Graphs with prescribed degree sequence

- Consider a random graph of \( n \) vertices
- Fix the degree of vertex \( i \) by \( d_i \)
- What do we know about the graph?
- How many graphs are possible?
  (Try for \( n = 4 \) and all degrees are 2)
- Is it always possible to create such a graph?

Easier case: Multigraph with prescribed degree sequence, self-loops add two to the degree

Still not all sequences are possible. Indeed \( \ell_n = \sum_{i=1}^{n} d_i \) should be even

One way to create a random graph is:
- draw \( d_i \) half edges pointing out of vertex \( i \) for all \( i \)
- pair the \( \ell_n \) half edges uniformly at random

The number of ways to pair up \( \ell_n \) half-edges is
\[
(\ell_n - 1)!! = (\ell_n - 1) \times (\ell_n - 3) \times \cdots \times 3 \times 1
\]

Not all pairings lead to different graphs

Let \( X_{ij} = X_{ji} \) be the number of edges between \( i \) and \( j \) in the multigraph and \( X_{ii} \) the number of self-loops
- \( d_i = X_{ii} + \sum_{j=1}^{n} X_{ij} \) (So, \( X_{ii} \) is counted twice)

Number of different matchings of half-edges which lead to graph \( G \) described by \( \{X_{ij}\}_{i,j} \) is
\[
\frac{\prod_{i=1}^{n} d_i!}{(\prod_{i=1}^{n} 2^{X_{ii}}) \prod_{i=1}^{n} \prod_{j=1}^{n-1} X_{ij}!}
\]

We see that all simple graphs have equal probability

Some conditions (7.5)

Let \( F_n(x) = n^{-1} \sum_{i=1}^{n} \mathbb{1}(d_i \leq x) \), and \( D_n \) the corresponding random variable
- \( D_n \) converges in distribution to some specified distribution \( D \) with distribution function \( F(x) \)
  - In this case this means \( \lim_{n \to \infty} F_n(x) = F(x) \) for all \( x \geq 0 \)
- \( \lim_{n \to \infty} \mathbb{E}(D_n) = \mathbb{E}(D) \)
- Sometimes we assume \( \lim_{n \to \infty} \mathbb{E}(D_n^2) = \mathbb{E}(D^2) \)

An example: Create degree sequence through drawing i.i.d. random variables
Exercise: Show that the conditions are satisfied (sometimes under extra conditions). What conditions are needed?
What if the graph is not simple?

- We might erase the self-loops and merge the multiple edges
- We might condition on the random graph being simple

Note that in multi-graph the expected number of self-loops is

\[ \frac{1}{2} \sum_{i=1}^{n} d_i (d_i - 1) \frac{\ell_n - 1}{\ell_n} \]

The number of multiple edges is roughly the square of this

Thm 7.8 and Prop 7.9

- Let \( \mu = \mathbb{E}(D) \) and \( \nu = \mathbb{E}((D)(D - 1))/\mu \)
- The number of self-loops is asymptotically Poisson with mean \( \nu/2 \)
- The number of multiple edges is asymptotically Poisson with mean \( \nu^2/4 \), independently.
- Number of simple random graphs with given degree sequence is roughly

\[ e^{-\nu/2-\nu^2/4} \frac{(\ell_n - 1)!!}{\prod_{i=1}^{n} d_i!} \]

Theorem (Cor 7.12)

If the degree sequence satisfies all conditions in 7.5 and \( \ell_n \) is even. Then an event \( A \) occurs with high probability in the uniform simple random graph with prescribed degree distribution, if it occurs with high probability in the corresponding configuration model multi-graph.

Proof:

\[ \mathbb{P}(A^c \text{ in simple graph}) = \mathbb{P}(A^c \text{ in CM | CM graph is simple}) \]

\[ = \frac{\mathbb{P}(A^c \text{ in CM and CM graph is simple})}{\mathbb{P}(\text{CM graph is simple})} \leq \frac{\mathbb{P}(A^c \text{ in CM})}{\mathbb{P}(\text{CM graph is simple})} \]

and result follows

Components: playing the physicist’s game

- Assume that \( n \) vertices have i.i.d. degrees with distribution \( D \), represented by half edges. Let \( p_k = \mathbb{P}(D = k) \)
- In case \( \ell_n \) is odd, add one half-edge to vertex \( n \)
- pair half-edges uniformly at random one by one (but in a nice order)
- What is the size of the component of vertex 1?
The number of neighbors of vertex 1 is distributed as $D$.

The degree of a neighbor of vertex 1 is not distributed as $D$.

Let $\mathbb{P}(\tilde{D} = k) = g_k = kp_k / \mathbb{E}(D)$.

Let 

$$\nu = \mathbb{E}(\tilde{D} - 1) = \sum_{k=1}^{\infty} (k-1) g_k = \sum_{k=1}^{\infty} \frac{(k-1)kp_k}{\mathbb{E}(D)} = \frac{\mathbb{E}(D(D-1))}{\mathbb{E}(D)}$$

The degrees of neighbors of 1 are assumed to be distributed as $\tilde{D}$ and independent.

Question: Why is this not correct?

The expected number of vertices at distance 2 of vertex 1 is $\mathbb{E}(D) \times \nu$.

What does $\tilde{D}$ look like?

- Exercise in pairs: Assume $\mathbb{P}(D = k) = \frac{\lambda^k}{k!} e^{-\lambda}$, give distribution of $\tilde{D}$?
- If $\mathbb{P}(D = k) = ck^{-\gamma}$, for $\gamma > 2$, then $\mathbb{P}(\tilde{D} = k) = \frac{k^{-\gamma-1}}{\sum_{k=1}^{\infty} k^{-\gamma}}$.
- Note that if $D$ has infinite variance, then $\tilde{D}$ has infinite mean.
- Let $G(x) = \mathbb{E}(x^D) = \sum_{k=0}^{\infty} p_k x^k$.

$$G(x) = \mathbb{E}(x^{\tilde{D} - 1}) = \sum_{k=1}^{\infty} g_{k-1} x^{k-1} = \sum_{k=1}^{\infty} \frac{kp_k}{\mathbb{E}(D)} x^{k-1} = \sum_{k=0}^{\infty} \frac{p_k}{\mathbb{E}(D)} \frac{d}{dx} x^k = \frac{d}{dx} \mathbb{E}(D) \Bigr|_{y=1}$$

As long as the number of paired half-edges is small, the degree distribution of the vertex of a uniformly chosen half-edge is roughly $\tilde{D}$.

Furthermore, the probability of choosing a half-edge belonging to a vertex with all its half-edges previously unpaired is high as well.

Use two stage branching process: number of children of generation 0 vertex is distributed as $D$.

Number of children of other vertices in process distributed as $\tilde{D} - 1$.

Probability of extinction of branching process with offspring distribution defined by $g_k$ is given by smallest solution of

$$\xi = \sum_{k=1}^{\infty} g_k \xi^k$$

Survival probability of the two-stage branching process approximating the cluster of vertex 1 is

$$1 - \sum_{k=1}^{\infty} p_k \xi^k$$

This survival probability is positive if $\xi < 1$ and 0 if $\xi = 1$.

$\xi < 1$ if and only if $\nu = \sum k g_k = \mathbb{E}(D(D-1))/\mathbb{E}(D) > 1$ (or $g_1 = 2p_2 / \mathbb{E}(D) = 1$, i.e. if $p_2 = 1$).
Theorem (Thm. 10.1)
Assume that $D_n \rightarrow D$ in distribution and $\mathbb{E}(D_n) \rightarrow \mathbb{E}(D)$. Furthermore $p_2 < 1$.

- If $\nu > 1$, then there exists $\zeta \in [0, 1]$ and $\xi \in [0, 1]$, such that
  \[\frac{|C_{\max}|}{n} \rightarrow \zeta \text{ in probability}\]
  and $n^{-1}$ times the number of vertices of degree $k$ in $C_{\max}$
  converges in probability to $p_k(1 - \xi^k)$
  and $n^{-1}$ times the number of edges in $C_{\max}$ converges in
  probability to $\mathbb{E}(D)(1 - \xi^2)/2$
  the proportion of vertices and edges in the second largest
  component converges in probability to 0
- If $\nu < 1$, then the proportion of vertices and edges in $C_{\max}$
  converges in probability to 0.

Questions to think about

- What happens if $p_2 = 1$?
- Why is in the sub-critical case the largest cluster size not necessarily $\Theta(\log n)$?