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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, October 26, 2013, 2012

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1. The equation of the tangent line is given by $y = f(1) + f'(1)(x-1)$. Since $f(1) = 2$ and $f'(x) = 2xe^{x^2-1} + \ln(2x-1) + \dfrac{2x}{2x-1} + 3x^2$ then $f'(1) = 7$ and $y = 2 + 7(x-1)$, i.e. $y = 7x - 5$.

2. We want to find $T = f(0) + f'(0)x + \dfrac{1}{2}f''(0)x^2 + \dfrac{1}{3!}f'''(0)x^3$. Since $f'(x) = 2e^{2x} + \dfrac{1}{x+1}$, $f''(x) = 4e^{2x} - \dfrac{1}{(x+1)^2}$ and $f'''(x) = 8e^{2x} + 2\dfrac{1}{(x+1)^3}$, then $f(0) = 1$, $f'(0) = 3$, $f''(0) = 3$ and $f'''(0) = 10$.
The answer is then $T = 1 + 3x + \dfrac{3}{2}x^2 + \dfrac{5}{3}x^3$.

3. The series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{(a+1)^2}\right)^n$ is convergent if $\dfrac{1}{(a+1)^2} < 1$, which means that $(a+1)^2 > 1$, i.e. $a^2 + 2a > 0$.
The inequality $a(a+2) > 0$ is satisfied for $a < -2$ and for $a > 0$.
The series $\displaystyle\sum_{n=0}^{\infty}\left(\frac{1}{a^2+2}\right)^n$ is convergent if $\dfrac{1}{a^2+2} < 1$, which is always true since $a^2 + 2 \geq 2 > 1$.
The answer is thus $a < -2$ and $a > 0$.
The sum of the first series is $\dfrac{1}{1 - \frac{1}{(a+1)^2}} = \dfrac{(a+1)^2}{(a+1)^2 - 1} = \dfrac{(a+1)^2}{a^2+2a}$, while the sum of the second series is $\dfrac{1}{1 - \frac{1}{a^2+2}} = \dfrac{a^2+2}{a^2+2-1} = \dfrac{a^2+2}{a^2+1}$. The equality of the sums, $\dfrac{(a+1)^2}{a^2+2a} = \dfrac{a^2+2}{a^2+1}$, reduces to $(a^2+2a+1)(a^2+1) = (a^2+2)(a^2+2a)$, i.e. $a^4 + 2a^3 + a^2 + a^2 + 2a + 1 = a^4 + 2a^2 + 2a^3 + 4a$, which is the same as $1 = 2a$.
Hence the sums are equal only for $a = \dfrac{1}{2}$.

4. Calculation of the determinants reduces the equation to $-1 + 2x^3 - 4x^2 = x^2 - 2$, which is the same as $2x^3 - 5x^2 + 1 = 0$.
We are first looking for possible rational solutions $\dfrac{p}{q}$, where p is a divisor of the free term 1 and q is a divisor of the coefficient at x^3. So $p = \pm 1$, while $q = \pm 1, \pm 2$. Th list of possible rational roots is: $\dfrac{p}{q} = \pm 1, \pm\dfrac{1}{2}$.
The control shows now that $x = \dfrac{1}{2}$ is a solution.
Polynomial division gives now $(2x^3 - 5x^2 + 1) : (x - \frac{1}{2}) = 2x^2 - 4x - 2$, i.e. $2x^3 - 5x^2 + 1 = (x - \frac{1}{2})(2x^2 - 4x - 2)$, so what is remaining is to solve the equation $2x^2 - 4x - 2 = 0$ and the solutions are $x = 1 \pm \sqrt{2}$.
Thus the answer is $x_1 = \dfrac{1}{2}$, $x_2 = 1 - \sqrt{2}$, $x_3 = 1 + \sqrt{2}$.

5. (a) $\displaystyle\lim_{x\to\infty}\frac{x\ln x}{x + \ln x}$ = ... since this is an expresion of the $\dfrac{\infty}{\infty}$-type, we use de L'Hopitals rule ...= $\displaystyle\lim_{x\to\infty}\frac{\ln x + 1}{1 + \frac{1}{x}} = \infty$.
(b) $\displaystyle\int_0^1 (x+1)e^{x^2+2x}\,dx$ = ... substitution: $x^2 + 2x = t$ gives $2(x+1)dx = dt$, i.e $(x+1)dx = \dfrac{1}{2}dt$ and, while x varies from 0 to 1, t varies from 0 to 3 ... $\displaystyle\int_0^3 \frac{1}{2}e^t\,dt = \frac{1}{2}e^t\Big|_0^3 = \frac{1}{2}(e^3 - e^0) = \frac{1}{2}(e^3 - 1)$.

6. From the equation for the tangent line for $x = 1$ we find that $y = -1$, meaning that the point $(1, -1)$ lies on the curve. Letting then $x = 1$ and $y = -1$ into the equation gives $-a + b + 1 = 0$.
Differentiation of the equation gives now $2axy^3 + 3ax^2y^2y' + 3bx^2y^2 + 2bx^3yy' + y' = 0$. If we let then $x = 1$ and $y = y(1) = -1$ we get $-2a + 3ay' + 3b - 2by' + y' = 0$. Now we should remember that y' equals the slope of the tangent line, which means that $y' = -2$. Hence we get $-2a - 6a + 3b + 4b - 2 = 0$, i.e. $-8a + 7b - 2 = 0$. Solving the system of equations: $-a + b + 1 = 0$ and $-8a + 7b - 2 = 0$ we find that$a = -9$ and $b = -10$.

7. (a): The inner part of the triangle. Finding stationary points: $f'_x = 4y^2 - 2xy^2 - y^3 = y^2(4 - 2x - y) = 0$ and $f'_y = 8xy - 2x^2y - 3xy^2 = xy(8 - 2x - 3y) = 0$. Since the x-coordinate and y-coordinate are positive for all points (x, y) within the triangle (i.e. $x > 0,\ y > 0$) then we need to solve the system $4 - 2x - y = 0$ and $8 - 2x - 3y = 0$. The only solution to this liner system is $x = 1$ and $y = 2$. Thus we have the first stationary point $P_1 = (1, 2)$ which obviously lies within the triangle.
(b): Edges:
(b1): The side of the triangle between the points $(0, 0)$ and $(6, 0)$ can be described as $y = 0$ and $0 < x < 6$. Thus, for all points (x, y) on this side of the triangle the value of the function is $f(x, y) = 0$.
(b2): The side of the triangle between the points $(0, 0)$ and $(0, 6)$ can be described as $x = 0$ and $0 < y < 6$. On this side the function is again a 0-function $f(x, y) = 0$.
(b3): Finding the equation for the line through $(0, 6)$ and $(6, 0)$: $y = ax+b$. Since both points $(0, 6)$ and $(6, 0)$ are on the line then $6 = a \cdot 0 + b$ and $0 = a \cdot 6 + b$. From this we obtain $b = 6$ and $a = -1$. Thus the points of this side of the triangle can be described as $y = -x + 6$, where $0 < x < 6$. On this side the function can be rewritten as $f(x, y) = 4x(-x+6)^2 - x^2(-x+6)^2 - x(-x+6)^3 = x(-x+6)^2(4-x-(-x+6)) = -2x(-x+6)^2$. This is a function $g(x)$ of one single variable x. Finding stationary points reduces then to finding solutions to the equation $g'(x) = 0$.
$g'(x) = -2(-x+6)^2 - 2x \cdot (-2)(-x+6) = -2(-x+6)(-x+6-2x) = -2(-x+6)(-3x+6)$. Since $0 < x < 6$ the only solution we get is $x = 2$. Then $y = -2 + 6 = 4$ and we have the second stationary point $P_2 = (2, 4)$.
Finally we add the vertices of the triangle: $P_3 = (0, 0),\ P_4 = (6, 0),\ P_5 = (0, 6)$.
Now we are ready to compare the values: $f(P_1) = 4$, $f(P_2) = -64$, $f(P_3) = 0$, $f(P_4) = 0$ and $f(P_5) = 0$. The smallest value of the function is thus -64 while the largest is 4.

Paul