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Course MM1005

Lecture 1:Review of elementary algebra

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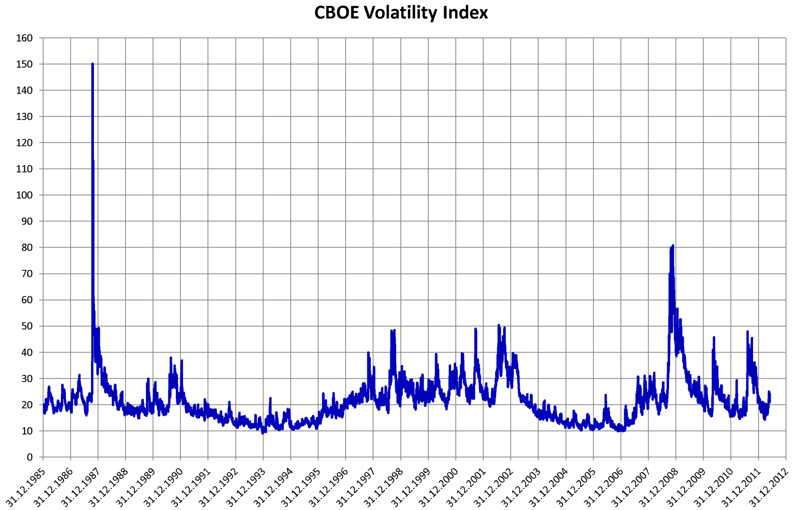
Logistic Information

- **Book:** Sydsaeter & Hammond, *Essential Mathematics for Economic Analysis (6:th ed)*
- **Time and Place:** Check in scheman.su.se
- **Final Exam:** (Prel.) Wednesday 28/9 (v39), starting 14:00. Re-exam (omtentamen) is on Wednesday 9/11 (v45), starting at 8:00

Disclaimer

We are not going to do this:



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Because you would need to know this:

Definition of VIX

VIX expresses volatility in percentage points. It is calculated as 100 times the square root of the expected 30-day variance of the rate of return of the forward price of the S&P 500 index.

$$\text{VIX} = 100\sqrt{\text{forward price of realized cumulative variance}}$$

Suppose the forward price F_t of the S&P index under a risk neutral measure Q follows

$$\frac{dF_t}{F_t} = \sigma_t\ dW_t \text{ so that } d\ln F_t = -\frac{\sigma_t^2}{2}\ dt + \sigma_t\ dW_t.$$

Here, the instantaneous volatility function σ_t is stochastic.

When one computes the differential of a function of F_t, an additional drift term $-\frac{\sigma_t^2}{2}dt$ in $d\ln F_t$ arises from the Ito lemma, where

$$\frac{1}{2}\sigma_t^2 F_t^2 (dW_t)^2 \frac{\partial^2}{\partial F_t^2}\ln F_t = -\frac{\sigma_t^2}{2}\ dt.$$

So what are we going to do?

- Functions of one variable.
- Derivatives, integration.
- Optimization.
- Same for functions in two variables.
- Linear algebra: Systems of linear equations. Determinants.
- Geometric sums.

It is a lot of material and the pace will be quick. This class cover part of the mathematical background that prestigious schools think you need in order to enter their programs:

`https://www.imperial.ac.uk/business-school/programmes/msc-risk-management/admissions/self-assessment-maths-test/`

Are they crazy?

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the kth row of $\mathbf{\Lambda}$. Then

$$\frac{\partial \ln L_S(\mathbf{X}_i, i=1,\ldots,n)}{\partial \boldsymbol{\beta}} = \sum_{i=1}^{n} \frac{1}{R} \sum_{r=1}^{R} \frac{-1}{\sigma_\varepsilon^2}(y_{it} - \mathbf{x}_{it}'(\boldsymbol{\beta} + \mathbf{\Lambda}\mathbf{v}_{ir}))(-\mathbf{x}_{it}) = \mathbf{0},$$

$$\frac{\partial \ln L_S(\mathbf{X}_i, i=1,\ldots,n)}{\partial \boldsymbol{\lambda}_k} = \sum_{i=1}^{n} \frac{1}{R} \sum_{r=1}^{R} \frac{-1}{\sigma_\varepsilon^2}(y_{it} - \mathbf{x}_{it}'(\boldsymbol{\beta} + \mathbf{\Lambda}\mathbf{v}_{ir}))(-x_{itk}\mathbf{v}_{ir}) = \mathbf{0}. \quad \textbf{(9-58)}$$

[The derivatives with respect to all the rows of $\mathbf{\Lambda}$ can be collected in one expression by replacing $(-x_{itk}\mathbf{v}_{\mathbf{ir}})$ with $(\mathbf{x}_{it} \otimes \mathbf{v}_{ir})$.] Note that σ_ε^2 does not affect the solutions to the likelihood equations in (9-58). To see this, multiply both equations by $-\sigma_\varepsilon^2$ and the same solution emerges independently of σ_ε^2. Thus, we can view the solution as the counterpart to least squares, which we might call minimum simulated sum of squares. Once the simulated sum of squares is minimized with respect to $\boldsymbol{\beta}$ and $\mathbf{\Lambda}$, then the solution for σ_ε^2 can be obtained via the likelihood equation

$$\frac{\partial \ln L_S(\mathbf{X}_i, i=1,\ldots,n)}{\partial \sigma_\varepsilon^2} = \sum_{i=1}^{n} \frac{1}{R} \sum_{r=1}^{R} \frac{-T}{2\sigma_\varepsilon^2} + \frac{\sum_{t=1}^{T}(y_{it} - \mathbf{x}_{it}'(\boldsymbol{\beta} + \mathbf{\Lambda}\mathbf{v}_{ir}))^2}{2\sigma_\varepsilon^4} = 0.$$

Multiply both sides of this equation by $-2\sigma_\varepsilon^4/T$ to obtain the equivalent condition,

$$\sum_{i=1}^{n} \frac{1}{R} \sum_{r=1}^{R} \left[\sigma_\varepsilon^2 - \frac{\sum_{t=1}^{T}(y_{it} - \mathbf{x}_{it}'(\boldsymbol{\beta} + \mathbf{\Lambda}\mathbf{v}_{ir}))^2}{T}\right] = 0.$$

This implies that the solution for σ_ε^2 is obtained by

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{R} \sum_{r=1}^{R} \frac{\sum_{t=1}^{T}(y_{it} - \mathbf{x}_{it}'(\boldsymbol{\beta} + \mathbf{\Lambda}\mathbf{v}_{ir}))^2}{T}$$

Section 1
Lecture 0

Lecture Goal and Outcome

Goal: Review of elementary algebra (should be at the Matte 2b/c level)

Learning Outcome: At the end of the lecture you will be able to give meaning to expressions of the form

$$a^q,$$

where a is a positive real number and q is any rational number.

Study Tip: This should be a review of topic you encountered in your high school math classes (Should be Matte 2 b or c). Skim through chapter 2 of the book and try the difficult exercises. If you have problem read more carefully the text and exercise more.

Why you should care

Powers appears in many instance in Economics:

$$K_t = K_0 \left(1 + \frac{p}{100}\right)^t$$

Need for negative exponents: How much should I invest today if I want K kr when I retire in 40 years?

Need for rational exponents: How much money will I have in 30 months?

Why you should care II

Here is a recurring formula appearing in studies of national economic growth:

$$Y = 2{,}262(K)^{0{,}203}(L)^{0{,}763}(1{,}02)^{t}$$

where

- Y is the national product
- K is the capital invested in stock
- L is the labor
- t is time

Lecture Plan

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- Numbers Sets (natural numbers, integers, rational and real numbers). Operations and rules. Operation with fractions.
- Powers (with rational and integer exponents). Rules for computations.
- Algebraic expressions. Computing and simplifying algebraic expressions.
- Binomial coeffients.

Section 2
Numbers Sets

Some sets

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- Natural Numbers $\mathbb{N} := \{0, 1, 2, \ldots\}$
- Integers $\mathbb{Z} := \{\ldots, -1, 0, 1, 2, \ldots\}$
- Rational numbers $\mathbb{Q} := \left\{ \frac{a}{b} \,|a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0 \right\}$
  Attention: They are not numbers with a finite decimal expansion!
  Example:$\frac{1}{3} = 0{,}3333 \cdots$
- The real numbers $\mathbb{R} = \{1, 0, 2\sqrt{2}, -\frac{2}{5}, \pi, e, e^{0,33}\}$

Notation

- $x \in \mathbb{Z}$ means "x is an integer"
- $b \notin \mathbb{Z}$ means "b is not an integer"
- $\mathbb{R} \backslash \mathbb{Q} := \{x \in \mathbb{R} \mid x \notin \mathbb{Q}\}$.
- $\mathbb{R}^* := \mathbb{R} \backslash \{0\}$
- $\mathbb{R}^+$, $\mathbb{R}^-$, $\mathbb{R}^{\geq 0}$, $\mathbb{R}^{||0||}$

Rules of operations

- $a + b = b + a$
- $ab = ba$
- $(a + b) + c = a + (b + c)$
- $(ab)c = a(bc)$
- $a(b + c) = ab + bc$

Attention: Order matters: $3 + 4 \cdot 5 =$

Operations with fractions

- $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$
- $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$
- $\frac{a}{b} = \frac{ac}{bc}$ for $c \neq 0$
- $\frac{a}{b} / \frac{c}{d} = \frac{ad}{bc}$ for $c \neq 0$

Examples

- $\frac{1}{2} + 1 + \frac{1}{3} + \frac{1}{4} =$
- $\left(\frac{1}{2} - \frac{1}{3}\right) / \left(\frac{3}{4} - \frac{1}{2}\right) =$
- Simplify $\frac{153}{42}$

Questions?

Section 3
Powers

Powers with natural exponents

Let a be a real number and n a natural number.

$$a^n := \underbrace{a \cdot a \cdots a}_{n \text{ times}} \qquad \text{if } n > 0$$

$$a^0 := 1$$

We say that a is the base and n the exponent

Powers with integer exponents

Let a be a **non zero** real number and n an integer.

$$a^0 := 1$$

$$a^n := \underbrace{a \cdot a \cdots a}_{n \text{ times}} \qquad \text{if } n > 0$$

$$a^n := \underbrace{\frac{1}{a} \cdot \frac{1}{a} \cdots \frac{1}{a}}_{-n \text{ times}} \qquad \text{if } n < 0$$

Rules

- $(a^n)^m = a^{n \cdot m}$
- $a^n \cdot a^m = a^{n+m}$
- $a^n / a^m = a^{n-m}$
- $(ab)^n = a^n b^n$

Rational exponents

Now we take a a positive real number q a positive integer. We define

$$a^{\frac{1}{q}} := \sqrt[q]{a}$$

That is $a^{\frac{1}{q}}$ is the unique positive real number b such that $b^q = a$

Common misconception

$$\sqrt{4} = \pm 2$$

This is false! We have that $\sqrt{4} = 2$. However we have that $x^2 = 4$ has two solutions: $\pm\sqrt{4} = \pm 2$,

Questions?

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Rational exponents II

Now we take a a positive real number $\frac{p}{q}$ a rational number (with $q > 0$).

$$a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (\sqrt[q]{a})^p$$

Alternatively

$$a^{\frac{p}{q}} = (a^p)^{\frac{1}{q}} = (\sqrt[q]{a^p})$$

Rules

The same computational rules hold for rational exponents! That is, if $a \in \mathbb{R}^+$ and $m,\ n \in \mathbb{Q}$ we have:

- $(a^n)^m = a^{n \cdot m}$
- $a^n \cdot a^m = a^{n+m}$
- $a^n / a^m = a^{n-m}$
- $(ab)^n = a^n b^n$

Examples

Compute the following expressions:

$$\frac{2^{\frac{1}{4}} \cdot 8^{\frac{1}{4}} \cdot 4^{\frac{1}{4}}}{2 \cdot 2^{-\frac{1}{2}}}$$

$$3\left(\sqrt{3}+2\sqrt{\frac{1}{3}}\right)^2$$

Section 4
Algebraic expressions

Algebraic expressions

An algebraic expressoin is anything that can be created with letters ($a, b, c \cdots x, y, z$) operations sign (=+-/$\times$) powers and real numbers.

- $E = mc^2$
- $K_t = K_0(1 + \frac{p}{100})^t$
- $x^2 + y^2 + b$

You can do operation with algebraic expressions and the same rule for computing holds as for operation between numbers.

Examples

- $(a+b)^2 = a^2 + 2ab + b^2$
- $a^2 - b^2 = (a+b)(a-b)$
- Expand $(1 - y^{-2})(y^2 - 2y^{-1} + 3)$
- Compute $(1+\sqrt{7})(1-\sqrt{-7})$
- Simplify
$$\frac{8\sqrt[3]{x^2}\sqrt[4]{y}\sqrt{\frac{1}{z}}}{-2\sqrt[3]{x}\sqrt{y^5}\sqrt{z}}$$
- Factor $e^{3x} + 2e^{2x} + e^x$
- Factor $7x^3y - 28xy$
- Simplify
$$\frac{30x^2 - 30a}{x+a}$$

Questions?

Section 5
Binomial Coefficients

Factorial

Let n a natural number, we define the factorial of n as:

$$n! = 1 \cdot 2 \cdots (n-1) \cdot n \quad \textbf{if } n > 0$$

$$0! = 1$$

Binomial coefficients

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Given n and k two natural numbers the binomial coefficient "n choose k" is denoted by

$$\binom{n}{k}$$

and compute the number of ways we can pick k distinct object out of n (the order does not count).

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{if } k \leq n$$

$$\binom{n}{k} = 0 \quad \text{if } k > n$$

Useful Identities

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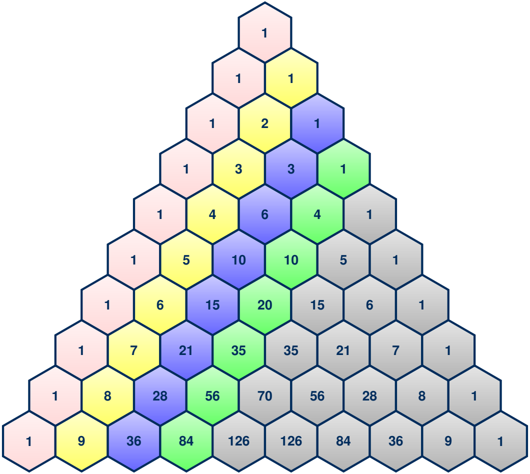
$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$$

Pascal Triangle



Newton binomial theorem

Binomial Theorem

Given n a natural number we have that

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \cdots + \binom{n}{n-1}ab^{n-1} + \binom{n}{1}b^n$$
$$= \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

- $(a+b)^2 =$
- $(a+b)^5 =$

Thank you for your attention!