HOMEWORK 1
DUE THURSDAY SEPTEMBER 28 AT 10:15

Note: You may email me your homework if you prefer, but I will only accept typed pdf submissions via email. You may collaborate with other students, but make sure to write up your solutions on your own.

Throughout this homework set, let k be a field. Let $E_{i,j}$ be the matrix with "1" in the (i, j) entry and "0" elsewhere. For all $i \neq j$, let $\epsilon_{i,j} = I_n + E_{i,j}$ and $U_{i,j} = \{I_n + aE_{i,j} | a \in k\}$.

The transpose of a matrix A is denoted ${}^tA$.

Problem 1 (Some computations in $GL(n)$ to get you warmed up)**.**

(a) Show that $\varphi : A \mapsto {}^tA^{-1}$ is an automorphism of $GL(n, k)$. (This automorphism is known as "inverse transpose"; the order of the operations does not matter in this case.)

(b) Show that φ is not inner when $n > 1$ and $\mathrm{Card}(k) > 3$.

(c) For all $i \neq j$, show that $U_{i,j}(k)$ is isomorphic to k as groups (where k is viewed as a group under addition and $U_{i,j}(k)$ is a group under matrix multiplication).

(d) Compute/explain the effect of φ on

(i) the subgroup of diagonal matrices

(ii) the subgroup of upper-triangular matrices

(iii) $\epsilon_{i,j}$

(e) Describe the fixed points of φ.

Problem 2 (Analogous computations in M_n)**.** *Let $M_n(k)$ denote the ring of all $n \times n$ matrices with entries in k. Recall that the bracket $[\cdot, \cdot]$ on $M_n(k)$ is given by $[A, B] = AB - BA$.*

(a) Show that $d\varphi : A \mapsto -{}^tA$ is a linear automorphism of $M_n(k)$ (as vector space over k) which preserves the bracket. (As the notation suggests, we will later see that $d\varphi$ can be obtained from φ by "differentiation".)

(b) Show that $d\varphi$ is not inner for all k. (In this case "inner" means of the form "bracket with X" for some $X \in M_n(k)$ i.e. $A \mapsto [A, X]$.)

(c) Compute/explain the effect of $d\varphi$ on

(i) the subgroup of diagonal matrices

(ii) the subgroup of upper-triangular matrices

(iii) $E_{i,j}$

(d) Describe the fixed points of $d\varphi$.

The relationship between the objects in Problem 1 and those in Problem 2 is a special case of the correspondence between algebraic groups and their Lie algebras: $M_n(k)$ is the Lie algebra of $GL(n, k)$.

Later in the course, we will see the notion of a *pinning* of a *based root datum*. It turns out that the diagonal subgroup of $GL(n)$ together with the subset of all the $U_{i,j}$ given by

$$\{U_{1,2}, \dots, U_{n-1,n}\}$$

determines a based root datum of $GL(n)$. We will see that choosing an element of each $U_{i,i+1}$ gives a pinning. A standard choice is $\epsilon_{i,i+1} \in U_{i,i+1}$.

Given any automorphism ϕ' of $GL(n)$, there exists $X \in GL(n)$ such that $X\phi' X^{-1}$ leaves the pinning $\{\epsilon_{1,2}, \dots \epsilon_{n-1,n}\}$ stable (it may not fix the pinning pointwise).

Problem 3. *In the case that $\varphi =$ "inverse transpose", find $X \in GL(n)$ such that $X\varphi X^{-1}$ leaves the above pinning stable.* Hint: *If you don't know where to start, try the 2×2 and 3×3 cases first.*

Problem 4. *Let V be an n-dimensional vector space over k. Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^n$ be two bases of V. Let X denote the change of basis matrix defined by $Xe_i = f_i$ for all i. Suppose $Q : V \times V \to k$ is a bilinear form (recall this means Q is linear in each variable). The matrix A of Q in the basis $\{e_i\}$ is defined as the matrix whose (i, j)-entry is $Q(e_i, e_j)$. Show that the matrix B of Q in the basis $\{f_i\}$ satisfies*

$$B = {}^tXAX.$$

E-mail address: wgoldring@math.su.se.

Hint: *One way to do this is by directly applying the definition of matrix multiplication several times. Another is to first show that, if two vectors* u, v *are expressed in the basis* $\{e_i\}$*, then the value* $B(u, v)$ *is given by matrix multiplication as* $B(u, v) = {}^t uAv$*.*

The following problem gives a special example of a "flag variety", one of the topics that will hopefully be studied in the second half of the course.

Problem 5. *Recall that* $(n-1)$*-dimensional projective space* $\mathbf{P}^{n-1}(k)$ *is defined by taking the quotient of* $k^n \setminus \{0\}$ *by the equivalence relation* $(a_1, \dots, a_n) \sim (b_1, \dots, b_n)$ *if there exists* $\lambda \in k^*$ *such that* $\lambda a_i = b_i$ *for all* i*. The equivalence class of* $(a_1, \dots, a_n)$ *is often written* $[a_1, \dots, a_n]$ *and known as "homogeneous coordinates".*

(a) *Describe an action of* $GL(n, k)$ *on* $\mathbf{P}^{n-1}(k)$*.*
(b) *Show that this action is transitive.* Hint: *Study the orbit of* $[1, 0, \dots, 0]$*.*
(c) *Describe the stabilizer of* $[1, 0, \dots, 0]$*.*