3.2.2. Spearman's $\rho \in [-1, 1]$ between X and Y is a rank correlation. It is calculated similarly as Pearson's ρ, but we use ranked variables $F_1(X)$ and $F_2(Y)$. It is expressed as:

$$\rho = 12 \int_x \int_y F_1(x) F_2(y) dF(x, y) - 3$$

Spearman's ρ is invariant under strictly monotone transformations of X and Y. It is considered to measure the degree of monotonic dependence, on the other hand Pearson's ρ only measures degree of linear dependence. Spearman's ρ between two copula variables U and V is:

$$\rho = 12 \int \int uv dC(u, v) - 3$$

3.2.3 Kendall's τ

Definition 3.2.3. Kendall's τ (or coefficient of concordance), $\tau \in [-1, 1]$ is given by:

$$\tau = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$$

where $(X_1, Y_1) \sim F$ and $(X_2, Y_2) \sim F$ are independent pairs of random variables. Kendall's τ is a rank correlation. It does not depend directly on the values and thus is invariant under strictly monotone transformations (Gruber (2014)). Kendall's τ between two copula variables U and V is:

$$\tau = 4 \int \int C(u, v) dC(u, v) - 1$$

3.2.4 Tail Dependence

Definition 3.2.4. Let $X = (X_1, X_2)^T$ be a two dimensional random vector with marginal distribution functions F_1 and F_2. The coefficient of **upper tail dependence** of X is defined as:

$$\lambda_U = \lim_{\varepsilon \to 1} P(X_1 > F_1^{-1}(\varepsilon) | X_2 > F_2^{-1}(\varepsilon))$$

Definition 3.2.5. Let $X = (X_1, X_2)^T$ be a two dimensional random vector with marginal distribution functions F_1 and F_2. The coefficient of **lower tail dependence** of X is defined as:

$$\lambda_L = \lim_{\varepsilon \to 0} P(X_1 \leq F_1^{-1}(\varepsilon) | X_2 \leq F_2^{-1}(\varepsilon))$$

3.3   Elliptic Copulas

Regular vines are used together with two-dimensional copulas, that is why we use only two dimensions to describe different copula families. However, it is common to model with multivariate copulas. Elliptic copulas are a popular choice, but they have some restrictions. First of all, elliptic copulas can only be used to model symmetric dependency structures. Multivariate Gaussian is criticized, because it does not have tail dependence. Moreover, Student t copula has a single degrees of freedom parameter for possibly a large dimension.

Table 3.1: Types of Bivariate Elliptic Copula

Name	Formula	Parameter	Tail Dependence
Gaussian	$C_\rho^{\mathrm{N}} = \Phi_R\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right)$	$\rho = sin(\pi\tau/2)$	No tail dependence
Student-t	$C_{\nu,\rho}^t = t_\nu(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2))$	$\rho = sin(\pi\tau/2)$	$2t_{\nu+1}(-\sqrt{\frac{(\nu+1)(1-\rho)}{1+\rho}})$

where τ is Kendall's τ, Φ^{-1} is the inverse cumulative distribution function of a standard normal, Φ_R is the joint cumulative distribution function of a bivariate normal distribution with mean vector zero and covariance matrix equal to the correlation matrix R, t_ν is Student-t cumulative distribution function with ν degrees of freedom.

Figure 3.2 illustrates dependence structure differences for Gaussian and Student t copulas with the same dependence parameter (linear correlation ρ).
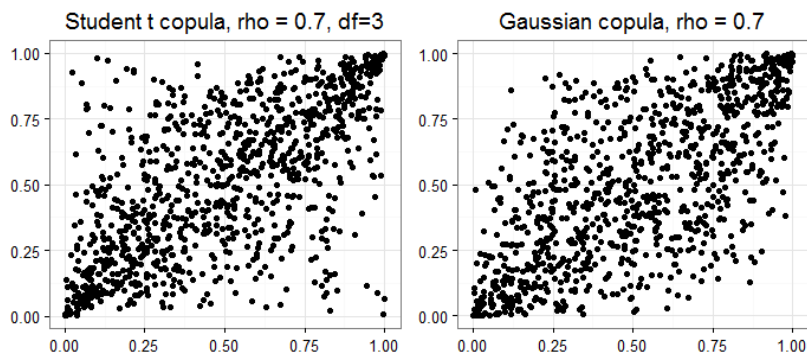
Figure 3.2: Elliptic Copulas

For elliptical distributions (also elliptical copulas Gaussian or Student t) the natural choice of dependence measure is a linear correlation. However, dependence properties that can be applied to elliptical distributions cannot always be applied to non-elliptical cases. We will further present such cases in the next section.

3.4   Archimedean Copulas

Archimedean bivariate copulas can be expressed in the following way (McNeil et al. (2005), p.220):

$$C(u_1, u_2) = \psi^{-1}(\psi(u_1) + \psi(u_2))$$

where ψ is a generator function $\psi : [0,1] \to [0,\infty]$. Here we only consider such ψ that is a strictly decreasing, continuous, convex function, satisfying $\psi(1) = 0, \psi(0) = \infty$ (provided that the parameters are within the ranges specified in Table 3.2).

The condition $\psi(0) = \infty$ means that ψ is a so called strict generator. The Archimedean copulas can be defined in a broader sense, however, we do not need this level of complexity, as we only use copulas where the generator is strict.

Table 3.2: Types of Bivariate Archimedean Copula

Name	Generator $\psi_\theta(t)$	Kendall's τ	Parameters	(λ_L, λ_U)
Clayton	$\frac{1}{\theta}(t^{-\theta} - 1)$	$\frac{\theta}{\theta+2}$	$\theta \geq 0$	$(2^{-1/\theta}, 0)$
Gumbel	$(-\ln(t))^\theta$	$1 - \frac{1}{\theta}$	$\theta \geq 1$	$(0, 2 - 2^{\frac{1}{\theta}})$
Frank	$-\ln(\frac{e^{-\theta t}-1}{e^{-\theta}-1})$	$1 - \frac{4}{\theta} + 4\frac{D_1(\theta)}{\theta}$	$\theta \in \mathbb{R}$	$(0,0)$
Joe	$-\ln(1 - (1-t)^\theta)$	$1 + \frac{4}{\theta^2}\int_0^1 (t\ln(t))(1-t)^{2(1-\theta)\theta}dt$	$\theta \geq 1$	$(0, 2 - 2^{\frac{1}{\theta}})$

Archimedean copulas allow modelling asymmetric dependencies and in general a variety of dependence structures. We are only using a bivariate Archimedean copulas and avoid their main disadvantage which appears in higher dimensions (parameters can be restrictive, because some correlation matrix entries are forced to be equal).

3.5 Pair Copula Constructions (PCC)

Modelling with standard multivariate copulas is not flexible. First of all, there is only a few choices for high dimensional multivariate copulas. Most popular are Student-t and Gaussian copulas, both of which can not be used to model asymmetric dependence structure that can often appear in data. Moreover, Student t copula, for example, has a single degrees of freedom parameter for possibly large number of variables. Archimedian copulas can handle asymmetry, however, they are criticized for having only one or two parameters and that becomes a big constraint especially in high dimension. That is why here we discuss pair copula approach which allows us to model with bivariate copulas. It is advantageous, because in bivariate case there is a wide selection of families to choose from.

Joint probability distribution function can be expressed with bivariate(or pair) copula densities and marginal densities and this method is called Pair Copula Constructions (first proposed by Joe (1996)). In order to derive PCC we are using two main results: conditional probability and an equation 3.2 that was derived from Sklar's Theorem.

PCC illustration in three dimensions

We will first illustrate Pair Copula Constructions in three dimensions to get the idea of this approach before generalizing it for higher dimensions. Suppose we want to express joint probability density function $f(x_1, x_2, x_3)$ using only marginal densities $f_1(x_1), f_2(x_2), f_3(x_3)$ and bivariate copulas. $f(x_1, x_2, x_3)$ can be divided for example into:

$$f(x_1, x_2, x_3) = f_{2|13}(x_2|x_1, x_3)f_{1|3}(x_1|x_3)f_3(x_3)$$

Using definition of conditional probability and equation 3.2, $f_{1|3}(x_1|x_3)$ can be written as:

$$f_{1|3}(x_1|x_3) = \frac{f_{1,3}(x_1,x_3)}{f_3(x_3)} \overset{3.2}{=} \frac{f_1(x_1)f_3(x_3)c_{13}(F_1(x_1),F_3(x_3))}{f_3(x_3)} = f_1(x_1)c_{13}(F_1(x_1),F_3(x_3))$$
(3.3)

Similarly,

$$
\begin{aligned}
f_{2|13}(x_2|x_1,x_3) &= \frac{f(x_1,x_2,x_3)}{f_{1,3}(x_1,x_3)} = \frac{f_{2,3|1}(x_2,x_3|x_1)f_1(x_1)}{f_{3|1}(x_3|x_1)f_1(x_1)} = \frac{f_{2,3|1}(x_2,x_3|x_1)}{f_{3|1}(x_3|x_1)} \\
&\overset{3.2}{=} \frac{f_{2|1}(x_2|x_1)f_{3|1}(x_3|x_1)c_{23|1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1);x_1)}{f_{3|1}(x_3|x_1)} \\
&= f_{2|1}(x_2|x_1)c_{23|1}(F(x_2|x_1),F_{3|1}(x_3|x_1);x_1) \\
&= f_2(x_2)c_{12}(F_1(x_1),F_2(x_2))c_{23|1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1);x_1)
\end{aligned}
$$

Although copula $c_{23|1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1);x_1)$ **depends** on conditioning variable x_1, for practical purposes we can use instead **simplified PCC**, which was shown by Hobæk Haff et al. (2010) to be a good approximation of PCC. For simplified PCC, we assume that pair copulas are independent of conditioning variables. Therefore, in this example simplified PCC would be:

$$
\begin{aligned}
f(x_1,x_2,x_3) =& f_{2|13}(x_2|x_1,x_3)f_{1|3}(x_1|x_3)f(x_3) \\
=& f_2(x_2)c_{12}(F_1(x_1),F_2(x_2))c_{23|1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1)) \\
& \cdot f_1(x_1)c_{13}(F_1(x_1),F_3(x_3))f_3(x_3) \\
=& f_1(x_1)f_2(x_2)f_3(x_3) && (marginals) \\
& \cdot c_{13}(F_1(x_1),F_3(x_3))c_{12}(F_1(x_1),F_2(x_2)) && (unconditional\, pairs) \\
& \cdot c_{23|1}(F_{2|1}(x_2|x_1),F_{3|1}(x_3|x_1)) && (conditional\, pair)
\end{aligned}
$$

This simplified PCC assumption is necessary for inference with PCC. This way we can have a flexible model, where pair copulas (in our case $c_{23|1}$, c_{13}, c_{12}) can be chosen separately. As a result, it is possible to model a variety of dependence structures, which also can be done in high dimensions applying the same strategy. Therefore, given any d dimensional density, we can decompose it into pair copula densities and marginal densities. It can be written as (Joe (1996)):

$$f(x_1,\ldots,x_d) = \prod_{j=1}^{d-1}\prod_{i=1}^{d-j} c_{i,(i+j)|(i+1),\ldots,(i+j-1)} \prod_{k=1}^{d} f_k(x_k)$$
(3.4)

However, the decomposition is not unique. For example, we could expand $f(x_1,x_2,x_3)$ to:

$$
\begin{aligned}
f(x_1,x_2,x_3) =& f_{1|23}(x_1|x_2,x_3)f_{2|3}(x_2|x_3)f_3(x_3) \\
=& f_1(x_1)f_1(x_2)f_1(x_3) && (marginals) \\
& \cdot c_{12}(F_1(x_1),F_2(x_2))c_{23}(F_1(x_1),F_3(x_3)) && (unconditional\, pairs) \\
& \cdot c_{13|2}(F_{1|2}(x_1|x_2),F_{3|2}(x_3|x_2)) && (conditional\, pair)
\end{aligned}
$$

Introduced by Bedford & Cooke (2001), regular vines can be used to organize simplified pair copula constructions. This method is called regular vine copula. In the next chapter we will present theoretical background for regular vine copulas.

4

Regular Vine Copulas

Regular vine copula is a simplified pair copula construction. In the previous chapter we saw that PCCs do not have a unique solution. Regular vine copula is a method to construct one solution of a simplified pair copula construction such that it would be as close as possible to real multivariate distribution that we aim to model and also capture the dependency structure between the variables in an efficient way.

4.1   Regular Vine

Regular vine is a concept known in graph theory and we will be using it to capture most important dependencies between variables that we want to model. We will introduce some theoretical background for regular vines, however, limiting graph theory to a level necessary for this thesis.

Definition 4.1.1. Let N be nodes and E be edges. Tree is a graph $T = (N, E)$ that is connected and has no cycles.

Table 4.1: Example: Trees
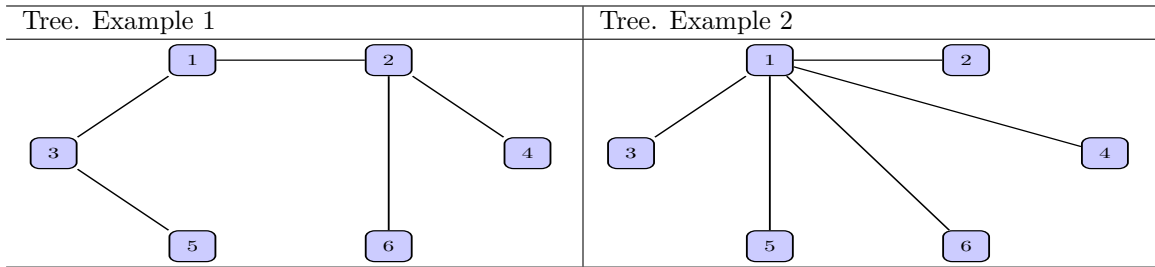
Table 4.1 shows how the examples of trees. In other words, trees are connected acyclic graphs.

Definition 4.1.2. A d-dimensional **regular vine** is a sequence of $d - 1$ linked trees with nodes N_i and edges E_i. $V = \{T_1, \ldots, T_{d-1}\}$

1. Tree T_1 has nodes $N_1 = \{1, \ldots, d\}$ and edges E_1.
2. For $i \geq 2$ tree T_i has nodes $N_i = E_{i-1}$.
3. For $i = 2, \ldots, d-1$ and $\{a, b\} \in E_i$ it must hold that $|a \cap b| = 1$ (Proximity condition)

Proximity condition means that if there is an edge in tree $T_i (where \quad i \geq 2)$ connecting two nodes then those two nodes are edges in tree T_{i-1} and they share a common node (note

that nodes in one tree are edges in the next tree). A 6-dimensional regular vine example is shown in Table 4.3.

4.2   Regular Vine Copula Specification

Regular vine copula can be fully described by vine structure matrix, the set of bivariate copulas, copula parameters and marginal distribution functions. More formally:

Definition 4.2.1. (Bedford & Cooke (2002)) **R-Vine copula specification** can be defined in the following way: (F, V, B) is an R-Vine copula specification if

1. $F = (F_1, \ldots, F_n)$ is a vector of continuous invertible distribution functions
2. V is an R-Vine on n elements
3. $B = \{B_e | i = 1, \ldots, n - 1; e \in E_i\}$ where B_e is a bivariate copula and E_i is the edge set of tree T_i of the R-Vine V.

Thus we have assigned a copula to every edge of the trees.

Structure selection will be described in section 4.2.1, a method how to select copulas - in 4.2.2 and copula parameters in section 4.2.3.

4.2.1   Structure Selection

The number of all possible r-vines increases dramatically in high dimensions. For example, in 6 dimensions there are 23,040 different regular vines and in 9 dimensions already more than 380 billion (number found in Gruber (2014)). Therefore, to be able to work in higher dimensions we rather trust heuristic methods used in Graph Theory to determine the structure of r-vine.

The r-vine of our interest is the one that captures the dependency structure most accurately. The main idea when finding r-vine is to prioritise strongest dependencies in the first trees. We use this method, because copulas specified in first tree often have the greatest influence and dependence tends to be the strongest in Tree 1 (Dißmann (2010), p. 50). For the precision of the model, the strongest dependencies are typically the most important and vice versa (copula distribution functions for parameters close to independence are very similar). This type of modelling has drawbacks, for example the solution is not necessarily global optimum, because each tree is analysed separately. This stepwise tree-by tree inference is a *sequential method.* However, it is a computationally fast and effective method comparing to alternatives. More about alternative ways to model with regular vine copulas can be found in Gruber (2014), where the author reviewed different regular vine copula application examples and discussed the advantages and disadvantages of each.

We are using Kendall's τ to measure the dependence and solving the following optimization problem for each tree in order to find a so called **maximum spanning tree** (a tree that maximizes cumulative pairwise dependencies, see example in Table 4.2).

$$ max \sum_{\text{edges } e_{ij} \text{ in spanning tree}} |\hat{\tau}_{ij}| \tag{4.1} $$

where the weights of edges are the absolute values of pairwise Kendall's τ $\hat{\tau}_{ij}$. See illustrative example in Table 4.2 of finding a maximum spanning tree and see further example of how to

find Regular Vine in Table 4.3

Table 4.2: Example: Finding Maximum Spanning tree

Graph	Explanation
	Choose arbitrary node. Say we choose node 1. Connect node 1 to one of the remaining nodes, based on (absolute) maximum Kendall's τ value. In this case, it is node 3.
	We are looking for an edge with an (absolute) maximum Kendall's τ value connecting one of the remaining nodes (2,4,5,6) to either node 3 or 1 that were chosen in previous step. It is an edge connecting nodes 2 and 3. We can now remove edge between 1 and 2, as tree should not have closed loop according to definition.
	Similarly, we are looking for an edge with an (absolute) maximum Kendall's τ value connecting one of the remaining nodes (4,5,6) to either 1, 2 or 3.
	As before, we are looking for an edge with an (absolute) maximum Kendall's τ value connecting 4 or 5 to one of the nodes 1, 2, 3 or 6.
	After connecting the last remaining node to a graph, we have found a maximum spanning tree.

The maximum spanning tree that we found encodes all largest dependencies based on kendall's τ dependency measure. This tree will be the first tree in the regular vine and all the remaining trees will depend on the structure of this first tree. Please see in Table 4.3 how to obtain further trees of the regular vine.
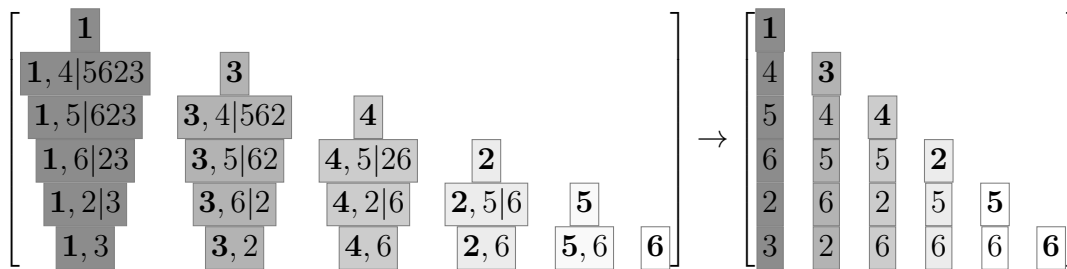
Table 4.3: Example. Finding Regular Vine

	Trees	Explanation
1	1 —1,3— 3 —2,3— 2 —2,6— 6 —4,6— 4; 6 —5,6— 5	This is Maximum Spanning Tree(see example 4.2). To find the remaining trees of R-vine, we give every edge a name based on neighbouring nodes. Names of edges will be nodes in tree 2.
2	1,3 —1,2|3— 2,3 —3,6|2— 2,6 —2,4|6— 4,6; 2,6 —2,5|6— 5,6	Edges are named depending on neighbouring nodes. We connect nodes (1,3), (2,3), (2,6). The remaining two nodes (5,6) and (4,6) should be connected to graph depending on their corresponding Kendall's τ values with an equivalent method we used to find maximum spanning tree (example is shown in table 4.2). The resulting tree is shown here.
3	1,2|3 —1,6|2,3— 3,6|2 —3,5|2,6— 2,5|6 —4,5|2,6— 2,4|6	We connect nodes (1,2|3), (3,6|2). The remaining two nodes (2,5|6) and (2,4|6) should be connected depending on Kendall's τ values(see example 4.2). The resulting tree is shown here.
4	1,6|2,3 —1,5|2,3,6— 3,5|2,6 —3,4|2,5,6— 4,5|2,6	Edges in tree 3 are nodes in tree 4. Nodes in Tree 4 are connected in the same sequence as they were in tree 3.
5	1,5|2,3,6 —1,4|2,3,5,6— 3,4|2,5,6	In 6 dimensions there are 5 trees. This is a final tree.

From the structure of regular vine, we can obtain regular vine structure matrix. In order to determine structure matrix, we start looking at tree 5. See Table 4.4 and further explanation on how to obtain structure matrix.

Table 4.4: Example: Finding R-Vine Structure Matrix

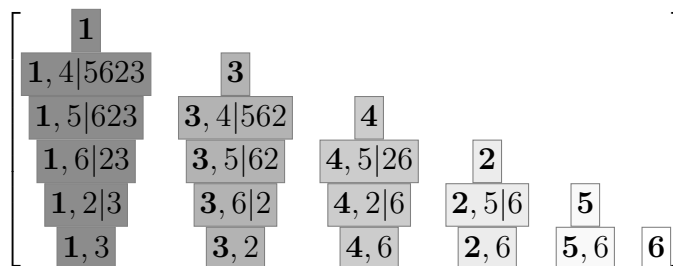|	Trees	Explanation
5		We are looking at edges at each tree in R-vine. We start at looking at edge in the last tree (1,4|2,3,4,6). We choose one of the conditioned variables 1 or 4. It does not matter whether we choose 1 or 4, because structure matrix is not unique. Suppose we choose 1. Then we look at further trees and identify edges where *one of the conditioned variables is 1* and mark them. (All those edges are marked in the same dark gray shade).
4		In previous step we already marked the edge (1,5|2,3,6), since it has conditioned variable 1. In this step we should look at the remaining edge (3,4|2,5,6). We randomly choose one of the conditioned variables 3 or 4. Suppose we choose 3. As before, we note edges in the remaining trees that has conditioned variable 3.
3		Similarly, we identify the remaining edge 4,5|2,6. Suppose we choose 4 and mark edges in remaining trees that has conditioned variable 4.
2		Again, we identify the remaining edge 2,5|6. Suppose we choose 2 and mark edges in remaining trees that has conditioned variable 2.
1		The only edge not marked in the previous steps is (5,6). Suppose we choose 5.

Furthermore, we will note edges in a structure matrix in the order of chosen conditioned variables in Table 4.4. Columns will encode in order in which we chose conditioned variables and rows will encode the Tree. For instance, Column 1 will encode the edges with conditioned variable chosen in Tree 5; remember, we chose variable 1. We note 1 in a first diagonal entry. The next step is to encode the edges that had conditioned variable 1, namely $(1, 4|5, 6, 2, 3)$; $(1, 5|6, 2, 3)$; $(1, 6|2, 3)$; $(1, 2|3)$; $(1, 3)$ in the order from Tree 5 to Tree 1. The matrices shown below illustrates the procedure described. We will not use the matrix on the left hand side, as it is just a visualisation of how we encode r-vine structure.

$$
\begin{bmatrix}
\mathbf{1} & & & & & \\
\mathbf{1},4|5623 & \mathbf{3} & & & & \\
\mathbf{1},5|623 & \mathbf{3},4|562 & \mathbf{4} & & & \\
\mathbf{1},6|23 & \mathbf{3},5|62 & \mathbf{4},5|26 & \mathbf{2} & & \\
\mathbf{1},2|3 & \mathbf{3},6|2 & \mathbf{4},2|6 & \mathbf{2},5|6 & \mathbf{5} & \\
\mathbf{1},3 & \mathbf{3},2 & \mathbf{4},6 & \mathbf{2},6 & \mathbf{5},6 & \mathbf{6}
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\mathbf{1} & & & & & \\
4 & \mathbf{3} & & & & \\
5 & 4 & \mathbf{4} & & & \\
6 & 5 & 5 & \mathbf{2} & & \\
2 & 6 & 2 & 5 & \mathbf{5} & \\
3 & 2 & 6 & 6 & 6 & \mathbf{6}
\end{bmatrix}
$$

The matrix on the right hand side is a **structure matrix** that represents the regular vine structure. Structure matrix is not unique. For instance, if in Table 4.4 in the first step we would choose variable 4 instead of 1, the matrix would be different, however, it would still encode the same regular vine structure.

4.2.2 Copula Selection

After we have determined the r-vine structure, we can fit pair copulas to all edges in r-vine (edges represent conditional and unconditional variable pairs). In the matrix below we wrote down all the edges that we had in the r - vine in the example from Table 4.3.

$$
\begin{bmatrix}
\mathbf{1} & & & & & \\
\mathbf{1},4|5623 & \mathbf{3} & & & & \\
\mathbf{1},5|623 & \mathbf{3},4|562 & \mathbf{4} & & & \\
\mathbf{1},6|23 & \mathbf{3},5|62 & \mathbf{4},5|26 & \mathbf{2} & & \\
\mathbf{1},2|3 & \mathbf{3},6|2 & \mathbf{4},2|6 & \mathbf{2},5|6 & \mathbf{5} & \\
\mathbf{1},3 & \mathbf{3},2 & \mathbf{4},6 & \mathbf{2},6 & \mathbf{5},6 & \mathbf{6}
\end{bmatrix}
$$

For edges in tree $T_k \in V$ we can select bivariate copulas B_k. For instance, (in Table 4.3) Tree 4 has two edges: $\{(1,5|623)$ and $(3,4|562)\}$. For each of those edges we will fit bivariate copulas $c_{1,5|6,2,3}(F_1(x_1|x_6, x_2, x_3), F_5(x_5|x_6, x_2, x_3))$ and $c_{3,4|5,6,2}(F_3(x_3|x_5, x_6, x_2), F_4(x_4|x_5, x_6, x_2))$. For determining copula family we can use AIC (Akaike Information Criterion).

Definition 4.2.2. (Akaike (1973)) Let L be the maximum likelihood function value and k be the number of parameters in the model. AIC is defined as:

$$
\text{AIC} = -2\ln(L) + 2k
$$

We choose model with minimum value of AIC. This model selection method rewards goodness of fit of a model and penalises increasing the number of parameters. Another possibility is to use Bayesian Information Criterion (BIC) instead of AIC. Both AIC and BIC are using maximum likelihood, however, AIC depends on sample size and BIC does not. There is also a question if likelihood based model selection methods appropriately take into account tail dependence. The problem with maximum likelihood is that it mostly fits the distribution in the "middle" and its tail has little impact. On the other hand, there seems to be no better alternative than AIC (Brechmann (2010), Section 5.4) and it is deemed to be more practical than BIC (Burnham & Anderson (2004)). Hence we used the following approach:

- Estimate parameters for each copula family using bivariate maximum likelihood estimation.
- Compute AIC. AIC of a bivariate copula family c with parameter or parameters θ is defined as (Schepsmeier et al. (2015)):

$$\text{AIC} = -2\ln(\prod_{i=1}^{N} c(u_{i,1}, u_{i,2}|\theta)) + 2k$$

  where $(u_{i,1}, u_{i,2})$ are observations, $k = 1$ for pair copulas with one parameter (e.g. Gaussian, Clayton etc.) and $k = 2$ for pair copulas with two parameters (e.g. t, BB1, etc.)
- Choose family with minimum AIC value

After choosing the best fitting copula families for the conditional and unconditional variable pairs determined by the edges in r-vine, we can proceed estimating the parameters of the copulas.

4.2.3 Parameter Estimation

Gruber (2014) reviews the literature for regular vine dependence modelling, where different modelling choices are compared in terms of their accuracy and computational time. Bayesian approach can be used to simultaneously select a structure and parameters of the regular vine copula, instead of step-wise procedure. We do not choose Bayesian approach due to the long computational time. Although Bayesian approach can be more accurate, stepwise model selection and maximum likelihood estimation of the parameters is much faster. For a fully Bayesian model selection it takes two hours to select a model(for 500 observations in 6 dimensions), while step-wise frequentist approach takes several seconds. Time consideration is especially important for backtesting procedure in model validation stage, where model will have to be re-tested several times.

Moreover, after we specified the regular vine, we can obtain copula density, which is a product of all bivariate copulas selected for edges in r-vine. In our example, r-vine copula density would be:

$$
\begin{aligned}
c(F_1(x_1)\ldots F_n(x_n)) =& c_{1,3}c_{3,2}c_{4,6}c_{2,6}c_{5,6} && \text{(Tree 1)}\\
& c_{1,2|3}c_{3,6|2}c_{4,2|6}c_{2,5|6} && \text{(Tree 2)}\\
& c_{1,6|23}c_{3,5|62}c_{4,5|26} && \text{(Tree 3)}\\
& c_{1,5|623}c_{3,4|562} && \text{(Tree 4)}\\
& c_{1,4|5623} && \text{(Tree 5)}
\end{aligned}
$$

where we denote $c_{1,3}$ as $c_{1,3}(F_1(x_1), F_3(x_3))$, or $c_{1,2|3}$ as $c_{1,2|3}(F_{1|3}(x_1|x_3), F_{2|3}(x_2|x_3))$ etc. Generally, we can write:

Theorem 4.2.1. R-vine copula density is uniquely determined as:

$$c(F_1(x_1), \ldots, F_d(x_d)) = \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)}(F(x_j(e)|x_D(e)), F(x_k(e)|x_D(e))) \qquad (4.2)$$

where $e = j(e), k(e)|D(e)$ are edges in E_i; $j(e), k(e)$ are conditioned nodes; $D(e)$ is the conditioning set; $c_{j(e),k(e)|D(e)}$ is bivariate copula density (Bedford & Cooke (2001), Brechmann & Joe (2015))

5

Marginal Time Series

Copulas are static models and we cannot model seasonality or similar trends solely through their use. That is why data should be independent and identically distributed for modelling with copulas. Log-returns have periods of higher variance and are time dependant. We have to remove the seasonality, trends and dependencies from the return series. For that we can use ARMA-GARCH model to obtain independent and identically distributed residuals and proceed modelling those residuals with copulas.

5.1 ARMA(m,n)-GARCH(p,q) Model

We can describe the return series as a process:

$$r_t = \mu_t + \varepsilon_t$$

where r_t is log return at time t, μ_t is expected value of conditional r_t, ε_t is a white noise series. Furthermore, μ_t can be treated as a constant, or modelled with ARMA model. ARMA model contains two smaller models: Autoregressive AR(m) model is written on the left hand side of the equation 5.1 and Moving Average MA(n) is on the right hand side of 5.1. With ARMA(m,n) model r_t has the following expression (Tsay (2010), p. 56):

$$r_t - \sum_{i=1}^{m} a_i r_{t-i} = c + \varepsilon_t - \sum_{i=1}^{n} b_i \varepsilon_{t-i} \tag{5.1}$$

$$r_t = c + \sum_{i=1}^{m} a_i r_{t-i} + \varepsilon_t - \sum_{i=1}^{n} b_i \tag{5.2}$$

where c is some constant, $\{a_1, \ldots, a_m\}$ are AR(m) parameters, $\{b_1, \ldots, b_n\}$ are MA(n) parameters.

Robert Fry Engle developed ARCH model in 1982 (Engle (1982)). Tim Bollerslev proposed an extension to ARCH model, which is called the generalized ARCH (GARCH) model (Bollerslev (1986)). GARCH (p,q) is used for modelling volatility σ_t. It is expressed as (Tsay (2010), p.114):

$$\varepsilon_t = \sigma_t z_t \quad \text{where } Z_t \sim D(0,1) \tag{5.3}$$

$$\sigma_t^2 = w + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2 \tag{5.4}$$

where $w, \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ are parameters of GARCH model, z_t are i.i.d standardized random variables and $D(0, 1)$ is some distribution with zero mean and unit variance. That distribution can be determined analysing quantile-quantile plots. The common choices include Standard Normal distribution, Skewed Normal, Student - t, Skewed Student t etc. After extracting the residuals from ARMA-GARCH model, we can perform Box Ljung test to assess whether the residuals are independent and identically distributed. If this is true, then we can proceed modelling with r-vine copulas.

ARMA-GARCH model lags can be chosen according to AIC criterion (Brockwell & Davis (1991)). We estimate AIC for all possible combinations of ARMA GARCH lags $m, n, p, q \leq 2$ and select the best model.

5.2   Box Ljung Test

Box Ljung statistic Q can be calculated with the following formula (Ljung & Box (1978)):

$$Q = n\,(n+2) \sum_{i=1}^{h} \frac{\hat{\rho}_i^2}{n-i}$$

where n is sample size, $\hat{\rho}_i$ is the sample autocorrelation at lag i and h is the number of lags tested. Under the null hypothesis statistic Q is $\chi^2_{(h)}$ distributed. We would reject null hypothesis (that data is randomly distributed) at some chosen significance level α if $Q > \chi^2_{1-\alpha,h}$, where $\chi^2_{1-\alpha,h}$ is the α-quantile of the chi-squared distribution with h is degrees of freedom.

6

Model Implementation

In this section we will show how to implement a model using real data.

6.1 Data Description

The presented theory will be tested with data consisting of commodities (Gold and Oil), foreign exchange rates (United States Dollar to Euro), equity (Euro Stoxx 50 equity index) and bonds (European Union Government and Corporate Bond Indexes).
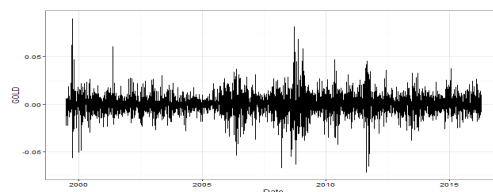
Gold

Date available from: 1999-01-04.
Source: Datastream (2016)
Unit: Euros per Troy Ounce - official price.
Abbreviation: GOLD

Daily Prices are set in US dollars per fine troy ounce. Euro prices are indicative prices for settlement only (The London Bullion Market Association (2016)).

Oil

Date available from: 1970-01-30
Source: Datastream (2016)
Unit: Price of Barrel Crude Oil in USD
Abbreviation: OIL
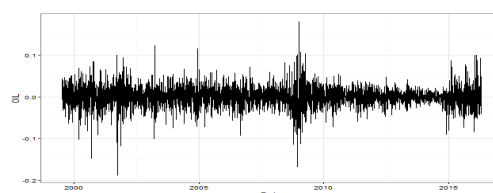
Foreign Exchange Rates

Date available from: 1957-12-31.
Source: Datastream (2016)
Unit: United States Dollar to Euro exchange rate
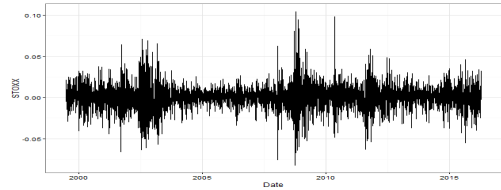**Abbreviation:** USEUR
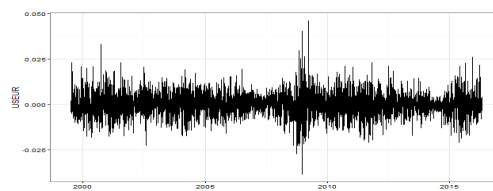
Equity

Date available from: 1997-06-30.
Source: Datastream (2016)
Unit: Price Index
Abbreviation: STOXX

EURO Stoxx 50 index was chosen to account for the equity price movements. The index consists of 50 stocks from 12 countries in Eurozone. Since the index contains several companies, the price movements are not specific to one sector, but to overall businesses in Eurozone. It reflects the biggest and most liquid stocks.
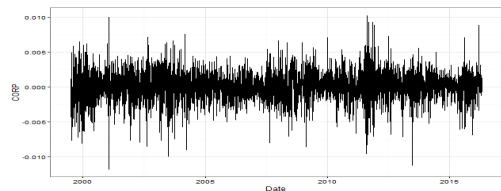
Euro Corporate Bond Index, 5-7 years maturity

Date available from: 1999-07-02
Source: Datastream (2016)
Unit: Clean Price Index
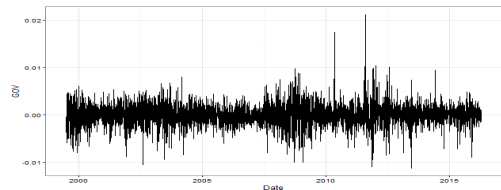**Abbreviation:** CORP

Euro Government Bond Index, 5-7 year maturity

Date available from: 1998-05-01
Source: Datastream (2016)
Unit: Clean Price Index
Abbreviation: GOV

Clean price is a more accurate measure of true index value. It does not depend on coupon dates, while dirty price depends on it. Clean price can change due to changing interest rates and/or bond issuer's credit rating.

Furthermore, we analyse data from 1999-07-02 until the 2016-04-20, which is 4343 trading days. We will estimate the model for the first 1000 days, which is a period from 1999-07-02 to 2003-05-20 and then backtest it on the remaining days.

6.2 Backtesting

.

Measuring the risk can be viewed as a statistical issue. We are using historical data in order to estimate the the parameters of r-vine copula model and the performance of the model will also depend on how close the estimated parameters are to their true values. We can predict risk measure VaR_α for different α values and use backtesting procedure to analyse if the model predictions are accurate. If the forecast of VaR_α is accurate, events where returns are smaller than VaR_α values should occur independently and proportion of such events should be roughly equal to α. We will determine the portfolio dependence structure using r-vine and assume that the structure does not change for the following day.

Backtesting Procedure

1. Convert time series to daily log returns

$$r_{t,j} = \ln\left(\frac{S_{t,j}}{S_{t-1,j}}\right)$$

where S_t is price of an asset at day t in $[1, 4343]$ and $j = \{1, \ldots, 6\}$ determines each of the six assets. We can see marginal distributions in Figure 6.1. It can be expected that bond indexes are less volatile than the remaining variables. Moreover, we get a total of 4342 log returns.
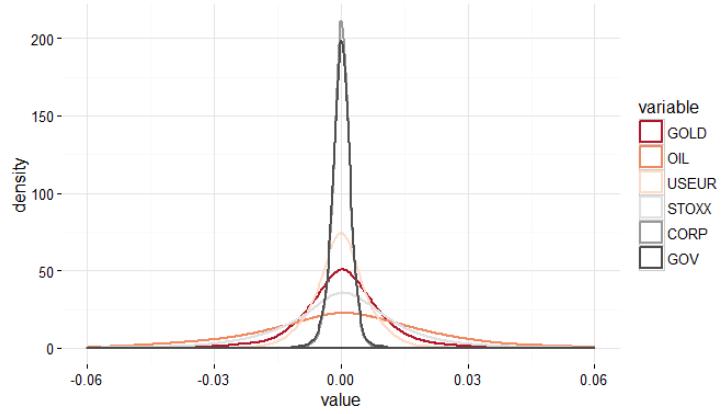

Figure 6.1: Distribution of log-returns

2. Fit univariate ARMA(p,q) GARCH(m,n) models separately for $r_{t,1}, \ldots, r_{t,6}$, where $t \in [1, 1000]$. Lags p, q, m, n can be determined using AIC criterion. Moreover, we use appropriate standardized residuals distribution, which we can determine by investigating quantile-quantile plots and estimate the optimal parameters of ARMA(m,n) GARCH(p,q) model. The results can be seen in Table 6.2. Using the pre-determined parameters we convert the remaining log returns with equations given below. For t in $[1001, 4342]$ and $j = \{1, \ldots, 6\}$ we apply the following:

$$r_{t,j} = c_j + \sum_{i=1}^{m} a_{i,j} r_{t-i,j} + \varepsilon_{t,j} - \sum_{i=1}^{n} b_{i,j} \varepsilon_{t-i,j}$$

$$\varepsilon_{t,j} = \sigma_{t,j} z_{t,j} \quad \text{where } \mathbf{z}_j \sim D(0,1)$$

$$\sigma_{t,j}^2 = w_j + \sum_{i=1}^{p} \alpha_{i,j} \varepsilon_{t-i,j}^2 + \sum_{i=1}^{q} \beta_{i,j} \sigma_{t-i,j}^2$$

3. Convert standardized residuals $z_{t,1}, \ldots, z_{t,6}$ to standard uniform variables $u_{t,1}, \ldots, u_{t,6}$ with probability integral transformation. Using the fitted marginal cumulative distribution

functions $F_1, \ldots, F_6$ that are described in Table 6.2, for t in $[1, 4342]$ we apply the following equation:

$$u_{t,j} = F_j(z_{t,j})$$

Backtest with constant models

4. Fit an r-vine copula model for $u_{t,1}, \ldots, u_{t,6}$ using model selection procedure described for r-vines for t in [1:1000]. Using the regular vine structure determined, we want to predict a distribution for days in the interval [1001,4342].

5. Simulate **one** sample $\mathbf{u}_1^{sim}, \ldots, \mathbf{u}_6^{sim}$ using fitted r-vine copula model in the previous step, where number of simulations $sim = 10^5$. This one sample is the same for all days in [1001,4342] and in Step 6 and onwards is equivalent to $\mathbf{u}_{t+1,1}^{sim}, \ldots, \mathbf{u}_{t+1,6}^{sim}$. For multivariate student t copula we simulate a similar sample.

Backtest with re-fitted models

4. Fit an r-vine copula model for $\mathbf{u}_{(t-500):t,1}, \ldots, \mathbf{u}_{(t-500):t,6}$ using model selection procedure described for r-vines, where $(t - 500) : t$ denotes a rolling window of 500 days. Using the regular vine structure determined up to time t, we want to predict a distribution at time $(t + 1)$ in [1001,4342].

5. Simulate sample $\mathbf{u}_{t+1,1}^{sim}, \ldots, \mathbf{u}_{t+1,6}^{sim}$ using fitted r-vine copula model in the previous step, where number of simulations $sim = 3000$. Repeat steps 4 and 5 for all days in chosen backtesting period $(t + 1)$ in [1001,4342]. Similarly, we can fit multivariate t copula for comparison with regular vine copula and simulate a similar sample.

6. Backtranform uniform variables $\mathbf{u}_{t+1,1}^{sim}, \ldots, \mathbf{u}_{t+1,6}^{sim}$ to standardized residuals using the inverse of cumulative distribution function.

$$\mathbf{z}_{t+1,j}^{sim} = F_j^{-1}(\mathbf{u}_{t+1,j}^{sim})$$

7. Use ARMA GARCH model coefficients to forecast one day ahead for each asset.

$$\sigma_{t+1}^2 = w + \alpha_1 \varepsilon_t^2 + \beta_1 \sigma_t^2$$
$$\boldsymbol{\varepsilon}_{t+1}^* = \sigma_{t+1}\mathbf{z}_{t+1}^{sim} \quad \text{where } \mathbf{z}_{t+1}^{sim} \sim D(0,1) \text{ is calculated in Step 6}$$
$$\mathbf{r}_{t+1}^* = c + \boldsymbol{\varepsilon}_{t+1}^* + a_1 r_t - b_1 \varepsilon_t$$

8. For assets $j = 1, \ldots, 6$ portfolio return at time $t + 1$ is:

$$\mathbf{X}_{t+1} = \sum_{j=1}^{6} \mathbf{r}_{t+1,j}^*$$

$\mathbf{X}_{t+1}$ is a vector. We have as many observations for $\mathbf{X}_{t+1}$ as the number of simulations.

9. Compute $\text{VaR}_\alpha(\mathbf{X}_{t+1})$

10. Compare the values of $\text{VaR}_\alpha(\mathbf{X}_{t+1})$ to returns r_{t+1} for each day $(t+1)$ in [1001,4342]. To test whether VaR prediction is good we can use Kupiec's Proportion of Failures Test and Christoffersen's Independence Test. Kupiec's Proportion of failures test allows us to evaluate

whether number of returns that exceeded VaR is expected (Kupiec (1995)). If the proportion of exceedances differs significantly from $\alpha \times 100\%$ then we can suspect that model might not be accurate. Moreover, Christoffersen's Independence Test evaluates if those exceedances occurred independently from one another (Christoffersen (1998)). Proportion of failure and independence tests can be joined together in order to examine the accuracy of VaR model. We will test two null hypotheses: firstly that the exceedances are independent and secondly that the exceedances are both independent and identically distributed.

6.3 R Commander Implementation

We have implemented the model in R commander. Here we describe the packages that we used to obtain the results.

- **ggplot2** is a package that provides tools to visualise data (Wickham (2009))
- **VineCopula** package contains algorithms related to regular vine copulas. We used function RVineStructureSelect() that specifies family, parameter and structure matrices of regular vine copula. Moreover, we used RVineSim() to simulate samples with wanted regular vine specification (Schepsmeier (2015))
- **rugarch** this package implements ARMA-GARCH methods and finds the parameters, we used functions ugarchspec() in order to specify the needed model and ugarchfit() to find the model parameters and other details. Function ugarchfit() also estimates p-values for those parameters, checks for convergence in distribution and provides useful plots, such as quantile-quantile plots or autocorrelation functions. Moreover, we used function VaRtest() that implements Kupiec and Christoffersen coverage tests (Ghalanos (2015))
- **GEVStableGarch** gsSelect() function was used to find ARMA GARCH lags based on AIC (Sousa et al. (2015))
- **QRM** This package contains Frank, Gumbel, Clayton, Normal and Student t copulas. It implements the fitting of the copulas. In case of Student t copula, we can choose to calculate dependence either with Kendall's τ or Spearman's ρ and the function fit.tcopula() maximizes log likelihood for the degrees of freedom parameter. Furthermore, we simulated samples with fitted student t copula parameters using function rcopula.t() (Bernhard & McNeil (2016))

6.4 ARMA GARCH Analysis of Time Series

We have fitted the ARMA GARCH model in order to obtain i.i.d time series. We will demonstrate an example for one of the assets in our chosen portfolio how this was done. We have chosen to analyse gold.

First of all, we had to identify the lags of ARMA(p,q) GARCH(m, n) model. This was done by evaluating AIC for all combinations of lags up to order $p, q, m, n \leq 2$. We found that order $p = 0, q = 1, m = 1, n = 1$ had the AIC $= -6.65$ and this was the best model. Moreover, we have fitted the ARMA(0,1) GARCH(1,1) model and obtained the following parameters:

Table 6.1: ARMA GARCH parameter estimates for gold

	Estimate	Std. Error	t value	Pr($>$|t|)
a_1	-0.085548	0.030997	-2.7599	0.005783
ω	0.000006	0.000001	7.2563	0.000000
α_1	0.081146	0.009658	8.4020	0.000000
β_1	0.859413	0.019408	44.28	0.000000
ν	4.159774	0.489894	8.4912	0.000000

The p-values in Table 6.1 are all significant (for significance level of 0.05), meaning that with 95% confidence the parameters are non zero. Furthermore, we have tested standardized residuals for serial correlation with Ljung-Box Test. For time lag of one day the test gave p-value $p = 0.516$, which means that we cannot reject the null hypothesis (our null hypothesis is that there is no serial correlation). Furthermore, we have tested different time lags and it shows a similar result.
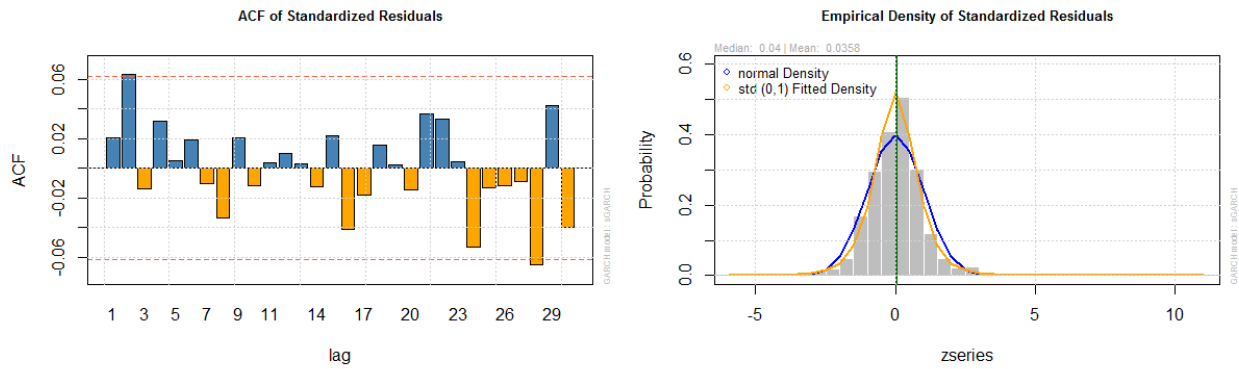
Figure 6.2: Gold auto-correlation function and fitted density plots

Serial correlation can also be observed with auto-correlation function. Auto-correlation function tests whether there are patterns in data. In Figure 6.2 we can observe that autocorrelation function twice exceeds the limits of significance, however, that can happen randomly when testing many lags. Moreover, in Figure 6.2 we can also notice that student t distribution seems to be a better fit than normal distribution.
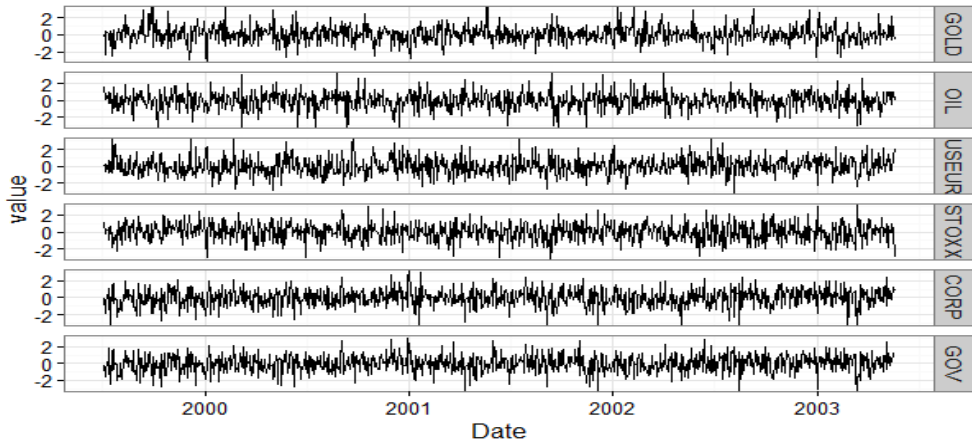
Figure 6.3: Standardized Residuals, 1000 days

Similarly, ARMA-GARCH model was fitted to oil, USD-Euro exchange rate, equity index, corporate and government bonds. Standardized residuals for the first 1000 days that are plotted in Figure 6.3. We can observe that there does not seem to be any volatility clustering, or moving average. We tested if we obtained independent identically distributed variables using Box Ljung test. The Box-Ljung test results, fitted distributions, quantile-quantile plots and ACF functions for all analysed variables are summarized in Table 6.2
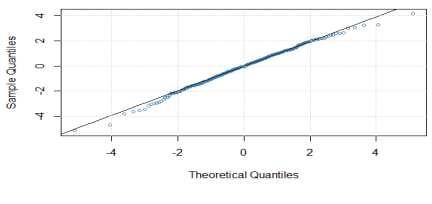

(a) Residual distribution

(b) Standardized Residual distribution
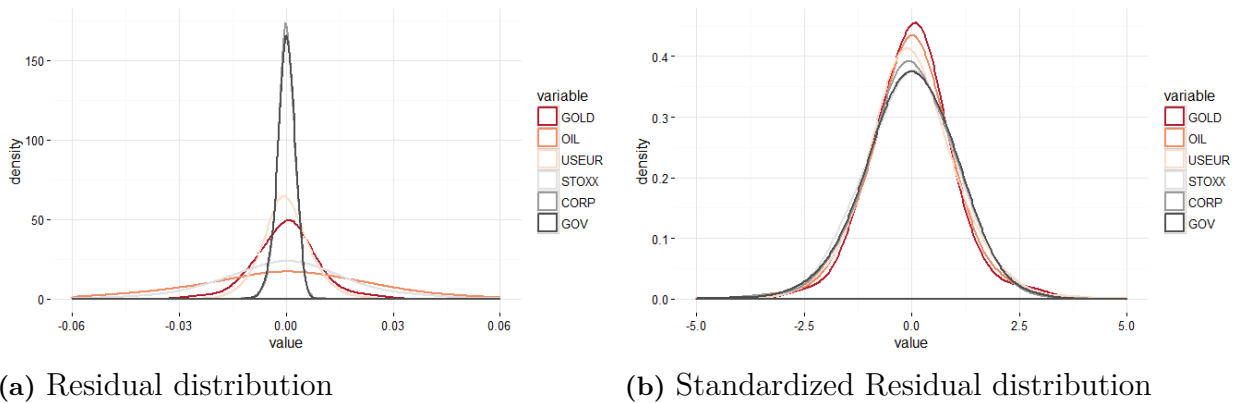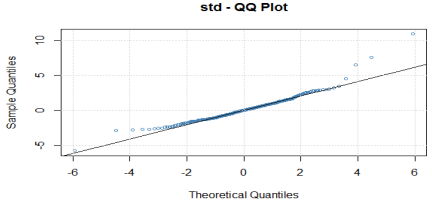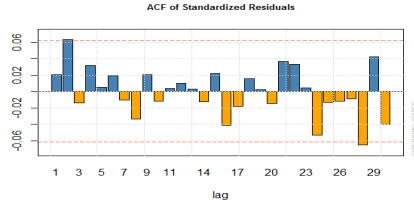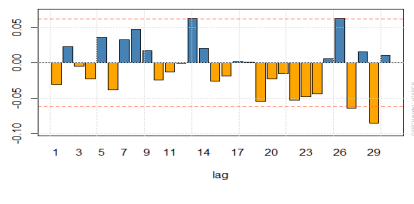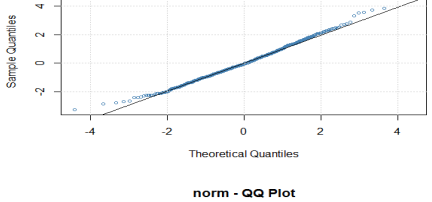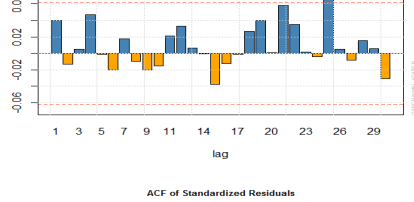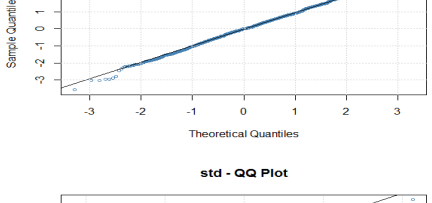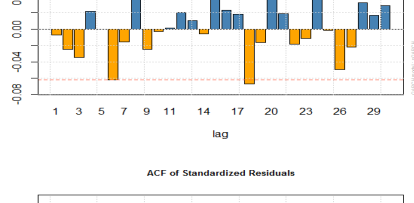
Figure 6.4: ARMA-GARCH residual distribution

In Figure 6.4 we see distributions of the residuals for each instrument. Residual distribution in some cases is identical to log return distribution of each asset. This is due to the fact that for most assets we had ARMA(0,0) model (except for gold), therefore $r_t = \varepsilon_t$.
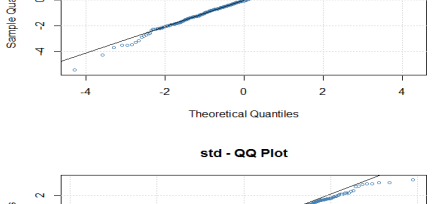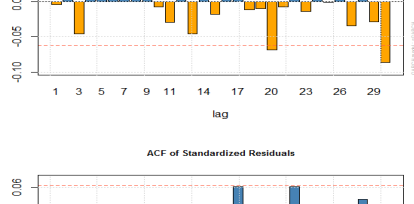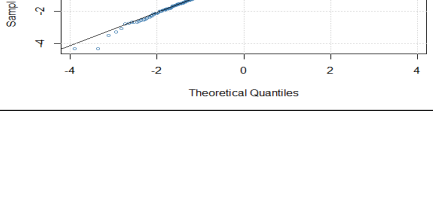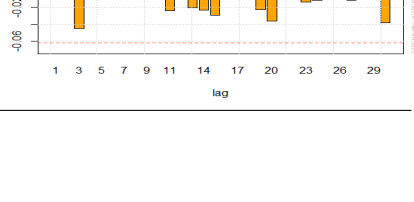
Table 6.2: ACFs, Q-Q plots and description of marginal time series

Description	Quantile-Quantile plot	ACF of standardized residuals
Gold ARMA(1,0)-GARCH(1,1) Distribution: $t_{\nu=4.16}$ Box Ljung $p = 0.51$		
Oil ARMA(0,0)-GARCH(1,1) Distribution: $t_{\nu=5.4}$ Box Ljung $p = 0.33$		
Useur ARMA(0,0)-GARCH(1,1) Distribution: $t_{\nu=7.8}$ Box Ljung $p = 0.21$		
Stoxx ARMA(0,0)-GARCH(2,1) Distribution: $N(0,1)$ Box Ljung $p = 0.81$		
Corp ARMA(0,0)-GARCH(1,1) Distribution: $t_{\nu=8.6}$ Box Ljung $p = 0.89$		
Gov ARMA(0,0)-GARCH(1,1) Distribution: $t_{\nu=12.8}$ Box Ljung $p = 0.36$		

7
Results

7.1 Regular Vine Specification

We have fitted ARMA GARCH parameters for marginal time series and transformed observations to uniform variables as described in Section 6.2 for the first 1000 days. Moreover, we have found regular vine structure and fitted the pair copulas as described in Chapter 4. Regular copula specification is provided in the Table below 7.1.

Table 7.1: R-vine copula specification for 1000 days

Structure						Family				
1						0				
5	4					0	1			
2	5	3				0	0	0		
6	2	5	2			2	2	0	2	
4	6	2	5	5		2	2	0	2	
3	3	6	6	6	6	34	2	2	0	2

Parameter 1					Parameter 2				
−					−				
−	0.12				−	−			
−	−	−			−	−	−		
−0.08	−0.25	−	0.46		13.47	8.4	−	19.6	
−1.18	−0.34	0.36	−	0.92	−	10.13	6.64	−	3.8

Where in family matrix the integers define the following pair copulas:
0 - Independence copula
1 - Gaussian Copula
2 - Student t copula
34 - rotated Gumbel copula (270 degrees). Rotations of 90 and 270 degrees allow to model the negative dependence, thus the corresponding parameter is negative (Brechmann & Czado (2013), p.8).

Equivalently, we fitted the multivariate student t copula based on the first 1000 days. The estimated degrees of freedom $\nu = 1.67$ and the correlation matrix ρ is shown below:

$$\rho = \begin{bmatrix} -0.01 & & & & \\ -0.26 & -0.01 & & & \\ 0.02 & -0.03 & -0.35 & & \\ -0.07 & -0.01 & 0.34 & -0.26 & \\ -0.07 & -0.03 & 0.36 & -0.33 & 0.92 \end{bmatrix}$$

7.2 Model Consistency

We have simulated a sample of size 4342 (same size as the real data) of uniform variables $\mathbf{u}_1, \ldots, \mathbf{u}_6$ with a regular vine structure given in Table 7.1. Moreover, we transformed $\mathbf{u}_1, \ldots, \mathbf{u}_6$ to log returns using the ARMA GARCH parameters that were estimated for the first 1000 observations and we have kept the real data volatility. This way we have obtained the data with constant regular vine structure, constant GARCH parameters and constant standardized residual distribution. We applied the backtesting procedure in order to investigate if the model was consistent.
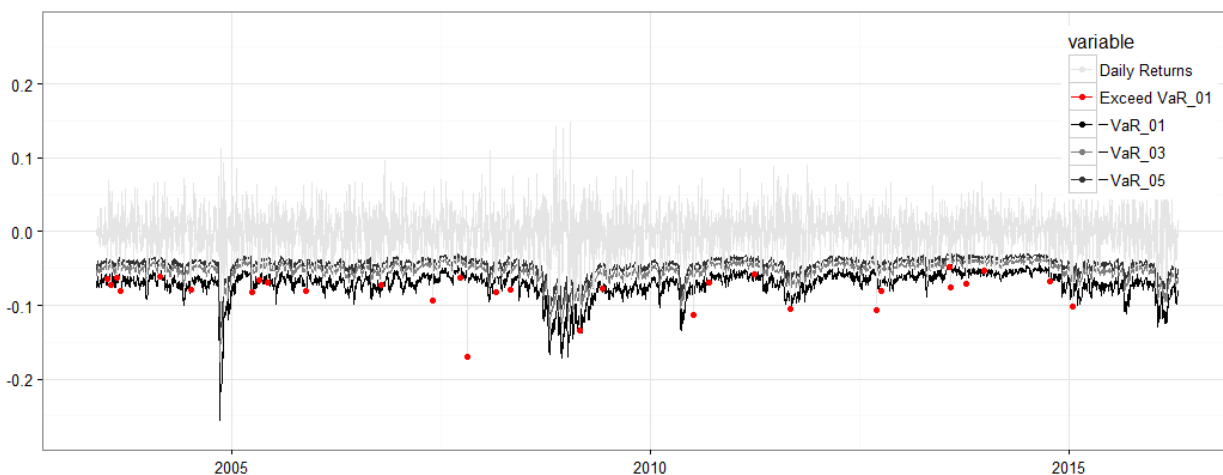

Figure 7.1: Backtest: model consistency

We tested VaR with Kupiec and Christoffersen coverage tests in order to determine if the simulated data exceeded VaR independently and in a right proportion. The results are summarized below in Table 7.2.

Table 7.2: VaR test, model consistency

VaR_α	Expected exceedances	Actual exceedances	H0 "Correct Exceedances"	H0 "Correct Exceedances & Indep"
$VaR_{0.05}$	167	185	$p = 0.16$	$p = 0.32$
$VaR_{0.03}$	100	106	$p = 0.56$	$p = 0.58$
$VaR_{0.01}$	33	30	$p = 0.55$	NA

For $\alpha = \{0.05, 0.03, 0.01\}$ tests' results show that we cannot reject the hypotheses that exceedances are independent and that the proportion of exceedances is correct. Thus we can confirm that the model is consistent.

7.3 Serial Correlation of Standardized Residuals

We have calculated standardized residuals using ARMA GARCH parameters that we obtained from the first thousand days. After some time we can expect that parameters need to be updated. Standardized residuals for the full data-set are shown in Figure 7.2.
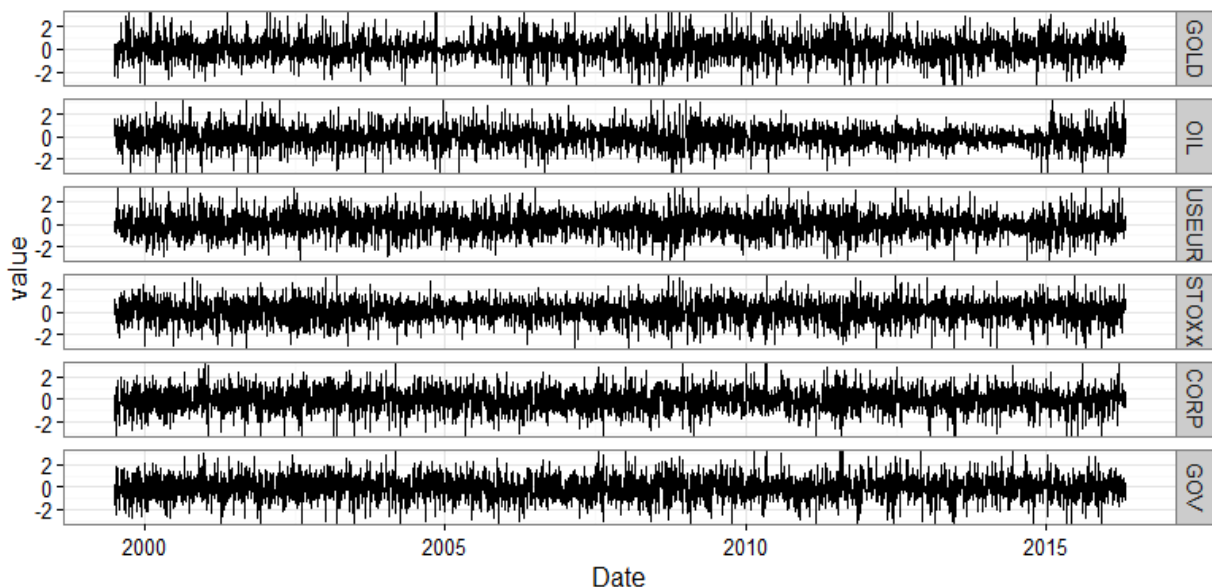
**Figure 7.2:** Box Ljung test result for standardized residuals

Table 7.3 summarizes Box Ljung test results that can help us to identify if there is any serial correlations in the data. We are able to observe that care should be taken in regards to the observations after approximately the 3000th. However, between the 1st and the 3000th observation the data does not seem to contain serial correlations.

Box Ljung	0-1000	1000-2000	2000-3000	3000-4000	1-4000	1-3000
GOLD	$p = 0.51$	$p = 0.71$	$p = 0.06$	$p = 0.0004$	$p = 0.026$	$p = 0.38$
OIL	$p = 0.33$	$p = 0.12$	$p = 0.47$	$p = 0.14$	$p = 0.56$	$p = 0.28$
USEUR	$p = 0.21$	$p = 0.73$	$p = 0.14$	$p = 0.2$	$p = 0.42$	$p = 0.14$
STOXX	$p = 0.81$	$p = 0.1$	$p = 0.14$	$p = 0.52$	$p = 0.07$	$p = 0.09$
CORP	$p = 0.89$	$p = 0.64$	$p = 0.62$	$p = 0.19$	$p = 0.92$	$p = 0.59$
GOLD	$p = 0.36$	$p = 0.49$	$p = 0.35$	$p = 0.02$	$p = 0.01$	$p = 0.13$

Table 7.3: Independence of time series

7.4 Regular Vine Copula and Student t Copula Comparison

We analysed how densities of portfolio return distribution simulated with r-vine differ from the ones simulated with student t copulas. We fitted copula models for the first 1000 observations and simulated two samples of size 10^5, one with regular vine copula and one with student t copula. Then we back-transformed those samples to return distribution for one day ahead using ARMA-GARCH parameters. The two densities are shown in Figure 7.3.
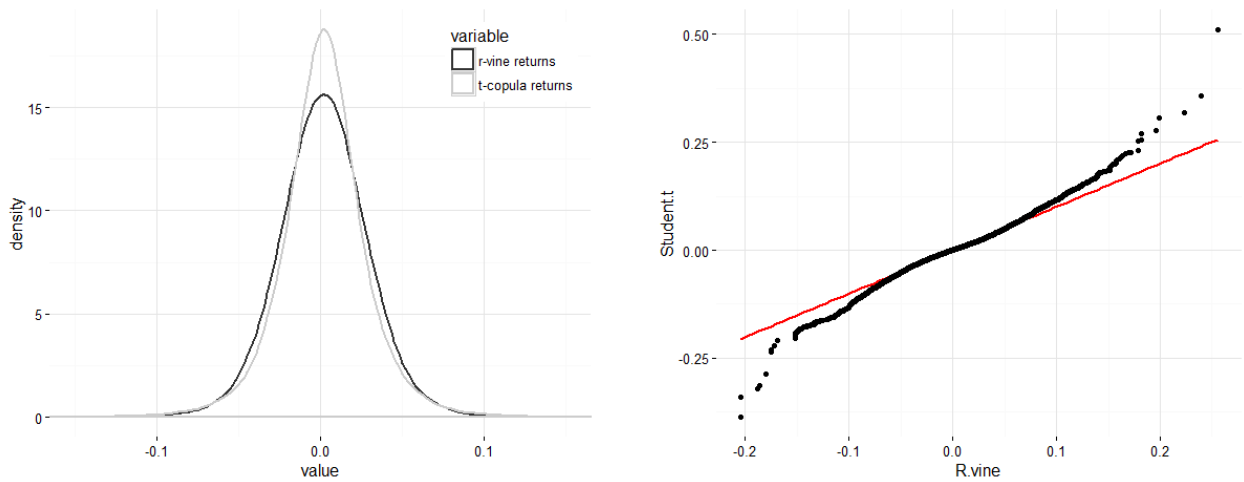
(a) Densities of predictive portfolio 1-step ahead

(b) Q-Q plot between empirical predictive portfolio distributions

Figure 7.3: Densities simulated with r-vine and t-copula

We can observe in Figure 7.3 that density simulated with student t copula has fatter tails and is steeper in the middle than the one simulated with r-vine.

7.4.1 Backtesting Fixed Models

Here we investigated the accuracy of VaR predictions when we do not update ARMA GARCH parameters or copula models. We have estimated ARMA GARCH parameters, regular vine copula parameters and multivariate student t copula parameters for the first thousand days and kept them fixed. We can expect models to perform worse as time goes by. Due to the fact that we kept the models constant, we could increase the number of simulations to 10^5 and get more precise estimates of VaR.
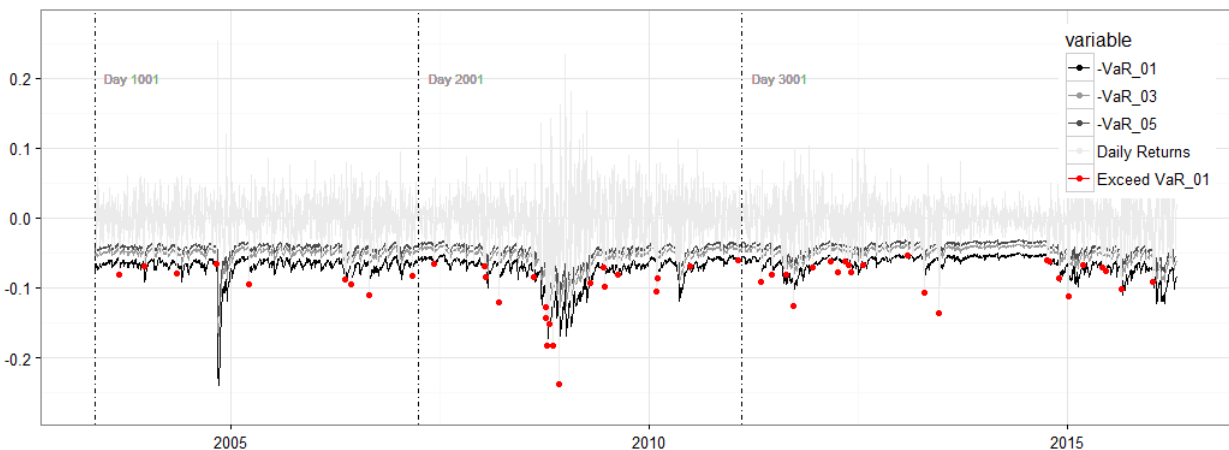

Figure 7.4: Backtest: fixed models, regular vine copula

Table 7.4: VaR test, fixed models, regular vine copula

VaR_α	Expected exceedances	Actual exceedances	H0 "Correct Exceedances"	H0 "Correct Exceedances & Indep"
From 2003-05-20 to 2016-04-19 (1001-4342 days)				
$VaR_{0.05}$	167	236	$p = 2.4 \times 10^{-7}$	$p = 4.5 \times 10^{-7}$
$VaR_{0.03}$	100	148	$p = 5.9 \times 10^{-6}$	$p = 2.2 \times 10^{-5}$
$VaR_{0.01}$	33	52	$p = 0.002$	$p = 0.01$
From 2003-05-20 to 2007-03-29 (1001-2000 days)				
$VaR_{0.05}$	50	49	$p = 0.88$	$p = 0.91$
$VaR_{0.03}$	30	28	$p = 0.71$	$p = 0.91$
$VaR_{0.01}$	10	10	$p = 1$	NA
From 2007-03-30 to 2011-02-10 (2001-3000 days)				
$VaR_{0.05}$	50	91	$p = 8.5 \times 10^{-8}$	$p = 5.8 \times 10^{-7}$
$VaR_{0.03}$	30	55	$p = 3.2 \times 10^{-5}$	$p = 0.0002$
$VaR_{0.01}$	10	19	$p = 0.01$	$p = 0.026$
From 2011-02-11 to 2016-04-19 (3001-4342 days)				
$VaR_{0.05}$	67	96	$p = 6.3 \times 10^{-4}$	$p = 4.5 \times 10^{-4}$
$VaR_{0.03}$	40	65	$p = 2.6 \times 10^{-4}$	$p = 7.7 \times 10^{-4}$
$VaR_{0.01}$	13	23	$p = 0.017$	NA

Table 7.4 summarizes Kupiec and Christoffersen coverage tests results for VaR for chosen periods of time. We can observe that Value-at-Risk predictions with r-vine were accurate in the period from 2003-05-20 to 2007-03-02. However, during the period of global financial crisis r-vine copula model did not perform sufficiently. Figure 7.4 illustrates the predictions of VaR and exceedances for $\alpha = 0.01$ level.
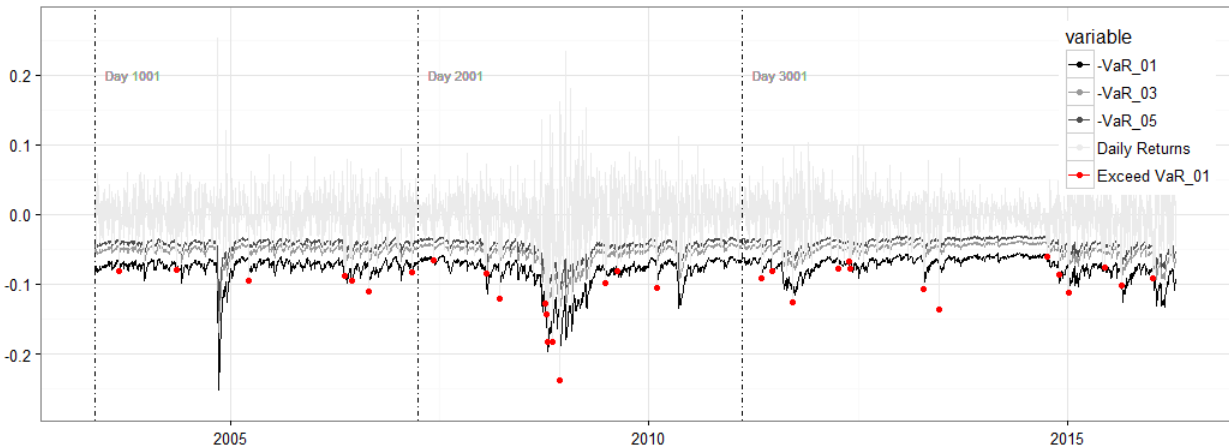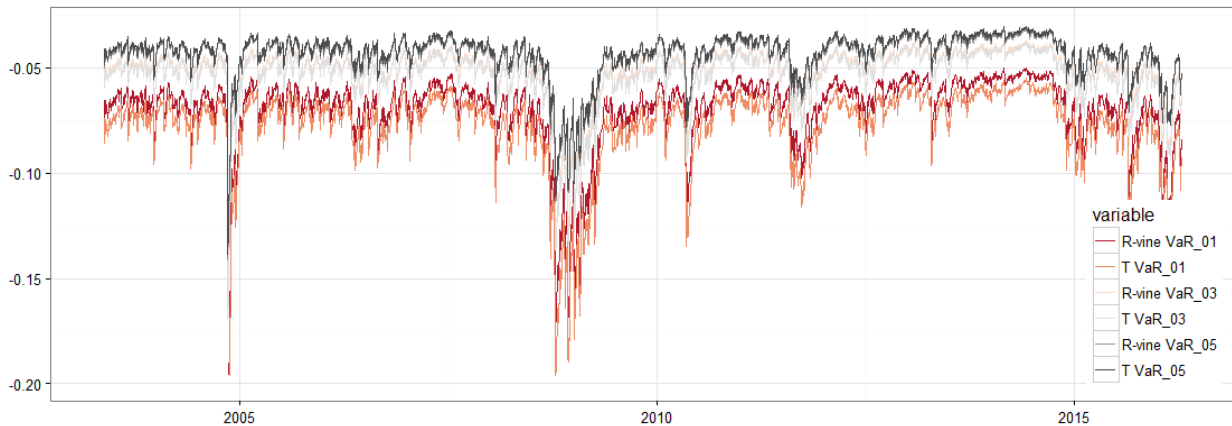

Figure 7.5: Backtest: fixed models, student t copula

Moreover, we used multivariate student t copula to simulate VaR predictions in the same way as with r-vine model. The results are summarized in Table 7.5 and Figure 7.5.

Table 7.5: VaR test, fixed models, student t copula

VaR_α	Expected exceedances	Actual exceedances	H0 "Correct Exceedances"	H0 "Correct Exceedances & Indep"
		From 2003-05-20 to 2016-04-19 (1001-4342 days)		
$VaR_{0.05}$	167	248	$p = 1.8 \times 10^{-9}$	$p = 1.7 \times 10^{-9}$
$VaR_{0.03}$	100	138	$p = 2.9 \times 10^{-4}$	$p = 5.6 \times 10^{-4}$
$VaR_{0.01}$	33	33	$p = 0.94$	NA
		From 2003-05-20 to 2007-03-29 (1001-2000 days)		
$VaR_{0.05}$	50	53	$p = 0.18$	$p = 0.68$
$VaR_{0.03}$	30	25	$p = 0.9$	$p = 0.57$
$VaR_{0.01}$	10	8	$p = 0.51$	NA
		From 2007-03-30 to 2011-02-10 (2001-3000 days)		
$VaR_{0.05}$	50	92	$p = 4.3 \times 10^{-8}$	$p = 3 \times 10^{-7}$
$VaR_{0.03}$	30	51	$p = 4 \times 10^{-4}$	$p = 0.002$
$VaR_{0.01}$	10	11	$p = 0.75$	NA
		From 2011-02-11 to 2016-04-19 (3001-4342 days)		
$VaR_{0.05}$	67	103	$p = 2.8 \times 10^{-5}$	$p = 7.3 \times 10^{-6}$
$VaR_{0.03}$	40	62	$p = 0.001$	$p = 0.002$
$VaR_{0.01}$	13	14	$p = 0.87$	NA

Furthermore, we analysed how VaR estimates differ for both models. We found that models had very similar VaR estimates for $\alpha = \{0.05, 0.03\}$, but for $\alpha = 0.01$ multivariate t copula estimated higher VaR than regular vine. Figure 7.6 illustrates VaR estimated with student t and regular vine copula.

Figure 7.6: VaR_α simulated with regular vine and student t copula models

7.4.2 Backtesting Re-fitted Models

We tested the model with standardized residuals shown in Section 7.3. We chose rolling window of 500 days to fit the regular vine and student t copula models and simulated a sample size of 3000 in order to calculate VaR_α. Figure 7.7 illustrates VaR predictions simulated with regular vine copula and Figure 7.8 - with student t copula.
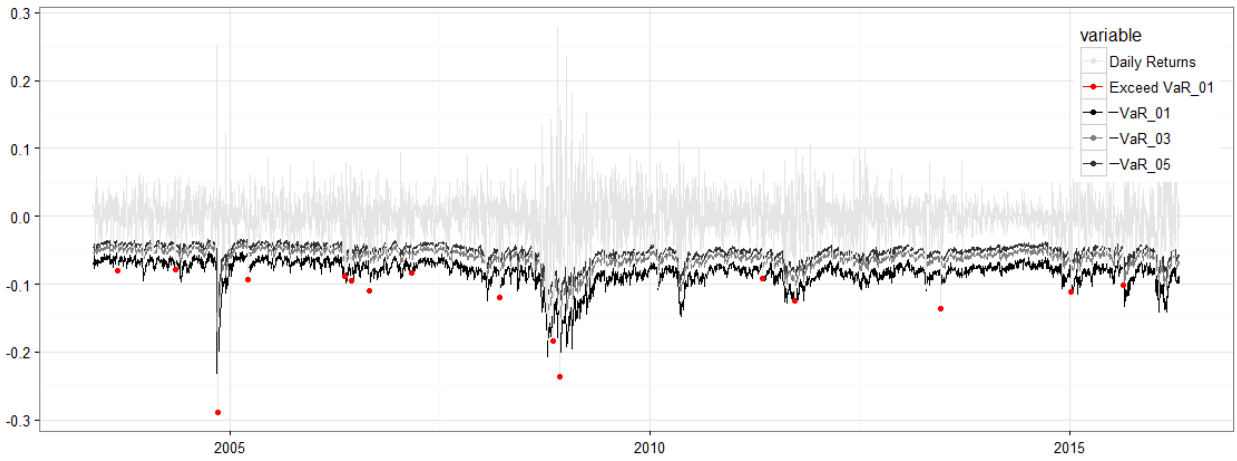
Figure 7.7: Backtest: rolling window for r-vine copula

Furthermore, we tested VaR with Kupiec and Christoffersen coverage tests and the results are summarized in Tables 7.6 and 7.7 for regular vine and student t copula models respectively.

Table 7.6: VaR test, regular vine copula with rolling window

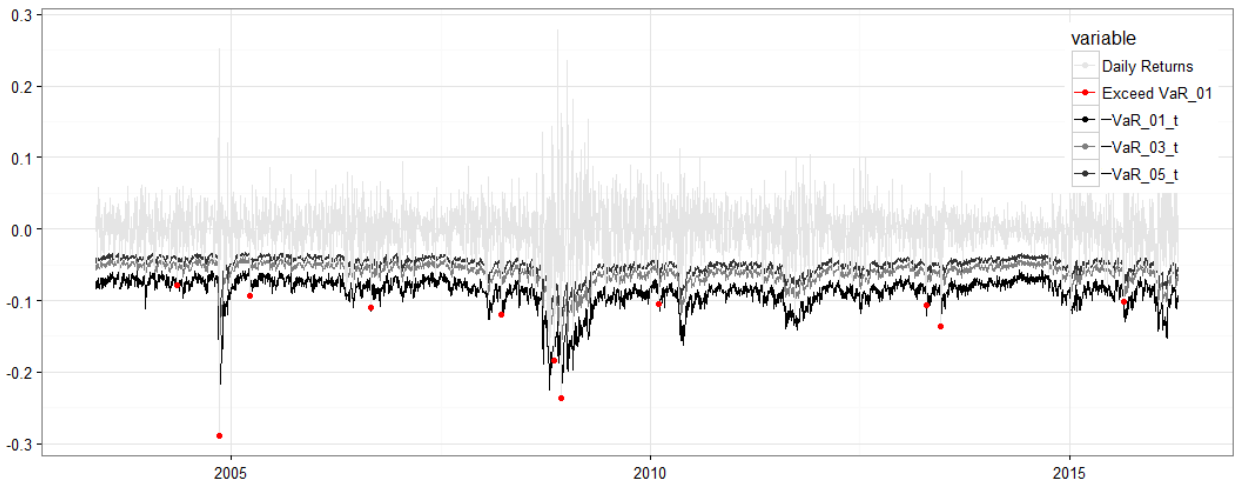| VaR_α	Expected exceedances	Actual exceedances	H0 "Correct Exceedances"	H0 "Correct Exceedances & Indep"
$VaR_{0.05}$	167	136	$p = 0.01$	$p = 0.038$
$VaR_{0.03}$	100	66	$p = 0.0002$	$p = 0.0009$
$VaR_{0.01}$	33	16	$p = 0.0007$	NA

Figure 7.8: Backtest: rolling window for multivariate student t copula

We can observe that neither model seems to perform well. Although multivariate student t copula VaR predictions at $\alpha = 0.05$ level (Table 7.7) seems to be right, for $\alpha = \{0.01, 0.03\}$ levels the model does not produce correct VaR values.

Table 7.7: VaR test, student t copula with rolling window

VaR_α	Expected exceedances	Actual exceedances	H0 "Correct Exceedances"	H0 "Correct Exceedances & Indep"
$VaR_{0.05}$	167	152	$p = 0.22$	$p = 0.35$
$VaR_{0.03}$	100	70	$p = 0.001$	$p = 0.005$
$VaR_{0.01}$	33	11	$p = 6 \times 10^{-6}$	NA

For backtesting copula models we needed to refit data many times. In the backtesting procedure where we refitted the copula models, due to computational time we restricted number of simulations. We chose to simulate samples of size 3000. When we measure value at risk, we are looking at the tail of the distribution, where observations are sparsely distributed, therefore there exists an error for the value at risk estimator $\widehat{\text{VaR}}_\alpha^{3000}$.
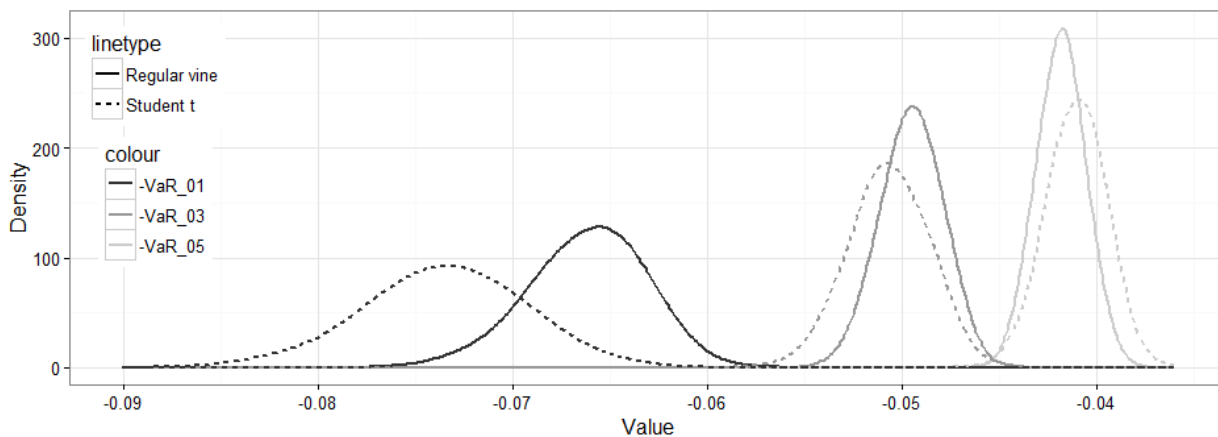
Figure 7.9: Density of VaR estimator based on different levels of α and different copula families for simulation size 3000

We have estimated $\widehat{\text{VaR}}_\alpha^{3000}$ for one day 2000 times, both with regular vine and with multivariate student t copula for different α levels. The distribution of resulting $\widehat{\text{VaR}}_\alpha^{3000}$ densities are shown in Figure 7.9. We can observe that the smaller the α - the larger the error.

8

Discussion and Conclusion

We have applied regular vine copula model and analyzed how accurately it can predict VaR_α for $\alpha = \{0.01, 0.03, 0.05\}$. First of all, we have tested whether the model is consistent by backtesting it on a simulated data that had constant ARMA GARCH model and dependency structure simulated with regular vine. We have found that the model was consistent and accurate in predicting VaR_α values based on Kupiec and Christoffersen coverage tests results (Table 7.2).

Moreover, we have backtested the model with real data using regular vine and multivariate student t copulas. First backtest was done with the fixed copula models (Section 7.4.1) and the second one with a rolling window of 500 days for the copula models (Section 7.4.2).

For the fixed models we found that copula models failed to predict VaR_α for $\alpha = \{0.03, 0.05\}$. On the other hand, multivariate Student t copula captured the right proportion of exceedances for $\alpha = 0.01$ while the regular vine did not. It could be due to a heavier tail in case of student t copula, which was compared to regular vine copula in Figure 7.3. We can however expect that dependency structure changed during global financial crisis of 2008 and in Figure A.1 we can observe that in fact correlation diverged from the level estimated in the period between the 1st and the 1000th day for some variables. We have divided the analysis into three parts, pre-crisis, during and after. We could see that models performed well under normal market conditions, but failed to re-adjust during and after the global financial crisis. After the crisis we could see in Table 7.3 that standardized residuals were not independent, therefore the results could have been distorted due to changed ARMA GARCH parameters rather than the copula models: we found that we should reject the hypothesis of independence for two out of six instruments after 3000th observation (2011-02-11).

Furthermore, we discovered that the backtest for copula models with rolling window did not adjust to global financial crisis, similarly as the fixed models. However, the results may have been affected by the limitations of our study that are illustrated in Figure 7.9. We could observe that $\widehat{VaR}_\alpha^{3000}$ estimator was not exact and varied.

In order to improve the result, we could extend the copula models by adding time variation for dependence parameters, similarly as was done by Patton (2006). That could allow for dependency models to quickly adjust to changing market conditions.

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A
Appendix 1

A.1 Time Variation of Dependence Parameter

For the dependence parameter (such as Kendall's τ), we can either assume that it is constant over time, or model it with some type of Moving Average model. Patton (2006) found that dependence parameter varies significantly in bivariate copula model, and follows a process similar to ARMA. Therefore assumption of constant dependence is strong.

In case of R vines, the challenge is that if the dependence parameter varies then also r-vine structure varies over time.

We have used the following equations found in Patton (2006) to model time varying correlation parameter ρ in order to understand how dependency varied for unconditional pairs of assets that had student t copula dependency structure:

$$\rho_t = \Lambda_1(\omega + \alpha \frac{1}{q} \sum_{i=1}^{q} \Phi^{-1}(u_{t-i})\Phi^{-1}(v_{t-i}) + \beta \rho_{t-1})$$

$$\Lambda_1(x) = \tanh(x) = \frac{1 - e^{-x}}{1 + e^{-x}}$$

$$\tau_t = \frac{2}{\pi} \arcsin(\rho_t)$$

Where Φ^{-1} is an inverse CDF of student t distribution with corresponding degrees of freedom, τ_t is Kendall's τ at time t, ω, α, β are some constants to be determined by optimization similarly as in ARMA-GARCH model, Λ_1 is a modified logistic transformation that keeps ρ_t in (-1,1), u, v are standard uniform variables.

The Figure A.1 below shows correlation variation, where red line represents correlation value between given assets from the 1st to the 1000th day. We can observe that dependence between assets varies and therefore depending on a period of time chosen the regular vine structure would change as well.
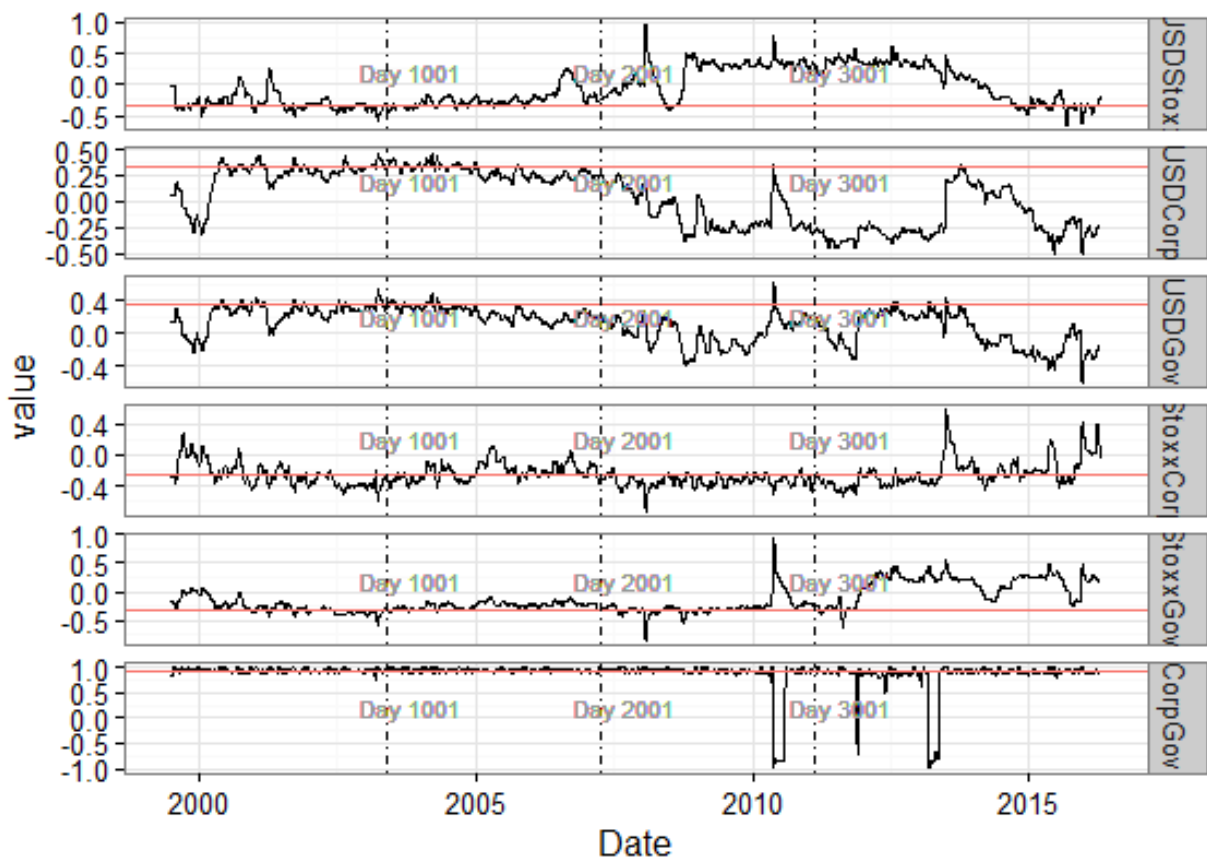
Figure A.1: Time variation of linear correlation parameter ρ_t