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Examination in
Mathematics for Economic and Statistical Analysis
Master Program; 7,5 ECTS
30th October 2015

Time: 09:00-14:00
Instructions:
- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Only **non-graphic** calculators are allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. JUSTIFY your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Mark clearly where is your final answer putting $\boxed{\text{A BOX}}$ around it.

Grades: Each solved problem is awarded by up to 10 points. At least 35 points are necessary for the grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

$$\text{a)} \quad \lim_{x\to+\infty}\left(\frac{x^2}{x+2}-x\right), \qquad\qquad \text{b)} \lim_{x\to 0^+}\frac{e^{x^2}-1}{x^2\ln x}.$$

Solution a) Observe that

$$\lim_{x\to+\infty}\left(\frac{x^2}{x+2}-x\right)=\lim_{x\to+\infty}\frac{-2x}{x+2}=\lim_{x\to+\infty}\frac{-2}{1+2/x}=\boxed{-2.}$$

b) We have an indetermination of the type $0/0$, so by L'Hôpital's rule

$$\lim_{x\to 0^+}\frac{1-e^{x^2}}{x^2\ln x}=\lim_{x\to 0^+}\frac{-2xe^{x^2}}{2x\ln x+x}=\lim_{x\to 0^+}\frac{-2e^{x^2}}{2\ln x+1}=\boxed{0.}$$

2. Consider the function $f(x)=\frac{1}{3}x^3+\frac{1}{2}x^2-2x+\frac{1}{2}$.
   (a) Find the critical points;
   (b) Find the intervals on which f is increasing and decreasing;
   (c) Give the maximum and minimum value of f on the interval $[0,2]$.

Solution (a) We have

$$f'(x)=x^2+x-2=(x-1)(x+2),$$

so $f'(x)=0$ when $x=1$ and $x=-2$. So the critical points are

$$\boxed{(1,f(1))=(1,-2/3) \text{ and } (-2,f(-2))=(-2,23/6).}$$

(b) $f'(x)$ is positive on $(-\infty,-2)$ and $(1,\infty)$, hence $\boxed{f \text{ is increasing on these intervals}}$. $f'(x)$ is negative on $(-2,1)$, hence $\boxed{f \text{ is decreasing on this interval}}$.

(c) We calculate

$$f(0)=\frac{1}{2} \qquad f(1)=-\frac{2}{3} \qquad f(2)=\frac{7}{6}$$

so $\boxed{\text{the minimum is at } x=1}$ and $\boxed{\text{the maximum is at } x=2.}$ (Note that $f(-2)=23/6$ is *not* the maximum since $-2\notin[0,2]$)

3. The equation $x^2y+3y^3=7$ implicitly defines y as a function of x: $y=y(x)$.

   (a) Find the value $y(2)$.

   (b) Find the equation of the tangent line the curve at the point $P=(2,y(2))$.

Solution (a) Plugging in $x=2$, we have

$$4y+3y^3=7$$

or

$$3y^3+4y-7=0.$$

Checking for integer solutions with $y\in\{\pm 1,\pm 7\}$, we see that $y=1$ is a solution.

(b) Differentiating implicitly gives

$$2xy+x^2y'+9y^2y'=0;$$

Plugging the point $P=(2,1)$ gives

$$4+4y'+9y'=0$$

and hence $y'(2)=-\dfrac{4}{13}$.

Hence an equation for the tangent line at this point is $\boxed{y=-\dfrac{4}{13}x+\dfrac{21}{13}.}$

4. Calculate the integrals

$$\text{a)}\quad \int_0^2 \frac{x^2+4x+2}{x+1}\mathrm{d}x; \qquad\qquad \text{b)}\quad \int x^3e^{x^2}\mathrm{d}x.$$

Solution a) To solve the integral we will first make a substitution $u=x+1$, the integral then becomes

$$\int_1^3 \frac{(u-1)^2+4(u-1)+2}{u}\mathrm{d}u=\int_1^3\frac{u^2+2u-1}{u}\mathrm{d}u=\int_1^3\left(u+2-\frac{1}{u}\right)\mathrm{d}u.$$

If we integrate this we get

$$\left[\frac{1}{2}u^2+2u-\ln(u)\right]_1^3=8-\ln(3).$$

So the answer is $\boxed{8-\ln(3).}$

b) To solve the integral we first make the substitution $u=x^2$, then we get that $\mathrm{d}x=\frac{1}{2x}\mathrm{d}u$ and the integral becomes

$$\int\frac{1}{2}ue^u\mathrm{d}u.$$

If we use integration by parts we get

$$\int\frac{1}{2}ue^u\mathrm{d}u=ue^u-e^u+C.$$

After substituting $u=x^2$ the final answer becomes

$$\boxed{\int x^3e^{x^2}\mathrm{d}x=\frac{1}{2}x^2e^{x^2}-\frac{1}{2}e^{x^2}+C.}$$

5. Find all the possible solutions of the following system of linear equations

$$\begin{aligned} x_1 + \ x_2 + \ x_3 + \ x_4 &= 5; \\ x_1 + 2x_2 + 2x_3 + 2x_4 &= 10; \\ x_1 + 2x_2 + 3x_3 + 3x_4 &= 15; \\ x_1 + 2x_2 + 3x_3 + 4x_4 &= 20. \end{aligned}$$

First we will write the system of equations in matrix form and we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 2 & 10 \\ 1 & 2 & 3 & 3 & 15 \\ 1 & 2 & 3 & 4 & 20 \end{array}\right)$$

We will use Gaussian elimination to find the solutions. First we subtract the first row from the second, third and fourth row and we get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 1 & 2 & 2 & 10 \\ 0 & 1 & 2 & 3 & 15 \end{array}\right)$$

Then we subtract the second row from the third and the fourth row and get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 & 10 \end{array}\right)$$

After that we subtract the third row from the fourth row and get

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

From this we conclude that the solutions of our system of equations is $\boxed{x_4 = 5 \text{ and } x_1 = x_2 = x_3 = 0}$.

6. Compute the Taylor polynomial of degree 2 for the function $g(x) = \sqrt{4+6x+2x^2}$, centered at the point $a = 0$.

Solution The solution is given by

$$P(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2$$

Now $g'(x) = \frac{6+4x}{2\sqrt{4+6x+2x^2}}$ and $g''(x) = -\frac{(6+4x)^2}{4(4+6x+2x^2)^{3/2}} + \frac{2}{\sqrt{4+6x+2x^2}}$. Hence

$$g(0) = 2, \qquad g'(0) = \frac{3}{2} \qquad \text{and} \qquad g''(0) = -\frac{1}{16}.$$

Thus $\boxed{P(x) = 2 + \frac{3}{2}x - \frac{1}{16}x^2.}$

7. Let $F(x,y) = x^2 + xy + y^2 + x + 2y + 14$. Show that F has a minimum at a certain point $(x_0, y_0) \in \mathbb{R}^2$, find it, and calculate the minimum value $F(x,y)$ for $(x,y) \in \mathbb{R}^2$.

Solution The system

$$0 = F_x = 2x + y + 1. \qquad 0 = F_y = x + 2y + 2,$$

yields that

$$x = 0 \qquad y = -1,$$

is the only stationary point. Observe that

$$F_{xx}(x,y) = 2 > 0 \qquad F_{yy}(x,y) = 2 > 0, \qquad \forall (x,y) \in \mathbb{R}^2,$$

and that $F_{xy}(x,y) = F_{yx}(x,y) = 1$. Thus, the determinant of the Hessian is

$$D(x,y) = F_{xx}F_{yy} - F_{xy}^2 = 4 - 1 = 3 > 0, \qquad \forall (x,y) \in \mathbb{R}^2.$$

Hence, $\boxed{(0,-1) \text{ is a global minimum point for } F}$ by one of theorems seen in the lectures (Theorem 13.2.1 in the course-book). Moreover

$$\boxed{\min_{(x,y)\in\mathbb{R}^2} F(x,y) = F(0,-1) = 13.}$$

GOOD LUCK!

The corrected exams will be handed out on Wednesday, November 10 2015, at 10:30, in the room next to the coffee shop, house 5, and after that in room 204, house 6.