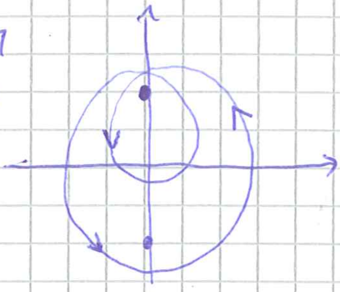
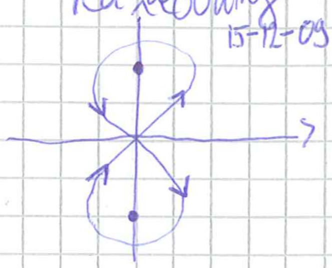
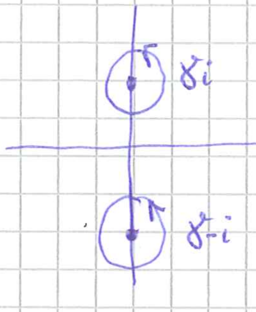
Räkneövning 15-12-09

(7) $\displaystyle\int_{\Gamma_k} \frac{dz}{z^2+1}$

Γ_1 

Γ_2 

Vi kan beräkna $\displaystyle\int_{\gamma_i} \frac{dz}{z^2+1}$ och $\displaystyle\int_{\gamma_{-i}} \frac{dz}{z^2+1}$ med



och sedan räkna hur ofta vi går runt polställen $+i$ och $-i$ med Γ_1 och Γ_2

$$\int_{\gamma_i} \frac{dz}{z^2+1} = \int_{\gamma_i} \frac{\frac{1}{z+i}}{z-i}\, dz = 2\pi i \left.\frac{1}{z+i}\right|_{z=i} = \pi$$

$$\int_{\gamma_{-i}} \frac{dz}{z^2+1} = \int_{\gamma_{-i}} \frac{\frac{1}{z-i}}{z+i}\, dz = 2\pi i \left.\frac{1}{z-i}\right|_{z=-i} = -\pi$$

Γ_1 går 2 gånger runt $+i$ i positiv riktning
1 gång runt $-i$ i positiv riktning

$$\Gamma_1 \rightsquigarrow 2\gamma_i + \gamma_{-i} \Rightarrow \int_{\Gamma_1} \frac{dz}{z^2+1} = 2\int_{\gamma_i} \frac{dz}{z^2+1} + \int_{\gamma_{-i}} \frac{dz}{z^2+1} = \pi$$

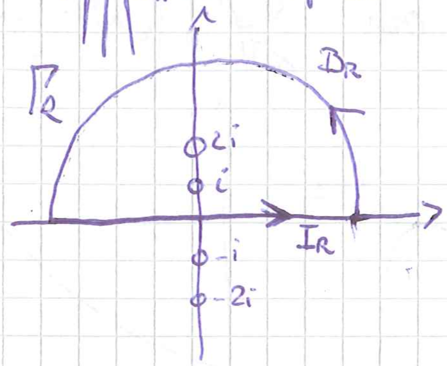
Γ_2 går 1 gång runt $+i$ i positiv riktning
1 gång runt $-i$ i negativ riktning

$$\Gamma_2 \rightsquigarrow \gamma_i - \gamma_{-i} \Rightarrow \int_{\Gamma_2} \frac{dz}{z^2+1} = \int_{\gamma_i} \frac{dz}{z^2+1} - \int_{\gamma_{-i}} \frac{dz}{z^2+1} = 2\pi$$

Räkneövning 15-05-06

(8) $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$

uppfatter det som en del av en komplex kurvintegral:



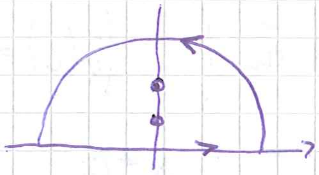
$I_R = [-R, R] \subseteq \mathbb{R}$

$B_R = \{ z = Re^{it};\ t \in [0, \pi] \}$

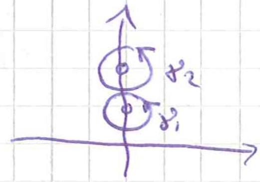
$\Gamma_R = I_R \cup B_R$

•) låt R vara godtyckligt:

$$\int_{\Gamma_R} \frac{1}{(z-i)(z+i)(z-2i)(z+2i)}\, dz =$$



$\cong$



$$= \int_{\gamma_1} \frac{\overbrace{\frac{1}{(z+i)(z-2i)(z+2i)}}^{=f}}{z-i}\, dz + \int_{\gamma_2} \frac{\overbrace{\frac{1}{(z-i)(z+i)(z+2i)}}^{=g}}{z-2i}\, dz$$

$$\overset{\text{Cauchy}}{=} 2\pi i\, f(i) + 2\pi i\, g(2i) = 2\pi i \frac{1}{2i(-i)3i} + 2\pi i \frac{1}{i \cdot 3i \cdot 4i} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

•) det är klart att $\int_{I_R} \dots dz \longrightarrow \int_{-\infty}^{\infty} \dots dx$ då $R \to \infty$

•) visa att $\int_{B_R} \dots dz \longrightarrow 0$ då $R \to \infty$

$$\int_{B_R} \frac{1}{(z^2+1)(z^2+4)}\, dz = \begin{bmatrix} z = Re^{it} \\ dz = iRe^{it} dt \end{bmatrix} = \int_0^{\pi} \frac{1}{(R^2 e^{2it}+1)(R^2 e^{2it}+4)}\, iRe^{it}\, dt$$

R stort, c litet

obs: $|R^2 e^{2it} + c| = |R^2 e^{2it} - (-c)| \geq \big| |R^2 e^{2it}| - |-c| \big| = |R^2 - c| \geq \frac{R^2}{2}$

$$\to \left| \int_{B_R} \frac{1}{(z^2+1)(z^2+4)}\, dz \right| \leq R \int_0^{\pi} \frac{1}{|R^2 e^{2it}+1|\,|R^2 e^{2it}+4|}\, dt \leq R \int_0^{\pi} \frac{1}{\frac{R^2}{2} \cdot \frac{R^2}{2}}\, dt = \frac{4R}{R^4} \pi \xrightarrow[R\to\infty]{} 0$$