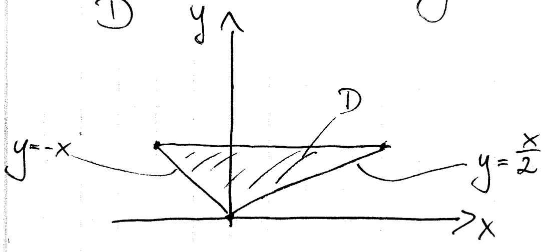
Räkneövning 15-03-26

6.15 $\iint_D \frac{1}{1+(x-2y)^2}\,dx\,dy$, D... triangeln med hörn $(0,0), (2,1), (-1,1)$



parametrisering av området

y går från 0 till 1

$\Rightarrow$ x går från $-y$ till $2y$

$$\int_0^1 \left(\int_{-y}^{2y} \frac{1}{1+(x-2y)^2}\,dx \right) dy = \int_0^1 \Big[\arctan(x-2y)\Big]_{-y}^{2y}\,dy$$

$$= \int_0^1 \underbrace{\arctan(0)}_{=0} - \arctan(-3y)\,dy =$$

$$= \int_0^1 \arctan 3y\,dy = \begin{bmatrix} t = 3y & 1 \rightsquigarrow 3 \\ \frac{dt}{dy} = 3 & 0 \rightsquigarrow 0 \end{bmatrix}$$

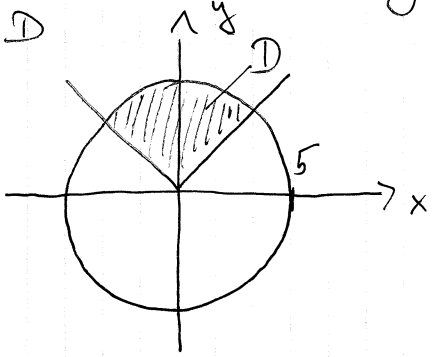
$$= \frac{1}{3}\int_0^3 \arctan t\,dt = \begin{bmatrix} f = \arctan t & f' = \frac{1}{1+t^2} \\ g' = 1 & g = t \end{bmatrix}$$

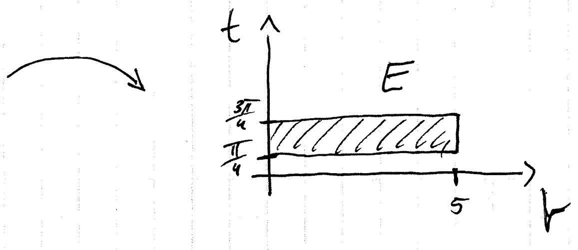
$$= \frac{1}{3}\Big[t\arctan t\Big]_0^3 - \frac{1}{3}\frac{1}{2}\int_0^3 \frac{2t}{1+t^2}\,dt =$$

$$= \arctan 3 - \frac{1}{6}\Big[\ln(1+t^2)\Big]_0^3 =$$

$$= \arctan 3 - \frac{1}{6}\ln 10$$

6.22 $\iint_D x^2 e^{x^2+y^2}\,dx\,dy$, $D=\{(x,y): x^2+y^2\le 25,\ y\ge |x|\}$





polära koordinater

$x = r\cos t \qquad r\in[0,5]$

$y = r\sin t \qquad t\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

$$\frac{d(x,y)}{d(r,t)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial t} \end{pmatrix} = \begin{pmatrix} \cos t & -r\sin t \\ \sin t & r\cos t \end{pmatrix} \Rightarrow \left|\frac{d(x,y)}{d(r,t)}\right| = r$$

$$\iint_D x^2 e^{x^2+y^2}dx\,dy = \iint_E r^2\cos^2 t\, e^{r^2\cos^2 t + r^2\sin^2 t}\, r\,dr\,dt =$$

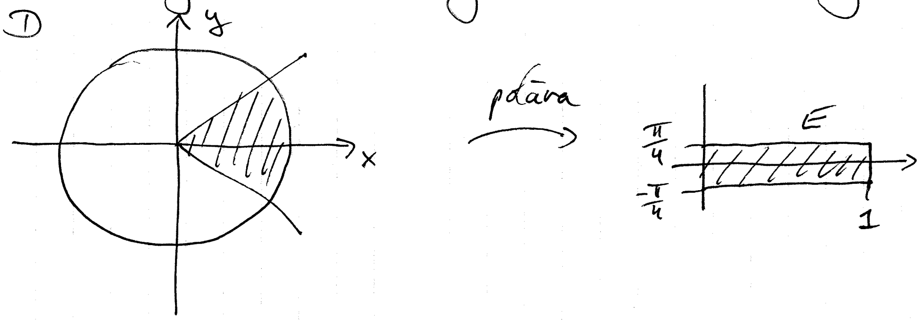
$$= \int_0^5 r^3 e^{r^2}dr \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos^2 t\,dt = \ldots =$$

$$= \left[\frac{1}{2}e^{r^2}(r^2-1)\right]_0^5 \cdot \left[\frac{1}{2}(t+\sin t\cos t)\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} =$$

$$= \left(\frac{1}{2}e^{25}(24) - \frac{1}{2}(-1)\right)\cdot\left(\frac{1}{2}\left(\frac{3\pi}{4} + \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4} - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\right)\right)$$

$$= \left(12e^{25}+\frac{1}{2}\right)\left(\frac{\pi}{4}-\frac{1}{2}\right)$$

6.23 $\iint\limits_D (x^2-y^2)\, e^{2xy}\, dx\, dy \qquad D=\{x^2+y^2\le 1,\ -x\le y\le x,\ x\ge 0\}$



$$\int_0^1 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (r^2\cos^2 t - r^2\sin^2 t)\, e^{2r^2\cos t\sin t}\, r\, dr\, dt =$$

$$= \int_0^1 r^3 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos^2 t - \sin^2 t)\, e^{2r^2\cos t \sin t}\, dt\, dr$$

$$\cos^2 t - \sin^2 t = \cos 2t \qquad 2\cos t \sin t = \sin 2t$$

$$= \int_0^1 r^3 \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2t\, e^{r^2 \sin 2t}\, dt\, dr =$$

$$= \int_0^1 r^3 \left[\frac{1}{r^2}\frac{1}{2} e^{r^2\sin 2t}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dr =$$

$$= \int_0^1 \frac{r}{2}\left(e^{r^2}-e^{-r^2}\right) dr = \begin{bmatrix} r^2 = s & 1\to 1 \\ 2r = \frac{ds}{dr} & 0\to 0 \end{bmatrix}$$

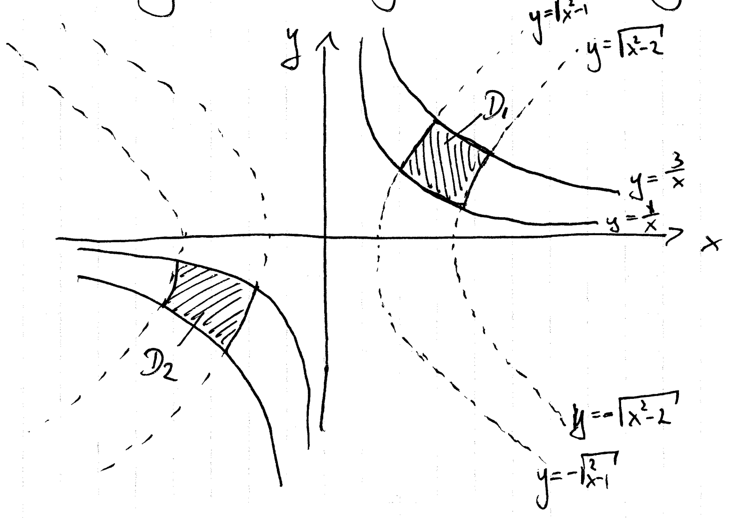
$$= \int_0^1 \frac{1}{4}\left(e^s - e^{-s}\right) ds = \frac{1}{4}\left(e^s + e^{-s}\right)\Big|_0^1 =$$

$$= \frac{1}{4}\left(e + \frac{1}{e} - 1 - 1\right) = \frac{e - 2 + \frac{1}{e}}{4}$$

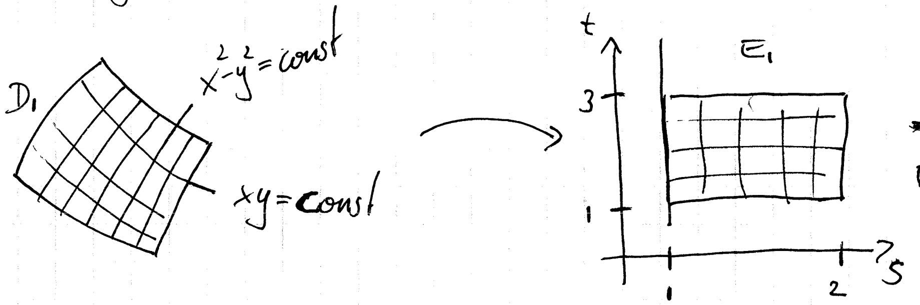
Ex 16 $\iint_D (x^4 - y^4)\,dx\,dy$

D... område i xy-planet som begränsas av

$x^2 - y^2 = 1 \quad x^2 - y^2 = 2 \quad xy = 1 \quad xy = 3$



variabelbyte:



$s = x^2 - y^2$
$t = xy$

$$\frac{d(s,t)}{d(x,y)} = \begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix} \qquad \left|\frac{d(s,t)}{d(x,y)}\right| = 2(x^2+y^2)$$

$$\left|\frac{d(x,y)}{d(s,t)}\right| = \frac{1}{\left|\frac{d(s,t)}{d(x,y)}\right|} = \frac{1}{2(x^2+y^2)}$$

$$\iint_{D_1} x^4 - y^4\,dx\,dy = \iint_{E_1} (x^2-y^2)(x^2+y^2)\frac{1}{2(x^2+y^2)}\,ds\,dt =$$

$$= \int_1^2\int_1^3 \frac{s}{2}\,dt\,ds = \frac{1}{2}\left[\frac{s^2}{2}\right]_1^2 \left[t\right]_1^3 = \frac{1}{2}\left(2-\frac{1}{2}\right)(3-1) = \frac{3}{2}$$

D_2: variabelbyte $x = -s \quad y = -t$ avbildar D_2 till D_1, $\frac{d(x,y)}{d(s,t)} = 1$

$$\Rightarrow \iint_{D_2} x^4 - y^4\,dx\,dy = \iint_{D_1} (-s)^4 - (-t)^4\,ds\,dt = \iint_{D_1} s^4 - t^4\,ds\,dt = \ldots = \frac{3}{2}$$