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Sketch of the solutions to the examination paper in Mathematics for Economic and Statistical Analysis, Master Program, October 2, 2013

1. If $\dfrac{3x^3-5x^2-8x-2}{x-a} = Ax^2+Bx+C$ then $3x^3-5x^2-8x-2 = (x-a)(Ax^2+Bx+C)$. Thus a is a (rational) root of $3x^3-5x^2-8x-2=0$. If then $a=\dfrac{p}{q}$, the number p is a divisor of 2 while q is a divisor of 3. The set of divisors of 2 is $\{\pm 1, \pm 2\}$ and divisors of 3 are $\{\pm 1, \pm 3\}$. We need then to check eight possible roots: $\pm 1, \pm\dfrac{1}{3}, \pm 2, \pm\dfrac{2}{3}$. Control shows that $a=-\dfrac{1}{3}$ is a root of the equation. Finally, the polynomial division: $(3x^3-5x^2-8x-2):(x-a)$ gives the result: $3x^3-5x^2-8x-2=(x-a)(3x^2-6x-6)$. So $A=3, B=C=-6$.

2. This geometric series with $r=\dfrac{2x^2-3}{5}$ is convergent if $-1<r<1$, i.e. only if $-1<\dfrac{2x^2-3}{5}<1$. Thus $-5<2x^2-3<5$, and then $-1<x^2<4$. Since square is never negative we may rewrite it as $0\le x^2<4$. This inequality is satisfied for $-2<x<2$.
The sum equals $\dfrac{1}{1-\frac{2x^2-3}{5}} = \dfrac{5}{8-2x^2}$. The equation $\dfrac{5}{8-2x^2}=\dfrac{5}{6}$ implies $8-2x^2=6$, i.e. $x^2=1$. The answer is thus $x=\pm 1$.

3. $\displaystyle\int_1^2 \frac{2x}{\sqrt{x^2-1}}\,dx = \ldots$ substitution $x^2-1=u$ gives $2xdx=du$ and when x varies from 1 to 2 then u goes from 0 to 3.
Thus we have ... $\displaystyle\int_0^3 \frac{du}{\sqrt{u}} = \lim_{c\to 0+}\int_c^3 \frac{du}{\sqrt{u}} = \lim_{c\to 0+} 2\sqrt{u}\Big|_c^3 = \lim_{c\to 0+}(2\sqrt{3}-2\sqrt{c}) = 2\sqrt{3}$. The integral converges.

4. Letting $x=0$ into our expression $x^2e^{y(x)}+x(y(x))^3-(e^x+1)y(x)+(x+1)^2-3=0$, we will get $0+0-(e^0+1)y(0)+1-3=0$, i.e. $-2y(0)-2=0$. Thus, $y(0)=-1$ and $P=(0,-1)$.
Implicit differentiation of the expression gives now $2xe^{y(x)}+x^2e^{y(x)}y'(x)+(y(x))^3+3x(y(x))^2y'(x)-e^xy(x)-(e^x+1)y'(x)+2(x+1)=0$. Let then $x=0$ and $y(0)=-1$. We will get $(-1)^3-e^0(-1)-(e^0+1)y'(0)+2=0$, which is $-1+1-2y'(0)+2=0$. Hence, $y'(0)=1$.
The slope a of the tangent line at $P: y=ax+b$ is then $a=1$. Thus the line is $y=x+b$. Since P is on this line then, letting $x=0,\ y=-1$, we will have $-1=0+b$, which implies $b=-1$. The equation of the tangent line is then $y=x-1$.

5. The determinant for the coefficient matrix equals 51. After substituting the first column of this matrix with the right-hand-side column and calculating the new determinant we get 102. Thus $x=\dfrac{102}{51}=2$.
In similar manner we find that $y=\dfrac{-153}{51}=-3$ and $z=\dfrac{51}{51}=1$.

6. We begin by identifying the stationary points and thus we solve the equations $h'_x=0$ and $h'_y=0$, i.e. $3x^2y-\dfrac{1}{3}y^3+9=0$ and $x^3-xy^2=x(x^2-y^2)=x(x-y)(x+y)=0$.
From the second equation we may conclude that we have three possibilities: $x=0,\ \ y=x$ or $y=-x$.
If $x=0$ then the first equation implies the $-\dfrac{1}{3}y^3+9=0$, i.e. $y^3=27$. That gives $y=3$ and we have the first stationary point $P_1=(0,3)$.
If $y=x$ then the first equation reads $3x^3-\dfrac{1}{3}x^3+9=0$ i.e. $\dfrac{8}{3}x^3+9=0$. This says that $x^3=-\dfrac{27}{8}$, so $x=-\dfrac{3}{2}$ and we have $P_2=(-\dfrac{3}{2},-\dfrac{3}{2})$.

Finally, if $y = -x$ then the first equation gives $-3x^3 + \frac{1}{3}x^3 + 9 = 0$ i.e. $-\frac{8}{3}x^3 + 9 = 0$. This implies that $x^3 = \frac{27}{8}$, so $x = \frac{3}{2}$ and then $P_3 = (\frac{3}{2}, -\frac{3}{2})$.
Let now $A = h''_{xx} = 6xy$, $B = h''_{xy} = 3x^2 - y^2$ and $C = h''_{yy} = -2xy$.
For P_1: $A = 0$, $B = -9$ and $C = 0$. Hence $AC - B^2 = -81$ and thus P_1 is a saddle point.
For P_2: $A = \frac{27}{2}$, $B = \frac{9}{2}$ and $C = -\frac{9}{2}$. Hence $AC - B^2 = -81$ and P_2 is also a saddle point.
For P_3: $A = -\frac{27}{2}$, $B = \frac{9}{2}$ and $C = \frac{9}{2}$. Hence $AC - B^2 = -81$ and P_3 is again a saddle point.

7. (a) Since $\ln z$ is defined only for positive z then $x^2 - 1$ must be positive. $x^2 - 1 > 0$ can be written as $(x - 1)(x + 1) > 0$ and thus $x < -1$ or $x > 1$. As intervals, the answer can be written as $(-\infty, -1)$ and $(1, \infty)$.
(b) Since $f(x)$ is not defined for $x = 0$, the graph of $f(x)$ doesn't meet the y-axis. On the other hand $f(x) = 0$ implies $x^2 - 1 = 1$, which means $x = \pm\sqrt{2}$. These are the intersection points between the graph and the x-axis.
(c) Since $f'(x) = \frac{2x}{x^2 - 1}$ and $x^2 - 1$ is positive for all x where the function is well defined, then the sign of $f'(x)$ depends only on the factor $2x$. Hence, $f'(x) < 0$ (the function is decreasing) for all $x < -1$ and $f'(x) > 0$ (the function is increasing) for all $x > 0$.
(d) We need to evaluate $\lim_{x\to-\infty} f(x)$, $\lim_{x\to-1^-} f(x)$, $\lim_{x\to1^+} f(x)$ and $\lim_{x\to\infty} f(x)$. The first and the last are equal ∞ while the remaining two are equal $-\infty$.

Paul