# Course MM1005

## Lecture 2: Introduction to functions, linear and quadratic functions

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# Questions?

# Lecture Goal and Outcome

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**Goal:** Review the concept of one variable function, with special enphasis given to linear and quadratic functions (this is a deepening of concepts you should have seen in the Matte 2b/c classes)

**Learning Outcome:** At the end of the lecture you will be able to solve problem like the following:

**Problem**

A firm produces a commodity and receives \$100 for each unit sold. The cost of producing and selling $x$ units is $20x + 025x^2$ dollars.

(a) Find the production level that maximizes profits.

(b) If a tax of \$10 per unit is imposed, what is the new optimal production level?

(c) Answer the question in (b) if the sales price per unit is p, the total cost of producing and selling $x$ units is $\alpha x + \beta x^2$, and the tax per unit is $\tau$ where $\tau \leq p - \alpha$.

# Why you should care

**Mathematical Modelling** = Describe the real world with the help of **functions**. If you do not understand functions you cannot understand models (or how they are made).

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This is a mathematical model

$$Y = 2{,}262(K)^{0{,}203}(L)^{0{,}763}(1{,}02)^t$$

where

- $Y$ is the national product
- $K$ is the capital invested in stock
- $L$ is the labor
- $t$ is time

# Lecture Plan

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- Rudiments of set theory
- Intervals & Absolute values
- Functions (graphs, linear and quadratic functions).

# Section 1
# Set Theory

# Set Notation

A set is a collection of elements.

**Example**

$$\begin{aligned} P &= \{\text{red, yellow, blue}\} \\ &= \{\text{primary colors}\} \\ &= \{c \mid c \text{ is a primary color}\} \end{aligned}$$

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The symbol $\emptyset$ denotes the set with no element, aka the empty set.

# Subsets

Given two sets $A$ and $B$ we say that $A$ is a subset of $B$ (and we write $A \subseteq B$) if every element of $A$ is also an element of $B$.

## Example

- $$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$$
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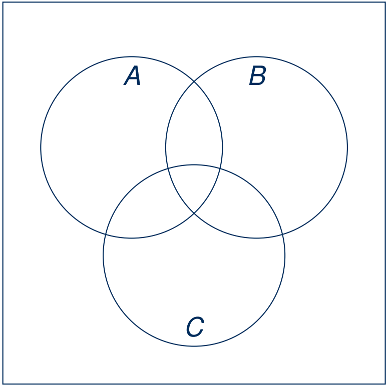
- $$\mathbb{Z} \not\subseteq \mathbb{R}^+$$

If $A$ is a subset of $B$ and $A$ is not equal to $B$, we say that $A$ is a proper subset of $B$, and we write $A \subset B$ or $A \subsetneq B$

# Union

Given two sets $A$ and $B$, their union is

$$A \cup B := \{x \mid x \in A, \text{OR } x \in B\}$$



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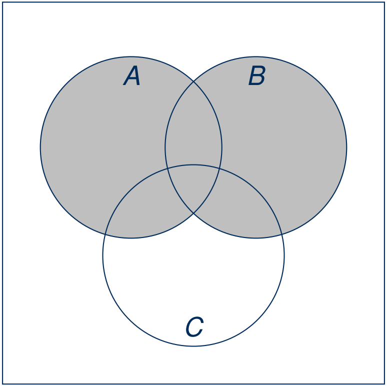

# Intersection

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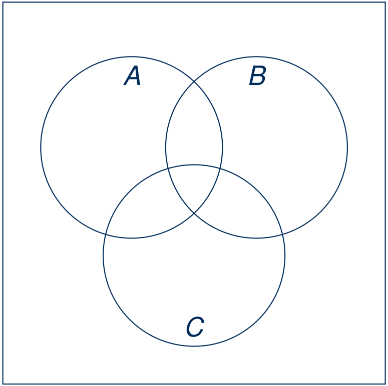

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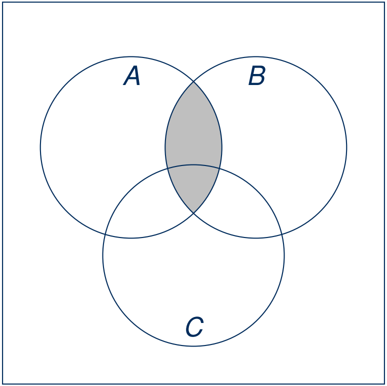

# Excision

Given two sets $A$ and $B$, we have that

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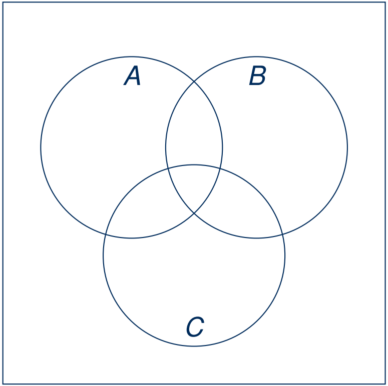

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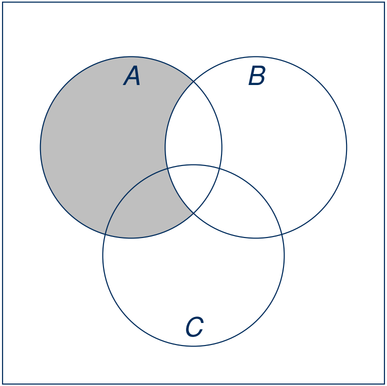
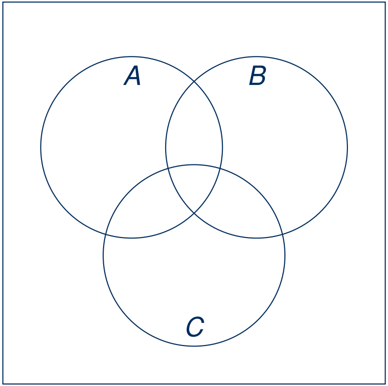


# Complementary

Given a sets $A$ (living inside a universe set $U$), its complementary is

$$A^c := \{x \in U \mid x \notin A\}$$



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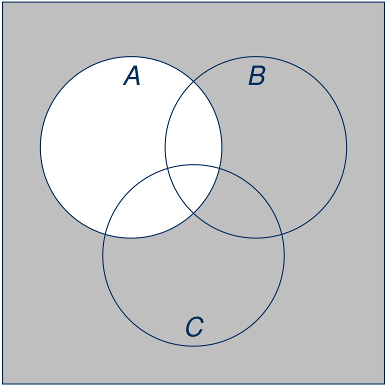

# Questions?

# Section 2
# Intervals and absolute value

# Intervals

Let $a$ and $b$ two real numbers with $a \leq b$. We have the finite intervals with extrema $a$ and $b$

- $[a, b] =$
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We have also infinite intervals

- $[a, +\infty) = \{x \in \mathbb{R} \mid a \leq x\}$
- $(a, +\infty) = \{x \in \mathbb{R} \mid a < x\}$
- $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$
- $(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$
- $(-\infty, +\infty) = \mathbb{R}$

# Examples

- $[-2, 3) \cap (-1, 100] =$
- $(-1, 1)^c =$
- $(-1, 7) \cup (-3, 5) =$

# Absolute value

Given $x$ a real number we have that

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- $|-100| =$
- $|1000| =$
- $|\pi| =$
- Find all the real $x$ such that $|2x - 6| < 1$

# Important example

We can use the absolute value to express rooth of powers:

$$\sqrt{a^2} = |a|$$

# Length

Given two real numbers *a* and *b* their distance is |a-b|.

Given a finite interval $[a, b]$ (or $(a, b]$, or $(a, b)$, or $[a, b)$) we have that its length is

$$|a - b| = b - a$$

# Questions?

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# Section 3
# Functions

# Functions

A function $f$ is an application that assign to any element of a set $D$ (called domain of $f$) a unique element of a set $B$. The range of $f$ is the set of all possible value that $f$ can take.

**Example**

It is very common to view many quantities (population, capital, radioactivity) as functions of time.

- Table of a function.
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$$K(t) = K_0 \left(1 + \frac{p}{100}\right)^t$$

# Natural/Maximal domain

Find the (maximal) domain of the following functions

- $f(x) = (\sqrt{1-x} - 2)^{-1}$
- $g(t) = \sqrt{1 - |t|}$

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Composition of Functions: We apply several functions one after the other.

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## Example

Let $f(x) = x^2$ and $g(x) = 2x + 3$. Compute

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## Example

Let $f(x) = x^2$ and $g(x) = 2x + 3$. Compute

- $f \circ g(x)$
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What do you notice?

# Questions?

# Linear functions

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$$f(x) = ax + b$$

Where $a$ and $b$ are two real numbers.

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- Domain?
- Range?
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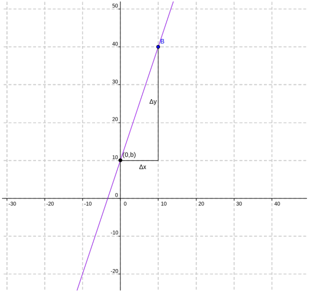


The number $b$ is the $y$-intercept.
The number $a$ is called the slope. We have that

$$a = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$$

# Exercise

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The number of employees at a company is predicted to grow linearly the next 10 years. Last year, there were 100 people at the company, and this year, there are 122. How many are predicted to work at the company 6 years from now?

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The number of employees at a company is predicted to grow linearly the next 10 years. Last year, there were 100 people at the company, and this year, there are 122. How many are predicted to work at the company 6 years from now?

Strategy: find the linear functions that describes the number of employees as a function of time.

# Questions?

# Quadratic functions

$$f(x) = ax^2 + bx + c$$

Where $a$, $b$, and $c$ are real numbers, $a \neq 0$.

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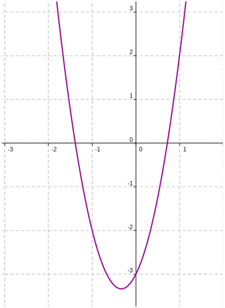
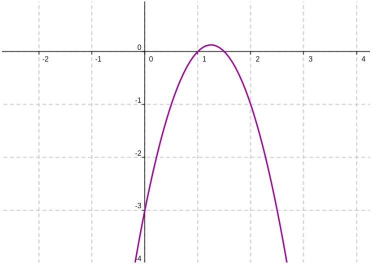


$a > 0$



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# The range of quadratic functions

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$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a},$$

Thus, we have that $f(x) = ax^2 + bx + c$ has

- a minimum point in $x = -\frac{b}{2a}$, when $a > 0$, and
- a maximum point in $x = -\frac{b}{2a}$, when $a < 0$, and

So we have that

$$R_f = \begin{cases} \left\{y \in \mathbb{R} \mid y \geq -\frac{b^2-4ac}{4a}\right\} & \text{if } a > 0 \\ \\ \left\{y \in \mathbb{R} \mid y \leq -\frac{b^2-4ac}{4a}\right\} & \text{if } a < 0 \end{cases}$$

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so we have that $f(x) = ax^2 + bx + x = 0$ if, and only if

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Find the zeroes of $f(x) = x^2 - 3x - 3$

# Back to our problem

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## Problem

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- Find the profit as a function of the units produced (Hint: it will be a quadratic function).
- Find the maximum of the function.

# Questions?

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## Thank you for your attention!