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Examination in
Mathematics for Economic and Statistical Analysis
Master Program; 7,5 ECTS
30th September, 2015

Time: 09:00-14:00

Instructions:
- During the exam you CAN NOT use any textbook, class notes, or any other supporting material.
- Only **non-graphic** calculators are allowed during the exam.
- In all your solutions show your reasoning, explaining carefully what you are doing. JUSTIFY your answers.
- Use natural language, not just mathematical symbols.
- Use clear and legible writing. Write preferably with a ball-pen or a pen (black or dark blue ink).
- Mark clearly where is your final answer putting A BOX around it.

Grades: Each solved problem is awarded by up to 10 points. At least 35 points are necessary for the grade E, 42 for D, 49 for C, 56 for B and 63 for A. Note that the problems are not ordered according to the difficulty!

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1. Calculate the limits

$$\text{a)} \quad \lim_{x\to+\infty} \frac{1+x\ln x}{x+2\ln x} \qquad\qquad \text{b)} \lim_{x\to 1} \frac{6x^2+4x-10}{9x^2-1}$$

Solution a) This is of indeterminate form $\frac{\infty}{\infty}$, so we apply L'Hôpital's rule to get

$$\lim_{x\to+\infty} \frac{1+x\ln x}{x+2\ln x} = \lim_{x\to+\infty} \frac{\frac{d}{dx}(1+x\ln x)}{\frac{d}{dx}(x+2\ln x)} = \lim_{x\to+\infty} \frac{\ln x + x\frac{1}{x}}{1+\frac{2}{x}} = \lim_{x\to+\infty} \frac{1+\ln x}{1+\frac{2}{x}} = \boxed{+\infty.}$$

b) The function

$$\frac{6x^2+4x-10}{9x^2-1}$$

is continuous at 1, so

$$\lim_{x\to 1} \frac{6x^2+4x-10}{9x^2-1} = \frac{6+4-10}{9-1} = \boxed{0.}$$

2. Consider the function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$.
   (a) Find the critical points;
   (b) Find the intervals on which f is increasing and decreasing;
   (c) Is there any inflection point? Argument your answer.
   (d) Give the maximum and minimum value of f on the interval $[0,3]$.

Solution (a) Observe that

$$f'(x) = x^2 + x - 6 = (x-2)(x+3) = 0 \Leftrightarrow x = 2 \quad \text{or} \quad x = -3.$$

So, the critical points are $\boxed{x = -3 \text{ and } x = 2.}$

(b) Using the relation between the sign of the derivative and the monotonicity of the function we have the situation reflected in the following table:

	$x < -3$	$x \in (-3,2)$	$x > 2$
$f'(x)$	$+$	$-$	$+$
$f(x)$	$\uparrow$	$\downarrow$	$\uparrow$

Than is, since $f'(x) > 0$ for $x < -3$ and $x > 2$, and $f'(x) < 0$ for $x \in (-3,2)$ we have that f is increasing on $(-\infty,-3) \cup (2,+\infty)$ and decreasing on $(-3,2)$.

(c) Since f is continuous in a compact interval for being a polynomial, the Extreme Value Theorem yields that the minimum and maximum for f in $[0,3]$ exist. Moreover, we also know that these have to are among the critical points in $(0,3)$ or the extremes of the interval. By the previous part of the problem, the only critical point that lies in $(0,3)$ is $x=2$ and $f(2)=\frac{8}{3}+2-12+8=\frac{2}{3}$. Moreover, $f(0)=8$ and $f(3)=9+\frac{9}{2}-18+8=3+\frac{1}{2}$. Hence $\boxed{f(2)=\frac{2}{3} \text{ is the minimum value of } f \text{ on } [0,3]}$ and $\boxed{f(0)=8 \text{ is the maximum value of } f \text{ on } [0,3].}$

3. Calculate the integrals

$$\text{a)} \quad \int \frac{2x^3-2x^2+1}{x-1}\mathrm{d}x \qquad\qquad \text{b)} \quad \int_0^1 \frac{t\ln(t^2+1)}{t^2+1}\mathrm{d}t$$

Solution a) Making the polynomial division one obtains that

$$\frac{2x^3-2x^2+1}{x-1}=\frac{2x^2(x-1)+1}{x-1}=2x^2+\frac{1}{x-1}.$$

Hence

$$\int \frac{2x^3-2x^2+1}{x-1}\mathrm{d}x=\int 2x^2\mathrm{d}x+\int \frac{1}{x-1}\mathrm{d}x=\boxed{\frac{2}{3}x^3+\ln|x-1|+C, \quad C\in\mathbb{R}.}$$

b) Making the substitution $y=\ln(t^2+1)$, then $\mathrm{d}y=\frac{2t}{t^2+1}\mathrm{d}t$. Hence

$$\int_0^1 \frac{t\ln(t^2+1)}{t^2+1}\mathrm{d}t=\frac{1}{2}\int_{0=y(0)}^{\ln 2=y(1)} y\mathrm{d}y=\boxed{\frac{\ln^2 2}{4}.}$$

4. The expression

$$x^2\ln y+y^3e^{-x}=8,$$

defines y as a function of x: $y=y(x)$.

(a) Find the value $y(0)$.

(b) Find the equation of the tangent line to $y(x)$ at the point $P=(0,y(0))$.

Solution (a) Evaluating the expression in $x=0$, we have that

$$y(0)^3=8 \Leftrightarrow \boxed{y(0)=2.}$$

(b) The equation of the tangent line to $y(x)$ at the point $P=(0,y(0))$ is given by

$$t(x)=y(0)+y'(0)x=2+y'(0)x.$$

Differentiating implicitly on x, we obtain that

$$2x\ln y+x^2\frac{y'}{y}+3y^2y'e^{-x}-y^3e^{-x}=0 \Leftrightarrow y'=\frac{(y^3e^{-x}-2x\ln y)y}{x^2+3y^3e^{-x}},$$

which yields

$$y'(0)=\frac{y^4(0)}{3y^3(0)}=\frac{2}{3}.$$

Hence the solution is $\boxed{y(x)=2+\frac{2x}{3}.}$

5. Let $F(x,y)=x^4+y^4-36xy$. Find all stationary points for this function and determine whether they are local maximums, minimums, or saddle points.

Solution Observe that the system

$$0 = F_x = 4(x^3 - 9y). \qquad 0 = F_y = 4(y^3 - 9x),$$

yields that

$$y = \frac{x^3}{9}.$$

Substituting in the other equation we have that

$$\frac{x^9}{9^3} - 9x = 0 \Leftrightarrow x = 0 \quad \text{or} \quad x^8 = 9^4 = 3^8 \Leftrightarrow x = 0 \quad \text{or} \quad x = \pm 3.$$

Hence, there are three stationary points: $\boxed{X_1 = (0,0), X_2 = (3,3) \text{ and } X_3 = (-3,-3).}$ Observe that

$$F_{xx} = 12x^2, \qquad F_{xy} = F_{yx} = -36, \qquad F_{yy} = 12y^2.$$

Hence, the determinat of the Hessian is

$$D(x,y) = F_{xx}F_{yy} - F_{xy}^2 = 36(4x^2y^2 - 1).$$

Evaluating on the stationary points we obtain that

x	y	F_{xx}	D	Type
0	0	0	$-36 < 0$	Saddle point
3	3	$12 * 9 = 104 > 0$	$36(4 * 81 - 1) > 0$	Local Minimum
-3	-3	$= 104 > 0$	$36(4 * 81 - 1) > 0$	Local Minimum

That is $\boxed{X_1 \text{ is a saddle point and } X_2, X_3 \text{ are local minima.}}$

6. Compute the Taylor polynomial of degree 2 for the function $g(x) = \sqrt[3]{x + 1000}$, centered at the point $a = 0$. Use this polynomial to find an approximate value of $\sqrt[3]{1003}$.

Solution The solution is given by

$$P(x) = g(0) + g'(0)x + \frac{g''(0)}{2}x^2$$

Now $g'(x) = \frac{(x+10^3)^{\frac{-2}{3}}}{3}$ and $g''(x) = -\frac{2(x+10^3)^{\frac{-5}{3}}}{9}$. Hence

$$g(0) = 10, \qquad g'(0) = \frac{10^{-2}}{3} \qquad \text{and} \qquad g''(0) = -2\frac{10^{-5}}{9}.$$

Thus $\boxed{P(x) = 10 + \frac{10^{-2}}{3}x - \frac{10^{-5}}{9}x^2.}$

Now

$$\sqrt[3]{1003} \approx P(3) = 10 + 10^{-2} - 10^{-5} = 10^{-5}(10^6 + 10^3 - 1) = 10^{-5}(1001000 - 1) = \boxed{10{,}00999.}$$

NB: The real value is approximatedly **10.00999**1663...

7. Solve the following system of linear equations using Cramer's Rule:

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 4x_3 & = & 2 \\ & & -x_2 & - & 3x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & 6x_3 & = & 2 \end{array}$$

Solution Cramer's Rule gives the unique solution

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}}{|\mathbf{A}|} \qquad x_2 = \frac{\begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}}{|\mathbf{A}|} \qquad x_3 = \frac{\begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}}{|\mathbf{A}|}$$

provided that $|\mathbf{A}| \neq 0$.

So we compute:

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{vmatrix} = 1\begin{vmatrix} -1 & -3 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} 0 & -3 \\ 2 & 6 \end{vmatrix} + 4\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix}$$
$$= 1(-6+6) - 2(0-(-6)) + 4(0+2) = -12+8 = -4$$

Similar determinant calculations for the numerators yield

$$\begin{vmatrix} 2 & 2 & 4 \\ 1 & -1 & -3 \\ 2 & 2 & 6 \end{vmatrix} = -8$$
$$\begin{vmatrix} 1 & 2 & 4 \\ 0 & 1 & -3 \\ 2 & 2 & 6 \end{vmatrix} = -8,$$
$$\begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & 1 \\ 2 & 2 & 2 \end{vmatrix} = 4,$$

so

$$\boxed{(x_1, x_2, x_3) = \left(\frac{-8}{-4}, \frac{-8}{-4}, \frac{4}{-4}\right) = (2, 2, 1)}$$

is the solution.
