MM1005 - Fall 2021

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Lecture 1 Functions

All real numbers ($1, 2, 3, \frac{10}{7}, \pi, e, \ldots$)
$-1, -\frac{7}{2}, -2e, -e^2, \ldots$

Def. A function $f : D \to \mathbb{R}$ is a rule that assigns to each element x in D a unique real value $f(x)$

$$x \longmapsto f(x)$$

- D is called the domain of definition of f
- The set of all possible values $f(x)$ when x is an element of D ($x \in D$ for short) is called the range of f

Examples

- $f : \{\text{All students}\} \longrightarrow \text{Height}$

  $\text{Karl} \longmapsto f(\text{Karl}) \equiv \text{Karl's height}.$

- $R(p) = p^2$

$$p \longmapsto p^2$$
$$3 \longmapsto 3^2 = 9$$
$$1 \longmapsto 1^2 = 1$$

$R : \mathbb{R} \longrightarrow \mathbb{R}$
$p \longmapsto p^2$

Given a function $f: D \to \mathbb{R}$ we define its graph as all the points in the plane of the form $(x, f(x))$ where $x \in D$

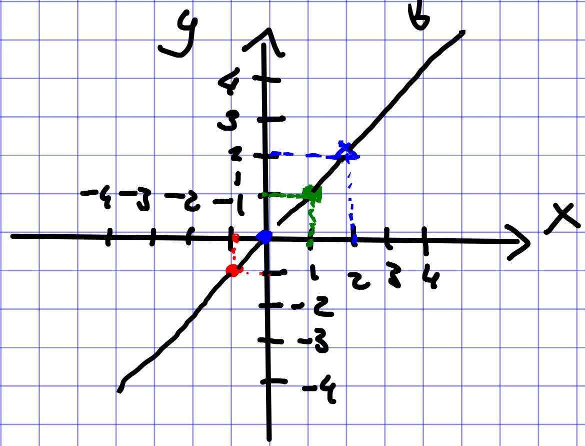
Example

a)

x	$f(x)$
-1	$f(-1) = -1$
-2	$f(-2) = -2$
0	0
1	1
2	2

$$\left[\begin{array}{l} f: \mathbb{R} \to \mathbb{R} \\ f(x) = x \end{array}\right]$$

graph of f

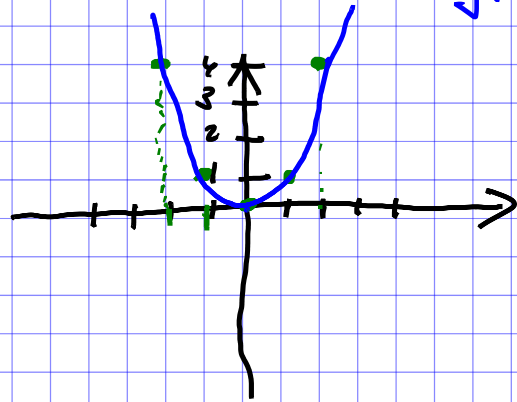


b)

x	$f(x)$
-2	$f(-2) = (-2)^2 = 4$
-1	$f(-1) = (-1)^2 = 1$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$

$$f: \mathbb{R} \to \mathbb{R}$$
$$x \mapsto x^2$$

Graph of $f(x) = x^2$



A polynomial of degree n $(= 0, 1, 2, 3, \cdots)$ with coefficients $a_n, a_{n-1}, \dots, a_0$ with $a_n \neq 0$ is an expression of the type

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 .$$

Example

1) $x^3 + 2x^2 + 3x + 1$ Polynomial of degree 3.

2) $2x^4 + 3x + 2$ polynomial of degree 4

3) $2x + 3$ polynomial of degree 1.

(Note $x^1 = x$ and $x^0 = 1$)

$2021 = 2021 \cdot x^0$ polynomial of degree 0

4) $x^2 + 2x + 1$ polynomial of degree 2

Affine functions (Linear) [Polynomials of degree 1]

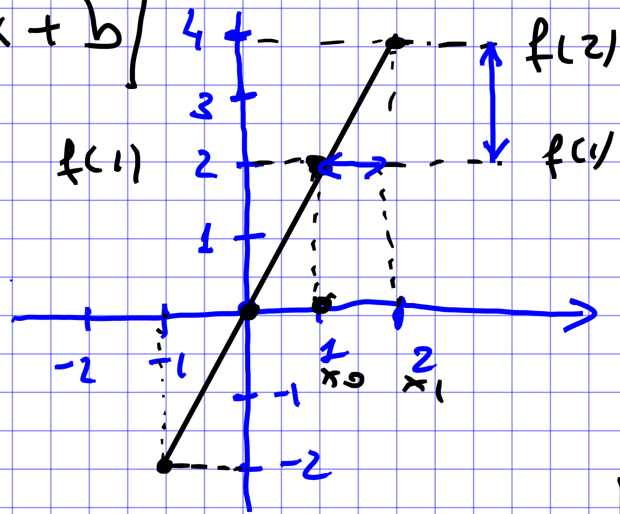
Given $a, b \in \mathbb{R}$ (fixed) we define $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = a \cdot x + b$$

Example

$f(x) = 2x$

x	$f(x)$
-1	$2(-1) = -2$
0	$2 \cdot 0 = 0$
1	$2 \cdot 1 = 2$
2	$2 \cdot 2 = 4$



Examples

1) $2x = 2x + 0$
2) $3x + 1$
3) $4x + 2$

Note that for $x_0 < x_1$

$$f(x_1) - f(x_0) = (a x_1 + b) - (a x_0 + b)$$
$$= a(x_1 - x_0)$$

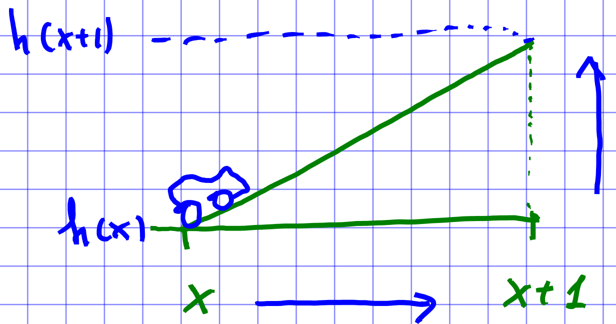
$x_0 = 1$
$x_1 = 2$

$f(2) - f(1) =$
$= (a \cdot 2 + b) - (a \cdot 1 + b)$
$= 2 \cdot a + \boxed{b} - a - \boxed{b}$
$= a + 0 = a$

So it follows that

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = a \quad \text{is } \underline{\text{constant}}$$

This constant is called the <u>slope</u> of the line



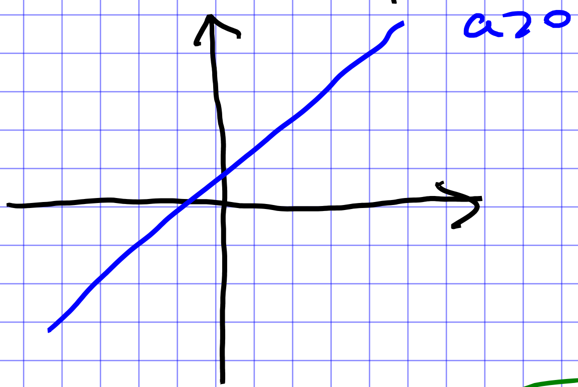
We model a road like a line:

$$f(x) = ax + b$$

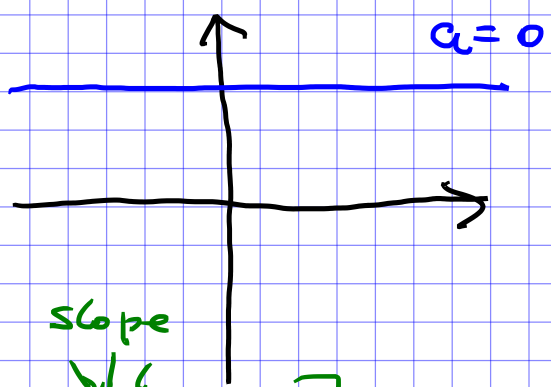
Then

$$a = \frac{h(x+1) - h(x)}{(x+1) - x} = \frac{h(x+1) - h(x)}{1} = \frac{10}{100}$$

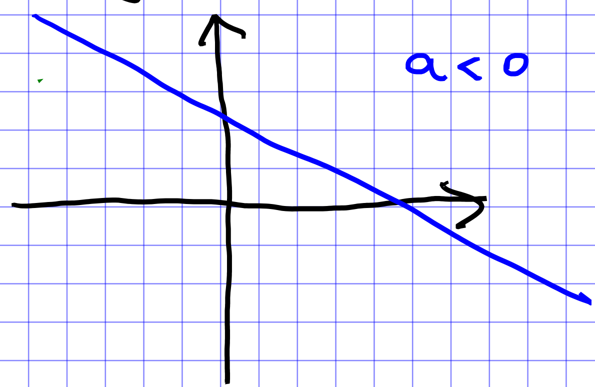
Positive slope



Zero slope



Negative slope

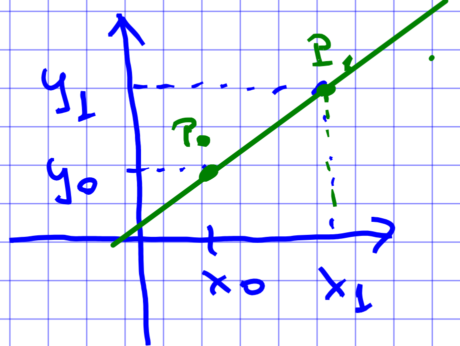


$$\Big[\; C(x) = \overset{\text{slope}}{a}\,x + b$$

C being the cost function $\Big]$

Example Find the line on the plane that goes through two points $P_0 = (x_0, y_0)$ and $P_1 = (x_1, y_1)$

- We know that the slope is constant
- The equation is of the type $f(x) = ax + b$
- the graph goes through the points P_0 & P_1 i.e.



$$y_0 = f(x_0) = a x_0 + b$$
$$y_1 = f(x_1) = a x_1 + b$$

$\underset{f(x_0)}{(x_0, y_0)}, \underset{f(x_1)}{(x_1, y_1)}$ lie on the graph of f

[x_0, x_1, y_0, y_1 given.]

[How can we find a, b?]

Note that

1) $y_1 - y_0 = a(x_1 - x_0)$ so $\left[a = \dfrac{y_1 - y_0}{x_1 - x_0} \right]$

2) $[y_0 = a x_0 + b] \implies \left[b = y_0 - a x_0 \right]$

Hence

$$\left[f(x) = a x + (y_0 - a x_0) = \boxed{a}(x - x_0) + y_0 \right]$$

(line with slope a, going through (x_0, y_0)
$(y_0 = f(x_0))$

Thus

- $\boxed{f(x) = a(x-x_0)+y_0}$ is the equation of the line with slope a that goes through the point (x_0, y_0)

- $f(x) = \dfrac{y_1-y_0}{x_1-x_0}(x-x_0)+y_0$ is the equation of the line through two points $P_0=(x_0,y_0)$ & $P_1=(x_1,y_1)$

Example: Find the line that goes trough the point $\boxed{P=(-2,3)}$ and has $\boxed{\text{slope } -1}$

$f(x) = a x + b$

So we know that $f(x) = (-1)x+b$
$= -x+b$

Since the line goes through P, it has to satisfy that

$3 = f(-2) = (-1)(-2)+b = 2+b$

$\Rightarrow b = 3-2 = 1$

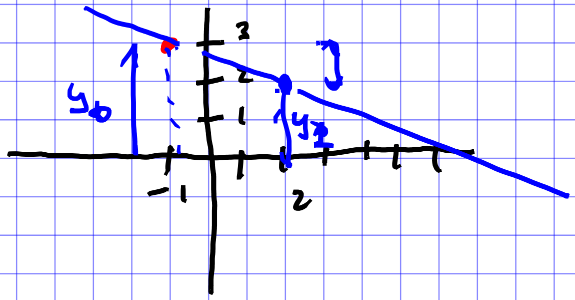
$f(x) = -x+1$

$f(x) = \underbrace{a}_{-1}(x-\underbrace{x_0}_{-2})+\underbrace{y_0}_{3}$

$f(x) = (-1)(x-(-2))+3$
$= -(x+2)+3 = -x-2+3$
$= -x+1$

Find the line passing through the points

$P_0 = (-1, 3) = (x_0, y_0) \qquad P_1 = (2, 2) = (x_1, y_1)$



Step 1: Find the slope $a = \dfrac{y_1 - y_0}{x_1 - x_0}$

Step 2: Reduce ourselves to finding the line of slope a passing throug P_0

Slope $\quad a = \dfrac{y_1 - y_0}{x_1 - x_0} = \dfrac{2-3}{2-(-1)} = \dfrac{-1}{2+1} = -\dfrac{1}{3}.$

then $\quad f(x) = -\frac{1}{3}(x - x_0) + y_0 = -\frac{1}{3}(x - (-1)) + 3$

$= -\frac{1}{3}(x+1) + 3 = -\frac{1}{3}x - \frac{1}{3} + 3$

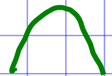
$= -\frac{1}{3}x + \underbrace{\left(3 - \frac{1}{3}\right)}_{\frac{3 \cdot 3 - 1}{3} = \frac{8}{3}} = -\frac{1}{3}x + \frac{8}{3}$

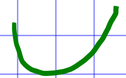
Quadratic functions (Polynomials of degree 2)

Given $a, b, c \in \mathbb{R}$ with $a \neq 0$

$$f(x) = ax^2 + bx + c$$

The graph is called a <u>parabola</u> that resembles

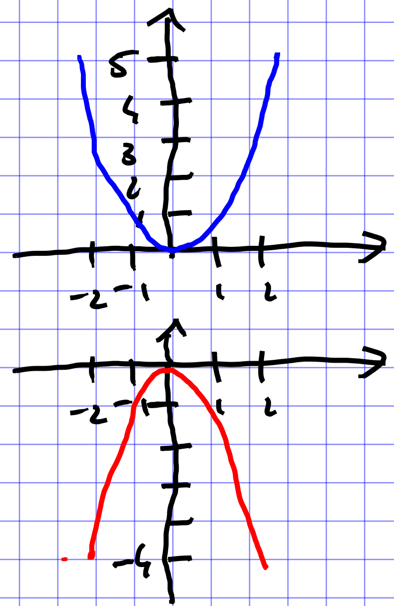
 (cave) if $a < 0$ (concave parabola)

 if $a > 0$ (convex parabola)

Examples

$\boxed{f(x) = 1x^2}$ $\quad$ $\boxed{g(x) = \overset{(-1)}{-}x^2}$

x	$f(x)$	$g(x)$
-2	4	-4
-1	1	-1
0	0	0
1	1	-1
2	4	-4



f convex

concave.

Profit = Revenue - Cost

- $R(p) = 10p - p^2$

$f(x) = ax^2 + bx + c$

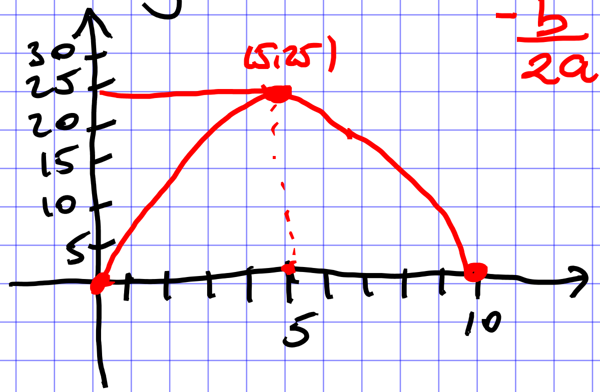
$a = -1$ $\quad c = 0$

$b = 10$

$R(x) = 10x - x^2$

$= -x^2 + 10x + 0$

What is the maximum revenue the company can bring in?



$-\frac{b}{2a} = \frac{-10}{2(-1)} = 5$

p	R(p)
0	0
10	0
5	$10 \cdot 5 - 5^2 = 50 - 25 = 25$

Note

1) $R(p) = (10 - p)p = 0$

if and only if $\quad 10 - p = 0 \iff p = 10$

$p = 0$ or $p = 10$

2) The graph of R is concave ∩

so R has a "maximum". value

Q: How to find it?

(Check)

Every function $f(x) = ax^2 + bx + c$ can be re-written

as

$$f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

See Appendix

2

Note that

1) $f\left(-\frac{b}{2a}\right) = c - \frac{b^2}{4a}$

2) If $a>0$, for all $x \in \mathbb{R}$

$$f(x) = \underbrace{a \cdot \underbrace{\left(x+\frac{b}{2a}\right)^2}_{\text{Positive}}}_{\text{Positive}} + \boxed{f\left(-\frac{b}{2a}\right)} \geq \boxed{f\left(-\frac{b}{2a}\right)}$$

So f attains its minimum value at $\boxed{-\frac{b}{2a}}$

3) If $a<0$

$$f(x) = \underbrace{\underbrace{a}_{<0} \cdot \overbrace{\left(x+\frac{b}{2a}\right)^2}^{\geq 0}}_{\text{Negative.}} + f\left(-\frac{b}{2a}\right) \leq f\left(-\frac{b}{2a}\right)$$

So f attains its maximum value at $-\frac{b}{2a}$

The point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$ is called the vertex of the parabola

In our problem above, the function

$$R(p) = 10p - p^2$$

attains its maximum value at

$$p_0 = \frac{-b}{2a} = 5$$

In that case

$$R(p_0) = 25$$ (The maximum revenue is 25 and is attained at $p=5$)

To solve an equation of the type

$(a \neq 0)$ $$ax + b = 0 \iff ax = -b \iff \left[x = -b/a \right]$$

To solve an equation of the type

$(a \neq 0)$ $$\left[ax^2 + bx + c = 0 \iff (*) \left[x = \frac{-b \pm \sqrt{b^2 - 4a \cdot c}}{2a} \right] \right]$$ (See Appendix)

$$\iff \left[\begin{array}{l} p = b/a \\ q = c/a \end{array} \right] \left[x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q} \right]$$

Example

ax^2+bx+c $a=1$ $c=1$ $b=2$

Solve $\boxed{x^2+2x+1=0}$ $\iff$ $x=\frac{-2\pm\sqrt{2^2-4\cdot 1\cdot 1}}{2\cdot(1)}=$

(Find all $x\in\mathbb{R}$ such that)

$$=\frac{-2\pm\sqrt{4-4}}{2}=\frac{-2\pm\sqrt{0}}{2}$$

$$=\frac{-2}{2}=-1$$

$\cdot A,B\in\mathbb{R}.$

$(A+B)^2=A^2+2A\cdot B+B^2$

$A^2-B^2=(A+B)(A-B)$

$$x^2+2x+1=x^2+2\cdot 1\cdot x+1^2=(x+1)^2=(x+1)(x+1)$$

So

$$x^2+2x+1=0\iff(x+1)(x+1)=0\iff x+1=0\iff x=-1.$$

Appendix

1) Rewrite ax^2+bx+c as $a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$

Note that

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x\right)+c$$

$$\star \qquad = a\left(x^2+2\frac{b}{2a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c$$

$$=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)$$

2) The solutions of $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

Note that by ($\star$) above

$$ax^2+bx+c=0 \Leftrightarrow a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)=0$$

$$\Leftrightarrow \quad a\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a}-c=\frac{b^2-4ac}{4a}$$

$$\Leftrightarrow \quad x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$\Leftrightarrow \quad x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$